



Schrödinger operators on graphs and geometry I: Essentially bounded potentials

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Abstract

The inverse spectral problem for Schrödinger operators on finite compact metric graphs is investigated. The relations between the spectral asymptotics and geometric properties of the underlying graph are studied. It is proven that the Euler characteristic of the graph can be calculated from the spectrum of the Schrödinger operator in the case of essentially bounded real potentials and standard boundary conditions at the vertices. Several generalizations of the presented results are discussed.

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1. Introduction

The theory of differential operators on metric graphs is a rapidly developing area of modern mathematical physics. Interest to these problems can be explained not only by important applications in the theory of quantum wave guides and nanoelectronics, but by discovered interesting phenomena putting these problems in the area between ordinary and partial differential operators. Indeed methods originally developed for both areas are successfully applied to study the problems on metric graphs. The main aim of this paper is to study the relation between the spectrum of a Schrödinger operator on such a graph and geometric properties of the graph. This question

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has already been studied for Laplace operators with standard boundary conditions at the vertices (see (1) below) and it was proven that the spectrum of the Laplace operator determines the total length, the number of connected components and the Euler characteristics of the underlying graph (see Definition 3). To establish the relation between the spectrum and the Euler characteristics one used so-called trace formula (see (2)) connecting the spectrum of the Laplacian with the set of periodic orbits on the graph. Using this relation effective formulas for the Euler characteristic can be proven (see (6) and (7)). The main goal of the current paper is to generalize these results to the case of Schrödinger operators on graphs. Such operators are not uniquely determined by the underlying graphs (like Laplacians with standard boundary conditions), but depend on the choice of real valued potential and may be other than the standard boundary conditions at the vertices. In the current article we are going to confine ourselves to the case of essentially bounded potentials and standard boundary conditions. The case of L_1 potentials and most general symmetric boundary conditions will be considered in the following publication, since these problems can be treated by similar methods.

To obtain a connection between the spectrum of a Schrödinger operator and the Euler characteristic of the graph we use trace formula. A similar formula was first proven by J.-P. Roth [33] using the heat kernel expansion, but we are going to use the trace formula in the form (2) first presented (without a proof) by J.-P. Roth as well. The formula we are going to use was first given by T. Kottos and U. Smilansky, but without paying attention to the fact that the secular equation describing the spectrum using vertex and edge scattering matrices in general does not determine the correct multiplicity of the eigenvalue zero. Correction of this inaccuracy allowed us to prove that two isospectral graphs (the corresponding Laplacians have the same spectra) have the same Euler characteristic and provide an effective formula for it. In the current article we not only prove a slightly different formula for the Euler characteristic, but give an alternative proof of this formula without any use of the trace formula, but only for graphs with the edges being integer multiples of one length to be called the basic length. We hope that this approach provides a new insight in the spectral asymptotics for such operators.

The new formula for the Euler characteristic obtained in this paper is valid not only for Laplacians but for Schrödinger operators with essentially bounded potentials and standard boundary conditions. To prove this fact we show first that the Euler characteristics is determined by the asymptotics of the eigenvalues only. Then it remains to prove that the spectra of a Laplacian and the corresponding Schrödinger operator are asymptotically close and therefore the formula for the Euler characteristic originally proven for Laplacians gives the correct result if the spectrum of the Laplacian is just substituted with the spectrum of the Schrödinger operator.

Current paper extends the recent article [26] and therefore correct references and historical remarks can be found there. On the other hand, it is impossible not to mention that the theory of differential operators on graphs has been grown from the pioneering works by B. Pavlov, N. Gerasimenko [15,16], P. Exner, P. Šeba [13] and Y. Colin de Verdière [11,12]. Recent interest in this subject was initiated by papers by V. Kostrykin and R. Schrader [19–21]. Several articles have been devoted to differential operators on trees—special class of graphs without cycles. It is possible to state that this problem is fully understood now [1,3,4,9,27,34,36,37]. On the other hand, very few papers are devoted to operators on arbitrary graphs (with cycles) and here we would like to mention J. von Below [5], R. Carlson [10], L. Friedlander [14], B. Gutkin and U. Smilansky [18], V. Kostrykin and R. Schrader [22], J. Boman, P. Kurasov, F. Stenberg and M. Nowaczyk [6,24–26,30,31]. The results presented in this article can be considered as a natural generalization of the classical results on inverse spectral and scattering problems for

ordinary differential equations [7,8,28]. Similar questions for discrete operators on graphs have been discussed by S.P. Novikov [29] and Y. Colin de Verdière [12].

2. Graph Laplacians: basic notations and trace formula

We are going to consider here only finite compact metrics graphs, i.e. metric graphs formed by a finite number of compact intervals.

Definition 1. A finite compact metric graph $\Gamma = \Gamma(\mathbf{E}, \sigma)$ consists of a finite set \mathbf{E} of compact intervals $\Delta_j, j = 1, 2, \dots, N$, called edges, and a partition σ of the set $\mathbf{V} = \{x_j\}$ of endpoints x_j of the edges, $\mathbf{V} = \bigcup_m V_m$. The equivalence classes $V_m, m = 1, 2, \dots, M$, will be called vertices, and the number of elements in V_m will be called the valence of V_m .

The distances on the edges and identification of the end points belonging to the same equivalence class induce naturally the distance on Γ and allows one to introduce the space $L_2(\Gamma)$ of square integrable functions on the graph with the standard scalar product $\langle f, g \rangle = \int_{\Gamma} f(x)g(x) dx$. This space is independent of the connectivity of the graph, since

$$L_2(\Gamma) = \bigoplus_{n=1}^N L_2(\Delta_n).$$

Definition 2. The Laplace operator $L(\Gamma)$ is the operator of negative second derivative in $L_2(\Gamma)$ defined on the domain of functions f from the Sobolev space $\bigoplus_{n=1}^N W_2^2(\Delta_n)$ satisfying standard boundary conditions at the vertices

$$\begin{cases} \sum_{x_j \in V_m} \partial_n f(x_j) = 0; \\ f \text{ is continuous at } V_m; \end{cases} \quad m = 1, 2, \dots, M, \tag{1}$$

where $\partial_n f(x_j)$ denotes the normal derivative of the function f at the end point x_j :

$$\partial_n f(x_j) = \begin{cases} f'(x_j) & \text{if } x_j \text{ is the left end point,} \\ -f'(x_j) & \text{if } x_j \text{ is the right end point.} \end{cases}$$

These boundary conditions reflect the connectivity of the graph in the sense, that two graphs formed by the same set of edges determine different standard boundary conditions if their vertex structures are different. It is important that the graph Γ determines the operator $L(\Gamma)$ uniquely. One of important characteristics of graphs is their Euler characteristic.

