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Solutions of Linear Nonautonomous Functional Differential Equations Which Are Exponentially Bounded for $t \to -\infty$

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1. INTRODUCTION

For $\gamma \in R^1$, denote by $\mathscr{L}(\gamma)$ the set of such solutions of

$$\frac{dx}{dt}(t) = A(t)\, x(t) + B(t)\, x(t-1), \tag{1.1}$$

which are defined on R^- (the set on nonnegative reals) and fulfil

$$\limsup e^{\gamma t} \mid x(t)| < \infty \tag{1.2}$$

($x(t) \in R^n$; $\mid x(t)|$ is the norm of $x(t)$; $A(t)$, $B(t)$ are $n \times n$-matrices). It was proved in [2] that $\mathscr{L}(\gamma)$ is a finite-dimensional linear space provided that A, B are locally integrable, $\mid B \mid$—the norm of B—is locally square integrable, and

$$\int_t^{t+1} \mid A(\tau)|\, d\tau \leqslant a, \qquad \int_t^{t+1} \mid B(\tau)|^2\, d\tau \leqslant b^2 \qquad \text{for} \quad t \leqslant -1, \tag{1.3}$$

a, b being any positive reals. Moreover, there were established estimates of dim $\mathscr{L}(\gamma)$ in terms of γ, a, b. In this paper, these results are extended to the linear functional differential equation

$$\frac{dx}{dt}(t) = F(t)\, x_t \tag{1.4}$$

($x_t \in C(\langle -1, 0\rangle \to R^n)$, $x_t(\sigma) = x(t+\sigma)$ for $\sigma \in \langle -1, 0\rangle$). The procedure is as follows: $x : R^- \to R^n$ is a solution of (1.1) fulfilling (1.2) iff $\{x_s \mid s = 0, -1, -2,...\}$ fulfils

$$x_{s+1} = Q_s x_s\,, \qquad s = -1, -2, -3,..., \tag{1.5}$$

(Q_s being the shifting operator of (1.3); cf. Definition 2.1) and

$$\limsup_{s \to -\infty} e^{\gamma s} \| x_s \| < \infty \tag{1.6}$$

and to this problem Corollary 3.1 of [3] applies. In Section 2, there are summed up some auxiliary results on equation (1.4), and in Section 3, there are established estimates of the shifting operators of (1.4) and it is proved that these operators are uniformly completely continuous (cf. [3, Note 2.2]). In Section 4, it is proved that dim $\mathscr{X}(\gamma)$ is finite and there is found an estimate for dim $\mathscr{X}(\gamma)$ (cf. Theorems 4.1 and 4.2). In the special case of Eq. (1.1), it need not be assumed that $|B|^2$ is locally integrable and (1.3) may be replaced by

$$\int_t^{t+1} |A(\tau)|\, d\tau \leqslant a, \qquad \int_t^{t+1} |B(\tau)|\, d\tau \leqslant b \qquad \text{for} \quad t \leqslant -1, \tag{1.7}$$

a, b being any positive reals. Moreover, if $e^{a+\gamma} \cdot b$ is sufficiently large, $be^{a+\gamma} \cdot bn(a + \gamma + \lg b + \lg n)$ is an upper bound for dim $\mathscr{X}(\gamma)$. If there exists such a continuous function ζ, $\zeta(0) = 0$ such that

$$\int_\sigma^\tau |A(\lambda)|\, d\lambda \leqslant \zeta(\tau - \sigma) \qquad \text{for} \quad \tau - 1 \leqslant \sigma \leqslant \tau \leqslant 0, \tag{1.8}$$

then 6 $e^\gamma bn(\gamma + \lg b + \lg n)$ is an upper bound for dim $\mathscr{X}(\gamma)$ provided that $e^\gamma b$ is large enough.

It was shown in [2]—under the assumption of (1.3)— that $e^{2\gamma}(1 + ae^a)^2 b^2 n$ is an upper bound for dim $\mathscr{X}(\gamma)$, $e^\gamma(1 + ae^a)b$ being sufficiently large. This is a better result with respect to n in comparison with the estimates from this paper, but a worse result with respect to γ, a, b.

