Two-dimensional analysis of bar-and-joint assemblies on a sphere: Equilibrium, compatibility and stiffness

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Abstract

This paper discusses a 2D truss formulation with spherical angular variables for bar-and-joint assemblies existing in 3D on the surface of a sphere. Based on the principle of potential energy, coupled problems of equilibrium and compatibility, as well as of stiffness and prestress stability are investigated. Finally, the results are illustrated by numerical examples.

1. Introduction

Among all idealised structures used in engineering problems, the space truss is one of the simplest ones. This kind of structures, as known, contains straight bars connected by frictionless ball joints to each other and to a foundation, while the joints can be subjected to concentrated external load. The necessary conditions of their rigidity are well known from Maxwell (1864), while linear algebraic computational methods built upon the equilibrium and compatibility matrices of such trusses (e.g. Szabó and Roller, 1971) are also widely used. In some cases, it may be convenient to deal with structures reduced to two dimensions. The previous methods and theories can then be applied easily for planar trusses obtained by the removal of one of the three Cartesian coordinates after a suitable rotation of the coordinate system if necessary.

There are other cases, however, where a structure exhibits two-dimensional behaviour by its intrinsic properties, although the surface it lies in is not a plane. Analogously to planar trusses, a pin-jointed structure fitted to a spherical surface can also be considered a two-dimensional truss if (a) any node and any link between them lies on that given spherical surface, (b) any node (pin) allows only a rotation about the axis perpendicular to the local tangent plane to the surface. For a better understanding, such trusses can be imagined in the first approach as a set of strings of zero torsional stiffness and zero bending stiffness about any axis tangent to the sphere but nonzero normal stiffness and nonzero bending stiffness about any radial axis. These strings are tied together at the joints on the spherical surface and each string is tightened to the same surface, following hence the trajectory of a great circle. The resultant assembly resembles a net with a rigid ball inside, where tangential forces are transmitted by the curved strings (truss members) while all radial force components are continuously balanced at the spherical surface by the rigid ball. For the sake of computational analogy with space or plane trusses, this model will be extended so that spherical truss members can undergo compression as well. An adequate (but structurally not very promising) realisation of this extended ball-and-net model can be a set of truss members constrained between two concentric spheres of nearly equal radius, where the truss members have (sufficient) stiffness against bending about a radial axis.

The reason for dealing with trusses only on a spherical and not on a general curved surface is the simple geometry of a sphere, coupled with the property that if such a spherical assembly behaves like a finite mechanism, any of its compatible configurations remains on the same sphere (conversely, any two-dimensional rigid truss element on a general surface is constrained to given positions because of the variable curvature of that surface). In addition to these technical advantages, spherical trusses are in fact applied in mechanical engineering (Chiang, 2000), in structural engineering, e.g. at deployable-retractable domes (Kovács, 2000), but some applications will also be shown in this paper where spherical trusses are used just as a mechanical model for spherical packing and covering problems.

This paper aims to derive the matrix equations of the spherical truss analysis in two spherical coordinates under the assumption

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of small displacements, with special interest on problems concerning rigidity and prestress stability. This approach is justified by the fact that in some applications it is more straightforward to consider spherical rather than Cartesian coordinates. Additionally, because of the reduction in the number of variables by one-third, savings in computational work are also expected.

2. Equations and matrices in a truss model

In accordance with the engineering usage, we define the system of equilibrium equations in the form

\[ \mathbf{A} \cdot \mathbf{s} = \mathbf{p}, \]

where \( \mathbf{A} \), \( \mathbf{s} \) and \( \mathbf{p} \) are the equilibrium matrix, the vector of bar forces and the vector of nodal load components, respectively. Note that some authors, such as Szabó and Roller (1971) define \( \mathbf{A} \) through the relationship \( \mathbf{A} \cdot \mathbf{s} + \mathbf{p} = \mathbf{0} \). By adoption of the theory of small displacements, the system of compatibility equations is written in a similar form as follows:

\[ \mathbf{C} \cdot \mathbf{d} = \mathbf{e}_c + \mathbf{e}_s, \]

where \( \mathbf{C} \), \( \mathbf{d} \), \( \mathbf{e}_c \), and \( \mathbf{e}_s \) denote the compatibility matrix, the vector of nodal displacement components, the vector of elastic bar elongations and the vector of elongations due to kinematic load, respectively (the overall elongation, \( \mathbf{e} \) is the sum of \( \mathbf{e}_c \) and \( \mathbf{e}_s \)). Eqs. (1) and (2) are linked by the material law. Assuming a linearly elastic behaviour, it takes the form

\[ \mathbf{F} \cdot \mathbf{s} = \mathbf{e}_c, \]

where \( \mathbf{F} \) is the flexibility matrix with values \( \frac{E_l}{E_0 A_0} \) in its main diagonal and zero otherwise (\( \frac{E_l}{E_0 A_0} \) and \( A_0 \) denote the stress-free length with no kinematic load (referred to as ‘unloaded length’ henceforth), Young’s modulus and cross-sectional area of the main diagonal bar, respectively). For the sake of simplicity, let us consider first an example in a two-dimensional Cartesian coordinate system. Fig. 1 shows only the kth truss member of the whole assembly, together with adjacent nodes \( N_a \) and \( N_e \). The position of the bar is described by an angle \( \eta_{m} \), measured at \( N_a \) from the positive direction of axis \( x \) (therefore, \( \eta_{m} = \eta_{m} \pm \pi \)). Since all displacements are small, a truss member of length \( l_k \) in its current position is ‘nearly’ parallel to its unloaded position with a length of \( l_k \).