Definition 3. The Euler characteristic of a (not necessarily connected) graph Γ formed by M vertices and N edges is

$$\chi(\Gamma) = M - N.$$

For connected graphs the Euler characteristic determines the number g of generators in the fundamental group on Γ

$$g = 1 - \chi.$$

One of the main tools to study the spectral properties of graph Laplacians is the trace formula, which connects the spectrum of the Laplace operator on a finite compact graph with the set of periodic orbits on it. The first formula of this kind was proven by J.-P. Roth [33]. The formula we are going to use first appeared in the papers by B. Gutkin, T. Kottos and U. Smilansky [18,23], and later by P. Kurasov and M. Nowaczyk [24,26]. We refer to [26] for the proof of the following theorem:

Proposition 1 (Trace formula, Theorem 2 from [26]). *Let Γ be a compact metric graph with Euler characteristic χ and the total length \mathcal{L} , and let $L(\Gamma)$ be the corresponding Laplace operator. Then the following two trace formulae establish the relation between the spectrum $\{k_n^2\}$ of $L(\Gamma)$ and the set \mathcal{P} of closed paths on the metric graph Γ*

$$\begin{aligned}
 u(k) &\equiv 2m_s(0)\delta(k) + \sum_{k_n \neq 0} (\delta(k - k_n) + \delta(k + k_n)) \\
 &= \chi\delta(k) + \frac{\mathcal{L}}{\pi} + \frac{1}{\pi} \sum_{p \in \mathcal{P}} l(\text{prim}(p))S(p) \cos kl(p),
 \end{aligned}
 \tag{2}$$

and

$$\begin{aligned}
 \sqrt{2\pi}\hat{u}(l) &= 2m_s(0) + \sum_{k_n \neq 0} 2 \cos k_n l \\
 &= \chi + 2\mathcal{L}\delta(l) + \sum_{p \in \mathcal{P}} l(\text{prim}(p))S(p)(\delta(l - l(p)) + \delta(l + l(p))),
 \end{aligned}
 \tag{3}$$

where

- $m_s(0)$ is the multiplicity of the eigenvalue zero;²
- p is a closed path on Γ ;
- $l(p)$ is the length of the closed path p ;
- $\text{prim}(p)$ is one of the primitive paths for p ;
- $S(p)$ is the product of all vertex scattering coefficients along the path p .

By a closed path on a metric graph we understand any continuous closed path, which does not turn back in the interior of any edge, but which may turn back at any vertex. It is clear that if the graph has neither loops nor two edges with the same endpoints, then every closed path is uniquely determined by the sequence of vertices it comes across. Cyclic permutation of the sequence of vertices does not change the closed path, i.e. paths are viewed as geometric sets and do not have the start and end points. By a primitive path $\text{prim}(p)$ we denote any closed continuous path such that the path p can be obtained by repeating the path $\text{prim}(p)$. With each vertex V_m we associate

² It is equal to the number C of connected components in accordance with Theorem 1 from [26].

the vertex scattering matrix formed by reflection and transition coefficients which are determined by the valence of the vertex only

$$(S_v^m)_{ij} = \begin{cases} \frac{2}{v_m}, & i \neq j, \\ \frac{2-v_m}{v_m}, & i = j, \end{cases} \tag{4}$$

where v_m is the valence of V_m , i.e. the number of elements in the equivalence class V_m . Metric graphs can be viewed as billiards where point particles are moving along the edges. Coming to the vertices the particles may be either reflected or transmitted to a neighboring edge. Each closed path is just a periodic trajectory for such a particle. Therefore to each path one associates the product $S(p)$ of all scattering coefficients along this path which represents in some sense the probability for the particle to take this particular trajectory.

Formulas (2) and (3) allow one to prove that two Laplace operators on compact finite metric graphs having the same spectrum have also the same

- number of connected components;
- total length;
- Euler characteristic.

(Uniqueness Theorem 1, [26].) The number of connected components is equal to the multiplicity of the zero eigenvalue (Theorem 1 from [26]). The total length of the graph determines the asymptotics of the eigenvalues (Weyl’s law)

$$\lim_{n \rightarrow \infty} \sqrt{\lambda_n}/n = \pi/\mathcal{L}. \tag{5}$$

This formula can be proven if one takes into account that the Laplace operator on Γ is a finite rank perturbation (in the resolvent sense) of the orthogonal sum of Laplace operators on separated edges. The same asymptotics holds for Schrödinger operators and can be proven using the same method (see e.g. [35] where this formula is proven even for weighted Laplacians).

To calculate the Euler characteristic the following formula has been derived (see (4.1) from [26]):

$$\chi = 2m_s(0) + \lim_{t \rightarrow \infty} \sum_{k_n \neq 0} \frac{2t}{k_n} \sin \frac{k_n}{t} \left(2 \cos \frac{k_n}{t} - 1 \right), \quad k_n^2 = \lambda_n. \tag{6}$$

In what follows we shall need a modification of this formula.

Theorem 1. *Let Γ be a compact metric graph and $L(\Gamma)$ be the corresponding Laplace operator. Then the Euler characteristic $\chi(\Gamma)$ is uniquely determined by the spectrum $\{\lambda_n\}$ of the Laplace operator $L(\Gamma)$*

$$\begin{aligned} \chi &= 2m_s(0) + 2 \lim_{t \rightarrow \infty} \sum_{k_n \neq 0} \cos k_n/t \left(\frac{\sin k_n/2t}{k_n/2t} \right)^2 \\ &= 2m_s(0) - 2 \lim_{t \rightarrow \infty} \sum_{k_n \neq 0} \frac{1 - 2 \cos k_n/t + \cos 2k_n/t}{(k_n/t)^2}. \end{aligned} \tag{7}$$

Proof. We present here a proof of this theorem based on trace formula (2), another direct proof for graphs with rationally depended lengths will be given in the following subsection.

