Weak AGM postulates and strong Ramsey Test: A logical formalization

Laura Giordano a, Valentina Gliozzi a,∗, Nicola Olivetti b

a Dipartimento di Informatica, Università del Piemonte Orientale “A. Avogadro”, Via Bellini 25/g, 15100 Alessandria, Italy
b Dipartimento di Informatica, Università di Torino, C.so Svizzera 185, 10149 Torino, Italy
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Abstract

We reformulate AGM postulates for belief revision systems that may contain conditional formulas. We show that we can establish a mapping between belief revision systems and conditionals by means of the so called Ramsey Test, without incurring Gärdenfors triviality result. We then derive the conditional logic BCR from our revision postulates by means of a strong version of the Ramsey Test. We give a sound and complete axiomatization of this logic with respect to its standard selection-function models semantics, and we prove its decidability. We finally show that there is an isomorphism between belief revision systems and selection function models of BCR via a representation theorem. The logic BCR provides a logical formalization of belief revision in the language of conditional logic.

Keywords: AGM postulates; Belief revision systems; Ramsey Test; BCR; Selection function models; Conditional logics

∗ Corresponding author.
E-mail addresses: laura@mfn.unipmn.it (L. Giordano), gliozzi@di.unito.it (V. Gliozzi), olivetti@di.unito.it (N. Olivetti).

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1. Introduction

In the last fifteen years many authors [4,7,13,16,18,21,29] have considered the problem of establishing a correspondence between belief change and conditional logics.

On the one side, belief change concerns the study and the formalization of a crucial capacity of intelligent agents, namely the capacity to change their own beliefs in the face of new information. Belief change has been widely studied in the last twenty years [2,3,7,15,17,33] and several theories have been proposed. The starting point is the work by Alchourrón, Gärdenfors and Makinson in the Eighties [2,7] who proposed a set of postulates for belief revision, universally known as AGM-theory; more recent work includes theories of iterated belief revision by Darwiche and Pearl [3] and by Lehmann [17], and of belief update proposed by Katsuno and Mendelzon in [15]. Given a representation of the belief state of an agent, the various theories define the properties of a belief change operator that maps a belief state and a formula into a new, consistent, belief state.

On the other side, conditional logics were introduced in the Sixties by Stalnaker [34] and Lewis [19], and have received renewed attention in Artificial Intelligence. The reason is that conditional logics provide a formal theory of hypothetical reasoning, which is a kind of reasoning central in many areas of Artificial Intelligence, such as knowledge representation, non-monotonic reasoning, reasoning about actions, planning and natural language analysis. For a review of the existing conditional logics and of their application areas, we refer to Nute [26,27], Ginsberg [11], and Nejdl [24].

Establishing a correspondence between the two research areas is interesting from two points of view. From the point of view of belief change, a correspondence could provide a useful formal tool to study properties of belief change itself. Indeed, conditional logic could provide a language to represent and reason about belief change within the object language, and in this way, it could be used to study properties of classes of belief change operators. From the point of view of conditional reasoning, establishing such a correspondence could provide a semantics for conditional sentences in terms of belief change, capturing in this way an idea by the philosopher F.P. Ramsey [7,12,18,28,34]. In [28] Ramsey proposed the following criterion to decide whether to accept a conditional sentence “if A, then B”: hypothetically add the antecedent A to your stock of beliefs by changing it as little as possible in order to preserve consistency, and then consider whether the consequent B follows; if B follows, accept the conditional, otherwise reject it. This is undoubtedly a very intuitive criterion. The problem is how to formalize it.

A first formalization of a possible relation between conditional sentences and belief change has been proposed by Gärdenfors in [7]. Gärdenfors devises a correspondence between conditional sentences and a specific kind of belief change, namely belief revision which is the kind of belief change that occurs when an agent acquires new information about a “static” world (as opposite to the so-called “belief update” that occurs when an agent changes its stock of beliefs as a consequence of a change in the world [15]).

Gärdenfors’s proposal, known as the Ramsey Test, is formulated as follows:

\[
\text{RT : } A > B \in K \quad \text{if and only if} \quad B \in K \ast A, 
\]

where > is a conditional operator (the meaning of \( A > B \) is “if A, then B”), \( K \) is a belief set (i.e., a deductively closed set of formulas) and \( \ast \) is the belief revision operator that takes
a belief set and a formula and adds the formula to the belief set by minimally changing it in order to preserve consistency.

Unfortunately, Gärdenfors has shown that his formulation leads to a triviality result according to which there is no significant belief revision system compatible with the Ramsey Test.

Gärdenfors’s negative result has stimulated a wide debate and literature aiming to reconcile the two sides of the coin: belief revision and conditional logic. The proposals in this direction can be distinguished into a few categories. Some authors maintain that the Ramsey Test should link conditionals and another kind of belief change, namely belief update [13]. Other authors suggest avoiding the triviality result by adding some preconditions to the Ramsey Test [21,29]. Others, finally, propose to exclude conditional formulas from belief sets, ascribing them a different epistemic status [18].

In this paper, we show that the correspondence postulated by RT can be safely assumed (and the triviality result avoided) if we weaken some postulates for belief revision. We weaken the postulates in the sense that we do not insist that conditional formulas, introduced in the belief sets by the Ramsey Test, must be subject to the same principles that rule the revision of non-conditional beliefs. We show that the triviality result can be avoided by simply putting some restrictions on the revision postulates in order to make a distinction between the postulates that apply to all beliefs (whence also to conditional beliefs), and those that only apply to non-conditional beliefs. More precisely, we show that the triviality result can be avoided by putting some restrictions on the application of the Minimal Change Principle. According to this principle, unnecessary losses of information are to be avoided, and when revising a belief set by a new formula, as much as possible of the old belief set should be preserved.

Our restrictions on the AGM postulates imply that the Minimal Change Principle does not apply to conditional formulas. This is supported by intuition. Learning new information may change our expectations and plausibility judgements about the world. Even if consistent, new information may change our revision strategies and conditional beliefs.

This simple and intuitive limitation on the range of applicability of some postulates allows us to avoid the triviality result. We will see in Section 3.2.3 that this limitation is in accordance with other theories of rational epistemic change, such as theories belonging to the Bayesian tradition [7].

Our weakened revision postulates are compatible not only with (RT) above, but also with a strong version of the Ramsey Test, formulated as follows:

(SRT)

- if $B \in K \ast A$, then $A > B \in K$ and
- if $B \notin K \ast A$, then $\neg(A > B) \in K$.

The conditional logic deriving from the strong Ramsey Test and our weakened revision postulates is a standard conditional logic with a possible world semantics.

The paper is organized as follows: Section 2 recalls the AGM theory of belief revision. Section 3 presents the Ramsey Test, provides an analysis of the triviality result, and introduces our solution. Section 4 defines a notion of conditional belief revision system which
contains weakened AGM postulates. In Section 5 conditional belief revision systems are shown to be compatible with the Ramsey Test, so that there are non-trivial conditional belief revision systems satisfying the Ramsey Test (RT). Section 6 argues in favor of a stronger version of the Ramsey Test (SRT), which also includes negative conditionals. Then Section 7 develops a conditional logic BCR, which derives from the weakened AGM postulates and the Strong Ramsey Test. A sound and complete axiomatization for the logic BCR is provided, and the logic is proved to be decidable. Section 8 presents a Representation Theorem establishing the relation between BCR-models and belief revision. The concluding Section 9 discusses related work.

2. AGM theory of belief revision

The best-known theory of belief revision is the one proposed by Alchourrón, Gärdenfors and Makinson [2], known as AGM theory. AGM theory makes strong assumptions on the nature of the belief structures treated by belief revision: they are deductively closed sets of classical logic formulas, called belief sets.

The revision operator is denoted by \( \ast \). The expansion of a belief set \( K \) by a formula \( A \) is the belief set \( K + A = Cn_{PC}(K \cup \{A\}) \). The operator \( \ast \) satisfies the following postulates:

\begin{enumerate}
  \item \( (K \ast 1) \) \( K \ast A \) is a belief set;
  \item \( (K \ast 2) \) \( A \in K \ast A \);
  \item \( (K \ast 3) \) \( K \ast A \subseteq K + A \);
  \item \( (K \ast 4) \) if \( \neg A \notin K \), then \( K + A \subseteq K \ast A \);
  \item \( (K \ast 5) \) \( K \ast A \vdash_{PC} \bot \) only if \( \vdash_{PC} \neg A \);
  \item \( (K \ast 6) \) if \( A \equiv B \), then \( K \ast A = K \ast B \);
  \item \( (K \ast 7) \) \( K \ast (A \land B) \subseteq (K \ast A) + B \);
  \item \( (K \ast 8) \) if \( \neg B \notin K \ast A \), then \( (K \ast A) + B \subseteq K \ast (A \land B) \).
\end{enumerate}

Postulate \( (K \ast 1) \) claims that the result of the revision of a belief set is a belief set; \( (K \ast 2) \) is a success postulate claiming that the new formula belongs to the belief set obtained; \( (K \ast 3) \) and \( (K \ast 4) \) represent the Minimal Change Principle, according to which if the new formula is consistent with the belief set to be revised, then the result of the revision is the simple addition of the formula to the old belief set, no other change occurs; \( (K \ast 5) \) claims that the result of revision is consistent if the new formula is also consistent; \( (K \ast 6) \) claims that the result of revision does not depend on the syntactic form of the new formula; \( (K \ast 7) \rightarrow (K \ast 8) \) are a generalization of \( (K \ast 3) \rightarrow (K \ast 4) \).

**Definition 2.1 (Belief revision system).** A belief revision system is a pair \((K, \ast)\), where \( K \) is a set of belief sets closed under the revision operator \( \ast \) (thus if \( K \in K \), also \( K \ast A \in K \), for any classical formula \( A \)), and the operator \( \ast \) satisfies postulates \((K \ast 1) \rightarrow (K \ast 8)\).
3. Ramsey Test and triviality result

As mentioned in the Introduction, Gärdenfors suggested that the AGM theory for belief revision could provide an epistemic semantics for conditionals. His proposal aimed to be the formalization of an acceptability criterion for conditionals proposed by F.P. Ramsey. The criterion, which has been very influential in the analysis of conditionals (indeed, also Stalnaker’s logic stems from it), is formulated as follows: accept the conditional “if \( A \) then \( B \)” in your belief state if and only if the minimal change of your belief state needed to accept \( A \) also entails accepting \( B \).

In Gärdenfors’s theory this informal criterion is formalized by the so called Ramsey Test (RT for short):

\[
\text{(RT)}: \quad A > B \in K \iff B \in K * A,
\]

where \( > \) is the conditional operator.

Unfortunately, Gärdenfors’s formulation of the Ramsey Test leads to a well known triviality result, according to which the only belief revision systems compatible with the Ramsey Test are trivial, i.e., satisfy the following definition.

**Definition 3.1 (Trivial belief revision system).** A belief revision system \((K, \ast)\) is trivial if for any three classical formulas \( A, B, C \), pairwise disjoint (i.e., such that \( \vdash_{\text{PC}} \neg (A \land B), \vdash_{\text{PC}} \neg (B \land C), \vdash_{\text{PC}} \neg (A \land C) \)), there is no belief set \( K \in K \) consistent with all of them (such that \( \neg A \notin K, \neg B \notin K, \) and \( \neg C \notin K \)).

Gärdenfors’s negative result claims that:

**Theorem 3.2** (Triviality result [5, p. 85]). *There is no non-trivial belief revision system which satisfies AGM postulates \((K \ast 1)\)–\((K \ast 8)\) and (RT).*

Roughly speaking, trivial belief revision systems can be assimilated to complete belief revision systems, containing only complete belief sets (such that for any formula \( A \), either \( A \) or \( \neg A \) is in the belief set). More precisely, trivial belief revision systems are a generalization of complete belief revision systems. Like complete belief revision systems, trivial belief revision systems do not allow the representation of agents’ incomplete knowledge about the world. For this reason, they are not well suited to represent agents’ belief change in a realistic way.

3.1. Gärdenfors’s analysis of the triviality result

Gärdenfors, and most of the authors that have considered the problem after him, claim that the triviality result derives from the conflict between the Minimal Change Principle that rules belief revision, on the one side, and a direct consequence of the Ramsey Test, called the Monotonicity Principle, on the other side. The two principles can be formulated as follows.
• **Minimal Change Principle**: if a sentence $B$ is accepted in a given state of belief $K$, and if $A$ is consistent with $K$, then $B$ is still accepted in the revision of $K$ by $A$.

• **Monotonicity Principle**: if $K \subseteq K'$, then $K \star A \subseteq K' \star A$.

It can be easily proved that the Monotonicity Principle is a direct consequence of the Ramsey Test: let $K \subseteq K'$, and $B \in K \star A$. By one direction of (RT), $A > B \in K$. Since $K \subseteq K'$, also $A > B \in K'$, and, by the other direction of (RT), $B \in K' \star A$.

According to Gärdenfors’s analysis, in order to avoid triviality, one of the two principles must be abandoned. And, in his view, the Minimal Change Principle is hardly questionable, since it captures the rational principle of informational economy. Therefore, always in his view, the more questionable principle is the Monotonicity Principle.

However, the main weakness of Gärdenfors’s discussion of the two principles is that he overtops the fact that by the Ramsey Test belief sets also contain *conditional* formulas, and not only classical ones. Difficulties arise precisely from the presence of conditional formulas. As we will see in the next section, the Minimal Change Principle is hardly defensible when applied to these formulas.

Similarly, Gärdenfors’s arguments against the Monotonicity Principle do not hit the mark because they address a form of Monotonicity that is much stronger than the one which derives from the Ramsey Test, and which is used in the triviality result. The Monotonicity Principle criticized by Gärdenfors claims that given any two belief sets $K$ and $K'$, included in one another, the revision of the first belief set is included in the revision of the second one. Of course, this form of Monotonicity, when taken as a general property which holds for arbitrary belief sets $K$ and $K'$, is counterintuitive.