Let $\alpha, \beta \in R^1$, $\beta \neq 0$. For the characteristic roots z_k that correspond to equation

$$\frac{dx}{dt}(t) = \alpha x(t) + \beta x(t-1) \tag{1.9}$$

($x(t) \in R^1$), the following asymptotic formulas are well known (cf. [4, Section 12.9]):

$$z_k = \lg |\beta| - \lg(2k\pi) + i\left(2k\pi + \arg\beta - \frac{\pi}{2}\operatorname{sgn} k\right) + o(1), \tag{1.10}$$

k being an integer such that $|k|$ is large, i being the imaginary unit. For $\gamma \in R^1$ let $N(\gamma)$ be the number of such z_k that $\operatorname{Re} z_k \geqslant -\gamma$. It follows from (1.10) that for every $\epsilon > 0$ there exists a $\lambda(\epsilon) > 0$ such that $N(\gamma) \geqslant (1 - \epsilon)\, \pi^{-1} e^\gamma |\beta|$, provided that $e^\gamma |\beta| \geqslant \lambda(\epsilon)$. Hence, if $n = 1$, any upper bound for dim $\mathscr{X}(\gamma)$ in case of Eq. (1.1) must be greater than $(1 - \epsilon)\, \pi^{-1} e^\gamma b$, if $e^\gamma b$ is sufficiently large. For $n > 1$, replace α, β in (1.9) by α id, β id, id being the $n \times n$ identity matrix ($x(t) \in R^n$). It follows that any upper bound for dim $\mathscr{X}(\gamma)$ in the case of Eq. (1.1) must be greater than $(1 - \epsilon)\, \pi^{-1} e^\gamma bn$ if $e^\gamma b$ is sufficiently large.

2. Linear Functional Differential Equations

Let n be a positive integer, let C be the linear space of continuous functions from $\langle -1, 0\rangle$ to R^n; it is assumed that R^n is provided with some norm, the norm of y being denoted by $|y|$ for $y \in R^n$; $\|x\| = \sup_{t\in\langle -1,0\rangle} |x(t)|$ for $x \in C$. Let $\mathscr{L}$ be the linear space of linear maps from C to R^n with the usual norm. Let $R^- = \{t \in R^1 \mid t \leqslant 0\}$. Let $F : R^1 \to \mathscr{L}$ fulfil

$$F(\cdot)y \qquad \text{is measurable for any } y \in C, \tag{2.1}$$

$$\|F(\cdot)\| \qquad \text{is locally integrable.} \tag{2.2}$$

If $u : \langle s-1, T\rangle \to R^n$, $s < T \leqslant 0$ is continuous, define $u_t \in C$ for $t \in \langle s, T\rangle$ by $u_t(\sigma) = u(t+\sigma)$, $\sigma \in \langle -1, 0\rangle$. u is called a solution of

$$\frac{dx}{dt}(t) = F(t)\, x_t \tag{2.3}$$

(on $\langle s, T\rangle$), if it is continuous on $\langle s-1, T\rangle$, if the restriction of u to $\langle s, T\rangle$ is absolutely continuous, and if (2.3) is fulfilled almost everywhere on $\langle s, T\rangle$, x being replaced by u. $u : \langle -\infty, T\rangle \to R^n$ is a solution of (2.3), if the restriction of u to $\langle s-1, T\rangle$ is a solution of (2.3) for any $s \in (-\infty, T)$. As in the theory of ordinary differential equations $u : \langle s-1, T\rangle \to R^n$ is a solution of (2.3), if it is continuous and if

$$u(t) = u(s) + \int_s^t F(\sigma)\, u_\sigma\, d\sigma, \qquad t \in \langle s, T\rangle. \tag{2.4}$$

From (2.4) the existence theorem may be obtained (by means of successive approximations) in the following form: if $w \in C$, $s \in R^-$, then there exists $u : \langle s-1, 0\rangle \to R^n$ such that u is a solution of (2.3) and $u_s = w$.

If u fulfils (2.4), then

$$|u(t)| \leqslant \|u_s\| + \int_s^t \|F(\sigma)\| \cdot \|u_\sigma\|\, d\sigma, \qquad s \leqslant t \leqslant T,$$

and, by Gronwall's inequality,

$$|u(t)| \leqslant \|u_s\| \exp \int_s^t \|F(\sigma)\|\, d\sigma, \qquad s \leqslant t \leqslant T \tag{2.5}$$

((2.5) implies uniqueness of solutions.) Moreover, if $s \leqslant \tau \leqslant t \leqslant T$, then

$$\begin{aligned} |u(t) - u(\tau)| &= \left\| \int_\tau^t F(\sigma)\, u_\sigma\, d\sigma \right\| \\ &\leqslant \|u_s\| \left[\exp \int_s^t \|F(\sigma)\|\, d\sigma - \exp \int_s^\tau \|F(\sigma)\|\, d\sigma\right]. \end{aligned} \tag{2.6}$$

DEFINITION 2.1. For $s = -1, -2, \ldots$, define $Q_s : C \to C$ as follows: if $w \in C$, find the solution u of (2.4), $w : \langle s-1, 0\rangle \to R^n$ such that $w_s = w$ (u exists and is unique) and define $Q_s w = u_{s+1}$. [Q_s are called shifting operators of Eq. (2.4).]

