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Remarks on Well-Posedness Theorems for Damped Second-Order Systems

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Abstract—We consider an operator theoretic formulation for distributed damped second-order (in time) forced linear elastic systems. A brief summary of previous well-posedness results is presented along with new results which allow relaxed spatial regularity (which is important in smart material systems applications) on the forcing or input function. Extensions to nonlinear systems are also indicated. The results are presented in a variational format for easy development of finite element approximation methods. © 1999 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

In this note, we revisit the well-posedness results for damped second-order systems with unbounded input operators as discussed in [1; 2, Chapter 4]. In those references, we considered systems of the form

$$\begin{aligned} \ddot{w}(t) + A_2\dot{w}(t) + A_1 w(t) = f(t), \qquad \text{in } V_1^*, \\ w(0) = w_0, \qquad \dot{w}(0) = w_1, \end{aligned} \tag{1}$$

in the context of a Gelfand quintuple of Hilbert spaces

$$V_1 \hookrightarrow V_2 \hookrightarrow H \hookrightarrow V_2^* \hookrightarrow V_1^*,$$

where $A_i \in \mathcal{L}(V_i, V_i^*)$, $i = 1, 2$. Under certain assumptions on the stiffness and damping operators, A_1, A_2, respectively, we gave well-posedness results (which we shall state precisely below) under the assumption $f \in L^2(0,T;V_2^*)$. This assumption, while satisfied in some cases of strong damping or in applications with strengthened regularity in the input, does not hold for certain important classes of problems encountered in actuation of smart material structures. Here, we improve these results so as to include those classes of problems. First we recall the results of [1].

2. PREVIOUS RESULTS

We assume that the differential equation in (1) is defined via sesquilinear forms σ_1, σ_2 and make the following standing assumptions throughout this note (these are the same as in [1,2]). The sesquilinear forms $\sigma_i : V_i \times V_i \to \mathbb{C}$ are V_i continuous with

$$|\sigma_i(\varphi, \psi)| \le c_i |\varphi|_{V_i} |\psi|_{V_i}, \qquad \text{for all } \varphi, \psi \in V_i, \quad i = 1, 2.$$

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Thus, the operators $A_i \in \mathcal{L}(V_i, V_i^*)$ in (1) are given by

$$\sigma_i(\varphi, \psi) = \langle A_i\varphi, \psi\rangle_{V_i^*, V_i}, \qquad \varphi, \psi \in V_i,$$

where the duality pairings $\langle \cdot, \cdot\rangle_{V_i^*, V_i}$ are extensions by continuity of the H inner product $\langle \cdot, \cdot\rangle_H$ from $H \times V_i$ to $V_i^* \times V_i$. We further assume that σ_1 is symmetric and V_1 elliptic, satisfying for some $k_1 > 0$,

$$\operatorname{Re} \sigma_1(\varphi, \varphi) = \sigma_1(\varphi, \varphi) \geq k_1 |\varphi|_{V_1}^2, \qquad \text{for } \varphi \in V_1.$$

Moreover, σ_2 is V_2 coercive satisfying for some $k_2 > 0$,

$$\operatorname{Re} \sigma_2(\varphi, \varphi) + \lambda_0 |\varphi|_H^2 \geq k_2 |\varphi|_{V_2}^2, \qquad \text{for } \varphi \in V_2.$$

Under these conditions, the following theorem was proved in [1] and appears as Theorem 4.1 of [2].

THEOREM 1. *Suppose σ_1 is symmetric, V_1 continuous and V_1 elliptic, σ_2 is V_2 continuous and V_2 coercive. Then for $w_0 \in V_1$, $w_1 \in H$, $f \in L^2(0,T; V_2^*)$, system (1), which can equivalently be written*

$$\begin{gathered} \langle \ddot{w}(t), \varphi\rangle_{V_1^*, V_1} + \sigma_2(\dot{w}(t), \varphi) + \sigma_1(w(t), \varphi) = \langle f(t), \varphi\rangle_{V_2^*, V_2}, \qquad \text{for } \varphi \in V_1, \\ w(0) = w_0, \qquad \dot{w}(0) = w_1 \end{gathered} \tag{2}$$

has a unique solution w in the sense of $L^2(0,T;V_1)^* \cong L^2(0,T;V_1^*)$ satisfying $w \in C(0,T;V_1)$, $\dot{w} \in L^2(0,T;V_2) \cap C(0,T;H)$, $\ddot{w} \in L^2(0,T;V_1^*)$. This solution depends continuously on the initial data (w_0, w_1) and input f in the sense that the mapping $(w_0, w_1, f) \to (w, \dot{w})$ is continuous from $V_1 \times H \times L^2(0,T;V_2^*)$ to $L^2(0,T;V_1) \times L^2(0,T;V_2)$.