On the contrary, the kind of Monotonicity deriving from the Ramsey Test does not apply to any pair of belief sets: it only applies to *conditional* belief sets that contain conditional formulas via the Ramsey rule. In this case, the Monotonicity Principle only claims that whenever all formulas (including the conditional ones) of a belief set $K$ are included in the formulas of a belief set $K'$, then the revision of $K$ is included in the revision of $K'$. This is exactly the meaning that the Ramsey Test gives to conditional formulas: conditional formulas precisely express the ways in which we revise our beliefs in the face of new information. In this more restricted interpretation, the Monotonicity Principle is hardly questionable. If there are two belief sets containing the same conditional formulas, but such that their revision behaves differently, in what sense can we say that there is a correspondence between the evaluation of conditional sentences and belief revision?

The good news is that it is not necessary to choose between the two principles. Rather, as we will see in Section 3.2, triviality can be avoided by putting some limitations on the scope of the Minimal Change Principle, and by dropping a questionable assumption on which the triviality result lies.

As a last point of this section, before we present our analysis of the triviality result, and our solution, we briefly review the solutions to the triviality result proposed in the literature. The proposals can essentially be divided in three groups. The first two groups

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1 To be precise, this is only one half of the Minimal Change Principle, called the Preservation Principle. The Minimal Change Principle also includes the other half, according to which the only formulas accepted in the revision of $K$ by $A$ are those belonging to the deductive closure of $K \cup \{A\}$.
directly derive from Gärdenfors’ analysis of the triviality result in terms of the conflict between the Monotonicity Principle and the Minimal Change Principle. The first group of proposals finds a way out to triviality by wholly giving up the Minimal Change Principle, and thus considering another kind of belief change operator, called belief update, that does not satisfy the principle. To this group belong the proposals by Grahne [13], and Ryan and Schobbens [32]. As we will see, as a difference with this group of proposals, we do not wholly abandon the Minimal Change Principle but, rather, we only impose some restrictions on the scope of the Principle. The second group tries to find a way out of triviality by adopting a formulation of the Ramsey Test that no longer entails the Monotonicity Principle. To this group belong the proposals by Makinson [22] (who considers a nonmonotonic version of the Ramsey Test), and by Rott [29] (who proposes adding some preconditions to the Ramsey Test). It is shown by Gärdenfors [7] that these proposals do not solve the problem of triviality. A third group considers solving the problem of triviality by removing conditionals from belief sets. This is the approach proposed by Levi [18], and Arló Costa [1]. As we will see more extensively in the Conclusions, this approach is related to ours. As a difference, we do not remove conditionals from belief sets but, rather, we put restrictions on some of the revision postulates in such a way that they only apply to the non-conditional part of belief sets. To this third group can be related several, apparently different, proposals, such as Friedman and Halpern’s mapping between conditionals and belief revision [4], and Morreau’s proposal [23]. For a full discussion of related works from the literature, we refer to the final section of the paper (Conclusions and Related Work).

3.2. Our analysis of the triviality result

3.2.1. Gärdenfors’s proof of the triviality result

Let us now consider Gärdenfors’s triviality proof by explicitly stating the steps and assumptions in which it is articulated.

Let \((K, \ast)\) be a belief revision system satisfying the Ramsey Test. Let \(A\), \(B\) and \(C\) be three formulas pairwise disjoint and \(K \in K\) a belief set consistent with \(A\), \(B\) and \(C\).

1. Consider the three belief sets:
   
   \[- K + A;\]
   \[- K + (A \lor B);\]
   \[- K + (A \lor C).\]
   
   These belief sets belong to the belief revision system \((K, \ast)\), as a consequence of the closure of belief revision systems with respect to expansion.

2. Consider now the two belief sets \(K + (A \lor B)\) and \(K + A\). By the properties of expansion, \(K + (A \lor B) \subseteq K + A\). Therefore, the set of conditional formulas in \(K + (A \lor B)\) is included in the set of conditional formulas of \(K + A\) and, by the Monotonicity Principle (following from the Ramsey Test), the revision of \(K + (A \lor B)\) (by any formula) is included in the revision of \(K + A\) (by the same formula). The same holds for belief sets \(K + (A \lor C)\) and \(K + A\).

3. Consider then the revision of the three belief sets at point 1 by the formula \((B \lor C)\).
– Since \((B \lor C)\) is consistent with \(K + (A \lor B)\), by the Minimal Change Principle (postulate \((K * 4)\)), it follows that: \(K + (A \lor B) + (B \lor C) \subseteq K + (A \lor B) \ast (B \lor C)\). Since \(K + (A \lor B) + (B \lor C) = K + B\), it follows that \(B \in K + (A \lor B) \ast (B \lor C)\).

– By the same reasoning, it follows also that \(C \in (K + (A \lor C)) \ast (B \lor C)\).

(4) From step 2, and 3 it follows that both \(B\) and \(C\) are contained in \((K + A) \ast (B \lor C)\). But \(B\) and \(C\) are pairwise disjoint. This contradicts postulate \((K * 5)\) requiring that the revision of a belief set by a consistent formula leads to a consistent belief set.

(5) It follows that the starting hypothesis (that \(K \in K\) is consistent with \(A, B\) and \(C\) pairwise disjoint) cannot hold. Hence, the belief revision system is trivial.

3.2.2. Questioning the assumption according to which belief revision systems are closed with respect to the expansion operator

We have seen that in Gärdenfors’s view the crucial point of the proof lies on the conflict between the Monotonicity Principle and the Minimal Change Principle, respectively at points 2 and 3 of the proof. In our opinion, on the contrary, the more questionable point of the whole proof is the assumption, made at step (1), according to which any belief revision system containing a belief set \(K\), also contains its expansions \(K + A, K + (A \lor B)\) and \(K + (A \lor C)\). More generally, we question the assumption according to which all belief revision systems are closed with respect to the expansion operator. Without this assumption, the whole proof does not work, and triviality is avoided. The assumption, in turn, derives from postulates \((K * 3)\) and \((K * 4)\) capturing the Minimal Change Principle, together with closure under revision.\(^2\)

In this section, we provide some arguments and a simple example to show that closure under expansion in the presence of Ramsey Test conditionals conflicts with revision postulates. In the next section, we provide some intuitive reasons for weakening the Minimal Change Principle and the corresponding revision postulates \((K * 3)\) and \((K * 4)\). The weakened version of the postulates does no longer entail closure under expansion. Since closure under expansion and the Minimal Change Principle are strictly connected, the arguments of this and the next section are related.

Let us consider the following simple example. The example shows that the expansion of a belief set by a formula may lead to a set of formulas that cannot be a belief set in a belief revision system, for its revision would violate revision postulates. Thus, closure under expansion is incompatible with revision postulates.

**Example 3.3.** Let the language \(L\) only contain the propositional variable \(p\). Let

\[
K = CnPC\{\top, p > p, \neg p > \neg p, \top > \top, \\
p > \top, \neg p > \top, \bot > p, \bot > \neg p, \bot > \bot\}.
\]

\(^2\) This can be easily verified: for any \(K\) and \(A\), if \(\neg A \notin K\), by \((K * 3)\) and \((K * 4)\), \(K * A = K + A\). By closure under revision, \(K * A \in K\). Thus \(K + A \in K\).
Notice that $K$ is a legitimate belief set and that the conditional formulas it contains define the revision of $K$ under any formula of the language. In fact, in this language, any propositional formula is equivalent (in PC) to one of the four formulas: $p$, $\neg p$, $\bot$ and $\top$.

Consider now $K + p$. This new set of formulas cannot be a belief set in the belief revision system, as its revision does not satisfy all of the revision postulates. Indeed, by the definition of $+$, $K + p = Cn_{PC}(K \cup p)$. Hence, $K + p$ does not contain the conditional $(p \lor \neg p) > p$ (for $(p \lor \neg p) > p \not\in K$, and $(p \lor \neg p) > p \not\in Cn_{PC}(K \cup p)$). But then, by the Ramsey Test, $p \not\in (K + p) \ast (p \lor \neg p)$, and postulate $(K \ast 4)$ is violated.

The problem, we claim, is that the deductive closure with respect to classical logic (used in defining the notion of expansion) is too weak when dealing with Ramsey Test belief states. Manifestly, if classical logic is well suited to infer how the propositional part of a belief state evolves when acquiring new information, it is not well suited to infer how its conditional part evolves. Roughly speaking, if a conditional belief revision system is closed under expansion, then it contains several belief sets (obtained by the expansion of the same belief set) that differ in their propositional beliefs but agree in their conditional beliefs. By the Ramsey Test, these belief sets are revised in the same way. But this contradicts the obvious fact that, by the revision postulates, the propositional formulas of a belief set are relevant to determine the result of revision.

The triviality result stems from the conflict between closure under expansion, on the one side, and belief revision postulates, on the other side. Closure under expansion and belief revision postulates are compatible only in trivial belief revision systems because in these systems the closure with respect to expansion has essentially no effect (since there are not many formulas consistent with a belief set).

Maintaining that belief revision and belief expansion must lead to different results, even if the new information is consistent with the epistemic state, questions Levi’s identity. According to this identity, belief revision can be described in terms of a combination of contraction and expansion, as follows: $K \ast A = (K - (\neg A)) + A$, where $-$ is a contraction operator defined as in [7]. In case $A$ is consistent with $K$, by the properties of contraction, it holds that $K - (\neg A) = K$, hence Levi’s identity becomes: $K \ast A = K + A$. As we argued above, we do not want this identity for conditional epistemic states. Therefore, if Levi’s identity is well suited for belief sets not containing conditionals, it is no longer suited for conditional epistemic states. This is a consequence of the inappropriateness of expansion to deal with conditional epistemic states. Similar considerations were expressed by Rott in [30] and Morreau in [23].

3.2.3. Questioning the Minimal Change Principle when applied to conditional beliefs

As we have already mentioned, closure under expansion is a consequence of the Minimal Change Principle captured by postulates $(K \ast 3)$, $(K \ast 4)$. Considerations similar to the ones made in the previous section apply to the principle and the postulates.

The postulates, and the principle, require that, whenever new information is consistent with a belief set being revised, the revision of the belief set by the new information con-

3 We only consider unnested conditionals in order to keep the example simple.
sists of a simple expansion of the belief set. The principle, and the postulates, are well suited to dealing with the propositional part of belief sets, because they capture the rational principle of informational economy according to which information is not free and therefore unnecessary loss of information should be avoided. However, both the principle and the postulates are not well suited to rule the change of the conditional part of a belief set. Indeed, they are in contrast with the intuitive fact that the acquisition of new consistent information may lead to change in the revision strategies and hence the conditional beliefs associated to the belief set, as the following examples show.

### 3.2.4. Examples

**Example 3.4.** A very rich woman was murdered last night. Her nephews Spike, Adam and Linda are suspected. Among them, the detective believes that Spike is most probably the murderer, that Adam is a remote but believable possibility and that Linda is probably innocent. If he were to discover that Spike is innocent, he would suspect Adam. His belief set can be represented as follows:

\[
K = \{ \text{Spike, } \neg \text{Spike } > \text{Adam} \}
\]

thus: the murderer is Spike; if it was not Spike, it was Adam.

If the detective now acquired some evidence proving definitely Adam’s innocence, he would have to abandon his conditional belief \( \neg \text{Spike } > \text{Adam} \), and maybe adopt the new conditional belief \( \neg \text{Spike } > \text{Linda} \) (if Adam was not the murderer, then Linda was). Thus, the new belief set obtained by a consistent revision of \( K \) contains different conditional beliefs to the original belief set.

**Example 3.5.** The same woman as in Example 3.4 was murdered last night. This time, the main suspects are Mary and John, the woman’s neighbors. To resolve his doubts about who the murderer is, the detective decides to look for the gun, believing that if the gun is found in John’s room, then John is the culprit, and if it is found in Mary’s room, Mary is the culprit. His belief set can be formalized as follows:

\[
K = \{ (\text{John } \lor \text{Mary}), (\text{gun}_\text{John} > \text{John}), (\text{gun}_\text{Mary} > \text{Mary}) \}.
\]

Suppose now that the gun is found in John’s room, but with Mary’s fingerprints. The detective would conclude that, in spite of the fact that the gun was found in John’s room, Mary is the culprit and therefore abandon his conditional belief \( \text{gun}_\text{John} > \text{John} \).

The moral we can derive from this discussion is that the triviality result could be avoided by giving up the assumption according to which belief revision systems are closed with respect to the expansion operator, and by limiting the action of the Minimal Change Principle to the propositional part of belief sets. We shall see in the next sections that these limitations are indeed sufficient to avoid the triviality result.

As a last point of this section, we would like to show that our intuition is consistent with other theories of epistemic change, such as the Bayesian theories of rational epistemic change. It is shown in Gärdenfors [7] that these theories have a close relation with belief
revision theory: in the specific case in which the new incoming information is consistent with the current epistemic case, the rule of conditionalization, employed in Bayesian theories to describe the probability change, satisfies the properties of belief expansion (of which the properties of belief revision are a generalization). Belief revision theory can thus be seen as an extension of Bayesian theories of rational change in order to cope with the case in which the new information is inconsistent with the current epistemic state.

We can now restate the first example above in Bayesian terms, in order to show that our intuitions are confirmed in this framework. An epistemic state can be represented by a probability function \( p \) (expressing the believer’s confidence in a set of formulas). A formula \( A \) is said to be accepted relative to an epistemic state \( p \) just in case \( p(A) = 1 \). In this framework, conditional beliefs are expressed by conditional probabilities. Thus, the conditional belief that “if \( A \), then \( B \)” can be expressed by the conditional probability \( p(B | A) \), measuring the confidence in \( A \) on the supposition that \( B \) holds.

In the case \( p(B) \neq 0 \), the epistemic state deriving from knowing \( B \) is the probability function \( p_B \), obtained from \( p \) by conditionalization: for all (non-conditional) \( A \), \( p_B(A) = p(A | B) = \frac{p(A \land B)}{p(B)} \). As the following example shows, in general, the acquisition of some new information may entail a change in conditional probabilities.

**Example 3.6.** Let us consider Example 3.4 above. The detective thinks that Spike is most probably the murderer, so we let \( p(Spike) = 0.9 \). Furthermore, the detective thinks that if Spike is not the murderer, then Adam is. So, his belief set can be represented as follows:

\[
K = \{ p(Spike) = 0.9, p(Adam | \neg Spike) = 1 \}.
\]

When acquiring some new evidence definitely proving Adam’s innocence, the detective adds to his belief set the information that \( p(Adam) = 0 \). This leads him to change his conditional probability \( p(Adam | \neg Spike) \) as follows:

\[
\frac{p(Adam \land \neg Spike)}{p(\neg Spike)} = 0.
\]

This example shows that also in a probabilistic framework, learning new information may lead to a change of conditional probabilities. In [20], Lewis provides a formal proof of the fact that the rule of conditionalization (ruling the change of atomic probabilities) cannot be applied to conditional probabilities themselves, on pain of triviality. This result parallels the triviality result of belief revision.