Consider the function φ defined as

$$\varphi(l) = \begin{cases} l, & 0 \leq l \leq 1; \\ 2 - l, & 1 \leq l \leq 2; \\ 0, & \text{otherwise.} \end{cases}$$

This function and any scaled function $\varphi_t(x) = t\varphi(tx)$ satisfy the following properties:

$$\int_{-\infty}^{+\infty} \varphi_t(x) dx = 1 \quad \text{and} \quad \varphi_t(0) = 0.$$

It is clear then that the Euler characteristic can be calculated from the Fourier transform of the distribution u using formula (3) and taking the limit $t \rightarrow +\infty$, since the length of the shortest orbit is positive

$$\chi(\Gamma) = \sqrt{2\pi} \lim_{t \rightarrow +\infty} \hat{u}[\varphi_t].$$

In fact $\hat{u}[\varphi_t]$ is just equal to $\chi(\Gamma)/\sqrt{2\pi}$ for $t > 2/\min\{d_j\}$, where d_j are the lengths of Δ_j . The Fourier transform of the function φ_t can easily be calculated

$$\hat{\varphi}_t(k) = \frac{1}{\sqrt{2\pi}} e^{ik/t} \left(\frac{\sin k/2t}{k/2t} \right)^2.$$

Now the Euler characteristic may be calculated by applying the distribution u to the test function $\hat{\varphi}_t$

$$\chi(\Gamma) = \sqrt{2\pi} \lim_{t \rightarrow +\infty} u[\hat{\varphi}_t] = 2m_s(0) + 2 \lim_{t \rightarrow +\infty} \sum_{k_n \neq 0} \cos k_n/t \left(\frac{\sin k_2/2t}{k_2/2t} \right)^2.$$

The second formula (7) is a result of direct calculation. \square

The eigenvalues of $L(\Gamma)$ satisfy the Weyl asymptotic law, i.e. grow linearly with n and therefore the sum in (7) is absolutely converging in contrast to one in (6).

3. Graphs having basic length

In this section we are going to study a very special class of graphs with rationally depended lengths of edges, namely the graphs with all edge lengths being integer multiples of one and the same length Δ to be called the *basic length*

$$d_j = n_j \Delta, \quad n_j \in \mathbb{N}. \tag{8}$$

Without loss of generality we may assume that Δ is the largest such number and we introduce the *basic frequency*

$$\Omega = \frac{2\pi}{\Delta}. \tag{9}$$

Such graphs will be called *graphs with basic length Δ* in what follows.³

The spectrum of the Laplace operator on such a graph appears to be almost periodic with the period Ω if one considers the k -scale instead of the λ -scale ($k^2 = \lambda$).

Lemma 1. *Let k_n^2 be an eigenvalue of the Laplace operator $L(\Gamma)$ on a graph Γ with the basic length Δ . Then $(k_n + \frac{2\pi}{\Delta})^2$ is an eigenvalue of $L(\Gamma)$ as well. The multiplicities of the eigenvalues coincide if both k_n^2 and $(k_n + \frac{2\pi}{\Delta})^2$ are different from zero.*

Proof. To prove the first statement it is enough to show that if $\psi(x)$ is an eigenfunction of $L(\Gamma)$ corresponding to a certain eigenvalue k_n^2 , then there exists another function $\tilde{\psi}$, which is an eigenfunction corresponding to $(k_n + \frac{2\pi}{\Delta})^2$. Every eigenfunction ψ can be written as a combination of incoming waves

$$\psi(x) = a_{2j-1}e^{ik_n|x-x_{2j-1}|} + a_{2j}e^{ik_n|x-x_{2j}|}, \quad x \in [x_{2j-1}, x_{2j}].$$

Then the function $\tilde{\psi}$ given on each interval $[x_{2j-1}, x_{2j}]$ by

$$\tilde{\psi}(x) = a_{2j-1}e^{i(k_n+2\pi/\Delta)|x-x_{2j-1}|} + a_{2j}e^{i(k_n+2\pi/\Delta)|x-x_{2j}|} \tag{10}$$

is an eigenfunction corresponding to the eigenvalue $\lambda = (k_n + 2\pi/\Delta)^2$. Really, this function satisfies the necessary differential equation on the edges and is continuous at the vertices. Hence to show that $\tilde{\psi}$ is an eigenfunction it is enough to show that the sum of normal derivatives at each vertex is zero, which is obviously true, since the normal derivatives of ψ and $\tilde{\psi}$ are related by

$$\partial_n \psi(x_j) = \frac{k_n}{k_n + 2\pi/\Delta} \partial_n \tilde{\psi}(x_j).$$

If $k_n + 2\pi/\Delta = 0$, then it is well known that the corresponding point $\lambda = 0$ is always an eigenvalue of the Laplacian (Theorem 1 from [26]).

The second statement that the multiplicities of the two eigenvalues coincide follows from the fact that for $\lambda \neq 0$ every eigenfunction is uniquely determined by the coefficients $a_j, j = 1, 2, \dots, 2N$. Therefore the transformation $\psi \mapsto \tilde{\psi}$ is one-to-one. \square

This lemma allows us to describe the structure of the spectrum of $L(\Gamma)$ for graphs having basic length.

Theorem 2. *Let $L(\Gamma)$ be a Laplace operator on a connected graph Γ with the basic length Δ . Its spectrum is pure discrete*

$$\sigma(L(\Gamma)) = \{\lambda_n = k_n^2, k_n \geq 0\}_{n=0}^\infty$$

³ Similar graphs have been studied recently in [32].

and has the following properties:

- (1) Apart from the point $k = 0$ the set $\{k_n\}_{n=0}^\infty$ is invariant under right shifts by $\Omega = 2\pi/\Delta$.
- (2) The points k_n inside the interval $(0, 2\pi/\Delta)$ are symmetric with respect to the center of the interval, i.e. if $\lambda_n = k_n^2$ is an eigenvalue, then $(2\pi/\Delta - k_n)^2$ is also an eigenvalue.
- (3) The point $\lambda_0 = 0$ has multiplicity 1.
- (4) The points $\lambda = (\frac{2\pi}{\Delta}m)^2, m = 1, 2, \dots$, have multiplicity $2 - \chi$, where χ is the Euler characteristic of Γ .

Proof. The proof of statement (1) follows immediately from Lemma 1. To prove the second statement we use again Lemma 1 for $(-k_n)^2 = k_n^2$ to conclude that if k_n^2 is an eigenvalue, then $(-k_n + 2\pi/\Delta)^2$ is also an eigenvalue. The third statement is valid for arbitrary connected graphs (Theorem 1 from [26]).