The above theorem is adequate to treat many systems of interest. For example (see [2]), for a smart material structure such as clamped strongly damped beams or plates with piezoceramic actuators, one has $V_2 = V_1 = H_0^2$ and $V_2^* = V_1^* = H^{-2}$ while $f(t) \in H^{-2}$ for each t. However, for a similar structure with only weak internal damping (such as spatial hysteretic damping or so-called structural (square root) damping, see [2, p. 116]), one has $V_2^* = (H_0^1)^* = H^{-1}$ so that f will *not* be in $L^2(0,T;V_2^*)$.

3. NEW RESULTS

For the new results we present here, one can relax the spatial regularity on f at the expense of added regularity in time. Such results are quite useful in control of smart material structures such as those discussed in [2] where $f(t,x) = g(t)h(x)$ and g has some smoothness, e.g., $g \in C^1$ or at least $g \in H^1$.

The shifting of smoothness in the spatial variable to additional smoothness in the time variable is very much similar to classical results in the treatment of distributed systems using semigroup theory (e.g., see [3, Chapter 4.2]). In that case, one attempts to represent solutions of systems such as (1) or (2) in terms of a variation-of-parameters formula employing the homogeneous system semigroup and questions under what conditions such a representation provides a strong solution of the equation (1). Our results in this spirit can be stated as follows.

THEOREM 2. *Under the assumptions of Theorem 1 with $f \in L^2(0,T;V_2^*)$ replaced by $f \in H^1(0,T;V_1^*)$, we have existence of a unique solution of (2) which depends continuously in the sense that $(w_0, w_1, f) \to (w, \dot{w})$ is continuous from $V_1 \times H \times H^1(0,T;V_1^*)$ to $L^2(0,T;V_1) \times L^2(0,T;V_2)$.*

PROOF. We sketch the proof which differs from that for Theorem 1 (see [2, pp. 98–104]) in only one essential aspect which, of course, makes strong use of the modified conditions on f.

As in [2, p. 99], we choose $\{\xi_i\}_{i=1}^{\infty}$ a linearly independent total subset of V_1 and let $V_1^m = \text{span}\{\xi_1, \ldots, \xi_m\}$. We choose $w_{0m}, w_{1m} \in V_1^m$ such that $w_{0m} \to w_0$ in $V_1, w_{1m} \to w_1$ in H as $m \to \infty$. Let $w_m(t) \equiv \sum_{i=1}^m \eta_{im}(t)\xi_i$ be the unique solution of the m-dimensional linear system

$$\begin{gathered}\langle \ddot{w}_m(t), \xi_i\rangle + \sigma_2\langle \dot{w}_m(t), \xi_i\rangle + \sigma_1\langle w_m(t), \xi_i\rangle = \langle f(t), \xi_i\rangle_{V_1^*, V_1}, \\ w_m(0) = w_{0m}, \qquad \dot{w}_m(0) = w_{1m},\end{gathered} \tag{3}$$

for $i = 1, 2, \ldots, m$. Multiplying the equation in (3) by $\dot{\bar{\eta}}_{im}(t)$ and summing over i, we obtain

$$\langle \ddot{w}_m(t), \dot{w}_m(t)\rangle_H + \sigma_2\,\langle \dot{w}_m(t), \dot{w}_m(t)\rangle + \sigma_1\,\langle w_m(t), \dot{w}_m(t)\rangle = \langle f(t), \dot{w}_m(t)\rangle_{V_1^*, V_1}. \tag{4}$$

Using the fact that $\frac{d}{dt}\sigma_1(w_m(t), w_m(t)) = 2\,\text{Re}\,\sigma_1(w_m(t), \dot{w}_m(t))$, we may take the Re part of (4), integrate over t and obtain

$$\begin{aligned}&|\dot{w}_m(t)|_H^2 + \sigma_1(w_m(t), w_m(t)) + \int_0^t 2\,\text{Re}\,\sigma_2\,(\dot{w}_m(s), \dot{w}_m(s))\,ds \\ &= |\dot{w}_m(0)|_H^2 + \sigma_1(w_m(0), w_m(0)) + \int_0^t 2\,\text{Re}\,\langle f(s), \dot{w}_m(s)\rangle_{V_1^*, V_1}\,ds\end{aligned} \tag{5}$$

exactly as in the arguments of [2].