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4 Of course, there are several differences between Bayesian epistemic theories and belief revision theories. For instance, the representation of epistemic states provided by the two theories is different: bayesian theories allow the expression of degrees of belief in a proposition, whereas belief revision theories only allow the expression of whether a proposition is accepted or not. However, the relation that can be established between the two kinds of theories abstracts from these differences, and only concerns the belief change mechanisms.

5 We cannot let \( p(Spike) = 1 \) for, otherwise the conditional probability \( p(Adam | \neg Spike) \) would not be defined.
4. Weakened AGM postulates

Let $\mathcal{L}$ be a propositional language containing the constant $\top$. We define the language $\mathcal{L}_\triangleright^u$ as the unnested conditional extension of $\mathcal{L}$, i.e., the least set of formulas such that

- if $A \in \mathcal{L}$ then $A \in \mathcal{L}_\triangleright^u$;
- if $A \in \mathcal{L}$ and $B \in \mathcal{L}_\triangleright^u$, then $A \triangleright B \in \mathcal{L}_\triangleright^u$;
- if $A, B \in \mathcal{L}_\triangleright^u$, then $\neg A \in \mathcal{L}_\triangleright^u$ and $A \land B \in \mathcal{L}_\triangleright^u$.

We can introduce the other connectives $\rightarrow$, $\lor$, $\leftrightarrow$ by the standard propositional equivalences. Let $S$ be a set of formulas of $\mathcal{L}_\triangleright^u$, we define $Cn_{PC}(S) = \{A \in \mathcal{L}_\triangleright^u \text{ s.t. } S \models_{PC} A\}$; the relation $S \models_{PC} A$ means that $A$ is a propositional consequence of $S$, where conditional formulas are treated as propositional atoms.

We call an epistemic state any set of formulas of $\mathcal{L}_\triangleright^u$ that is deductively closed with respect to $Cn_{PC}$. A belief set is a deductively closed set of formulas of $\mathcal{L}$. We introduce the belief function $[\ ]$ that associates to each epistemic state $K$ its corresponding belief set

$$[K] = K \cap \mathcal{L}.$$  

A revision operator $*$ is any function that takes an epistemic state and a formula in $\mathcal{L}$ as input, and gives an epistemic state as output. The expansion of an epistemic state $K$ by a formula $A$ is the set $K + A = Cn_{PC}(K \cup \{A\})$.

**Definition 4.1** (Conditional belief revision system). A conditional belief revision system is a pair $(K, *)$, where $K$ is a set of epistemic states closed under the revision operator $*$. The operator $*$ satisfies the postulates:

$$(B \ast 1) \quad K \ast A \text{ is an epistemic state};$$
$$(B \ast 2) \quad A \in K \ast A;$$
$$(B \ast 3) \quad [K \ast A] \subseteq [K + A];$$
$$(B \ast 4) \text{ if } \neg A \notin [K], \text{ then } [K + A] \subseteq [K \ast A];$$
$$(B \ast 5) \quad K \ast A \vdash_{PC} \bot \text{ only if } \vdash_{PC} \neg A;$$
$$(B \ast 6) \text{ if } A \equiv B, \text{ then } K \ast A = K \ast B;$$
$$(B \ast 7) \quad [K \ast (A \land B)] \subseteq [(K \ast A) + B];$$
$$(B \ast 8) \text{ if } \neg B \notin [K \ast A], \text{ then } [(K \ast A) + B] \subseteq [K \ast (A \land B)];$$
$$(B \ast \top) \text{ for any } K \text{ consistent, } K \ast \top = K.$$  

Postulates $(B \ast 1)$, $(B \ast 2)$, $(B \ast 5)$, $(B \ast 6)$ are AGM postulates $(K \ast 1)$, $(K \ast 2)$, $(K \ast 5)$, $(K \ast 6)$. Postulates $(B \ast 3)$, $(B \ast 4)$, $(B \ast 7)$, $(B \ast 8)$ are the restriction of AGM postulates $(K \ast 3)$, $(K \ast 4)$, $(K \ast 7)$, $(K \ast 8)$ to belief sets. Postulates $(K \ast 3)$, $(K \ast 4)$ represent the Minimal Change Principle. The restriction of $(K \ast 3)$, $(K \ast 4)$ means that the Minimal Change Principle is restricted to the non-conditional part of epistemic states. A similar remark applies to $(K \ast 7)$ and $(K \ast 8)$ which are a generalization of $(K \ast 3)$, $(K \ast 4)$.$^6$ These restrictions are essentially the same introduced by Darwiche and Pearl

$^6$ Given $(B \ast \top)$, $(K \ast 3)$, $(K \ast 4)$ derive respectively from $(K \ast 7)$, $(K \ast 8)$ by taking $A = \top$. 
in [3] where they extend AGM theory for dealing with iterated belief revision. Postulate 
\((B \ast \top)\) expresses a rather intuitive property of revision, according to which the revision
of an epistemic state with a tautology \(\top\) does not affect the epistemic state. It comes for
free from the original AGM postulates \((K \ast 3), (K \ast 4)\); we have introduced it as we can
no longer derive it from our corresponding \((B \ast 3), (B \ast 4)\).

The restrictions we have put on AGM postulates have an impact on the closure proper-
ties of conditional belief revision systems. We have seen that AGM postulates entail that
belief revision systems are closed with respect to expansion (i.e., if \(K\) is the set of all the
epistemic states of a conditional belief revision system, then for any epistemic state \(K \in K\),
also \(K + A \in K\)). As we have seen, this property follows immediately from \((K \ast 3), (K \ast 4)\)
together with the closure with respect to the revision operator. In contrast, this property
cannot be derived from our modified postulates. In our setting, the closure with respect to
revision does not imply the closure with respect to expansion. As we have argued in the
introduction, the revision of an epistemic state with a formula \(A\), even when \(A\) is consistent
with the state, may affect the conditional formulas holding in that state in a way that is not
reflected by the simple expansion operation. Our postulates entail a weaker property: given
\(K\) and \(A\) of a belief revision system \(\langle K, \ast \rangle\), \(K\) contains an epistemic state \(K'\), such that
\([K + A]\) is the belief set associated with \(K'\).

5. Non-triviality of conditional belief revision systems

In this section, we show that our weakened AGM postulates \((B \ast 1)–(B \ast \top)\) are compat-
ible with the Ramsey Test. More precisely, we show that there are non-trivial conditional
belief revision systems satisfying \((RT)\).

The notion of non-trivial belief revision system can be extended straightforwardly to
conditional belief revision systems.

**Definition 5.1** (Non-trivial conditional belief revision system). A conditional belief revi-
sion system \(\langle K, \ast \rangle\) is non-trivial if there are three formulas \(A, B, C\) in \(L\), which are
pairwise disjoint (i.e., such that \(\vdash PC \neg (A \land B), \vdash PC \neg (B \land C), \vdash PC \neg (A \land C)\)), and an
epistemic state \(K \in K\) such that \(\neg A \notin [K], \neg B \notin [K], \) and \(\neg C \notin [K]\).

We show that to any AGM belief revision system we can associate a conditional belief
revision system that is belief conservative with respect to it, where:

**Definition 5.2** (Belief conservativity). A conditional belief revision system \(\langle K', \ast' \rangle\) is belief
conservative with respect to an AGM belief revision system \(\langle K, \ast \rangle\) just in case for any
\(K \in K\) there is a \(K'\) in \(K'\) such that \(K = [K']\).

Non-triviality follows from the fact that non-trivial AGM belief revision systems give
rise to non-trivial revision systems satisfying \((B \ast 1)–(B \ast \top), (RT)\).

In short, to any belief set of an AGM belief revision system \(\langle K, \ast \rangle\) we can associate an
episodic state that is obtained by extending the belief set by conditional formulas of \(L_u^o\).
We then collect all the epistemic states built from the belief sets of \(K\), and we define a new
belief revision operator on epistemic states. Finally, we show that the structure so obtained is belief conservative with respect to \((K, \ast)\) and satisfies \((B \ast 1)-(B \ast \top)\), \((RT)\).

Let \((nl)\) be the nesting level of a formula in \(L_u^\omega\), defined as: \(nl(A) = 0\) if \(A \in \mathcal{L}\); \(nl(A \land B) = \max(nl(A), nl(B))\); \(nl(\neg A) = nl(A)\); \(nl(A > B) = 1 + nl(B)\). By using the nesting levels just defined, we define a sublanguage of \(L_u^\omega\) ordered by increasing complexity of the formulas contained: the sublanguage \(L_0^\omega\) will contain only the formulas of \(L_u^\omega\) with nesting level \(0\), the sublanguage \(L_1^\omega\) will contain only the formulas of \(L_u^\omega\) with nesting level \(\leq 1\) and so on. Formally,

- \(L_0^\omega = \{A \in L_u^\omega : nl(A) = 0\}\);
- \(L_i^\omega = \{A \in L_u^\omega : nl(A) \leq i\}\).

Clearly, it holds that \(L_0^\omega \subseteq \cdots \subseteq L_{i-1}^\omega \subseteq L_i^\omega\).

For all belief sets \(K\) of \(K\), we define

- \(\phi(K, 0) = K\);
- \(\phi(K, i) = Cn_{PC}(\phi(K, i-1) \cup \{A > B : A > B \in L_i^\omega \land B \in \phi(K \ast A, i-1)\})\).

Finally, we let:

- \(\phi(K) = \bigcup_i \phi(K, i)\).

We define a conditional belief revision system by letting

- \(K' = \{\phi(K) : K \in K\}\);
- \(\phi(K) \ast' A = \phi(K \ast A)\).

**Proposition 5.3.** \((K', \ast')\) is belief conservative with respect to \((K, \ast)\).

**Proof.** We show that for all \(K \in K\), \(K = [\phi(K)]\).

\(\Rightarrow\) By definition, \(\phi(K, 0) = K\). Therefore, \(K \subseteq \phi(K)\). Furthermore, \(K \subseteq \mathcal{L}\), and \(K \subseteq [\phi(K)]\).

\(\Leftarrow\) We show that for any \(A \in \mathcal{L}\), if \(A \in \phi(K, i + 1)\), then \(A \in \phi(K, i)\). Suppose, reasoning by absurdity, that it is not so, and that \(A \in \phi(K, i + 1)\) but \(A \notin \phi(K, i)\). Then there are \(E_0 \ldots E_i \in \phi(K, i + 1) \setminus \phi(K, i)\), with \(E_0 \ldots E_i \in \mathcal{L}_{i+1}\), and \(D_0 \ldots D_j \in \phi(K, i)\) such that \(E_0 \ldots E_i, D_0 \ldots D_j \vdash_{PC} A\) but \(D_0 \ldots D_j \not\vdash_{PC} A\). However, this is not possible, since \(E_1 \ldots E_j\) can be considered as atoms as far as propositional derivability is concerned, and they do not occur neither in \(A\) nor in \(D_0 \ldots D_i\). Therefore, for any \(A \in \mathcal{L}\), if \(A \in \phi(K, i + 1)\), then \(A \in \phi(K, i)\). Hence, \(A \in \phi(K, 0) = K\). \(\square\)

**Proposition 5.4.** \((K', \ast')\) satisfies postulates \((B \ast 1)-(B \ast \top)\).

**Proof.** Let \(\phi(K) + A = Cn_{PC}(\phi(K) \cup \{A\})\).
(B * 1) By the definition of $\phi(K)$, $\phi(K) \subseteq L^0_\omega$. Furthermore, we show that for all $D_1 \ldots D_n$ and all $A \in L^0_\omega$, if $D_1 \ldots D_n \vdash_{PC} A$, and $D_1 \ldots D_n \in \phi(K)$, then $A \in \phi(K)$. This follows by the fact that if $D_1 \ldots D_n \in \phi(K)$, then there is an $i$ such that $D_1 \ldots D_n \in \phi(K, i)$. By definition, $\phi(K, i)$ is deductively closed.

(B * 2) By the properties of $\ast$. $A \in \phi(K \ast A)$. Therefore, $A \in \phi(K \ast A, 0)$, and $A \in \phi(K \ast A)$. By definition of $\ast'$, $A \in \phi(K) \ast' A$.

(B * 3) By Proposition 5.3 [\phi(K \ast A)] = K \ast A$, and by the definition of $\ast'$, also [\phi(K) \ast' A] = K \ast A$. We now show that $\phi(K) + A = K + A$. First of all, $\phi(K) + A = Cn_{PC}(\phi(K) \cup \{A\}) \cap \mathcal{L}$. Therefore, $\phi(K) + A$ is the set of non-conditional formulas that can be obtained by the propositional calculus from $\phi(K)$ and $A$. By considerations similar to the ones made in the proof of Proposition 5.3, we know that these are the formulas that can be obtained through the propositional calculus from $A$ and the non-conditional part of $\phi(K)$. Since the non-conditional part of $\phi(K)$ is $K$, it follows that $\phi(K) + A = Cn_{PC}(K \cup \{A\})$, and $\phi(K) + A = K + A$. (B * 3) follows from the identities just shown and from $(K \ast 3)$.

(B * 4) If $-A \notin \phi(K)$, then $-A \notin K$ (by Proposition 5.3). By $(K \ast 4)$, $K + A \subseteq K \ast A$. Moreover, we have shown just above that $[\phi(K) + A] = K + A$ and $[\phi(K) \ast' A] = K \ast A$, and we conclude that $[\phi(K) + A] \subseteq [\phi(K) \ast' A]$.