Fig. 1b shows an equilibrium force system at joint \( N_p \) with external load components \( P_{x}, P_{y} \). Member force \( s_{x} \) exerted at joint \( N_{h} \) (also drawn as a positive (tensile) force) can then be resolved into vertical and horizontal components via multiplication by \( \cos \eta_{m} \) and \( \sin \eta_{m} \), respectively. Their negatives must be equal to \( P_{x} \) and \( P_{y} \), consequently, the entries in \( \mathbf{A} \) are as follows:

\[ a_{x,k} = -\cos \eta_{m} \quad a_{y,k} = -\sin \eta_{m} \]

where subscript \( x \) refers to the force corresponding to the horizontal (displacement) component at \( N_{h} \), while \( y \) denotes the column that belongs to the kth bar (if joint \( N_a \) and the kth bar are not connected to each other, entries \( a_{x,k} \) and \( a_{y,k} \) are zero). Note that any arguments above can be extended to three dimensions by considering direction cosines of angles measured from \( y \) and \( z \) as well.

In Fig. 1c, both an unloaded position (dashed line) and the current position (solid line) of the truss member are displayed. If the displacement \( \Delta x \) is small enough, \( \eta_{m} \approx \eta_{m}^0 \) can be accepted, and the elongation of the member due to a positive vertical displacement at joint \( N_e \) can be approximated as \( \Delta x \approx -\Delta x \cos \eta_{m} \). Similar arguments hold for a horizontal displacement (Fig. 1d), hence for an entry of \( \mathbf{C} \), we obtain

\[ c_{k,x} = -\cos \eta_{m} \quad c_{k,y} = -\sin \eta_{m} \]

This can be generalized again for three dimensions, and Eqs. (4) and (5) constitute a proof of the well-known relationship \( \mathbf{C} = \mathbf{A}^{-1} \).

Consider now a stress-free great circle arc (of length \( l_k \)) between joints \( N_a \) and \( N_e \) as a member of a spherical truss; see Fig. 2a for illustration. In a spherical coordinate system, azimuth (\( \phi \)) and meridian (\( \theta \)) angles are used as independent geometrical variables; we measure \( \phi \) from a zero meridian towards East, while \( \theta \) has its zero value at the North pole (\( O \)). Note that the third common spherical variable (the radial distance) is now kept at a constant value \( R \). Like in the former example, direction angles (\( \eta_{m} \)) will be understood in a counter-clockwise sense, but measured from the local southern direction.

Entries of \( \mathbf{A} \) can be derived from Fig. 2b exactly as in planar trusses, except for that \( \eta_{m} \) is measured now from the meridian: equilibrium conditions and resolution of \( s_{m} \) yields that

\[ a_{x,m} = -\cos \theta \eta_{m} \quad a_{y,m} = -\sin \theta \eta_{m} \]

For Figs. 2c and d, similar arguments apply again as in the first example: spherical right triangles of a sufficiently small leg along the meridian or azimuth can be considered as ‘nearly’ planar objects, but the same trigonometrical relationships yield now
\[ c_{k\mu} = -\cos \eta_{k\mu}, \quad c_{\mu k} = -\sin \eta_{k\mu} \sin \theta_{\mu}. \]  

(7)

This result, compared to Eq. (6), means that matrices \( A \) and \( C \) cease to be transpose to each other. Its major disadvantage is that such equilibrium and compatibility matrices are incompatible with the energy principles of mechanics. In the following section, therefore, it is attempted to build a consistent truss theory based on energy considerations.

3. The Hellinger–Reissner principle

The Hellinger–Reissner principle provides an efficient tool (Washizu, 1982) to solve the problem posed above. It has already been successfully applied to a comprehensive description of the classical theory of space trusses by Tarnai and Szabó (2002). The method is based on the analysis of an energy functional \( \Pi_k \), called Hellinger–Reissner functional, whose first variation can be used to formulate both the equilibrium and compatibility conditions. \( \Pi_k \) consists of three terms: the negative of the work done by external (nodal) forces (written as a sum of two expressions below), the strain energy accumulated in the members and an additional expression related to physical or kinematic constraints (e.g., given length of a truss element) as follows:

\[ \Pi_k = -\sum_{\mu=1}^{m} R P_{\mu\mu} (\theta_{\mu} - \theta_{\mu}^0) - \sum_{\mu=1}^{m} R \sin \theta_{\mu} P_{\mu\mu} (\phi_{\mu} - \phi_{\mu}^0) 
+ \frac{1}{2} \sum_{k=1}^{n} \frac{E A_k}{l_k} (e_k)^2 + \sum_{k=1}^{n} \bar{F}_k A_k, \]  

(8)

where \( e_k \) refers to the elastic elongation of the \( k \)th truss member (we assume that the assembly is composed of \( m \) nodes and \( n \) curved members); zero superscripts of nodal coordinates indicate reference values (note that reference positions of two joints at the end of a given bar does not correspond necessarily to the unloaded length of that bar, the difference is contained by the kinematic load involved in \( \bar{F}_k \)).