Let $M(n)$ be the set of $n \times n$ matrices; $M(n)$ is a linear normed space, the norm being introduced in the usual way. For $\tau \in \langle -1, 0\rangle$, define $P_{s,\tau} : C \to R^1$ by $P_{s,\tau} w = (Q_s w)(\tau)$. By (2.5), $P_{s,\tau}$ is continuous and, by the representation of linear functionals on C, there exists $V_{s,\tau} : \langle -1, 0\rangle \to M(n)$ such that

$$V_{s,\tau} \quad \text{is of bounded variation,} \tag{2.7}$$

$$V_{s,\tau}(-1) = 0 \quad \text{and} \quad V_{s,\tau} \ \text{is left continuous at any } \lambda \in (-1, 0), \tag{2.8}$$

and

$$(Q_s w)(\tau) = P_{s,\tau} w = \int_{-1}^{0} V_{s,\tau}(d\lambda)\, w(\lambda) \tag{2.9}$$

(i.e., in components $(P_{s,\tau} w)_i = \sum_{j=1}^{n} \int_{-1}^{0} w_j(\lambda)\, d((V_{s,\tau}(\lambda))_{i,j})$.

Moreover, it may be deduced from (2.5) and (2.6) that

$$\operatorname{var} V_{s,\tau} \leqslant \exp \int_{s}^{s+1+\tau} \|F(\sigma)\|\, d\sigma \tag{2.10}$$

and

$$\operatorname{var}(V_{s,\tau} - V_{s,\vartheta}) \leqslant \exp \int_{s}^{s+1+\tau} \|F(\sigma)\|\, d\sigma - \exp \int_{s}^{s+1+\vartheta} \|F(\sigma)\|\, d\sigma \tag{2.11}$$

for $-1 \leqslant \vartheta \leqslant \tau \leqslant 0$, $s = -1, -2, \ldots$.

3. A SPECIAL SET Ω

DEFINITION 3.1. Let $M(n)$ be the set of $n \times n$ matrices, let $\mu, \nu \in R^1$, $0 < \nu \leqslant \mu$. Define $\omega(\mu, \nu)$ to be the set of such $Q : C \to C$ which can be represented in the following way: for any $\tau \in \langle -1, 0\rangle$, there exists $V_\tau : \langle -1, 0\rangle \to M(n)$ such that V_τ is of bounded variation and

$$\operatorname{var} V_\tau \leqslant \mu, \tag{3.1}$$

$$\begin{gathered} \text{there exists } \chi : \langle -1, 0\rangle \to R^1, \text{ continuous, nondecreasing,} \\ \chi(-1) = 0, \qquad \chi(0) = \nu, \qquad \operatorname{var}(V_\tau - V_\sigma) \leqslant \chi(\tau) - \chi(\sigma) \\ \text{for } -1 \leqslant \sigma \leqslant \tau \leqslant 0, \end{gathered} \tag{3.2}$$

$$(Qy)(\tau) = \int_{-1}^{0} V_\tau(d\lambda)\, y(\lambda) \qquad \text{[cf. (2.9)]}. \tag{3.3}$$

THEOREM 3.1. *$\omega(\mu, \nu) \subset \Omega(\{k_i\}, \{\rho_i\})$, k_i, ρ_i being defined by*

$$\begin{gathered} k_0 = 0, \qquad k_1 = n, \qquad k_i = (1 + 2^{i-2})n, \qquad i = 2, 3, \ldots, \\ \rho_0 = \mu, \qquad \rho_i = \nu 2^{-i+1}, \qquad i = 1, 2, 3, \ldots . \end{gathered} \tag{3.4}$$

(For the definition of $\Omega(\{k_i\}, \{\rho_i\})$ see [3, Definition 2.1] for $X = C$.)