(B * 5) If $\phi(K)$ is inconsistent, then there is an $i$ such that $\phi(K, i)$ is inconsistent. But if $\phi(K, i)$ is inconsistent, then also $\phi(K, i - 1)$ is inconsistent. Indeed, suppose $\phi(K, i) \vdash_{PC} \bot$ and $\phi(K, i - 1) \vdash \bot$. Then, by construction of $\phi(K, i)$, there must be a formula $A > B \in L_i$ such that $A > B \in \phi(K, i)$ and $-A > B \in \phi(K, i)$. But this is impossible, since only positive conditionals are inserted at each step. Therefore, if $\phi(K, i)$ is inconsistent, this only be because $\phi(K, i - 1)$ is inconsistent. We can repeat the reasoning until $\phi(K, 0) = K$. It follows that if $\phi(K) \vdash_{PC} \bot$, then $K \vdash_{PC} [\phi(K)] \bot$. Consider now $\phi(K) \ast' A = \phi(K \ast A)$. For what just shown, if $\phi(K) \ast' A \vdash_{PC} \bot$, then $K \ast A \vdash_{PC} \bot$, and by $(K \ast 5) \vdash_{PC} -A$.

(B * 6) By $(K \ast 6)$, if $A \equiv B$, then $K \ast A = K \ast B$, therefore $\phi(K \ast A) = \phi(K \ast B)$ and $\phi(K) \ast' A = \phi(K) \ast' B$.

(B * 7) From Proposition 5.3 $[\phi(K \ast (A \land B))] = K \ast (A \land B)$ and by the considerations made in the proof of $(B \ast 3)$, we can derive that $[\phi(K \ast A) + B] = (K \ast A) + B$. By $(K \ast 7)$, $K \ast (A \land B) \subseteq (K \ast A) + B$ from which we conclude that $[\phi(K \ast (A \land B))] \subseteq [\phi(K \ast A) + B]$.

(B * 8) If $-B \notin [\phi(K \ast A)]$, then by Fact 1, $-A \notin (K \ast A)$ and by $(K \ast 8)$, $(K \ast A) + B \subseteq K \ast (A \land B)$. Since $[\phi(K \ast A) + B] = (K \ast A) + B$ (as shown in the proof of $(B \ast 3)$), and $[\phi(K \ast (A \land B))] = K \ast (A \land B)$, it follows that $[\phi(K \ast A) + B] \subseteq [\phi(K \ast (A \land B))]$.

(B * 9) By $(K \ast 3)$ and $(K \ast 4)$, if $-\top \notin K$ (i.e., if $K$ is consistent), then $K = K + \top = K \ast \top$. It follows, by the construction of $\phi(K)$, that for any $K$ consistent, $\phi(K) = \phi(K \ast \top) = \phi(K) \ast' \top$. □

Proposition 5.5. $(K', \ast')$ satisfies $(RT)$.

Proof. If $B \in \phi(K) \ast' A$ then $B \in \phi(K \ast A)$, and $B \in \phi(K \ast A, i)$, for some $i$. By the definition of $\phi(K, i)$, it follows that $A > B \in \phi(K, i)$ and therefore $A > B \in \phi(K)$.
If $B \notin \phi(K) \ast A$, then $B \notin \phi(K \ast A)$ and by construction, $A \triangleright B \notin \phi(K)$. □

Putting together the previous propositions we obtain:

**Theorem 5.6.** Let $\langle K, \ast \rangle$ be an AGM revision system, then there is a conditional belief revision system $\langle K', \ast' \rangle$ that is belief conservative with respect to $\langle K, \ast \rangle$.

We can therefore conclude that:

**Theorem 5.7.** There is a non-trivial conditional belief revision system.

**Proof.** If a *conditional* belief revision system is belief conservative with respect to a non-trivial AGM belief revision system, then it is non-trivial. Therefore by Theorem 5.6 it is sufficient to show that there is a non-trivial AGM revision system. To this end, let the language $\mathcal{L}$ contain only the propositional variables $p_1, p_2, p_3, p_4$. Let $A = \neg p_2 \land \neg p_3 \land p_4$; $B = \neg p_3 \land \neg p_4 \land p_2$; $C = \neg p_2 \land \neg p_4 \land p_3$.

Clearly, $\vdash_{PC} \neg (A \land B)$, $\vdash_{PC} \neg (A \land C)$ and $\vdash_{PC} \neg (B \land C)$. Consider now an AGM belief revision system $\langle K, \ast \rangle$ such that (i) $\ast$ is the full meet revision operator (so that if $\neg A \notin K$, then $K \ast A = Cn_{PC}(K \cup \{A\})$, if $\neg A \in K$, then $K \ast A = Cn(A)$), (ii) it contains a belief set $K = Cn_{PC}(p_1) \in K$. An AGM system satisfying both (i) and (ii) exist, since $K$ can be obtained by successively revising any belief set first by $\neg p_1$ and then by $p_1$. It is non-trivial, since $K + A, K + B$ and $K + C$ are consistent. □

6. Strong Ramsey Test

Our weakened postulates for belief revision are also consistent with a stronger version of the Ramsey Test, namely:

(SRT)

- if $B \in K \ast A$, then $A \triangleright B \in K$ and
- if $B \notin K \ast A$, then $\neg (A \triangleright B) \in K$.

Indeed, the construction of $\phi(K)$ made in the previous section can be extended in order to include also negative conditional beliefs as follows.

- $\phi(K, 0) = K$;
- $\phi(K, 1) = Cn_{PC}(\phi(K, 0)) \cup \{A \triangleright B: A \triangleright B \in \mathcal{L}_1 \text{ and } B \in \phi(K \ast A, 0)\} \cup \{\neg (A \triangleright B): A \triangleright B \in \mathcal{L}_1 \text{ and } B \notin \phi(K \ast A, 0)\}$;
- $\phi(K, i + 1) = Cn_{PC}(\phi(K, i)) \cup \{A \triangleright B: A \triangleright B \in \mathcal{L}_{i+1} \text{ and } B \in \phi(K \ast A, i)\} \cup \{\neg (A \triangleright B): A \triangleright B \in \mathcal{L}_{i+1} \text{ and } B \notin \phi(K \ast A, i)\}$.

Finally, we let:
• \( \phi(K) = \bigcup_i \phi(K, i) \).

We can then define a \textit{conditional} belief revision system by letting

- \( K' = \{ \phi(K) : K \in K \} \);
- \( \phi(K) \ast' A = \phi(K \ast A) \).

We can show, similarly to what we have done in the previous Section 5.3, that:

**Proposition 6.1.** \( (K', \ast') \) is belief conservative with respect to \( (K, \ast) \).

In the following we shall study the conditional logic which stems from our weakened rationality postulates and the strong version of the Ramsey Test.

Our choice in favor of the Strong Ramsey Test is motivated by the fact that, when conditionals are introduced in epistemic states, their acceptance in a state \( K \) is determined by the behaviour of the revision operator \( \ast \), which is completely defined for a fixed belief revision system \( (K, \ast) \). In fact, for a given \( K \in K \) and \( A, B \in \mathcal{L} \), we have that either \( B \in K \ast A \) or \( B \notin K \ast A \). There is not a third choice. On the other hand, if we accept the weaker version of the Ramsey Test \( (RT) \) given above, it may occur that neither \( A > B \in K \) nor \( \neg(A > B) \in K \), that is, the conditional \( A > B \) is undetermined in the epistemic state \( K \).

However, we can notice that this is not a real nondeterminacy: given the fact that \( A > B \notin K \), according to \( (RT) \) it must be that \( B \notin K \ast A \), so that the nondeterminacy on \( A > B \) can only be resolved by adding \( \neg(A > B) \) to \( K \) (as advocated by the strong version of the Ramsey Test), and not by adding \( A > B \) to \( K \). In such a case, adding \( A > B \) to \( K \) would violate both the weak and the strong versions of the Ramsey test.

We believe that allowing an incomplete specification of Ramsey conditionals (so that neither \( A > B \) nor \( \neg(A > B) \) belong to \( K \)) can be meaningful if the revision operator is incompletely specified, so that we do not know whether \( B \in K \ast A \) or \( B \notin K \ast A \), but that it is inappropriate when \( \ast \) is given and completely determined. This incompleteness of epistemic states with respect to the conditional formulas has already been questioned by Friedman and Halpern in [4].

In essence, if we accept the weak version of the Ramsey Rule \( (RT) \) we are faced with a mismatch. On one side of the equivalence, two cases are possible:

- either \( B \in K \ast A \) or \( B \notin K \ast A \).

On the other side of the equivalence, three cases are possible:

- \( A > B \in K \) and \( \neg(A > B) \notin K \),
- \( A > B \notin K \) and \( \neg(A > B) \in K \),
- \( A > B \notin K \) and \( \neg(A > B) \notin K \).

Of the last three cases, the first one corresponds to the case \( B \in K \ast A \). The second one corresponds to the case \( B \notin K \ast A \). What about the third case? In the third case, as in the second one, it must be that \( B \notin K \ast A \), otherwise \( (RT) \) would be violated. Therefore, it is
not clear why one should make a distinction between three cases by using the conditional formulas, while only two cases are actually possible for the revision operator.

A strong version of the Ramsey Test, which deals also with negated conditionals, has been proposed by Levi [18], and Arló Costa [1] by defining the acceptability of both the positive and negative conditionals. However, as a difference from the rule (SRT), conditional sentences are excluded from belief sets in Levi’s and Arló Costa’s proposal.

7. The logic

In this section, we consider the conditional logic that derives from the Strong Ramsey Test and from our weakened postulates. Valid formulas are just those that belong to every epistemic state of every conditional revision system.\footnote{In this respect, our notion of validity is less strict than the one adopted by Gärdenfors in [7], according to which a formula $A$ is valid if $\neg A$ does not belong to any epistemic state of any belief revision system.}

\textbf{Definition 7.1 (Revision valid formulas).} A formula $A \in \mathcal{L}_u$ is revision-valid if for all revision systems $(\mathbf{K}, *)$, for all $K \in \mathbf{K}$ we have $A \in K$.

7.1. Axiomatization

The logic resulting from Definition 7.1, from the postulates $(B * 1) - (B * \top)$ and from (SRT) is defined as follows.

\textbf{Definition 7.2 (BCR).} The logic BCR is the smallest logic containing the following axioms and deduction rules:\footnote{In the following, when omitting parentheses, we shall assume that $>$ is more binding than $\rightarrow$.}

- (CLASS) All classical propositional axioms and inference rules;
- (CONS) all formulas $\neg(A > \bot)$ such that $A \in \mathcal{L}$ and $\nvdash_{PC} \neg A$;
- (ID) $A > A$;
- (DT) $(A \land C) > B \rightarrow A > (C \rightarrow B)$, where $B \in \mathcal{L}$;
- (CV) $\neg(A > \neg C) \land (A > B)) \rightarrow (A \land C) > B$ where $B \in \mathcal{L}$;
- (BEL) $(A > B) \rightarrow \top > (A > B)$;
- (REFL) $(\top > A) \rightarrow A$;
- (TRANS) $(A > B) \rightarrow A > (\top > B)$;
- (EUC) $\neg(A > B) \rightarrow (A > \neg(\top > B))$;
- (RCEA) if $\vdash A \leftrightarrow B$, then $\vdash (A > C) \leftrightarrow (B > C)$;
- (RCK) if $\vdash (A_1 \land \cdots \land A_n) \rightarrow B$, then $\vdash ((C > A_1) \land \cdots \land (C > A_n)) \rightarrow (C > B)$.

The logic BCR has strong similarities with Stalnaker’s logic C2. First, from (DT) and (REFL) we can derive (MP) $(A > B) \rightarrow (A \rightarrow B)$ restricted to the case of $A, B \in \mathcal{L}$. Moreover, if we assume the axiom $A \rightarrow (\top > A)$, from (CV) we derive (CS) $(A \land B) \rightarrow (A > B)$ again restricted to the case of $A, B \in \mathcal{L}$, from (EUC) we derive the usual (CEM)
\((A > B) \lor (A > \neg B)\) and axioms (TRANS) and (BEL) become tautological. Note that all axioms (ID), (CV), (MOD), (MP), (CS) and (CEM) belong to the axiomatization of Stalnaker’s logic \(C2\) (see [27]). A conditional formula \(\top > A\) can be regarded as a belief operator meaning that “\(A\) is believed”. Moreover, axioms (REFL), (EUC) and (TRANS) (the last two ones for \(A = \top\)) make the belief operator an S5 modality.

We shall see that the theorems of BCR are exactly the revision valid formulas. We first show that revision-validity is a conservative extension of the classical propositional calculus, this property is not entirely evident because of axioms (CONS).

**Lemma 7.3.** If \(A \in \mathcal{L}\) and \(A\) is revision-valid then \(\vdash_{PC} A\).

**Proof.** Using Theorem 5.6, we can easily see that there exists a conditional belief revision system \(\langle K, * \rangle\) containing a \(K \in K\) such that \([K] = \{ C \in \mathcal{L} : \vdash_{PC} C \} \).

By revision validity, we have that \(A \in [K]\). Thus \(\vdash_{PC} A\). \(\square\)

**Theorem 1.** All the theorems of BCR are revision-valid.

**Proof.** We check that given any conditional revision system \(\langle K, * \rangle\) and any \(K \in K\), if \(A\) is a theorem of BCR, then \(A \in K\). We recall that \(K\) is complete with respect to conditional formulas, i.e., for any \(A > B \in \mathcal{L}_C\) we have either \(A > B \in K\) or \(\neg(A > B) \in K\). By this fact, when we have to show that \(F \rightarrow G \in K\), where \(F\) is a boolean combination of conditional formulas, we can limit our consideration to the case when \(F \in K\), for, otherwise, we would have \(\neg F \in K\) and hence trivially \(F \rightarrow G \in K\) by deductive closure. Moreover, always by deductive closure, if we show that \(G \in K\), we have also shown that \(F \rightarrow G \in K\).

These steps are tacitly applied in the following proofs.