If the angles \( \theta, \phi \) are intended to be kept as independent variables, it is necessary to associate the moment of external forces with them in an energy-related expression. Thus, the first two terms both give the negative of the work of external moments with them in an energy-related expression. Thus, the first two terms both give the negative of the work of external moments with them in an energy-related expression. Thus, the first two terms both give the negative of the work of external moments with them in an energy-related expression.

\[ \frac{\delta \Pi_k}{\delta \theta_{\mu}} = -R (P_{\mu\mu} - P_{\mu\mu}^0), \quad \frac{\delta \Pi_k}{\delta \phi_{\mu}} = -R \sin \theta_{\mu} (P_{\mu\mu} - P_{\mu\mu}^0). \]

Additionally, it is also possible to consider \( A_k \) as internal force in the \( k \)th bar, proportional to the elastic elongation:

\[ e_k = \frac{A_k l_k}{E A_k}. \]  

(13)

If Eq. (8) is rewritten in terms of \( F_k \) using Eq. (13) as well, the Hellinger–Reissner functional assumes the following form:

\[ \Pi_k = -\sum_{\mu=1}^{m} M_{\mu\mu} (\theta_{\mu} - \theta_{\mu}^0) - \sum_{\mu=1}^{m} M_{\mu\mu} (\phi_{\mu} - \phi_{\mu}^0) - \frac{1}{2} \sum_{k=1}^{n} \bar{F}_k A_k \]  

(14)

According to the Hellinger–Reissner principle, stationarity of \( \Pi_k \) is a necessary and sufficient condition for the equilibrium and compatibility of the entire structure, so the first variation of \( \Pi_k \) must be zero:

\[ \sum_{\mu=1}^{m} \frac{\partial \Pi_k}{\partial \theta_{\mu}} \delta \theta_{\mu} + \sum_{\mu=1}^{m} \frac{\partial \Pi_k}{\partial \phi_{\mu}} \delta \phi_{\mu} + \sum_{k=1}^{n} \frac{\partial \Pi_k}{\partial A_k} \delta A_k = 0. \]

(15)

Since the variation can be made with respect to both displacement and force variables, it yields both the system of equilibrium equations:

\[ \sum_{k=1}^{n} \frac{\partial F_k}{\partial A_k} A_k - M_{\mu\mu} = 0, \quad \sum_{k=1}^{n} \frac{\partial F_k}{\partial \phi_{\mu}} A_k - M_{\mu\mu} = 0, \quad \mu = 1, \ldots, m \]

(16)

and the compatibility equations:

\[ \bar{F}_k = 0, \quad k = 1, \ldots, n. \]

(17)

It should be emphasized that Eqs. (16) and (17) also apply for large displacements. Analogously to that is done in Szabó and Roller (1971), the system of incremental matrix equations can also be derived for spherical trusses to follow large displacements of the assembly, e.g., by considering the total differential of both equations written above. However, care must be taken of external nodal forces, as the analogy holds only for their moments: an azimuthal force component \( P_{\mu\mu} = M_{\mu\mu}/R \) sin \( \theta_{\mu} \) depends also on the variable \( \theta_{\mu} \).

Considering only constant and linear terms in the Taylor expansion of Eq. (17) in the neighbourhood of reference coordinates \( \theta_{\mu}, \phi_{\mu}^0 (\mu = 1, \ldots, m) \), a linearized compatibility equation (for small displacements only) is obtained:

\[ \sum_{\mu=1}^{m} \left( \frac{\partial F_k}{\partial \theta_{\mu}} \delta \theta_{\mu} + \frac{\partial F_k}{\partial \phi_{\mu}} \delta \phi_{\mu} \right) + l_k - \bar{F}_k - e_k - t_k = 0, \]

(18)

(\( l_k \) is the current arc length, while \( e_k \) stands for the prescribed kinematic loads (e.g., elongation due to thermal effect).)

Let now a function \( F \) be defined as the sum of \( \bar{F}_k \) and the elastic elongation:

\[ F_k = R \arccos(\cos \theta_{\mu} \cos \theta_{\mu} + \sin \theta_{\mu} \sin \theta_{\mu} \cos(\phi_{\mu} - \phi_{\mu}^0)) 
- l_k - e_k - t_k. \]