Let us establish the last assertion. The proof we present here is a modification of the proof of Theorem 1 from [26]. Without loss of generality we assume that $k = \Omega = 2\pi/\Delta$. Then it is easy to construct $g + 1$ linearly independent eigenfunctions corresponding to this particular k , where g is the number of generators in the fundamental group for Γ . Let us denote by $\psi_0(x, k)$ the eigenfunction given by

$$\psi_0(x, k) = \cos k(x - x_{2j-1}), \quad x \in [x_{2j-1}, x_{2j}].$$

This function satisfies the differential equation on each edge, is continuous at all vertices (in fact it is equal to 1 at all vertices) and all normal derivatives are equal to zero, which implies that their sum at each vertex is also zero and the boundary conditions at all vertices are satisfied.

If Γ is not a tree, then it can be transformed to a tree T by removing exactly $g = 1 - \chi = N - M + 1$ edges denoted by $\Delta_1, \Delta_2, \dots, \Delta_{N-M+1}$ without loss of generality. Let us denote by l_i the shortest nontrivial closed path on $T \cup \Delta_i$ passing through Δ_i . Note that every such path comes across exactly one removed edge. Consider the functions ψ_i defined by

$$\psi_i(x, k) = \begin{cases} \pm \sin k(x - x_{2j-1}), & \text{provided } x \in \Delta_j \text{ and } \Delta_j \subset l_i; \\ 0, & \text{otherwise;} \end{cases}$$

where the sign depends on whether the path l_i runs along Δ_j in the positive (+) or in the negative (−) direction. The function ψ_i is not only continuous along the path l_i but its first derivative is continuous as well.

Each function ψ_i satisfies the eigenfunction equation, is continuous at all vertices (in fact equal to zero there) and the sum of normal derivatives at each vertex is zero (if the vertex is on the path l_i then only two normal derivatives are different from zero but cancel each other, if the vertex is not on the path, then all normal derivatives are zero).

It is clear that the functions $\psi_0, \psi_1, \dots, \psi_g$ are linearly independent and this implies that the multiplicity of the eigenvalue k is not less than $1 + g$. It remains to prove that these functions span the corresponding eigensubspace up.

Let $\psi(x, k)$ be any eigenfunction of $L(\Gamma)$ corresponding to $k = 2\pi/\Delta$. First of all it is clear that this function attains the same value at all vertices, say f_0 . The function $\psi(x, k) - f_0\psi_0(x, k)$ is an eigenfunction of $L(\Gamma)$ equal to zero at all vertices. Then the restriction of this function to the edge Δ_i is proportional to $\sin k(x - x_{2i-1})$

$$\psi(x, k) - f_0\psi_0(x, k)|_{\Delta_j} = f_i \sin k(x - x_{2i-1}).$$

Consider the function

$$\hat{\psi}(x, k) = \psi(x, k) - f_0\psi_0(x, k) - \sum_{i=1}^g f_i\psi_i(x, k),$$

which is an eigenfunction of $L(\Gamma)$ equal to zero at all vertices and on all edges $\Delta_i, i = 1, 2, \dots, g$. It follows that this function is supported by the tree T . It is easy to see that $\hat{\psi}(x, k)$ is identically equal to zero. First we note that $\hat{\psi}$ is equal to zero on all loose edges (satisfies zero Cauchy data at the loose end points). Then it follows that $\hat{\psi}$ is zero on all edges connected by at least one end point to only loose edges. Continuing in this way we conclude that $\hat{\psi}(x, k)$ is identically equal to zero, which implies

$$\psi(x, k) = f_0\psi_0(x, k) + \sum_{i=1}^g f_i\psi_i(x, k).$$

The last equality means that the spectral multiplicity of $\lambda = (2\pi/\delta)^2$ is equal to $1 + g = 2 - \chi$. The proof for $\lambda = (2\pi m/\delta)^2, m = 2, 3, \dots$, follows the same lines. \square

Now we are ready to give a direct proof of Theorem 1 formula (7) for the Euler characteristic under the additional condition that Γ has a basic length.

Proof of Theorem 1. (For graphs having basic length.) Assume first that the graph Γ is connected. Let us denote by $\omega_j^2, j = 1, 2, \dots, J$ the eigenvalues of $L(\Gamma)$ inside the interval $(0, \Omega)$ (see (9)). Then the limit on the right-hand side of (7) can be written as follows:

$$\begin{aligned} & 2 - 2 \lim_{t \rightarrow \infty} \sum_{k_n \neq 0} \frac{1 - 2 \cos k_n/t + \cos 2k_n/t}{(k_n/t)^2} \\ &= 2 - 2(1 + g) \lim_{t \rightarrow \infty} \sum_{m=1}^{\infty} \frac{1 - 2 \cos \Omega m/t + \cos 2\Omega m/t}{(\Omega m/t)^2} \\ &\quad - 2 \lim_{t \rightarrow \infty} \sum_{m=0}^{\infty} \sum_{j=1}^J \frac{1 - 2 \cos(\omega_j + \Omega m)/t + \cos 2(\omega_j + \Omega m)/t}{((\omega_j + \Omega m)/t)^2}, \end{aligned} \tag{11}$$

where we used that all points $k = m\Omega, m = 1, 2, \dots$, have multiplicity $1 + g$ and the point $k = 0$ has multiplicity 1. The first limit can be calculated using formula

$$\sum_{m=1}^{\infty} \frac{1 - 2 \cos m/t + \cos 2m/t}{(m/t)^2} = \frac{1}{2}, \tag{12}$$

since the limit obviously does not depend on the value of Ω . To prove this formula we use the sum ((1.443.3) from [17])

$$\sum_{m=1}^{\infty} \frac{\cos mx}{m^2} = \frac{\pi^2}{6} - \frac{\pi x}{2} + \frac{x^2}{4}$$

to obtain

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{1 - 2 \cos m/t + \cos 2m/t}{(m/t)^2} &= t^2 \left(\sum_{m=1}^{\infty} \frac{1}{m^2} - 2 \sum_{m=1}^{\infty} \frac{\cos m/t}{m^2} + \sum_{m=1}^{\infty} \frac{\cos m^2/t}{m^2} \right) \\ &= t^2 \left\{ \frac{\pi^2}{6} - 2 \left(\frac{\pi^2}{6} - \frac{\pi}{2t} + \frac{1}{4t^2} \right) + \frac{\pi^2}{6} - \frac{\pi}{2t} + \frac{1}{4t^2} \right\} = \frac{1}{2}. \end{aligned}$$