- (CLASS) All epistemic states are closed with respect to the propositional calculus by definition. They therefore include all tautologies of propositional calculus and they are closed with respect to propositional inference rules.
- (CONS) Let \(A \in \mathcal{L}\) suppose that \(\not\vdash \neg A\) then by (\(B * 5\)) \(\bot \not\in K * A\). By (SRT) we have \(\neg(A > \bot) \in K\).
- (ID) By (\(B * 2\)) \(A \in K * A\); thus by (SRT) \(A > A \in K\).
- (DT) Let \(A \land C > B \in K\), with \(B \in \mathcal{L}\); we have \(B \in K * (A \land C)\) by (SRT). Since \(B \in \mathcal{L}\), we have \(B \in [K * (A \land C)]\). By (\(B * 7\)), we have \(B \in [(K * A) + C]\). Thus \(B \in (K * A) + C\). But this implies that \(C \rightarrow B \in K * A\), so that by (SRT) \(A > (C \rightarrow B) \in K\).
- (CV) Assume that \(\neg(A > \neg C) \land (A > B) \in K\), with \(B \in \mathcal{L}\); then it must be \(\neg(A > \neg C) \in K\) and \((A > B) \in K\). Assume that \(K\) is consistent, for otherwise there is nothing to prove. Then \(A > \neg C \not\in K\). By (SRT) we have \(\neg C \not\in K * A\) and \(B \in K * A\). Since \(\neg C, B \in \mathcal{L}\), we have \(\neg C \not\in [K * A]\) and \(B \in [K * A]\). By (\(B * 8\)) \([(K * A) + C] \subseteq [K * (A \land C)]\). Since \(B \in [K * A]\), we also have \(B \in [(K * A) + C]\), thus \(B \in [K * (A \land C)]\). But this implies \(B \in K * (A \land C)\), thus by (SRT) we have \(A \land C > B \in K\).
- (BEL) Let \(A > B \in K\). By (\(B * \top\)), we have \(K = K * \top\). Thus \(A > B \in K * \top\). By (SRT) \(\top > (A > B) \in K\).
• (REFL) Let $\top > A \in K$. By (SRT) and $(B \star \top)$ we have $A \in K \star \top = K$. Thus $A \in K$.

• (EUC) Let $\neg (A > B) \in K$. Assume that $K$ is consistent, for otherwise there is nothing to prove. Then $A > B \notin K$. By (SRT) we have $B \notin K \star A$. By (SRT) we obtain $\neg (\top > A) \in K \star A$ and by (SRT) again $A > \neg (\top > B) \in K$.

• (TRANS) Let $A > B \in K$. By (SRT) we have $B \in K \star A$. By $(B \star \top)$, we have $B \in K \star A \star \top$. By (SRT) we obtain $\neg (\top > B) \in K$.

• (RCEA) Suppose that $A \leftrightarrow B$ is revision-valid, where $A, B \in L$. By the previous lemma we have, $\vdash_{PC} A \leftrightarrow B$. Let $A > C \in K$; by (SRT) we have $C \in K \star A$. By $(B \star 6)$, we have $C \in K \star B$; thus $B > C \in K$ by (SRT). The other half is symmetrical.

• (RCK) Let $(A_1 \land \cdots \land A_n) \rightarrow B$ be revision valid and $C > A_1 \in K, \ldots, C > A_n \in K$. By revision validity, we have $A_1 \land \cdots \land A_n \rightarrow B \in K \star C$. By (SRT) we have that $A_1, \ldots, A_n \in K \star C$. By deductive closure, we have $B \in K \star C$, so that $C > B \in K$ by (SRT). $\square$

The following corollary shows that BCR is a conservative extension of classical logic.

**Corollary 7.4.** Let $A \in L$, if $A$ is a theorem of BCR then $\vdash_{PC} A$.

### 7.2. Semantics

We show that BCR has a standard semantics in terms of selection function models for conditional logics. We then show a precise correspondence between revision systems and models of this logic. As a byproduct, we obtain the converse of the previous theorem, that is to say that only BCR theorems are valid with respect to conditional belief revision systems.

**Definition 7.5.** A BCR-structure $M$ has the form $(W, f, [[[ ]]])$, where $W$ is a non-empty set, whose elements are called possible worlds; $f$, called the selection function, is a function of type $L \times W \rightarrow 2^W$ that, given a formula $A$ and a possible world $w$, selects the most preferred $A$-worlds (with respect to $w$); and $[[ ]]$, called the evaluation function, is a function of type $\text{Var} \rightarrow 2^W$. The evaluation function $[[ ]]$ is extended to every $L_\uparrow$-formula by stipulating:

1. $[[A \land B]] = [[A]] \cap [[B]]$;
2. $[[\neg A]] = W - [[A]]$;
3. $[[A > B]] = \{w : f(A, w) \subseteq [[B]]\}$.

For $S \subseteq W$, we define $\text{Form}(S) = \{A \in L_\uparrow^w \mid S \subseteq [[A]]\}$ and $\text{Prop}(S) = \text{Form}(S) \cap L$. We assume that the selection function $f$ satisfies the following properties:

(S-ID) $f(A, w) \subseteq [[A]]$;

1. Using the standard boolean equivalences, we obtain $[[A \lor B]] = [[A]] \cup [[B]]$, $[[A \rightarrow B]] = (W - [[A]]) \cup [[B]]$, $[[\top]] = W$, $[[\bot]] = \emptyset$. 
For any $A, B \in L$, if $[[A]] = [[B]]$ then $f(A, w) = f(B, w)$.

Proposition (DT) $\text{Prop}(f(A \land C, w)) \subseteq \text{Prop}(f(A, w) \cap [[C]])$.

Proposition (CV) $f(A, w) \cap [[C]] \neq \emptyset \rightarrow \text{Prop}(f(A, w)) \subseteq \text{Prop}(f(A \land C, w))$.

Proposition (REFL) $w \in f(\top, w)$.

Proposition (TRANS) $x \in f(A, w) \land y \in f(\top, x) \rightarrow y \in f(A, w)$.

Proposition (BEL) $w \in f(\top, y) \rightarrow f(A, w) = f(A, y)$.

Proposition (CONS) For any $A \in L$, if $\not\vdash_{PC} \neg A$, then $f(A, w) \neq \emptyset$.

We say that a formula $A$ is true in a BCR-structure $M = \langle W, f, [[\ ]]\rangle$ if $[[A]] = W$. We say that a formula is BCR-valid if it is true in every BCR-structure. Given a BCR-structure $M$, a set of formulas $S$ and a formula $A$, we also introduce the following notation $S \models_M A$ to say that for all $w \in M$ if, for all $B \in S$, $w \in [[B]]$, then $w \in [[A]]$.

Note that, in a given BCR-structure $M$, we can define through the selection function $f$ an equivalence relation $\approx$ on the set of worlds $W$ as follows: for all $w, w' \in W$,

$$w \approx w' \iff w' \in f(\top, w).$$

The properties of $\approx$ being reflexive, transitive and euclidean come from the semantic conditions (REFL), (TRANS) and (EUC) on the selection function $f$ (and, more precisely, from the last two conditions by taking $A = \top$).

The relation $\approx$ determines equivalence classes among worlds. The intuition is that each equivalence class of worlds represents an epistemic state. Thus, the formulas believed in a world $w$ are the formulas $A$ holding in the equivalence class associated with $w$ (i.e., such that for all $w' \approx w, w' \in [[A]]$). The formulas believed in a world $w$ are the formulas $A$ such that $\top > A$ holds in $w$ (recall that $\top > A$ holds in $w$ just in case $f(\top, w) \subseteq [[A]]$). In general, we will use the notation $\top > A$ to mean that $A$ is believed, in contrast to the meaning of $A$ which is $A$ is true.

When a conditional $A > B$ is evaluated in a world $w$, the selection function selects the set $f(A, w)$ of the most preferred $A$-worlds with respect to $w$, and $B$ is evaluated in such a set of worlds. As a consequence of axiom (BEL), evaluating a conditional formula $A > B$ in a world is exactly the same as evaluating that formula in a different world in the same class. Axioms (EUC) and (TRANS) make $f(A, w)$ an equivalence class to which we can associate an epistemic state. Fig. 1 provides a representation of BCR semantic structures. Fig. 2 in the next section shows the kind of mapping we can establish between epistemic states and equivalence classes of worlds.

Notice that, since $(A > C) \lor \neg(A > C)$ is a tautology, from (EUC) and (TRANS) we can conclude $(A > (\top > C)) \lor (A > \neg(\top > C))$, that is, $C$ is either believed or not believed in the most preferred $A$-worlds. This is the conditional excluded middle, (CEM), restricted to belief formulas. While the presence of (CEM) in Stalnaker’s logic causes the selection function to select a single world (i.e., $f(A, w) = \{j\}$ for all $A$ and $w$), when (CEM) is restricted to belief formulas (as in our logic), it determines the unicity of the epistemic state associated to all worlds belonging to $f(A, w)$. When we evaluate a conditional, we want to move from one world, with an associated epistemic state, to other worlds with a different epistemic state.
The fact that in our logic epistemic states are represented by a set of \( \approx \)-equivalent worlds evokes Grove’s sphere semantics for belief revision [14], where a belief state is represented by a set of worlds (models). As a difference, Grove’s semantics is similar to the sphere semantics for conditionals, whereas our semantics for conditionals is defined in terms of selection function models. Defining an equivalent sphere semantics for our logic is not straightforward, as we lack, in their full generality, some axioms, which hold in sphere semantics, such as CV.

The axiomatization of BCR is sound and complete with respect to semantic introduced above.

In the following, for readability, we use the notation \( x \models A \) rather than \( x \in [ [ A ] ] \).

**Theorem 7.6** (Soundness). If a formula \( A \) is a theorem of BCR then it is BCR-valid.

**Proof.** One checks each axiom and then shows that rules (RCEA) and (RCK) preserve validity. Let \( M = \langle W, f, [ [ \cdot ] ]^M \rangle \) be a BCR structure.

(CLASS) Since the elements of \( W \) are classical interpretations.

(CONS) Let \( A \in \mathcal{L} \) and \( \not\vdash_{PC} \neg A \). By (S-CONS), \( f(A, w) \neq \emptyset \), thus \( w \models \neg(A > \bot) \).

(ID) By (S-ID) \( f(A, w) \subseteq [ [ A ] ]^M \). It follows that \( w \models A > A \).

(DT) Let \( w \in W, A, B, C \in \mathcal{L} \), and \( w \models A \land C > B \). Then \( B \in Prop(f(A \land C, w)) \). By (S-DT), we have \( B \in Prop(f(A, w) \cap [ [ C ] ]^M) \). Let \( y \in f(A, w) \): if \( y \in [ [ C ] ]^M \), then \( y \models B \), and hence also \( y \models \neg C \) and hence also \( y \models C \rightarrow B \).

(CV) Let \( w \in W, A, B, C \in \mathcal{L} \), and let \( w \models \neg(A \rightarrow \neg C) \) and \( w \models A > B \). We have that \( f(A, w) \cap [ [ C ] ]^M \neq \emptyset \), and \( B \in Prop(f(A, w)) \). From (S-CV) it follows that \( B \in Prop(f(A \land C, w)) \) and therefore that \( w \models (A \land C) > B \).

(REFL) Let \( w \in W, A, B, C \in \mathcal{L} \), and \( w \models \top > A \). Then \( f(\top, w) \subseteq [ [ A ] ]^M \). By (S-REFL), \( w \in f(\top, w) \). Therefore \( w \in [ [ A ] ]^M \) and \( w \models A \).

(EUC) Let \( w \in W \). Let \( w \models \neg(A > B) \). Then, there is \( y \in f(A, w) \) s.t. \( y \models \neg B \). By (S-EUC), for all \( z \in f(A, w), y \in f(\top, z) \). It follows that for all \( z \in f(A, w), z \models \neg(\top > B) \), and therefore \( w \models A > \neg(\top > B) \).
(BEL) Let \( w \in W \), and \( w \models A > B \). Then, \( f(A, w) \subseteq [[B]]^M \). By (S-BEL), for all \( y \in f(\top, w) \), \( f(A, y) = f(A, w) \). Therefore, \( f(A, y) \subseteq [[B]]^M \) and \( y \models A > B \). It follows that \( w \models \top > (A > B) \).

(TRANS) Let \( w \in W \) and \( w \models A > B \). Then, \( f(A, w) \subseteq [[B]]^M \). By (S-TRANS), for any \( y \in f(A, w) \) and \( z \in f(\top, y) \), \( z \in f(A, w) \). Therefore, \( z \in [[B]]^M \) and \( y \models \top > B \). It follows that \( w \models A > (\top > B) \).

(RCK) We show that if \( (A_1 \land \cdots \land A_n) \rightarrow B \) is valid, then also \( ((C > A_1) \land \cdots \land (C > A_n)) \rightarrow (C > B) \) is valid. Let \( w \in W \), \( w \models C > A_1, \ldots, w \models C > A_n \). Then \( f(C, w) \subseteq [[A_1]]^M \land \cdots \land f(A_n, w) \subseteq [[A_n]]^M \), hence \( f(C, w) \subseteq [[A_1]]^M \land \cdots \land [[A_n]]^M \). Since \( A_1 \land \cdots \land A_n \rightarrow B \) is valid, \( [[A_1]]^M \land \cdots \land [[A_n]]^M \subseteq [[B]]^M \). It follows that \( f(C, w) \subseteq [[B]]^M \), and \( w \models C > B \).

(RCEA) We show that if \( A \leftrightarrow B \) is valid, then \( A > C \leftrightarrow B > C \) is valid. If \( A \leftrightarrow B \) is valid, then \( [[A]]^M = [[B]]^M \). For all \( w \in W \), by (S-RCEA), \( f(A, w) = f(B, w) \). Therefore, \( w \models A > C \) if and only if \( w \models B > C \). \( \Box \)

**Theorem 7.7** (Completeness). If \( A \) is BCR-valid then it is a theorem of BCR.

**Proof.** We fix a language \( \mathcal{L}_ \rightarrow^u \) and we build up a canonical model \( M = \langle W, f, [[ \ ]] \rangle \) for \( \mathcal{L}_ \rightarrow^u \), such that for every \( \mathcal{L}_ \rightarrow^u \)-formula \( A \) we have

\( A \) is a theorem of BCR  \iff  \( [[A]] = W \).

We consider maximal consistent sets of formulas of BCR and we assume that the usual properties of maximal consistent sets are known (e.g. if \( X \) is maximally consistent, then \( D \in X \) or \( \neg D \in X \)). In particular, for any formula \( A \), we have that

**Fact 1.** \( A \) is a theorem of BCR if \( A \in X \) for every maximal consistent set \( X \).

We define \( M = \langle W, f, [[ \ ]] \rangle \), as follows

\( W = \{ X \subseteq \mathcal{L}_ \rightarrow^u \mid X \text{ is maximally consistent} \}, \)

\( f(B, X) = \{ Y \in W \mid \{ C \in \mathcal{L}_ \rightarrow^u \text{ s.t. } B > C \in X \} \subseteq Y \}, \)

\( [[p]] = \{ X \in W \mid p \in X \}. \)

One can prove the following facts.