(11)
\(k_k - f_k^p\) becomes zero and a formula similar to Eq. (2) is obtained. It is easy to verify now that \(A = C^i\), since
\[
a_{ij, k} = \frac{\partial F_k}{\partial x_i} a_{k, ij} = \frac{\partial F_k}{\partial \theta_j}
\]
from Eq. (16), and
\[
c_{ij, k} = \frac{\partial F_k}{\partial \phi_j} c_{k, ij} = \frac{\partial F_k}{\partial \phi_i}
\]
from Eq. (18). For a better comparability with the results under (6) and (7), general entries of both matrices will also be written here but this step requires some additional considerations. First, we recall that Eq. (12) can also be interpreted as a compound function, so can be derived according to the chain rule, e.g.
\[
\frac{\partial F_k}{\partial \mu} = \frac{dF_k}{d\cos(k/R)} \frac{\partial \cos(k/R)}{\partial \mu}
\]
Since
\[
\frac{d}{d\cos(k/R)} = \frac{-R}{\sin(k/R)}
\]
we have
\[
\frac{\partial F_k}{\partial \mu} = \frac{-R}{\sin(k/R)} \frac{\partial \cos(k/R)}{\partial \mu}
\]
that can be rewritten as
\[
\frac{\partial F_k}{\partial \mu} = \frac{-R}{\sin(k/R)} (-\sin \theta_k \cos \theta_k + \cos \theta_k \sin \theta_k \cos(\varphi_k - \varphi_k)).
\]
Another trigonometrical formula (for the length of arc ON, in triangle ON, \(N_N\)) says that
\[
\cos \theta_k = \cos \varphi_k \cos \varphi_k + \sin \varphi_k \sin \varphi_k \cos(\pi - \eta_{ij}),
\]
then writing Eq. (10) into (25) yields after simplification that the right-hand side of Eq. (24) equals \(-R \cos \eta_{ij}\). Similarly, if Eq. (12) is derived with respect to \(\varphi_k\), the result will be
\[
\frac{\partial F_k}{\partial \varphi_j} = \frac{R}{\sin(k/R)} \sin \varphi_k \sin \varphi_k \sin(\varphi_k - \varphi_k).
\]
The sine law in triangle ON, \(N_N\), can be used as follows:
\[
\frac{\sin(\pi - \eta_{ij})}{\sin \theta_k} = \frac{\sin(\varphi_k - \varphi_k)}{\sin(k/R)}.
\]
giving \(-R \sin \theta_k \sin \eta_{ij}\) on the right-hand side of Eq. (26). In summary,
\[
a_{\mu, k} = c_{k, \mu}, \quad a_{\mu, k} = c_{k, \mu} = -R \sin \eta_{ij}, \sin \theta_k.
\]
that is, the equilibrium matrix and the compatibility matrix are transpose to each other, but instead of resolutions, moments (one about the North-South axis and another about the horizontal central axis in the vertical plane of the node) of nodal forces should be written in Eq. (8), as well as in the vector \(p\) of external loads according to Eq. (16).

4. Stiffness

Unlike the engineering approach suggests, rigidity of graphs in the mathematical literature (e.g., Asimow and Roth, 1978; Connelly, 1980) is discussed without respect to any material properties: the bars of a corresponding truss should be considered perfectly rigid. Nevertheless, the definition of first-order rigidity (or infinitesimal rigidity) given by Connelly and Whiteley (1996) matches completely the case when an assembly is said to have material stiffness (see Guest (2006)) in the engineering usage: it means that the compatibility matrix has full rank. Such assemblies will be qualified in this paper as ones having first-order stiffness: ‘first-order’ because it can be identified by a linear matrix analysis; ‘stiffness’ instead of ‘rigidity’ in order to emphasize that non-rigid material is assumed (but at the same time we must suppose that if an assembly is stiff to the first order, the effect of strains on the original geometry is negligible). For abstract graphs, it seems to be a separate question to qualify the stability of the equilibrium since any disturbance of the original geometry is only possible if once the assembly has been identified as an infinitesimal mechanism, that is, infinitesimally non-rigid. For those assemblies, it is the paper of Connelly and Whiteley (1996) again which defines the higher-order rigidity on the basis of the self-stress stability. (It is often referred to as second-order rigidity if can be shown by a second-order approximation.) Note also the approach of Pellegrino and Calladine (1986), based on the positivety of the work of ‘product forces’ rather than stability criteria. By adoption of any non-rigid material behaviour, as it also happens in our linear elastic model, it becomes possible to introduce disturbances in (therefore, to analyse the stability of) any assembly of finite material stiffness. Hence, stiffness (rather than rigidity) of a spherical assembly will be analysed here within a general framework of stability problems, pointing out the first-order rigidity or material stiffness as a special case. If necessary, we can speak about stiffness against a given displacement, understood as the existence of stable equilibrium if the system is disturbed along the displacement only.

Consider a spherical truss for which compatibility conditions are satisfied: it yields \(F_k = e_k\) (see the definition of \(F_k\) at Eq. (12)). Now \(e_k\) can be eliminated from Eq. (8) as follows:
\[
\Pi_k = -\sum_{j=1}^{m} M_{ij} \left( \theta_k - \theta_j \right) - \sum_{j=1}^{m} M_{ij} \left( \varphi_k - \varphi_j \right) + \frac{1}{2} \sum_{k=1}^{n} E_{ik} \left( F_k \right)^2
\]
Note that the above form of \(\Pi_k\) (denoted by \(\Pi_k\)) depends only on the \(\theta_k, \varphi_k\) displacement parameters. It is known that an energy functional on a compatible displacement field has a local minimum at a stable equilibrium configuration. Let us analyse then the second variation of \(\Pi_k\) (for the sake of compactness, with a generalized notation \(i = 1, \ldots, 2m\)) instead of \(\theta_k, \varphi_k\) \(\mu = 1, \ldots, m)\):
\[
\delta^2 \Pi_k = \sum_{j=1}^{m} \sum_{k=1}^{n} \delta \Pi_k \delta_j \delta_k + \sum_{j=1}^{m} \sum_{k=1}^{n} \sum_{k=1}^{n} E_{ik} \left( F_k \right)^2 \frac{\partial^2 F_k}{\partial \delta_i \partial \delta_j} \delta \Pi_k \delta_j
\]
we recall that despite the different subscripting, the values \(\delta_i\) and \(\delta_j\) refer to a unique set of displacement \((\delta_k)\). With respect to Eqs. (13) and (20), it can be rearranged to the following form:
\[
\delta^2 \Pi_k = \sum_{j=1}^{m} \sum_{k=1}^{n} \left( \sum_{k=1}^{n} c_{k, j} \delta_k \right) \delta_k + \sum_{j=1}^{m} \sum_{k=1}^{n} A_{ik} \delta \Pi_k \delta_j
\]
where \(f_{i, k}\) corresponds to the kth diagonal element in the flexibility matrix \(F\). Notice that the first sum within the brackets should be referred to as \(k_{ik}\) being the given entry of the material stiffness matrix
\[
K = AF^{-1}C = C^i F^{-1}C
\]
This fact explains why the matrix denoted by the second sum within the brackets is called complementary stiffness matrix by Tarnai and Szabó (2002) (precisely, this matrix has originally been defined with negative sign, see the notes at Eq. (1)). We remark that some authors, e.g. Przemieniecki (1968) use the term geometrical stiffness matrix, with reference to that it represents the stiffness component due to the change in the original geometry; while in Guest (2006), it appears as the difference between the
stress matrix and a diagonal matrix of the tension coefficients (force over length) of the members. Mathematically speaking, it is the Hessian matrix $H$ of $\sum F_iA_i$, therefore the sum at the right-hand side can also be denoted by $h_{ij}$.