Proof. Let $\vartheta : \langle 0, \nu\rangle \to \langle -1, 0\rangle$ be such that $\chi \circ \vartheta(\lambda) = \lambda$ for $\lambda \in \langle 0, \nu\rangle$ (if χ is strictly increasing, then ϑ is the inverse). Let $Q \in \omega(\mu, \nu)$. Define

$$X^{(0)} = C, \qquad X^{(1)} = \{y \in C \mid (Qy)(-1) = 0\}, \tag{3.5}$$

$$X^{(i)} = \left\{y \in C \mid (Qy)\left(\vartheta\left(\frac{j\nu}{2^{i-2}}\right)\right) = 0 \qquad \text{for } j = 0, 1, \ldots, 2^{i-2}\right\}, \qquad i = 2, 3, \ldots . \tag{3.6}$$

It is easy to see that

$$\operatorname{codim}(X^{(1)} \mid C) \leqslant n = k_1, \qquad \operatorname{codim}(X^{(i)} \mid C) \leqslant n(2^{i-2} + 1) = k_i \quad \text{for } i = 2, 3, \ldots .$$

Let $y \in X^{(i)}$, $i \geqslant 2$, $\tau \in \langle -1, 0\rangle$. There exists an r, $r = 0, 1, \ldots, 2^{i-1}$, such that

$$\vartheta\left(\frac{r\nu}{2^{i-1}}\right) \leqslant \tau \leqslant \vartheta\left(\frac{(r+1)\nu}{2^{i-1}}\right).$$

Find an integer $\tilde{r}$ such that $2\tilde{r} = r$ or $2\tilde{r} = r + 1$. $(Qy)(\vartheta(\tilde{r}\nu/2^{i-2})) = 0$, as $y \in X^{(i)}$ and it follows from (3.2) and (3.3) that

$$\|(Qy)(\tau)\| = \left\|(\varphi y)(\tau) - (\varphi y)\left(\vartheta\left(\frac{\tilde{r}\nu}{2^{i-2}}\right)\right)\right\| \leqslant \left\|\chi(\tau) - \chi\left(\frac{\tilde{r}\nu}{2^{i-2}}\right)\right\| \leqslant \frac{\nu}{2^{i-1}} = \rho_i.$$

The cases $i = 1, 2$ are similar; the proof is complete.

LEMMA 3.1. *Let k_i, ρ_i be defined by* (3.4) *and let m, p be positive integers.*

If $m \leqslant pn$, *then* $[\Xi(m, p)]^{1/mp} = (g(m))^{1/mp}\mu.$ (3.7)

If $pn < m \leqslant 2pn$, *then* $[\Xi(m, p)]^{1/mp} = (g(m))^{1/mp}\mu^{np/m}\nu^{(m-np)/m}.$ (3.8)

If $pn(2^{r-2} + 1) < m \leqslant pn(2^{r-1} + 1)$, $\quad r = 2, 3, 4, \ldots,$ *then*

$$[\Xi(mp)]^{1/mp} = \nu \cdot (g(m))^{1/mp} \cdot \left(\frac{\mu}{\nu}\right)^{pn/m} \cdot 2^{-r+1+pn/m(2^{r-1}+r-2)}. \tag{3.9}$$

For the definition of $\Xi(m, p)$, see [3, Definition 2.2]; $g(m)$ is defined in [3, Definition 1.2]; formulas (3.7), (3.8), and (3.9) are obtained from (3.4).

LEMMA 3.2. *Put* $\Psi(\xi) = \xi/(\xi - 1)[\frac{1}{8}(\xi - 1)]^{1/\xi}$ *for* $\xi \geqslant 2$. *Let* $0 < \nu \leqslant \mu$ *and let* k_i, ρ_i *be defined by* (3.4), $m > 2np$, m, p *being positive integers. Then,*

$$[\Xi(m, p)]^{1/mp} \leqslant 4\nu \frac{np}{m} (g(m))^{1/mp} \cdot \left(\frac{\mu}{\nu}\right)^{np/m} \Psi\left(\frac{m}{np}\right). \tag{3.10}$$

Proof. By the first inequality in (3.9) $pn/m(2^{r-1} + 2) \leqslant 2$; hence,

$$[\Xi(m, p)]^{1/mp} \leqslant \nu(g(m))^{1/mp} \left(\frac{\mu}{\nu}\right)^{np/m} \cdot 2^{-1} \cdot 2^{(r-4)(-1+np/m)}.$$

By the second inequality in (3.9) $2^{r-4} \geqslant 8^{-1}(m/np - 1)$; hence,

$$2^{(r-4)(-1+pn/m)} \leqslant 8 \frac{np}{m} \cdot \frac{m}{np} \left(\frac{m}{np} - 1\right)^{-1} \left[8^{-1}\left(\frac{m}{np} - 1\right)\right]^{pn/m}$$

and (3.10) holds.