At this point, we consider the last term in (5) and differ from the estimates in [2]. Using the fact that $f \in H^*(0, T; V_1^*)$, we observe that

$$\begin{aligned}\int_0^t \langle f(s), \dot{w}_m(s)\rangle_{V_1^*, V_1}\,ds &= \int_0^t \frac{d}{ds}\langle f(s), w_m(s)\rangle - \left\langle \dot{f}(s), w_m(s)\right\rangle ds \\ &= \int_0^t -\left\langle \dot{f}(s), w_m(s)\right\rangle ds + \langle f(t), w_m(t)\rangle - \langle f(0), w_m(0)\rangle,\end{aligned}$$

where all $\langle\cdot,\cdot\rangle$ are interpreted in the duality pairing for $V_1^* \times V_1$ sense. Thus, we find

$$\begin{aligned}\left|\int_0^t \langle f(s), \dot{w}_m(s)\rangle\,ds\right| &\le \int_0^t \left(\frac{1}{2}\left|\dot{f}(s)\right|_{V_1^*}^2 + \frac{1}{2}|w_m(s)|_{V_1}^2\right) ds \\ &+\frac{1}{4\varepsilon}|f(t)|_{V_1^*}^2 + \varepsilon|w_m(t)|_{V_1}^2 + \frac{1}{2}|f(0)|_{V_1^*}^2 + \frac{1}{2}|w_m(0)|_{V_1}^2,\end{aligned} \tag{6}$$

where ε is chosen so that $k_1 - 2\varepsilon > 0$. Using (6) in (5) along with ellipticity and coercivity conditions on σ_1 and σ_2, respectively, we obtain

$$\begin{aligned}&|\dot{w}_m(t)|_H^2 + (k_1 - 2\varepsilon)|w_m(t)|_{V_1}^2 + \int_0^t 2k_2\,|\dot{w}_m(s)|_{V_2}^2\,ds \\ &\le |w_{1m}|_H^2 + (c_1 + 1)|w_{0m}|_{V_1}^2 + |f(0)|_{V_1^*}^2 + \frac{1}{2\varepsilon}|f(t)|_{V_1^*}^2 + \int_0^t \left|\dot{f}(s)\right|_{V_1^*}^2 ds \\ &\quad + \int_0^t \left\{2\lambda_0\,|\dot{w}_m(s)|_H^2 + |w_m(s)|_{V_1}^2\right\} ds.\end{aligned} \tag{7}$$

This estimate replaces (4.13) in [2]. Arguing as in [2], we see that (7) implies

$$\begin{aligned}&|\dot{w}_m(t)|_H^2 + (k_1 - 2\varepsilon)|w_m(t)|_{V_1}^2 + \int_0^t 2k_2\,|\dot{w}_m(s)|_{V_2}^2\,ds \\ &\le \mathcal{K}\left(|w_0|_{V_1}, |w_1|_H, |f|_{H^1(0,T;V_1^*)}\right) + \int_0^t \left(2\lambda_0\,|\dot{w}_m(s)|_H^2 + |w_m(s)|_{V_1}^2\right) ds.\end{aligned} \tag{8}$$

Use of the usual Gronwall arguments with (8) guarantees that $\{\dot{w}_m\}$ is bounded in both $C(0, T; H)$ and $L^2(0, T; V_2)$ while $\{w_m\}$ is bounded in $C(0, T; V_1)$, exactly as in [2]. The remainder of the proof follows precisely the arguments in [2] using these *a priori* bounds to extract subsequences that converge to the unique solution of (2). As in [2], one also uses the estimate (8) to establish the appropriate continuous dependence as stated in the theorem.

4. CONCLUDING REMARKS

The same type of arguments given here can be used to improve the results for nonlinear second-order systems given in [4]. The resulting enhanced applicability of the well-posedness theorems for nonlinear systems under the replacement of $f \in L^2(0,T;V_2)$ by $f \in H^1(0,T;V_1^*)$ is exactly the same as discussed above for linear systems. (The fact that one still requires the embedding $V_2 \hookrightarrow H$ be compact in the nonlinear theory of [4] does not affect the relevance and importance of the new results for nonlinear systems.)

The conditions and arguments given here can also be used for nonlinear systems with hysteresis (e.g., see [5]).

REFERENCES

1. H.T. Banks, K. Ito and Y. Wang, Well-posedness for damped second-order systems with unbounded input operators, *Differential and Integral Equations* **8**, 587–606 (1995).
2. H.T. Banks, R.C. Smith and Y. Wang, *Smart Material Structures: Modeling, Estimation and Control*, Masson/John Wiley, Paris/Chichester, (1996).
3. A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, (1983).
4. H.T. Banks, D.S. Gilliam and V.I. Shubov, Global solvability for damped abstract nonlinear hyperbolic systems, *Differential and Integral Equations* **10**, 309–332 (1997).
5. H.T. Banks and G.A. Pinter, Well-posedness for nonlinear systems with hysteresis, (to appear).