Now here are two cases depending on whether or not an arbitrary variation $\{\delta x_i\}$ of the displacement field belongs to the (right) nullspace of $C$, denoted by $N(C)$ (i.e., $\{\delta x_i\}$ defines or not an (inextensible) infinitesimal mechanism).

**Case 1:** $\{\delta x_i\} \notin N(C)$. It is then only possible for non-rigid assemblies. Because of the obvious positive definiteness of $F^2$, the left triple sum in Eq. (31), $\sum \sum \sum \delta x_i \epsilon_{ij} \delta x_j$, is positive and the assembly is said to have first-order stiffness against a motion along $\{\delta x_i\}$. It is just a remark that such assemblies can still have stability problems at a special load level. Assume that there is a set of internal forces with or without a state of self-stress or external forces in equilibrium for which the right sum, $\sum \sum \sum \delta x_i \epsilon_{ij} \delta x_j$, is nonzero. Since the parameter of the force system involved in $H$ can be set arbitrarily (together with the corresponding amount of kinematic load), it is always possible then to get a negative value for the expression of Eq. (31). In fact, it can be seen easily that such instability requires $A_k$ values to be in the order of magnitude of $EA_k$, which occurs only in an assembly having a ‘nearly’ degenerate configuration (see the example of shallow shells or arches).

**Case 2:** $\{\delta x_i\} \in N(C)$, the left triple sum in Eq. (31) is zero for such a $\{\delta x_i\}$ (it has no first-order stiffness against $\{\delta x_i\}$); consequently, only the positivity of the right total sum is to be verified. Now the assembly is said to have second-order stiffness against the same $\{\delta x_i\}$ if the expression $\sum \sum \sum \delta x_i \epsilon_{ij} \delta x_j$ is positive.

It is important to note that compatibility is affected by the internal forces in an elastic assembly. Thus, it may seem impossible to set the parameter of an external force system or state of self-stress arbitrarily in a given geometrical configuration. Nevertheless, the kinematic load involved in $F_k$ can always be chosen to satisfy the conditions of compatibility (the bars can be heated or cooled accordingly). That process is also equivalent to a gradual change in the unloaded length $l_0$, as soon as its influence on $F_k$ is negligible. In the following analysis, we will investigate the stability of the equilibrium position of a structure with given external forces; kinematic loads will always be assumed to be compatible with the current magnitude of internal forces.

We shall consider henceforth an assembly with $p$ independent infinitesimal mechanisms $\{\delta x_i\}$ (or $\{\delta d_i\}$) in $N(C)$. $r = 1, \ldots, p$ as a vector basis of an individual mechanism $\{\delta x_i\}$; that is, the variation of displacements is made within this linear space of mechanisms. Furthermore, the existence of at least one set of internal forces $\{A_k\}$ in equilibrium with or without external forces or kinematic load is assumed ( $\{A_k\}$ may contain states of self-stress as well). Obviously, any linear combination of $\{\delta d_i\}$, also gives an infinitesimal mechanism $\{\delta d_i\}$ as follows:

$$\{\delta d_i\} = \sum_{i=1}^{p} x_i \{\delta d_i\}.\quad (32)$$

Writing this back into Eq. (31), we have

$$\delta^2 W_k = \sum_{q=1}^{p} \sum_{r=1}^{p} x_q \left( \sum_{i=1}^{m} \sum_{j=1}^{m} \delta d_q \ h_{ij} \ δd_j \right) x_r, \quad (33)$$

which is a quadratic form with arbitrary constants $\{x_i\}$. At the same time, the bracketed double sum can be regarded as an entry $w_{ik}$ of a $p$-by-$p$ matrix $W$ that can be called the reduced complementary stiffness matrix or reduced geometrical stiffness matrix of the assembly. Thus, existence of second-order stiffness against any possible mechanism $\{\delta d_i\}$ is equivalent to the positive definiteness of $W$. Since the parameter of $\{A_k\}$ can be multiplied by $-1$, the final form of our statement reads: an assembly is stiff in second order iff there is a set $\{A_k\}$ of internal forces with or without external forces in equilibrium for which the reduced complementary stiffness matrix $W$ is sign definite.