THEOREM 3.2. *To every* $\epsilon > 0$ *and* $E \geqslant 1$ *there exists* $\lambda(\epsilon, E) > 0$ *such that the following assertion holds:*

Let $1 \leqslant \mu/\nu \leqslant E$, *let* k_i, ρ_i *be defined by* (3.4), $Q_j \in \Omega(\{k_i\}, \{\rho_i\})$, $j = -1, -2, \ldots$. *For* $c > 0$, *let* $Z(c)$ *be defined by* [3, *Definition* 3.1]. *If* $c\nu \geqslant \lambda(\epsilon, E)$, *then*

$$\dim Z(c) < (1 + \epsilon)\, 2ec\nu n \lg(c\nu n). \tag{3.11}$$

Proof. Let m be the whole part of $(1 + \epsilon)\, 2ec\nu n \lg(c\nu n)$ and let p be the whole part of $\frac{1}{2} \lg(c\nu n)$. It may be verified that $m > 2np$ if $c\nu \geqslant e$ so that (3.10) holds. Replace in the right side of (3.10) $(g(m))^{1/mp}$ by $m^{1/2p}$ (cf. [3, (1.16) and (1.20)]). Then, there may be found such a $\lambda = \lambda(\epsilon, E)$ that the right side of (3.10) is less than $1/c$, if $c\nu \geqslant \lambda$ and (3.11) follows by [3, Theorem 3.2].

4. ESTIMATES OF $\dim Z(\gamma)$

Equation (2.3) and its special case

$$\frac{dx}{dt}(t) = A(t)\, x(t) + B(t)\, x(t - 1). \tag{4.1}$$

will be discussed.

Let $C, F, Q_s, M(n)$, etc., have the same meaning as in Sections 2 and 3. It will be assumed that F fulfils (2.1) and (2.2) and that there exists $\kappa > 0$ such that

$$\int_s^{s+1} \|F(t)\| \, dt \leqslant \kappa \qquad \text{for} \quad s = -1, -2, \ldots . \tag{4.2}$$

$A : R^- \to R^n$, $B : R^- \to R^n$ are assumed locally integrable and

$$\int_s^{s+1} |A(t)| \, dt \leqslant a, \qquad \int_s^{s+1} |B(t)| \, dt \leqslant b, \tag{4.3}$$

a, b being any positive reals.

Definition 4.1. For any $\gamma \in R^1$, let $\mathscr{X}(\gamma)$ be the set of solutions u of (2.3), which are defined on R^- and fulfil

$$\limsup_{t \to -\infty} e^{\gamma t} \, |u(t)| < \infty.$$

Let $\mathscr{X}(\gamma, (4, 1))$ have the analogous meaning with respect to (4.1).

Definition 4.2. If $u : R^- \to R^n$ is a solution of (2.3), define

$$W(u) = \{u_s \mid s = -1, -2, \ldots\}.$$

Lemma 4.1. *The restriction of W to $\mathscr{X}(\gamma)$ is a bijection of $\mathscr{X}(\gamma)$ on $Z(e^\gamma)$ for any $\gamma \in R^1$.*

This follows from (4.2) and (2.5).

It follows from (4.2), (2.9), (2.10), and (2.11) that $Q_s \in \omega(e^\kappa, e^\kappa - 1)$, $s = -1, -2, \ldots$. Hence by Theorem 3.1, Lemma 4.1, and [3, Corollary 3.1] we obtain

Theorem 4.1. $\dim \mathscr{X}(\gamma) < \infty$ *for* $\gamma \in R^1$.

Moreover, Theorems 3.1 *and* 3.2 *imply*

Theorem 4.2. *Let* $\delta > 0$, $E = e^\delta(e^\delta - 1)^{-1}$. *If* $\kappa \geqslant \delta$, $e^\gamma(e^\kappa - 1) \geqslant \lambda(\epsilon, E)$, *then* $\dim \mathscr{X}(\gamma) < (1 + \epsilon)\, 2ee^\gamma(e^\kappa - 1)n \lg(e^\gamma(e^\kappa - 1)n)$.