To calculate the second limit let us use that the points ω_j are situated symmetrically with respect to the center of the interval $(0, \Omega)$

$$\begin{aligned} &\sum_{m=0}^{\infty} \sum_{j=1}^J \frac{1 - 2 \cos(\omega_j + \Omega m)/t + \cos 2(\omega_j + \Omega m)/t}{((\omega_j + \Omega m)/t)^2} \\ &= \frac{1}{2} \sum_{m=0}^{\infty} \sum_{j=1}^J \left\{ \frac{1 - 2 \cos(m + \omega_j/\Omega)/(t/\Omega) + \cos 2(m + \omega_j/\Omega)/(t/\Omega)}{((m + \omega_j/\Omega)/(t/\Omega))^2} \right. \\ &\quad \left. + \frac{1 - 2 \cos(m + (\Omega - \omega_j)/\Omega)/(t/\Omega) + \cos 2(m + (\Omega - \omega_j)/\Omega)/(t/\Omega)}{((m + (\Omega - \omega_j)/\Omega)/(t/\Omega))^2} \right\} \\ &= \frac{1}{2} \sum_{m \in \mathbb{Z}} \sum_{j=1}^J \frac{1 - 2 \cos(m + \omega_j/\Omega)/(t/\Omega) + \cos 2(m + \omega_j/\Omega)/(t/\Omega)}{((m + \omega_j/\Omega)/(t/\Omega))^2}. \end{aligned} \tag{13}$$

We are going to prove that the last sum is equal to zero using the formula

$$\sum_{m \in \mathbb{Z}} \frac{e^{i(m+\alpha)x}}{(m + \alpha)^2} = \frac{2\pi e^{2\pi i\alpha}}{1 - e^{2\pi i\alpha}} x - \frac{(2\pi)^2 e^{2\pi i\alpha}}{(1 - e^{2\pi i\alpha})^2}, \quad \alpha \notin \mathbb{Z}. \tag{14}$$

To prove this formula one may exploit the following idea: find a linear function $f(x) = ax + b$, $0 \leq x \leq 2\pi$, such that the series on the left-hand side of the formula is exactly the Fourier series for f in the orthogonal basis $e^{i(n+\alpha)x}$. The function f is represented by the following almost everywhere converging Fourier series

$$f(x) = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} f_m e^{i(m+\alpha)x}, \tag{15}$$

where

$$f_m = \int_0^{2\pi} (ax + b) e^{-i(m+\alpha)x} dx. \tag{16}$$

The function f may be chosen equal to

$$f(x) = \frac{e^{2\pi i\alpha}}{1 - e^{2\pi i\alpha}} x - \frac{2\pi e^{2\pi i\alpha}}{(1 - e^{2\pi i\alpha})^2}.$$

Then the Fourier coefficients are given by

$$f_m = \int_0^{2\pi} \left(\frac{e^{2\pi i \alpha}}{1 - e^{2\pi i \alpha}} x - \frac{2\pi e^{2\pi i \alpha}}{(1 - e^{2\pi i \alpha})^2} \right) e^{-i(m+\alpha)x} dx = \frac{1}{(m + \alpha)^2}.$$

Thus formula (14) is proven and it implies in particular that

$$\sum_{m \in \mathbb{Z}} \frac{1 - 2e^{i(m+\alpha)x} + e^{2i(m+\alpha)x}}{(m + \alpha)^2} = 0, \quad \text{provided } \alpha \notin \mathbb{Z}. \tag{17}$$

Really using (14) we have:

$$\begin{aligned} & \sum_{m \in \mathbb{Z}} \frac{1 - 2e^{i(m+\alpha)x} + e^{2i(m+\alpha)x}}{(m + \alpha)^2} \\ &= \sum_{m \in \mathbb{Z}} \frac{1}{(m + \alpha)^2} - 2 \sum_{m \in \mathbb{Z}} \frac{e^{i(m+\alpha)x}}{(m + \alpha)^2} + \sum_{m \in \mathbb{Z}} \frac{e^{2i(m+\alpha)x}}{(m + \alpha)^2} \\ &= -\frac{(2\pi)^2 e^{2\pi i \alpha}}{(1 - e^{2\pi i \alpha})^2} - 2 \left(-\frac{(2\pi)^2 e^{2\pi i \alpha}}{(1 - e^{2\pi i \alpha})^2} + \frac{2\pi e^{2\pi i \alpha}}{1 - e^{2\pi i \alpha}} x \right) - \frac{(2\pi)^2 e^{2\pi i \alpha}}{(1 - e^{2\pi i \alpha})^2} + \frac{2\pi e^{2\pi i \alpha}}{1 - e^{2\pi i \alpha}} 2x \\ &= 0, \end{aligned}$$

where to calculate the first sum we used that the series (15) at $x = 0$ converges to $\frac{1}{2}(f(+0) + e^{-2\pi 1\alpha} f(2\pi - 0))$. It turns out that

$$\sum_{m \in \mathbb{Z}} \frac{1 - 2 \cos(m + \alpha)x + \cos 2(m + \alpha)x}{(m + \alpha)^2} = 0, \quad \text{provided } \alpha \notin \mathbb{Z}, \tag{18}$$

and therefore the second sum in (11) (the sum (13)) is equal to zero. Finally we get

$$2 - 2 \lim_{t \rightarrow \infty} \sum_{k_n \neq 0} \frac{1 - 2 \cos k_n/t + \cos 2k_n/t}{(k_n/t)^2} = 2 - 2(1 + g) \frac{1}{2} + 0 = 1 - g = \chi.$$

Now it is straightforward to generalize this result to include not connected graphs to get (7). \square

Thus we have proven the formula for the Euler characteristic for graphs having basic length without any use of the trace formula. It might be important to find a similar proof for arbitrary graphs. Such alternative to the trace formula approach may provide a new insight on the structure of the spectral asymptotics for $L(\Gamma)$.

We would like to present here few explicit examples illustrating formulas (7).

- (1) *Single interval.* Let the graph Γ coincide with the interval $[0, \pi]$ (with separated end points). The Euler characteristic is $\chi = 1$. The spectrum of $L(\Gamma)$ is $\sigma(L) = \{n^2, n = 0, 1, 2, \dots\}$. Substituting $k_n = n, n = 0, 1, 2, \dots$, into formula (7) we get

$$\chi = 2 - 2 \lim_{t \rightarrow \infty} \sum_{n=1}^{\infty} \frac{1 - 2 \cos nt + \cos 2n/t}{(n/t)^2} = 2 - 1 = 1,$$

where we used formula (12).