**Fact 2.** For every formula \( A \in \mathcal{L}_ \rightarrow^u \) and \( X \in W \), \( A \in X \) iff \( X \in [[A]] \).

**Fact 3.** The structure \( M \) satisfies all conditions of Definition 7.5: namely, (S-ID), (S-RCEA), (S-DT), (S-CV), (S-REFL), (S-TRANS), (S-EUC), (S-BEL), (S-CONS).

Given \( X \in W \), \( A \in \mathcal{L} \), we let \( S_A(X) = \{ B \in \mathcal{L}_ \rightarrow^u \mid A > B \in X \} \).

- (S-ID), let \( Y \in f(A, X) \), we have \( S_A(X) \subseteq Y \); by (ID) \( A \in S_A(X) \), thus \( A \in Y \).
- (S-RCEA), let \( [[A]] = [[B]] \), where \( A, B \in \mathcal{L} \), then \( [[A \leftrightarrow B]] = W \). By Fact 2 and Fact 1, we have \( A \leftrightarrow B \) is a theorem of BCR, by (RCEA), for any formula \( C \),
(A > C) ↔ (B > C) is a theorem of BCR. Thus, using the previous notation, for every X ∈ W, S_A(X) = S_B(X). This implies f(A, X) = f(B, X).

- (S-DT), let B ∈ Prop(f(A ∧ C, X)), then A ∧ C > B ∈ X, by (DT) we have A > (C → B) ∈ X. Thus, C → B ∈ Prop(f(A, X)). This means that for every Y ∈ f(A, X) ∩ [[C]], we have C → B ∈ Y and C ∈ Y; it follows B ∈ Y. Thus f(A, X) ∩ [[C]] ⊆ [[B]], whence B ∈ Prop(f(A, X) ∩ [[C]]).

- (S-CV), let f(A, X) ∩ [[C]] ≠ ∅ and B ∈ Prop(f(A, X)). By hypothesis it cannot be A > ¬C ∈ X, thus we have ¬(A > ¬C) ∈ X; by hypothesis we also have A > B ∈ X. By (CV) we conclude that A ∧ C > B ∈ X. Thus B ∈ Prop(f(A ∧ C, X)).

- (S-REFL), We have to show that S_τ(X) ⊆ X. Let A ∈ S_τ(X), then τ > A ∈ X. By (REFL) A ∈ X.

- (S-TRANS), Let X ∈ f(A, Z), Y ∈ f(τ, X). We must show that Y ∈ f(A, Z). From the hypothesis we have that (i) S_A(Z) ⊆ X and (ii) S_τ(X) ⊆ Y and we must show that S_A(Z) ⊆ Y. To this regard, let B ∈ S_A(Z), we have A > B ∈ Z; by (TRANS) we get A > (τ > B) ∈ Z, thus τ > B ∈ S_A(Z), whence τ > B ∈ X by (i). We have B ∈ S_τ(X) and then B ∈ Y by (ii).

- (S-EUC), let X, Y ∈ f(A, Z), we must show that X ∈ f(τ, Y). From the hypothesis we have that (i) S_A(Z) ⊆ X and (ii) S_τ(Z) ⊆ Y and we must show that S_τ(Y) ⊆ X. To this regard suppose that B ∉ X, we prove that B ∉ S_τ(Y). Since B ∉ X, by (i) B ∉ S_A(Z). Thus A > B ∉ Z and then ¬(A > B) ∈ Z. By (EUC) we have A > ¬(τ > B) ∈ Z. Thus ¬(τ > B) ∈ S_A(Z), whence ¬(τ > B) ∈ Y by (ii). We have obtained τ > B ∉ Y, so that B ∉ S_τ(Y).

- (S-BEL) let X ∈ f(τ, Y), we show that f(A, X) = f(A, Y). By definition of f, we have to show that for any Z ∈ W, S_A(X) ⊆ Z iff S_A(Y) ⊆ Z.

We actually prove that in the hypothesis X ∈ f(τ, Y) we have A > B ∈ X iff A > B ∈ Y, from which the result follows. Let A > B ∈ Y, we have τ > (A > B) ∈ Y by (BEL). Thus A > B ∈ X. Conversely, if A > B ∉ Y, τ > (A > B) ∉ Y by (REFL). This implies that ¬(τ > (A > B)) ∈ Y, whence τ > ¬(τ > (A > B)) ∈ Y by (EUC). Thus, ¬(τ > (A > B)) ∈ X and also ¬(A > B) ∈ X by (BEL), whence A > B ∉ X.

- (S-CONS) let ̸∈_Pc ¬A and let X ∈ W. By (CONS) we have ¬(A > ⊥) ∈ X. Thus A > ⊥ ∉ X. We can conclude that f(A, X) ⊆ [[⊥]]. This means that f(A, X) ≠ ∅.

We have shown that M is a BCR structure. The theorem now follows immediately from Facts 1 and 2: if A is BCR valid, then in particular it is valid in M, i.e., [[A]] = W; by Fact 2, we have A ∈ X for every maximal consistent set. By Fact 1, we conclude that A is a theorem of BCR. □

We show that:

**Theorem 7.8.** The logic BCR is decidable.

For the proof see Appendix A.
Fig. 2. Representation of a conditional belief revision system in a BCR structure.

8. Representation theorem

We can establish a direct relationship between conditional revision systems and BCR models. They are essentially the same thing. On the one side, to any conditional belief revision system can be associated a BCR-model that represents it, in the sense that for any consistent epistemic state $K$ there is a possible world $w$ such that $\text{Form}(f(\top, w)) = K$ and $\text{Form}(f(A, w)) = K \ast A$. On the other hand, the theorem also shows that given a BCR-model, we can build a conditional revision system that contains the epistemic states represented by some equivalence class in $M$, and whose revision operator is defined in terms of the selection function as:

$$K \ast A = \text{Form}(f(A, w)) \quad \text{for } w \text{ such that } K = \text{Form}(f(\top, w)).$$

Fig. 2 shows the relation holding between belief revision systems and BCR-models.

Consider a structure $(K, *)$, where $K$ is a set of sets of $\mathcal{L}_L^\mathcal{D}$-formulas and $*$ is a mapping $K \ast \mathcal{L} \rightarrow K$. Then we have:

**Theorem 8.1** (Representation Theorem). $(K, *)$ is a conditional revision system if and only if there is a BCR model $M = (W, f, [\ ]])$, such that for all consistent $K \in K$ there is $w \in W$ for which it holds that:

$$K = \text{Form}(f(\top, w)) \quad \text{and} \quad K \ast A = \text{Form}(f(A, w)).$$

**Proof.** $(\Rightarrow)$ Let $(K, *)$ be a revision system, we define a BCR-structure $M = (W, f, [\ ]])$ as follows:

$$W = \{(K, w): w \in 2^{\mathcal{P}rop}, K \in K \text{ and } w \models [K]\};$$

$$C_K = \{(K', w) \in W: K' = K\},$$

$$[[p]] = \{(K, w) \in W: w \models_{PC} p\} \quad \text{for all propositional variables } p \in \mathcal{L};$$

$$f(A, (K, w)) = C_{K \ast A}.$$
We observe that if $K$ is consistent, then $K = K \ast \top$, whence $C_K = C_{K \ast \top}$. This implies $f(\top, (K, w)) = C_{K \ast \top} = C_K$. By making use of the properties of the revision operator $\ast$, we can show that $M_a$ is a BCR-structure.

- (S-ID) By definition of $f$, for all $(K', w') \in f(A, (K, w))$ we have $K' = K \ast A$ and $w' \models K \ast A$. By postulate $(B \ast 1)$ we have that $A \in K \ast A$, and hence $(K', w') \in [[A]]$.
- (S-RECA) If in $M_a$, $[[A]] = [[B]]$ holds, then it must be that $A \equiv B$ is valid, as the set $\{w': (K', w') \in [[A]]\}$ contains all and only the classical models of $A$. By postulate $(B \ast 6)$, we have that $K \ast A = K \ast B$. Hence, if $(K', w') \in f(A, (K, w))$, then $K' = K \ast A = K \ast B$ and from $w' \models K \ast A$ we can conclude $w' \models K \ast B$. Therefore, $(K', w') \in f(B, (K, w))$.
- (S-DT) Let $B \in Prop(f(A \land C, (K, w)))$. By definition, $f(A \land C, (K, w))$ contains all the classical models of $K \ast (A \land C)$. Hence, we have that: $B \in Prop(f(A \land C, (K, w)))$ if and only if for all $w'$, if $w' \models K \ast (A \land C)$, then $w' \models B$. Hence, for all $B \in \mathcal{L}$, $B \in Prop(f(A \land C, (K, w)))$ if and only if $B \in (K \ast (A \land C))$. Similarly, it is easy to see that for all $B \in \mathcal{L}$, $B \in Prop(f(A, (K, w)))$ if and only if $B \in (K \ast A) + C$. From $(B \ast 7)$ we have that $[K \ast (A \land C)] \subseteq [(K \ast A) + C]$, from which, given the above equivalences, the wanted property trivially follows.
- (S-CV) Let $f(A, (K, w)) \cap [[C]] \neq \emptyset$. By definition of $f$, $f(A, (K, w))$ contains all and only the classical interpretations of $K \ast A$. Hence, $\neg C \notin K \ast A$, and $\neg C \notin [K \ast A]$. By $(B \ast 8)$, we derive that $[(K \ast A) + C] \subseteq [K \ast (A \land C)]$. Let $B \in Prop(f(A \land C, (K, w))) \cap [[C]]$. Then, by definition of $f$, $B \in [(K \ast A) + C]$. Hence, by $(B \ast 8)$, $B \in [K \ast (A \land C)]$ and, by definition of $f$, $B \in Prop(f(A \land C, (K, w)))$.
- (S-TRANS) It follows from the fact that $f(\top, (K, w)) = C_K$.
- (S-TRANS) Let $(K'', w) \in f(\top, (K', w))$. Then, $(K'', w) \in C_{K' \ast \top} = C_{K''}$. It follows that $K'' = K'$. Similarly, if $(K', w) \in f(A, (K, w))$, then $(K', w) \in C_{K \ast A}$, and $K' = K \ast A$. By the two equivalences just stated, also $K'' = K \ast A$, thus, by definition of $f$, $(K'', w) \in f(A, (K, w))$.
- (S-EUC) If $(K', w), (K'', w) \in f(\top, (K', w))$, then $(K', w), (K'', w) \in C_{K' \ast \top} = C_K$. Thus, $K' = K'' = K$. Hence, $(K', w) \in C_{K''} = C_{K' \ast \top}$, and $K' \in f(\top, (K'', w))$.
- (S-BEL) If $(K, w) \in f(\top, (K', w))$, then $(K, w) \in C_{K' \ast \top} = C_{K'}$, thus $K = K'$. Hence, $K \ast A = K' \ast A$ and, by definition of $f$, $f(A, (K, w)) = f(A, (K', w))$.
- (CONS) If $\not\vdash_{PC} A$, then by $(B \ast 5)$. $K \ast A \not\vdash_{PC} \bot$. Hence, $C_{K \ast A} \neq \emptyset$. By definition of $f$, it follows that $f(A, (K, w)) \neq \emptyset$.

Let $K \in \mathcal{K}$ be consistent, so that there exists $(K, w) \in W$. We then prove that $K = Form(f(\top, (K, w)))$ and $K \ast A = Form(f(A, (K, w)))$.

We first prove the property for formulas of a subset $\mathcal{L}^*$ of $\mathcal{L}_a^*$ defined as follows:

- if $A \in \mathcal{L}$ then $A \in \mathcal{L}^*$,
- if $A \in \mathcal{L}$ and $B \in \mathcal{L}^*$ then $A \ast B \in \mathcal{L}^*$ and $\neg (A \ast B) \in \mathcal{L}^*$.

We prove the above property by induction on the structure of $A \in \mathcal{L}^*$. Let $A \in \mathcal{L}$ and $(K, w) \in W$, first observe that $(K, w) \models A$ iff $w \models A$. Now given $A \in \mathcal{L}$ suppose $A \in$
\( K \), then \( A \in [K] \). Let \((K, w') \in f(\top, (K, w)) = C_K\), we have \(w' \models [K]\), thus \( w' \models A\), whence \((K, w') \models A\). If \( A \notin K \) then \([K] \cup \{\neg A\}\) is propositionally consistent, let \( w' \) such that \( w' \models [K] \cup \{\neg A\}\). We have \((K, w') \not\models A\) and \((K, w') \in C_K = f(\top, (K, w))\).

For the induction step let \( A = B > C \), where \( B \in \mathcal{L} \) and \( C \in \mathcal{L}^*\). We have \( B > C \in K \) iff \( C \in K * B \). By the induction hypothesis, this holds iff \( C \in \text{Form}(f(\top, (K * B, w_1))) \) for some \( w_1 \models [K * B] \). We then have \( C \in \text{Form}(f(\top, (K * B, w))) \) iff \( C \in \text{Form}(C_{K * B * \top}) \) iff \( C \in \text{Form}(f(B, (K, w))) \) for any \( w \models [K] \). But this holds iff \((K, w) \models B > C\), that is, by the semantic properties, iff \( B > C \in \text{Form}(f(\top, (K, w)))\).

The case of \( A = \neg(B > C)\) is reduced to the previous one as \( K \) and \( C_K = f(\top, (K, w)) \) are complete with respect to conditional formulas.

We have shown that for \( A \in \mathcal{L}^* \) we have \( A \in K \) iff \( A \in \text{Form}(f(\top, (K, w))) \).