Theorem 4.2 applies to Eq. (4.1); in this case, more detailed results may be obtained, as an explicit formula is available for $V_{s,\tau}$ in (2.9).

Lemma 4.2. $Q_s \in \Omega(\{k_i\}, \{\rho_i\})$ *for* $s = -1, -2, \ldots$, Q_s *being the shifting operators of* (4.1), k_i *and* ρ_i *being defined by* (3.4) *with* $\mu = e^a(b + 1)$, $\nu = e^a b$.

Proof. Let $U : R^- \to M(n)$ be a fundamental matrix of $dx/dt = A(t)x$. Then,

$$(Q_s y)(t) = U(t+1+s)\, U^{-1}(s)\, y(0) + \int_0^{t+1} U(t+1+s)\, U^{-1}(\sigma+s)\, B(\sigma+s)\, y(\sigma-1)\, d\sigma. \tag{4.4}$$

By Gronwall's lemma

$$| U(\tau)\, U^{-1}(\lambda)| \leqslant e^a \qquad \text{for} \quad s \leqslant \lambda \leqslant \tau \leqslant s+1, \quad s = -1, -2, \ldots . \tag{4.5}$$

Put $\int_s^{s+1} | B(\sigma)|\, d\sigma = b_s$ and assume that $b_s > 0$. Find such $\vartheta_s : \langle 0, b_s \rangle \to \langle -1, 0 \rangle$ that $\int_{-1}^{\vartheta_s(\lambda)} | B(s+1+\sigma)|\, d\sigma = \lambda$ for $\lambda \in \langle -1, 0 \rangle$ and define linear subspaces $X^{(i)}$ of C by (3.5) and

$$X^{(i)} = \left\{ y \in C \mid (Q_s y) \left(\vartheta_s \left(\frac{j b_s}{2^{i-2}} \right) \right) = 0, j = 0, 1, \ldots, 2^{i-2} \right\}, \qquad i = 2, 3, \ldots .$$

Obviously, $\operatorname{codim}(X^{(i)} \mid C) \leqslant k_i$, and the estimate $\| Q_s y \| \leqslant \rho_i \| y \|$ for $y \in X^{(i)}$ is obtained from (4.3), (4.4), and (4.5). Theorem 3.2 and Lemma 4.2 imply

THEOREM 4.3. *Let* $\delta > 0$, $E = 1 + \delta^{-1}$, $\epsilon > 0$. *If* $b \geqslant \delta$, $e^{\gamma+a} b \geqslant \lambda(\epsilon, E)$, *then*

$$\dim \mathscr{X}(\gamma, (3, 1)) < (1+\epsilon)\, 2ee^{\gamma+a} bn(\gamma + a + \lg b + \lg n).$$

Let $\zeta : \langle 0, \infty) \to R^1$ be continuous and increasing, $\zeta(0) = 0$, $\zeta(\sigma) = a$ for $\sigma \geqslant 1$, and assume in addition that

$$\int_\sigma^\tau | A(\lambda)|\, d\lambda \leqslant \zeta(\tau - \sigma) \qquad \text{for} \quad \tau - 1 \leqslant \sigma \leqslant \tau \leqslant 0. \tag{4.6}$$

The following Lemma may be proved by a slight modification of the proof of Lemma 4.2.

LEMMA 4.3. $Q_s \in \Omega(\{k_i\}, \{\rho_i\})$ *for* $s = -1, -2, \ldots$, Q_s *being shifting operators of* (4.1), k_i *being defined by* (3.4), $\rho_0 = e^a(b+1)$, $\rho_1 = e^a b$, $\rho_i = (1+\eta) b \cdot 2^{-i+1} \exp(\zeta((1+\eta^{-1}) 2^{-i+2}))$, $i = 2, 3, \ldots$, $0 < \eta \leqslant 1$, η *being arbitrary.*

THEOREM 4.4. *There exists* $\lambda_1 > 0$, *which depends on* ζ, δ, ϵ *only*, $\delta > 0$, $0 < \epsilon \leqslant 1$, *such that if* $e^\gamma b \geqslant \lambda_1$, $a \leqslant \delta^{-1}$, *then*

$$\dim \mathscr{X}(\gamma, (3, 1)) \leqslant (1+\epsilon)\, 2ee^\gamma bn(\gamma + \lg b + \lg n).$$

The proof makes use of Lemma 4.3 and depends on estimates similar to those from Section 3 and is omitted.

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