- (2) *Simple circle.* Let the graph Γ coincide with the circle having length π , i.e. it can be treated as the interval $[0, \pi]$ with the end points identified. The Euler characteristic is $\chi = 0$. The spectrum of $L(\Gamma)$ is $\sigma(L) = \{n^2, n = 0, 2, 2, 4, 4, \dots\}$. Substitution $k_n = n$ into formula (7) gives

$$\chi = 2 - 4 \lim_{t \rightarrow \infty} \sum_{n=1}^{\infty} \frac{1 - 2 \cos 2nt + \cos 4n/t}{(n/t)^2} = 2 - 2 = 0,$$

where we again used formula (12).

- (3) *Symmetric star graph.* Let Γ be the star graph formed by m equal edges of the length π joined at one end point. The Euler characteristic is $\chi = 1$. The spectrum consists of simple eigenvalues $n^2, n = 0, 1, 2, \dots$, and eigenvalues $(1/2 + n)^2, n = 0, 1, 2, \dots$, having multiplicity $m - 1$. The formula (7) gives then

$$\begin{aligned} \chi &= 2 - 2 \lim_{t \rightarrow \infty} \sum_{n=1}^{\infty} \frac{1 - 2 \cos n/t + \cos 2n/t}{(n/t)^2} \\ &\quad - 2(m - 1) \lim_{t \rightarrow \infty} \sum_{n=1}^{\infty} \frac{1 - 2 \cos(n + 1/2)/t + \cos 2(n + 1/2)/t}{((n + 1/2)/t)^2} \\ &= 2 - 1 - 0 = 1, \end{aligned}$$

where we used formulas (12) and (17).

4. Spectral asymptotics and Euler characteristic

In this section we are going to show that the Euler characteristic is determined entirely by the asymptotics of the spectrum of the Laplace operator. The limit of each term in the series (7) does not depend on k_n

$$\lim_{t \rightarrow \infty} \frac{1 - 2 \cos k_n/t + \cos 2k_n/t}{(k_n/t)^2} = -1.$$

Taking this into account it is clear that changing of any finite number of eigenvalues does not affect the limit (7). We are going to prove that the same is true even if the number of changed eigenvalues is infinite, but under certain additional restrictions.

Lemma 2. Let k_n and k_n^0 be two real sequences satisfying the following conditions

$$|k_n - k_n^0| = O\left(\frac{1}{n}\right), \quad k_n^0 = \frac{\pi}{\mathcal{L}}n + O(1), \tag{19}$$

and the limit $\lim_{t \rightarrow \infty} \sum_{n=1}^{\infty} \cos k_n^0/t \left(\frac{\sin k_n^0/2t}{k_n^0/2t}\right)^2$ exists. Then the following limits coincide

$$\lim_{t \rightarrow \infty} \sum_{n=1}^{\infty} \cos k_n/t \left(\frac{\sin k_n/2t}{k_n/2t}\right)^2 = \lim_{t \rightarrow \infty} \sum_{n=1}^{\infty} \cos k_n^0/t \left(\frac{\sin k_n^0/2t}{k_n^0/2t}\right)^2. \tag{20}$$

Proof. Without loss of generality we assume that $\mathcal{L} = \pi$. It will be convenient to write estimates (19) in the form

$$|k_n - k_n^0| \leq A \frac{1}{n}, \quad |k_n - n| \leq B, \quad |k_n^0 - n| \leq B, \quad n = 1, 2, \dots, \tag{21}$$

with certain positive constants A and B . In addition we shall use the following notations

$$a_n(t) \equiv \cos k_n/t \left(\frac{\sin k_n/2t}{k_n/2t}\right)^2, \quad a_n^0(t) \equiv \cos k_n^0/t \left(\frac{\sin k_n^0/2t}{k_n^0/2t}\right)^2.$$

To prove lemma we are going to establish two estimates which will be suitable for terms with small and large indices respectively.

Estimate 1. (Suitable for small values of n .)

$$|a_n(t) - a_n^0(t)| \leq c \frac{(n + B)^2}{t^2}, \tag{22}$$

where c is a certain positive constant $c > 0$. Consider the function

$$f(\alpha) = \begin{cases} \cos 2\alpha \left(\frac{\sin \alpha}{\alpha}\right)^2, & \alpha \neq 0, \\ 1, & \alpha = 0. \end{cases}$$

The derivatives of f are

$$\begin{aligned} f'(\alpha) &= -2 \sin 2\alpha \left(\frac{\sin \alpha}{\alpha}\right)^2 + 2 \cos 2\alpha \frac{\sin \alpha}{\alpha} \frac{\alpha \cos \alpha - \sin \alpha}{\alpha^2}, \\ f''(\alpha) &= -4 \cos 2\alpha \left(\frac{\sin \alpha}{\alpha}\right)^2 - 8 \sin 2\alpha \frac{\sin \alpha}{\alpha} \frac{\alpha \cos \alpha - \sin \alpha}{\alpha^2} + 2 \cos 2\alpha \left(\frac{\alpha \cos \alpha - \sin \alpha}{\alpha^2}\right)^2 \\ &\quad + 2 \cos 2\alpha \frac{\sin \alpha}{\alpha} \frac{-\alpha^2 \sin \alpha - 2\alpha \cos \alpha + 2 \sin \alpha}{\alpha^3}, \end{aligned}$$

and we see that $f'(0) = 0$ and $f''(\alpha)$ is uniformly bounded. Hence Taylor’s formula gives

$$f(\alpha) - f(0) - f'(0)\alpha = f''(\xi) \frac{\alpha^2}{2}$$

and therefore

$$|f(\alpha) - 1| \leq \frac{1}{2} \max |f''(\alpha)| \alpha^2.$$

This implies that

$$|a_n(t) - 1| \leq \frac{1}{2} \max |f''(\alpha)| \frac{(n + B)^2}{4t^2}$$

and similar estimate (22) for the difference $|a_n(t) - a_n^0(t)|$ with $c = \frac{1}{4} \max |f''(\alpha)|$.

Estimate 2. (Suitable for large values of n .)

$$|a_n(t) - a_n^0(t)| \leq d \frac{t}{(n - B)^3}, \quad n > B, \tag{23}$$

where d is a certain positive constant $d > 0$. To prove the estimate we use that the function $\alpha^2 f'(\alpha)$ is uniformly bounded. Using the first mean value theorem we get

$$a_n(t) - a_n^0(t) = f(k_n/2t) - f(k_n^0/2t) = f'(\xi_n/2t)(k_n/2t - k_n^0/2t),$$

where ξ_n satisfies the same estimate as k_n and k_n^0 (see the second and third estimates in (21))

$$|\xi_n - n| \leq B.$$

For $n > B$, it follows that

$$|a_n(t) - a_n^0(t)| \leq \max |\alpha^2 f'(\alpha)| \frac{1}{(\frac{n-B}{2t})^2} A \frac{1}{2nt} \leq d \frac{t}{(n - B)^3}$$

with $d = 2A \max |\alpha^2 f'(\alpha)|$, which is exactly estimate (23).