For the evaluation of the entries in $W$, it is necessary to compute the second derivatives of $F_k$. It is done using the chain rule like in Eq. (21):

$$\frac{\partial^2 F_k}{\partial \delta d_i \partial \delta d_j} = \frac{\partial}{\partial \delta d_j} \left( \frac{\partial F_k}{\partial \delta d_i} \frac{\partial \cos(l_i/R)}{\partial \delta d_i} \right), \quad (34)$$

that is,

$$\frac{\partial^2 F_k}{\partial \delta d_i \partial \delta d_j} = \frac{\partial}{\partial \delta d_j} \left( \frac{\partial F_k}{\partial \delta d_i} \frac{\partial \cos(l_i/R)}{\partial \delta d_i} \right) + \frac{\partial F_k}{\partial \delta d_i} \frac{\partial^2 \cos(l_i/R)}{\partial \delta d_i \partial \delta d_j}. \quad (35)$$

For the second term (called $S_{ijk}$ henceforth), it follows from Eq. (22) that

$$S_{ijk} = \frac{-R \cos(l_i/R)}{\sin(l_i/R)} \frac{\partial^2 \cos(l_i/R)}{\partial \delta d_i \partial \delta d_j},$$

while the first term of Eq. (35), called $T_{ijk}$, can further be transformed into

$$T_{ijk} = \frac{\partial^2 F_k}{(\cos(l_i/R))^2} \frac{\partial \cos(l_i/R)}{\partial \delta d_i} \frac{\partial \cos(l_k/R)}{\partial \delta d_j}, \quad (37)$$

then referencing Eqs. (23) and (20), we have

$$T_{ijk} = \frac{\partial^2 F_k}{(\cos(l_i/R))^2} \frac{\sin(l_i/R)}{\sin(l_k/R)} \cdot \epsilon_{ikj}. \quad (38)$$

Regarding also the derivatives of both sides of Eq. (22):

$$\frac{\partial^2 F_k}{(\cos(l_i/R))^2} = \frac{-R \cos(l_i/R)}{\sin(l_i/R)} \cdot \epsilon_{ikj}, \quad (39)$$

the final form of $T_{ijk}$ reads

$$T_{ijk} = \epsilon_{ikj} \cdot \frac{-\cos(l_i/R)}{R \sin(l_k/R)} \epsilon_{kj}. \quad (40)$$

and from Eqs. (36) and (40), we obtain that

$$A_k \frac{\partial^2 F_k}{\partial \delta d_i \partial \delta d_j} = A_k \frac{\epsilon_{ikj} \cdot - \cos(l_i/R)}{R \sin(l_k/R)} \epsilon_{kj} + \frac{-R A_k \epsilon_{ikj} \cos(l_i/R)}{R \sin(l_k/R)} \frac{\partial^2 \cos(l_k/R)}{\partial \delta d_i \partial \delta d_j}. \quad (41)$$

We remark that the coefficient of $-c_{ikj}$ in Eq. (41),

$$t_{ikj} = \frac{A_k \cos(l_i/R)}{R \sin(l_k/R)} \frac{1}{R \tan(l_k/R)}, \quad (42)$$

is analogous to that is referred to as stress in Connelly and Whiteley (1996) and tension coefficient in Guest (2006). Consequently, after substitution into Eq. (31), the corresponding sum defines a stiffness matrix component that Guest (2006) describes as the difference between the modified material stiffness matrix and the material stiffness matrix. This component influences the first-order (material) stiffness when there is initial stress in some members; any diagonal entry of $F_k$ is then modified by subtraction of the corresponding value of $t_{ikk}$. The above-introduced stiffness component $(c_{ikj} \cdot t_{ikj} - c_{ikj})$, however, still does not affect the entries in $W$, once any $\{\delta d_i\}$ has been assumed to belong to the right nullspace of $C$. As a result, an entry of the matrix $W$ to be checked for sign definiteness is given now by the formula below:

$$w_{ij} = \sum_{l=1}^{m} \sum_{j=1}^{m} \delta d_q \ h_{ij} \left( - \frac{A_k \epsilon_{ikj} \cos(l_i/R)}{R \sin(l_k/R)} \frac{\partial^2 \cos(l_k/R)}{\partial \delta d_i \partial \delta d_j} \right) \delta d_j. \quad (43)$$

There are six different kinds of second derivatives depending on the variables of derivation; the six formulae are given below (if $\mu \neq \nu$):
\[
\frac{\partial^2 \cos(l/R)}{\partial \theta_\mu \partial \theta_\mu} = - \cos \theta_\mu \cos \theta_\mu - \sin \theta_\mu \sin \theta_\mu \cos (\varphi_\mu - \varphi_\mu) (= - \cos(l/R)).
\]

It might be interesting to analyse the terms of Eq. (41) in the case where \( R \to \infty \), and compare them to the results obtained from classical truss theory. It is easy to see for the first term that

\[
\frac{\partial^2 \cos(l/R)}{\partial \theta_\mu \partial \theta_\mu} = \sin \theta_\mu \sin \theta_\mu + \cos \theta_\mu \cos \theta_\mu \cos (\varphi_\mu - \varphi_\mu).
\]

\[
\frac{\partial^2 \cos(l/R)}{\partial \theta_\mu \partial \varphi_\mu} = \cos \theta_\mu \sin \theta_\mu \sin (\varphi_\mu - \varphi_\mu).
\]

\[
\frac{\partial^2 \cos(l/R)}{\partial \varphi_\mu \partial \varphi_\mu} = - \sin \theta_\mu \sin \theta_\mu \cos (\varphi_\mu - \varphi_\mu).
\]

\[
\frac{\partial^2 \cos(l/R)}{\partial \theta_\mu \partial \varphi_\mu} = \sin \theta_\mu \sin \theta_\mu \cos (\varphi_\mu - \varphi_\mu).
\]

(44)

The compatibility matrix is compiled according to Eqs. (24) and (26). After deletion of four columns pertaining to nodes 0 and 3, we have

\[
\mathbf{C} = (c_{ij}) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix},
\]

where column labels are \( \theta_1, \varphi_1, \theta_2 \) and \( \varphi_2 \), respectively.