To extend the property to \( \mathcal{L}^* \), we observe that every \( \mathcal{L}^* \)-formula \( A \) is equivalent to a set of formulas \( \bigvee C_i \) where \( C_i \in \mathcal{L}^* \). To see this observe that the following equivalences are valid:

\[
A > (B \land C) \equiv (A > B) \land (A > C) \quad \text{and} \quad A > (B \lor (\neg(C > D))) \equiv (A > B) \lor (A > (\neg(C > D))).
\]

These equivalences can be used to move out conjunction and disjunction on conditionals. It is easy to extend the property to disjunctions of the above form. Let \( C = \bigvee C_i \) where \( C_i \in \mathcal{L}^* \). We can write \( C \) as \( D \lor D' \) where \( D \in \mathcal{L} \) and \( D' \in \mathcal{L}^* - \mathcal{L} \). Suppose that \( C \in K \), but \( C \not\in \text{Form}(f(\top, (K, w))) \), thus we have \( D \not\in \text{Form}(f(\top, (K, w))) \) and \( D' \not\in \text{Form}(f(\top, (K, w))) \). Since \( f(\top, (K, w)) \) is complete on conditionals, and \( D' \) is a disjunction of conditionals, we have \( \neg D' \in \text{Form}(f(\top, (K, w))) \). By the previous result on \( \mathcal{L}^* \) formulas, we obtain that \( D \not\in K \) and \( \neg D' \in K \). But since \( C = D \lor D' \in K \) and \( K \) is deductively closed, it must be \( D \in K \) and we get a contradiction. The other direction is exactly analogous.

For the other identity, we have \( K * A = \text{Form}(f(\top, (K * A, w'))) \), for some \( w' \models [K * A] \). But \( \text{Form}(f(\top, (K * A, w'))) = \text{Form}(C_{K * A}) = \text{Form}(f(A, (K, w))) \).

\((\Leftarrow)\) Given a model \( M = (W, f, [[\cdot]]) \), let \( K_{\bot} = \mathcal{L}^*_\bot \) and

\[
K * A = \text{Form}(f(A, w)) \quad \text{for} \quad K = f(\top, w).
\]

We first observe that * is a function: if \( \text{Form}(f(\top, w)) = \text{Form}(f(\top, w')) \) then also \( \text{Form}(f(A, w)) = \text{Form}(f(A, w')) \) (because for any \( w, w \models A \iff \top > A > B \)).

We then check that \( (K_M, *_{M}) \) satisfies all the revision postulates including (SRT). Observe that for any \( S \subseteq W \), it holds \( \text{Cn}_{PC}(\text{Form}(S) \cup \{A\}) = \text{Form}(S \cap [[A]]) \). Thus, given \( K = \text{Form}(f(\top, w)) \) and a formula \( A \), we have \( K + A = \text{Form}(f(\top, w) \cap [[A]]) \). Moreover \([K] = \text{Prop}(f(\top, w))\). Clearly \( \text{Form}(f(\top, w)) \) is an epistemic state. Let \( K = f(\top, w) \).

- \((B * 2)\) \( A \in K * M A \).
  We have \( w \models A \models A \). Thus \( A \in \text{Form}(f(A, w)) = K * M A \).
- \((B * 3)\) \([K * M A] \subseteq [K + A] \).
  We have \([K * M A] = \text{Prop}(f(A, w)) \subseteq \text{Prop}(f(\top, w) \cap [[A]]) = [K + A] \) by (S-DT) as \( A \equiv A \land \top \).
• (B * 4) If \( \neg A \not\in [K] \) then \([K + A] \subseteq [K *_M A]\).
  Let \( \neg A \not\in K = \text{Form}(f(\top, w)) \), then \( w \not\models \top \rightarrow \neg A \), so that \( f(\top, w) \cap \{A\} \neq \emptyset \). By (S-CV), we have \( \text{Prop}(f(\top, w)) \subseteq \text{Prop}(f(\top \land A, w)) = \text{Prop}(f(A, w)) \). Thus also \([K + A] = \text{Prop}(f(\top, w) \cap \{A\}) \subseteq \text{Prop}(f(A, w)) = [K *_M A]\).

• (B * 5) \( K *_M A \models_{PC} \bot \) only if \( \not\models_{PC} \neg A \).
  If \( \not\models_{PC} \neg A \), we have \( f(A, w) \neq \emptyset \). Thus \( \bot \not\in \text{Form}(f(A, w)) = K *_M A \).

• (B * 6) If \( A \equiv B \), then \( K *_M A = K *_M B \).
  This follows from (S-RCEA).

• (B * 7) \([K *_M (A \land B)] \subseteq ([K *_M A] + B]\).
  We have \([K *_M (A \land B)] = \text{Prop}(f(A \land B, w)) \subseteq \text{Prop}(f(A, w) \cap \{B\}) = ([K *_M A] + B)\) by (S-DT).

• (B * 8) if \( \neg B \not\in [K *_M A] \), then \([K *_M A] + B \subseteq [K *_M (A \land B)]\).
  If \( \neg B \not\in [K *_M A] \), with \( B \in \mathcal{L} \), then \( \neg B \not\in K *_M A = \text{Form}(f(A, w)) \). Thus \( w \not\models A > \neg B \), whence \( f(A, w) \cap \{B\} \neq \emptyset \). But then by (S-CV) we have: \([K *_M A] + B = \text{Prop}(f(A, w) \cap \{B\}) \subseteq \text{Prop}(f(A \land B, w)) = [K *_M (A \land B)]\).

(B * \top) for any \( K \) consistent, \( K *_M \top = K \).
  It follows by definition of \(*_M\).

(SRT) if \( B \in K *_M A \) then \( A > B \in K \) and if \( B \notin K *_M A \) then \( \neg (A > B) \in K \).
  We have \( A > B \in K = \text{Form}(f(\top, w)) \) iff \( w \models \top \rightarrow A > B \) iff \( w \models A > B \) iff \( B \in \text{Form}(A, w) = K *_M A \). The other direction follows then by the fact that \( f(\top, w) \) is complete on conditionals. \(\square\)

The direction (\(\Leftarrow\)) of the previous theorem proves also the converse of Theorem 1.

**Theorem 8.2.** If \( A \) is revision-valid then \( A \) is BCR-valid, whence a theorem of BCR.

**Proof.** Let \( A \) be revision-valid. Let \( M \) be a BCR structure, then we consider the conditional revision system \((K_M, *_M)\) obtained from \( M \) as shown in the (\(\Leftarrow\))-part of the previous theorem. We have that \( A \in \text{Form}(f(\top, w)) \) for every \( w \in W \), but this implies that \( \{A\} = W \) (as \( w \in f(\top, w) \)). Thus \( A \) is true in \( M \). We have shown that \( A \) is BCR-valid. \(\square\)

9. Conclusions and related work

In this paper we have shown how the triviality result by Gärdenfors can be avoided by restricting some of the revision postulates to propositional formulas. In particular, the Minimal Change Principle is not applied to conditional formulas and, as a consequence, the revision of a belief set with a consistent formula does not result in the mere expansion of the set with that formula. Hence, the revision of a belief set with a consistent formula may produce a change in the current revision strategies, expressed by the conditional formulas in the belief set. We have argued that the restrictions on the postulates are supported by intuition.

By means of our restricted postulates and the Strong Ramsey Test, we have been able to define the conditional logic BCR. This logical system has a sound and complete ax-
iomatization with respect to its standard semantics in terms of selection function models of conditional logic. Moreover we have proven that this logic is decidable. We have then shown an isomorphism between BCR models and conditional revision systems. In this respect, we can claim that BCR gives an axiomatization of revision within the language of conditional logic. By this mapping we can think of using the logic BCR as a formal tool to prove properties of revision. In future research, we shall investigate how to express meaningful properties of revision operators in the language of BCR. Moreover we intend to study automated proof-methods for BCR, as an essential step towards automatizing reasoning about revision.

Our approach is related to other works in the area. We give a brief account. First, it can be observed that the kind of restrictions we have put on AGM postulates are essentially the same introduced by Darwiche and Pearl in [3] where they extend AGM theory for dealing with iterated belief revision. In [9,10] a conditional logic has been proposed to represent belief revision systems whose postulates are a small variant of Darwiche and Pearl’s ones. Though in this paper the problem of iterated belief revision has not been addressed, we can expect that the results presented in this paper still hold when we extend the AGM theory with new postulates for dealing with iterated revision. In fact, the logic BCR presented in this paper has strong similarities with the logic IBC introduced in [10] to deal with iterated revision, even if IBC does not have restrictions on the nesting of conditionals, whereas BCR does. More precisely, the following axioms are common to the two systems (in their respective language): (CLASS), (ID), (DT), (CV), (BEL), (REFL), (TRANS), (EUC), (RCEA), (RCK). IBC contains in addition axioms that formalize iterated revision postulates and those that formalize the modality □ (defined by □A ≡ ¬A > ⊥). On the other hand, IBC does not contain the axiom (CONS). The similarities between BCR and IBC suggest that we can reasonably expect that BCR can be extended to deal with iterated revision.

The triviality result by Gärdenfors has been widely investigated in the literature. A possible way out of the triviality result consists in considering a different notion of belief change, called “belief update” [13,15], which does not enforce the Minimal Change Principle (not even for propositional formulas). Grahne [13] has proposed a conditional logic which combines updates and counterfactuals and which does not entail triviality.

Ryan and Schobbens [32] have established a link between updates and counterfactuals, by regarding them as existential and universal modalities. The Ramsey rule is an axiomatization of the inverse relationship between the two sets of modalities.

Makinson [22] has analyzed the triviality result in the case that the inference operator is non-monotonic and he has proved that triviality still holds in this case.

To avoid triviality, Rott [29] has suggested weakening the Ramsey Test which, by the way, in Gärdenfors’s formulation, leads to the counterintuitive conclusion that if A ∈ K and B ∈ K then A > B ∈ K. However, Gärdenfors [6] has shown that Rott’s reformulation of the Ramsey Test does not avoid triviality.

Lindström and Rabinowicz in [21] discuss some alternative solutions to Gärdenfors negative result such as questioning preservation, weakening the Ramsey rule, leaving the conditional formulas out of the belief sets, and making the evaluation of conditionals dependent on the epistemic state. Our solution has some similarities with the last one, that they call the “indexical interpretation of conditionals”, according to which the truth value
of a conditional formula in a world depends not only on the world, but also on the epistemic state in which the conditional appears. We have seen that also in our logic the evaluation of a conditional in a world depends on the epistemic state associated to that world. The difference between our proposal and Lindström and Rabinowicz’s is that we do not need to introduce epistemic states as extra elements in our semantics. Moreover, Lindström and Rabinowicz [21] do not aim to define a conditional logic system as we have done in this work.

Levi [18] has proposed a way out of the triviality problem based on a strict separation between conditional and non-conditional beliefs. The triviality is avoided by assuming that conditional sentences cannot be members of belief sets, consequently the belief operator only applies to sets of propositional beliefs and therefore revision postulates are naturally restricted to propositional belief sets. In this respect our approach has some similarity with Levi’s approach (see also [8]). As a difference, we only restrict some of the postulates and we allow the occurrence of conditionals in epistemic states, including iterated conditionals whose acceptance (whence meaning) is not defined in Levi’s framework.

Levi’s approach has been further developed by Arló Costa in [1], where he examines the conditional axiomatization deriving from Levi’s formulation of the Ramsey Test. Arló Costa’s conditional logic is related to ours, in that all of his axioms are derivable from ours (whereas the inverse does not hold). There is nonetheless a main difference between his approach and ours. This difference lies in the objectives of the two approaches. As a matter of fact, Arló Costa does not aim to define a possible world semantics for the conditional logic, nor (a fortiori) he aims at establishing a representation result between belief revision systems and conditional models. On the contrary, as we have seen, this is a major part of our work.

Nejdl and Banagl’s proposal of a specific non-trivial belief revision operator [25] is also inspired by Levi’s proposal. In their approach, to each belief revision system is associated a family of semantic models, one for each epistemic state. The revision operator is external to models and applies to models. This is a substantial difference with our proposal, according to which we associate a single model to each belief revision system and we represent revision by the selection function, that is internal to the model. A major difference with our proposal is that [25] only deals with a specific non-trivial belief revision operator, and does not address the problem of defining a general correspondence between belief revision operators and models of conditional logic.

The idea of avoiding triviality by maintaining a separation between propositional beliefs and conditional formulas also underlies the proposal developed in [4], where a logical framework for modeling both revision and update is defined. In [4] Friedman and Halpern introduce the notion of belief change system (BCS), a generalization of Gärdenfors’s belief revision system. A BCS contains three components: the set of possible epistemic states, a belief assignment that maps each epistemic state to a set of beliefs, and a transition function which transforms epistemic states. The properties of the transition function are expressed in terms of conditional axioms. As a difference with standard conditional logics and with our approach, Friedman and Halpern’s semantics is built upon epistemic states rather than worlds as primitive semantic objects.

Similarly to our approach, Friedman and Halpern’s conditional language $L^\succ$ only allows objective formulas in the left hand side of a conditional. As a difference, their logic
\( \mathcal{L}^> \) is completely disjoint from the propositional language \( \mathcal{L} \). Moreover, their logic contains a belief operator \( \mathsf{B} \) in addition to the conditional operator.

As observed by Friedman and Halpern, their solution avoids the triviality problem by considering belief sets in \( \mathcal{L} \) and only revising with formulas in \( \mathcal{L} \). Instead, we allow for belief sets containing conditional formulas through the Ramsey Test (which is added as a new postulate), but we restrict some of the AGM postulates so that they do not hold for conditional beliefs.

As we have already mentioned, abandoning closure with respect to expansion has been advocated by other authors, such as Rott [30], and Morreau [23].

Rott’s analysis of Gärdenfors’s triviality result (see [30]) is close to our own analysis: similarly to us, he regards the closure under expansion as the unmotivated assumption to be rejected in order to avoid triviality. In this regard he makes a distinction between Addition and Expansion, the former is a nonmonotonic operation taking care of consistent revision, whereas the latter is the usual operation. Gardenfors’ argument leading to triviality is based on the wrong identification of the two operations. Rott proposes some postulates relating the two concepts without enforcing their identification. Unlike us, he accepts AGM postulates in their integrity, but replacing the expansion operator with addition. In a further paper [31], he proposes a conditional logic related to belief revision. The conditional logic is defined semantically by families of epistemic entrenchment relations with a preferential ordering. The logic obtained is related to the conditional logic VC. There might be connections with our logic, however it does not seem an easy task to compare the two semantics, as they are based on rather different primitive notions.