The limits appearing in (20) are equal if the following limit equals to zero

$$\lim_{t \rightarrow \infty} \sum_{n=1}^{\infty} \left| \cos k_n/t \left(\frac{\sin k_n 2t}{k_n/2t} \right)^2 - \cos k_n^0/t \left(\frac{\sin k_n^0 2t}{k_n^0/2t} \right)^2 \right| \equiv \lim_{t \rightarrow \infty} \sum_{n=1}^{\infty} |a_n(t) - a_n^0(t)|. \tag{24}$$

Let us split the infinite series into the finite sum of the first K elements and the remaining infinite series as

$$\sum_{n=1}^{\infty} = \sum_{n=1}^K + \sum_{n=K+1}^{\infty} .$$

To prove that the limit is zero it is enough to prove that for any $\epsilon > 0$ there exists $t_0 = t_0(\epsilon)$, such that for any $t > t_0(\epsilon)$ the number $K = K(\epsilon, t)$ can be chosen in such a way that both the finite and infinite sums are less than $\epsilon/2$.

The two sums can be estimated using (22) and (23) as

$$\sum_{n=1}^K |a_n(t) - a_n^0(t)| \leq \sum_{n=1}^K c \frac{(K+B)^2}{t^2} \leq c \frac{(K+B)^3}{t^2},$$

$$\sum_{n=K+1}^{\infty} |a_n(t) - a_n^0(t)| \leq \sum_{n=K+1}^{\infty} d \frac{t}{(n-B)^3} \leq \frac{d}{2} \frac{t}{(K-B)^2}.$$

Each of these sums is less than $\epsilon/2$ if the following two inequalities are satisfied

$$K(\epsilon, t) \leq \left(\frac{\epsilon t^2}{2c}\right)^{1/3} - B \quad \text{and} \quad K(\epsilon, t) \geq \sqrt{\frac{dt}{\epsilon}} + B.$$

Hence the series in (24) is less than ϵ if

$$\sqrt{\frac{dt}{\epsilon}} + B \leq \left(\frac{\epsilon t^2}{2c}\right)^{1/3} - B.$$

For any $\epsilon > 0$ there exists t_0 , such that for any $t > t_0$ the last inequality is satisfied and it is possible to choose integer $K(\epsilon, t)$, such that both the finite and infinite sums are less than $\epsilon/2$. For such t we have that the infinite series in (24) is less than ϵ . It follows that the limit in (24) holds. \square

This result will be used in the following section to derive formula for the Euler characteristic using the spectra of Schrödinger operators on graphs.

5. Schrödinger operators on compact finite graphs

Consider an arbitrary essentially bounded real valued function q on Γ

$$q \in L_{\infty}(\Gamma). \tag{25}$$

Then the operator Q of multiplication by the function q is bounded in $L_2(\Gamma)$

$$\|Qf\|_{L_2(\Gamma)}^2 = \int_{\Gamma} |q(x)f(x)|^2 dx = \sum_{j=1}^N \int_{\Delta_j} |q(x)f(x)|^2 dx$$

$$\leq \sum_{j=1}^N \|q\|_{L_{\infty}(\Delta_j)}^2 \|f\|_{L_2(\Delta_j)}^2 \leq \|q\|_{L_{\infty}(\Gamma)}^2 \|f\|_{L_2(\Gamma)}^2.$$

The Schrödinger operator with the potential q can be defined as the operator sum

$$S = L(\Gamma) + Q \tag{26}$$

and it is self-adjoint on the domain of the Laplace operator (sum of a self-adjoint and a bounded self-adjoint operators), i.e. on the set of functions from $W_2^2(\Gamma \setminus \mathbf{V})$ satisfying standard boundary conditions (1).

Theorem 3. Let Γ be a compact finite metric graph and $L(\Gamma)$ be the corresponding Laplace operator with the spectrum $\lambda_n(L) = k_n^2(L)$, $n = 0, 1, 2, 3, \dots$. Let q be an $L_\infty(\Gamma)$ function and $S = L(\Gamma) + Q$ —the corresponding Schrödinger operator. Then the spectrum of S is pure discrete $\lambda_n(S) = k_n^2(S)$, $n = 0, 1, 2, \dots$,⁴ and satisfy the asymptotic formula

$$k_n(S) = k_n(L) + O(1/n), \quad \text{as } n \rightarrow \infty. \tag{27}$$

Proof. The statement that the spectrum of S is pure discrete follows from the fact that any bounded perturbation of an operator with pure discrete spectrum is pure discrete as well [2].

It is clear that the Schrödinger operator S is bounded from below by $-\|Q\|$. Let us denote by $N_\Delta(L)$ and $N_\Delta(S)$ the number of eigenvalues in the interval Δ for the operators L and S respectively. Then the following inequalities hold

$$N_{[\lambda_0(L) - \|Q\|, \lambda_n(L) + \|Q\|]}(S) \geq n \quad \text{and} \quad N_{[\lambda_0(S) - \|Q\|, \lambda_m(S) + \|Q\|]}(L) \geq m,$$

which imply that

$$|\lambda_n(S) - \lambda_n(L)| \leq \|Q\|. \tag{28}$$

In the k -scale this means that $k_n(S)$ and $k_n(L)$ are asymptotically close in the sense that (27) holds. \square

We are able to establish now the formula connecting the spectrum of a Schrödinger operator on a graph with the Euler characteristic of the graph.