The only state of self-stress (from the left nullspace of \( C \); normalized) is \( A_0 = [1/\sqrt{3} \ 1/\sqrt{3} \ 1/\sqrt{3}]^\top \), while the right nullspace of \( C \) contains two infinitesimal mechanisms: \( \{d\theta_1\} = [1 \ 0 \ 0 \ 0] \) and \( \{d\varphi_3\} = [0 \ 0 \ 0 \ 1] \) (shown in Fig. 3B). Before computing directly the elements of \( w_\phi \), let the bracketed expression of Eq. (43), \( A_i S_{ij} \), be given in a matrix form, e.g. for the leftmost truss member (thus, \( i = j = 0, \varphi_0, \theta_1 \) and \( \varphi_1 \), respectively):

\[
\begin{bmatrix} A_i S_{ij} \end{bmatrix} = \begin{bmatrix} -R \sqrt{3} \\ 0 \\ -R \sqrt{3} \\ 0 \end{bmatrix} \begin{bmatrix} \cos \alpha & 0 & 1 \\ 0 & 1 & 0 \\ 0 & -2 \cos \alpha & 0 \\ -2 \cos \alpha & 0 & 1 \end{bmatrix}.
\]

Performing now the summation over \( k \) can be rewritten in a row form as follows:

\[
A_k / l \mathbf{X}^1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \mathbf{X}.
\]

Here, \( \mathbf{X}^1 = [x_0, y_0, x_e, y_e] \) and \( \mathbf{I} \) is a 2-by-2 identity matrix, giving the same result as found for plane trusses, e.g. in Guest (2006).

5. Examples

Both forthcoming structural examples are about infinitesimal inextensible mechanisms with one or more states of self-stress; the existence of their second-order stiffness is investigated. Since any infinitesimal mechanism belongs to the right nullspace of \( C \), it is sufficient to concentrate only on the (geometrical) stiffness component in Eq. (43), irrespective of the elastic or rigid material. We recall, however, that if the assemblies are considered with elastic properties, a specific kinematic load or a specific unloaded length \( l_0 \) (obtained as the difference of the loaded length and elastic elongation) is required for any member to maintain compatibility.

5.1. Example 1: kinematically indeterminate simple linkages

‘Straight’ linkages (whose nodes are incident to the same line) in a Euclidean space form an infinitesimal mechanism that can be stiffened by pre-stressing. Consider now a similar but spherical linkage, whose nodes are aligned to a given great circle of the sphere. For the sake of simplicity, a three-member linkage with equidistant nodes lying on the equator will only be analysed. The structure itself is shown in Fig. 3a, where empty circles (at the extremes) indicate supported nodes. Each bar has an angular dimension \( \alpha \), so, for each node \( N_\mu, \theta_\mu = \pi/2 \) and \( \varphi_\mu = -3\pi/2 + \mu \alpha \) (\( \mu = 0, \ldots, 3 \)).

Note that \( \alpha = \pi \) is excluded as it corresponds to three arches of indefinite position due to antipodal joints. The result in the first row coincides with the expectations that a straight spherical linkage of three or more members cannot be stabilized by prestress if its full length is somewhat bigger than the half length of a great circle. Interestingly, however, there are degenerate (self-overlapping) spherical linkages that account for stable equilibrium if their members are in compression, see rows 3 and 6.

We remark that a linkage of only two members produces stable equilibrium with compression even if its full angular dimension is
between $\pi$ and $2\pi$, i.e., is not self-overlapping (the proof proceeds exactly as above).

5.2. Example 2: a spherical covering problem

Solution of spherical circle packing and covering problems have already been mentioned among possible applications of spherical truss theory. The problem can be stated as follows: for a given positive integer $n$, what is the maximal radius of $n$ equal circles that can still be arranged on the surface of the unit sphere without overlapping (packing problem), or what is the minimal radius of $n$ equal circles that can still cover the unit sphere (covering problem). Both problems can be analysed by truss theory considering an associated graph (Fejes-Tóth, 1972). In the circle-covering problem, for instance, the graph is a bipartite graph that has two kinds of vertices: circle centres are of the first kind, while points where six circles (i.e., when their radius is minimal) is obtained by gradual concerted shrinking of the bars (radii), up to the point when the assembly gets tightened to the sphere. In this configuration, there are either tensile members or a better comparability with Fig. 4b–d, column labels typeset in bold face indicate the vertices of the first kind, underlined numbers refer to nodes on the equator.

The referred local (and incidentally, also the global) optimal arrangement of six circles. The condition given above, i.e., that a state of self-stress with pure tension is required for a local optimum is rather heuristic: in accordance of the previous sections of this paper, the stability of that state of self-stress should be shown.