Similarly to what we have done, Morreau [23] states that triviality can be avoided by giving up the closure of belief revision systems with respect to expansion. In spite of this similarity, Morreau’s approach is radically different from ours. First of all, Morreau considers belief revision systems that satisfy only some of the AGM postulates. In particular, he considers non-deterministic belief revision systems, that do not satisfy postulate (K*1). Second, and most important, Morreau establishes a very weak relation between belief revision systems and conditional models: although he shows that there exists a conditional model to which a belief revision system can be associated, the relation does not hold in general for all conditional models and belief revision systems. Furthermore, the relation established between conditional models and belief revision systems associates to each “belief state” a distinct revision function, making it difficult to cope with iterated belief revision (as it is impossible to impose conditions on iterated applications of different selection functions).

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Appendix A. Decidability

We show that:

**Theorem A.1.** The logic BCR is decidable.

To this purpose, we show that for any formula $F$, if there is a BCR-model $M$ and a world $w_0$ of $M$ such that $M, w_0 \models F$, then there is also a finite BCR-model $M^*$ containing $w_0$ such that $M^*, w_0 \models F$. This property is called the finite model property. The finite model property, together with the recursiveness of the axiomatization of the logic entails that the logic is decidable. Indeed, for any formula $F$, if it is a theorem of the logic, it will eventually be derived from the axioms and derivation rules. On the contrary, if it is not a theorem of the logic, by considering all the finite BCR-models we shall eventually find a finite BCR-model $M^*$ that falsifies it (i.e., that contains a world $w$ such that $M^*, w \models \neg F$).

The rest of the section is devoted to the proof of the finite model property.

Let $F$ be a formula, $M$ a BCR-model and $w_0$ a world in $M$ such that $M, w_0 \models F$. Intuitively, the new finite model $M^*$ is built from $(M, w_0)$ by considering only the portion of $M$ that is relevant to determine the truth value of $F$ in $w_0$. Furthermore, a sort of filtration is applied to the model so obtained: in $M$ we define $S$-equivalent two worlds if they agree on the evaluation of all the propositional variables of $F$. For each set of worlds selected by the selection function, we consider only one representing element for each $S$-equivalence class. The construction of the model is illustrated in Fig. A.1.

**A.0.1. Construction of the finite model**

Let $\text{Var}_F$ be the set of the propositional variables appearing in $F$, and $\mathcal{L}_F$ the boolean closure of $\text{Var}_F \cup \{\top\}$.

We define an $S$-equivalence relation $\equiv_{\text{Var}_F}$ over the set of worlds $W$ by stipulating that two worlds are $S$-equivalent if they evaluate in the same way the atoms in $\text{Var}_F$. Thus:

$$w \equiv_{\text{Var}_F} w' \text{ if and only if } w \models p \text{ if and only if } w' \models p.$$

We call *choice function* a function $\pi$ that picks one single representing-element of each $S$-equivalence class.

For any set $S$ of possible worlds in $W$, we denote by $S/ \equiv_{\text{Var}_F}$ the set of the $S$-equivalence classes of $S$, and with $\pi(S/ \equiv_{\text{Var}_F})$ the subset of $S$ containing exactly the worlds selected by $\pi$.

For any cluster $f(\top, w)$ with $w \in W$, fix $\pi(f(\top, w)/ \equiv_{\text{Var}_F})$.\(^{10}\)

Let $n$ be the nesting level of $F$ (as defined in Section 5). We build the model $M^* = (W_n, f^*, [\ ]^*)$ as follows (see Fig. A.1):

$$W_0 = \{w_0\} \cup \pi(f(\top, w_0)/ \equiv_{\text{Var}_F});$$

\(^{10}\) This amounts to fixing $\pi(f(A, w))$ for all $A \in \mathcal{L}$ and $w \in W$, since by (S-TRANS) and (S-EUC) for $w' \in f(A, w))$, $f(A, w) = f(\top, w')$. 

Fig. A.1. Construction of the finite model. The resulting model contains only a portion of the original model. Furthermore, filtration is applied to the sets of worlds selected by the selection function.

\[ W_i = W_{i-1} \cup \bigcup \{ \pi(f(A,w)/\equiv_{\mathit{Var}_F}) \text{ for } w \in W_{i-1} \text{ and } A \in \mathcal{L}_F \}; \]

\[ W_n = W_{n-1} \cup \bigcup \{ \pi(f(A,w)/\equiv_{\mathit{Var}_F}) \text{ for } w \in W_{n-1} \text{ and } A \in \mathcal{L}_F \}. \]

The valuation function \([ [ ] ]^*\) is defined as follows:

\([ [p] ]^* = [ [p] ] \cap W_n,\)

for all \(p \in \mathit{Var}_F\). It extends to more complex formulas in a standard way as:


where \(f^*\) is defined as follows.

Let \(w \in W_{n-1}\), then for any \(A \in \mathcal{L}_F,\)

\(f^*(A, w) = f(A, w) \cap W_n,\)
Let \( w \in W_n - W_{n-1} \). Let \( e(\mathcal{L}_F) \) be an enumeration of \( \mathcal{L}_F \), and \( e'(W_n) \) be an enumeration of \( W_n \), and \( D \in \mathcal{L}_F \) and \( w' \in W_{n-1} \) be the minimal elements (with respect to \( e \) and \( e' \)) s.t. \( w \in f(D, w') \).\(^{11}\) For any \( A \in \mathcal{L}_F \), we define \( f^* \) as follows:

\[
f^*(A, w) = \begin{cases}
  f^*(A \land D, w') & \text{if } f^*(D, w') \cap [[A]]^* \neq \emptyset, \\
  f^*(A, w_0) & \text{if } f^*(D, w') \cap [[A]]^* = \emptyset.
\end{cases}
\]

We can prove that \( M^*, w_0 \models F \). To this purpose, we first prove the two following lemmas that show which is the correspondence between the evaluation of formulas in the original model \( M \) and their evaluation in \( M^* \). From the definition of \([[\ ]]]^*\), it immediately follows that:

**Lemma A.2.** For all formulas \( G \in \mathcal{L}_F \), for all \( w \in W_n \), \( w \in [[G]]^* \) if and only if \( w \in [[G]]^* \).

This property extends to conditional subformulas of \( F \) as follows:

**Lemma A.3.** For all subformulas \( G \) of \( F \) with nesting level \( nl \leq i \), for all \( w \in W_{n-i} \), \( w \in [[G]]^* \) if and only if \( w \in [[G]]^* \).

**Proof.** By induction on the complexity of \( G \).

If \( G \) is an atom, this follows by definition of \([[\ ]]]^*\). If \( G \) is a boolean combination of formulas, the property follows straightforwardly by definition of \([[\ ]]]^*\) and by inductive hypothesis.

Let \( G = C > D \), with \( nl(C > D) > 0 \). Since \( nl(G) > 0 \), we only need to prove the property for the worlds in \( W_{n-i} \) with \( i > 0 \). Let us consider the two directions of the proof.

(\( \Leftarrow \)) Let \( w \in [[G]]^* \). Then \( f^*(C, w) \subseteq [[D]]^* \), i.e., \( f(C, w) \cap W_n \subseteq [[D]]^* \). By inductive hypothesis, we have that \( f(C, w) \cap W_n \subseteq [[D]]^* \), and hence that \( f(C, w) \subseteq [[D]]^* \) (recall that in our semantics, if \( D \) is conditional, either \( D \) holds in all worlds of \( f(C, w) \) or in none; if \( D \) is propositional, the property holds since \( W_n \) contains one representing element per each \( S \)-equivalence class of \( f(C, w) \)). Thus, \( w \in [[G]]^* \).

(\( \Rightarrow \)) In the other direction, if \( w \in [[G]]^* \), \( f(C, w) \subseteq [[D]]^* \), therefore also \( f(C, w) \cap W_n \subseteq [[D]]^* \). By inductive hypothesis, we have that \( f(C, w) \cap W_n \subseteq [[D]]^* \), and, by definition of \( f^* \), that \( f^*(C, w) \subseteq [[D]]^* \). Thus \( w \in [[G]]^* \). \( \square \)

As an immediate consequence of the lemmas, we have the following theorem.

**Corollary A.4.** \( M^*, w_0 \models F \).

We still have to prove that \( M^* \) is a BCR model, and that it is finite.

**Theorem A.5.** The model \( M^* = (W_n, f^*, [[\ ]]]^*) \) is a BCR-structure.

\(^{11}\) Taking \( D \) and \( w' \) minimal grants that for any \( x \) and \( y \in W_n - W_{n-1} \), if there is a \( z \) and a \( B \) s.t. \( x, y \in f(B, z) \) then for all \( A \), \( f^*(A, x) = f^*(A, y) \), thus \((S-BEL) \) holds.
Proof. (Sketch) The selection function $f^*$ satisfies the conditions (S-ID)–(S-CONS). We illustrate here only the proof for conditions (S-CV), (S-EUC), and (S-CONS). The other cases can be proven similarly. For each condition, we need to consider two cases, corresponding to the two cases of the above definition of $f^*$.

(S-CV)

1. Let $w \in W_{n-1}$. Let $f^*(A, w) \cap [[C]]^* \neq \emptyset$ and $B \in \text{Prop}(f^*(A, w))$. We have to prove that $B \in \text{Prop}(f^*(A \land C, w))$. From $f^*(A, w) \cap [[C]]^* \neq \emptyset$ in $M^*$, we derive by definition of $f^*$ and by Lemma A.2, that $f(A, w) \cap [[C]] \neq \emptyset$ in $M$. From $B \in \text{Prop}(f^*(A, w))$ we derive by definition of $f^*$ and by Lemma A.2 that $B \in \text{Prop}(f^*(A, w))$. By (S-CV) in the original model, it follows that $B \in \text{Prop}(f(A \land C, w))$. By definition of $f^*$ and Theorem A.2, we conclude that $B \in \text{Prop}(f^*(A \land C, w))$.

2. Let $w \in W_n - W_{n-1}$. We have to consider three cases.
   (a) Let $f^*(D, w') \cap [[A]]^* = \emptyset$ and $f^*(D, w') \cap [[A \land C]]^* = \emptyset$. In this case, $f^*(A, w) = f^*(A, w_0)$ and $f^*(A \land C, w) = f^*(A \land C, w_0)$. The property holds for it holds for $w_0$.
   (b) If $f^*(D, w') \cap [[A]]^* \neq \emptyset$ and $f^*(D, w') \cap [[A \land C]]^* \neq \emptyset$, then $f^*(A, w) = f^*(A \land D, w')$ and $f^*(A \land C, w) = f^*(A \land D) \land C, w')$ (by (S-RCEA)). The property holds for it holds for $w \in W_{n-1}$.
   (c) Let $f^*(D, w') \cap [[A]]^* \neq \emptyset$ but $f^*(D, w') \cap [[A \land C]]^* = \emptyset$. We show that (S-CV) trivially holds, since $f^*(A, w) \cap [[C]]^* = \emptyset$. First of all, notice that, by definition, $f^*(A, w) = f^*(A \land D, w')$. Second, notice that by (S-CV) for $w' \in W_{n-1}$, $\text{Prop}(f^*(D, w')) \subseteq \text{Prop}(f^*(A \land D, w'))$. Third, notice that by the hypothesis $f^*(D, w') \cap [[A \land C]]^* = \emptyset$, thus $\neg(A \land C) \in \text{Prop}(f^*(D, w'))$. From these three observations, it follows that $\neg(A \land C) \in \text{Prop}(f^*(A, w))$, and by (S-ID) and propositional calculus $\neg C \in \text{Prop}(f^*(A, w))$. Hence, $f^*(A, w) \cap [[C]]^* = \emptyset$, and (S-CV) trivially holds.

(S-RCEA)

1. Let $w \in W_{n-1}$
   (a) If $x \in f^*(A, w)$, then by definition of $f^*$, $x \in f(A, w)$. There are now two cases: either $x \in W_{n-1}$ or $x \in W_n - W_{n-1}$. In the first case, $f^*(\top, x) = f(\top, x) \cap W_n$. Hence, if $y \in f^*(\top, x)$, then $y \in f(\top, x)$, and by (S-RCEA) in $M$, it follows that $y \in f(A, w)$, thus $y \in f(A, w) \cap W_n = f^*(A, w)$. In the second case, if $y \in f^*(\top, w)$, then $y \in f(D, w')$ for $D, w'$ such that $x \in f(D, w')$. By (S-EUC) in $M$, $y \in f(\top, x)$, and by (S-RCEA) in $M$, it follows that $y \in f(A, w)$, hence $y \in f(A, w) \cap W_n = f^*(A, w)$.

2. Let $w \in W_n - W_{n-1}$.
   We consider two cases.
   (a) $f^*(D, w') \cap [[A]]^* \neq \emptyset$. If $x \in f^*(A, w)$, by definition of $f^*$, $x \in f^*(A \land D, w')$ for $D, w'$ such that $w \in f(D, w')$. 


We now have to distinguish two cases: either $x \in W_{n-1}$ or $x \in W_n - W_{n-1}$. In the first case, if $y \in f^*(\top, x)$, then $y \in f(\top, x)$. By (S-TRANS) applied to $M$, it follows that $y \in f(A \land D, w')$ and therefore $y \in f^*(A, w)$. If $x \in W_n - W_{n-1}$, then if $y \in f^*(\top, x)$, then $y \in f^*(A \land D, w')$ and therefore $y \in f^*(A, w)$.

(b) If $f^*(D, w') \cap [[A]]^* = \emptyset$, then $f^*(A, w) = f^*(A, w_0)$. The property holds, since it holds for $f^*(A, w_0)$.

(S-CONS)

1. Let $w \in W_{n-1}$. By (S-CONS) in the original model $M$, for any $A \in \mathcal{L}_F$, if $\not \vdash_{PC} \neg A$, then $f(A, w) \neq \emptyset$. Hence, also $\pi(f(A, w)) \neq \emptyset$ and by construction of $M^*$, also $f^*(A, w) = f(A, w) \cap W_n \neq \emptyset$.

2. Let $w \in W_n - W_{n-1}$. If $A \in \mathcal{L}_F$, then either $f^*(A, w) = f^*(A \land D, w')$ or $f^*(A, w) = f(A, w_0)$. In both cases, for what shown at the previous step, $f^*(A, w) \neq \emptyset$.

Theorem A.6. The model $M^*$ is finite.

Proof. By construction, since $n < \omega$ and $W_n$ is finite. □

References