Theorem 4. Let Γ be a finite compact metric graph and $L(\Gamma)$ —the corresponding Laplace operator (with standard boundary conditions). Let $q \in L_\infty(\Gamma)$ be a real valued potential and $S = L(\Gamma) + Q$ —the corresponding Schrödinger operator, where Q is the operator of multiplication by q . Then the Euler characteristic $\chi(\Gamma)$ of the graph Γ is uniquely determined by the spectrum $\lambda_n(S)$ of the operator S and can be calculated using the limit

$$\chi(\Gamma) = 2 \lim_{t \rightarrow \infty} \sum_{n=0}^{\infty} \cos \sqrt{\lambda_n(S)}/t \left(\frac{\sin \sqrt{\lambda_n(S)}/2t}{\sqrt{\lambda_n(S)}/2t} \right)^2, \tag{29}$$

where we use the following natural convention

$$\lambda_m = 0 \quad \Rightarrow \quad \frac{\sin \sqrt{\lambda_m(S)}/2t}{\sqrt{\lambda_m(S)}/2t} = 1. \tag{30}$$

Proof. Theorem 3 together with equality (20) implies that formula (29) gives the same result if the spectrum $\lambda_n(S)$ of the Schrödinger operator is substituted by the spectrum $\lambda_n(L)$ of the corresponding Laplace operator. Then formula (7) implies that the Euler characteristic of Γ is determined by (29). \square

⁴ The first several eigenvalues of S may be negative, but we are interested in the asymptotics of $\lambda_n(S)$ as $n \rightarrow \infty$. For large values of n the corresponding k_n are all positive reals.

The last two theorems state that two Schrödinger operators on graphs may have the same spectrum only if the underlying graphs have the same total length and Euler characteristic, in other words, if the graphs have the same size and complexity.

Uniqueness Theorem 1. *If two Schrödinger operators on finite compact metric graphs have the same spectrum, then the underlying graphs have the same*

- total length;
- Euler characteristic,

provided the potentials are essentially bounded and the boundary conditions at the vertices are standard (see (1)).

6. Generalizations and further developments

The results of this paper can be extended to the case of most general boundary conditions at the vertices. This problem appears to be more sophisticated than it may be expected. The main reason is that each vertex scattering matrix in general is not energy independent but tends to a certain limiting matrix S_v^∞ . The limiting matrix in its turn corresponds to certain symmetric boundary conditions, but these conditions may be incompatible with the connectivity of the graph Γ . In another words these new boundary conditions may not connect all edges joined at the vertex at the same time as the original conditions (corresponding to the energy dependent scattering matrix) do connect all the edges together.

Let us study the following elementary example. Consider the interval $\Delta_1 = [-\pi, \pi]$ turned into circle by joining together the end points $-\pi$ and π with the help of the following boundary conditions

$$\begin{cases} \psi(-\pi) = -\partial_n \psi(+\pi), \\ \psi(\pi) = -\partial_n \psi(-\pi); \end{cases} \tag{31}$$

which are obviously properly connecting, i.e. connect together the boundary values of the functions from both edges. These boundary conditions can be written in the conventional form [19]

$$A \begin{pmatrix} \psi(\pi) \\ \psi(-\pi) \end{pmatrix} + B \begin{pmatrix} \partial_n \psi(\pi) \\ \partial_n \psi(-\pi) \end{pmatrix} = 0$$

by choosing the matrices A and B equal to $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The corresponding vertex scattering matrix is

$$S_v(k) = -\frac{I - ikB}{I + ikB}$$

and it tends to the unit matrix $S_v^\infty = I$ as $k \rightarrow \infty$. The boundary conditions determining $S_v(k) = I$ are just Neumann boundary conditions $\partial_n \psi(+\pi) = 0 = \partial_n \psi(-\pi)$, which obviously disconnect the two endpoints. Therefore it is natural to call the boundary conditions (31) by *not asymptotically properly connecting*. Such boundary conditions result in that the spectral asymptotics for such graphs coincides with the one for graphs with boundary conditions having energy independent scattering matrices and different connectivity matrix, i.e. having different geometry.

We illustrate this idea by calculating the spectra of the operators appearing in the example under consideration. Let us denote by \tilde{L} the second derivative operator $-\frac{d^2}{dx^2}$ defined on the functions from $W_2^2[-\pi, \pi]$ and satisfying boundary conditions (31). These boundary conditions can be written as follows using standard derivative with respect to the variable $x \in [-\pi, \pi]$

$$\begin{cases} \psi(-\pi) = \psi'(+\pi), \\ \psi'(-\pi) = -\psi(\pi). \end{cases} \tag{32}$$

It is easy to see that the boundary conditions are invariant with respect to the change of the coordinate $x \mapsto -x$ and hence the operator \tilde{L} commutes with the symmetry operator $\mathcal{P}\psi(x) = \psi(-x)$. Therefore all eigenfunctions of \tilde{L} are either even or odd. The dispersion equations for even and odd functions can be obtained by substituting the Ansätze $\psi_s(x) = \cos kx$ and $\psi_a(x) = \sin kx$ into the boundary conditions

$$\tan k^s \pi = -\frac{1}{k^s}; \tag{33}$$

$$\cot k^a \pi = -\frac{1}{k^a}. \tag{34}$$

It is easy to see that the eigenvalues satisfy the following asymptotic conditions

$$k_n^s = n + O\left(\frac{1}{n}\right), \quad k_n^a = \frac{2n+1}{2} + O\left(\frac{1}{n}\right), \quad n = 0, 1, 2, \dots, \tag{35}$$

where $(k_n^s)^2$ and $(k_n^a)^2$ denote the eigenvalues for even and odd eigenfunctions respectively. These eigenvalues should be compared with the eigenvalues $k_n^{s0^2} = (n)^2$ and $k_n^{a0^2} = (\frac{2n+1}{2})^2$ for the Laplace operator on the interval $[-\pi, \pi]$ (with standard, i.e. Neumann boundary conditions at the end points $\psi'(-\pi) = 0 = \psi'(\pi)$). Let us try to reconstruct the Euler characteristic of the graph using formula (29). Lemma 2 implies that

$$\lim_{t \rightarrow \infty} \sum_{k_n=k_n^s, k_n^a} \cos k_n/t \left(\frac{\sin k_n/2t}{k_n/2t} \right) = \lim_{t \rightarrow \infty} \sum_{k_n^0=k_n^{s0}, k_n^{a0}} \cos k_n^0/t \left(\frac{\sin k_n^0/2t}{k_n^0/2t} \right) = 1, \tag{36}$$

which is in contradiction to the fact that the Euler characteristic of the circle is equal to 0. It follows that formula (7) in general is not valid for Schrödinger operators on graphs with general (self-adjoint) boundary conditions at the vertices. An extension of this formula for the case of general boundary conditions will be the subject of our following publication.

We decided to leave to the following publication another important generalization of the obtained results concerning Schrödinger operators with L_1 potentials. The proofs for $q \in L_1(\Gamma)$ and general boundary conditions at the vertices use similar methods and will be presented in a forthcoming publication.

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