The referred local (and incidentally, also the global) optimal configuration is shown in Fig. 4a. Sphere centres have a regular octahedral arrangement: these vertices of the first kind are plotted with double contour. The graph has a total of 14 vertices and 24 edges, meaning 28 kinematical degrees of freedom and 24 constraints. The three rigid body displacements were taken into account by three extra constraints instead of fixing of some nodes as in Example 1, in order to preserve some of the symmetry of the system. The left nullspace of the resultant 27-by-28 matrix $C$ contains two independent states of self-stress (triplets of values correspond to forces in bars attached to the same node of the second kind; the last triplet in brackets is due to the extra constraints):

\[
\{A_{k,1}\} = \frac{1}{\sqrt{12}}\begin{bmatrix}
111 & 000 & 111 & 000 & 000 & 111 \\
111 & 111 & 000 & 000 & 111 & 000 \\
111 & 000 & 111 & 000 & 000 & 111 \\
000 & 111 & 111 & 000 & 000 & 111 \\
000 & 000 & 111 & 111 & 000 & 000 \\
000 & 000 & 000 & 111 & 111 & 000 \\
\end{bmatrix}.
\]

\[
\{A_{k,2}\} = \frac{1}{\sqrt{12}}\begin{bmatrix}
000 & 111 & 000 & 111 & 000 & 000 \\
000 & 000 & 111 & 111 & 000 & 000 \\
000 & 000 & 000 & 111 & 111 & 000 \\
111 & 000 & 111 & 000 & 000 & 111 \\
111 & 111 & 000 & 000 & 111 & 000 \\
111 & 000 & 000 & 111 & 111 & 000 \\
\end{bmatrix}.
\]

It is easy to see that these states of self-stress correspond to the pre-stressing one of the two spherical tetrahedra ($1, 3, 6, 8$ and $2, 4, 5, 7$) crossing perpendicularly at the vertices of the first kind. Similarly, the right nullspace can be represented by the mechanisms as follows:

\[
\{\delta d_{1}\} = \begin{bmatrix}
0 & 0 & b & 0 & b & -a & 0 & -b & 0 & 0 & b & 0 & 0 & 0 & b & 0 & 0 & -b \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix},
\]

\[
\{\delta d_{2}\} = \begin{bmatrix}
0 & 0 & -a & 0 & b & 0 & b & 0 & -a & 0 & 0 & b & 0 & 0 & 0 & b & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix},
\]

where $a = 0.1889822365\ldots$, $b = 0.3273268353\ldots$, $c = 0.2314550249\ldots$, $d = 0.1291751051\ldots$, $e = 0.0913405927\ldots$ For a better comparability with Fig. 4b–d, column labels typeset in bold face indicate the vertices of the first kind, underlined numbers refer to nodes on the equator.

Through the same steps as in Section 5.1 (whose details are not presented here) one can get $W$ with the choice, e.g. $A_{k} = A_{k,1}$ as follows:

\[
W = \{w_{ij}\} = \begin{bmatrix}
0.3499271061 & 0 & 0 \\
0 & 0.1749635530 & 0 \\
0 & 0 & 0.0544969264 \\
\end{bmatrix}.
\]

Note that $W$ is already diagonal, which is due now just to the symmetry properties of the three base mechanisms $\{\delta d_{i}\}$. Being all values positive in the main diagonal, however, is a direct proof of the positive definiteness of $W$. Thus, $+A_{k,1}$ provides a stable equilibrium for the assembly even though it is kinematically indeterminate. In other words, none of the linear combinations of mechanisms $\{\delta d_{i}\}$ can be of second or higher order (see Gáspár and Tarnai, 1994).

6. Discussion

The comprehensive spherical truss theory introduced above has been developed on the basis of a simple ball-and-net model. The
formulation is based on the principle of potential energy, but unlike in the case of classical trusses, the work of external moments instead of that of external forces is considered. Detailed formulation has also been given for different components of the tangent stiffness matrix (the material stiffness matrix, a stiffness matrix component due to initial stresses and the stiffness matrix component depending on the changes in geometry). It provides a tool for the analysis of the prestress stability of a given equilibrium configuration, making also possible to detect that a given infinitesimal mechanism can or cannot be a finite mechanism as well.

The main benefit of the spherical description is a new approach that makes possible to perform mechanical analysis without converting spherical coordinates into Cartesian ones; another of its advantages is obviously the reduction in the number of both the geometrical unknowns and constraints by about one third through the use of only two angular variables. At the same time, some difficulties arise from the singularity at the poles (and consequently, from a numerical instability in their neighbourhood which is a possible subject of further investigations). It is also problematic because some highly symmetrical assemblies cannot be analysed just in their optimal orientation. Special care must also be taken of the wide set of derivative formulae, mainly when the Hessian matrix is compiled, although this task can easily be completed using symbolic programming techniques. In spite of the problems listed above, the applicability of this theory has been shown by two numerical examples, and the results agreed with those obtained from 3D truss analysis (Section 5.2) and confirmed some natural conjectures about stretched pin-jointed linkages, showing also an interesting example when the structural stability is ensured by a state of self-stress with each member in compression (Section 5.1).

Acknowledgement

Support provided by OTKA under Grant No. T 046846 is hereby gratefully acknowledged.

References