JOURNAL OF COMBINATORIAL THEORY, Series B 49, 287–294 (1990)

Some Localization Theorems on Hamiltonian Circuits

A. S. HASRATIAN

Department of Applied Mathematics, University of Yerevan, Yerevan, 375049, USSR

AND

N. K. KHACHATRIAN

Computing Center, Academy of Sciences of the Armenian SSR, Yerevan, 375014, USSR

Communicated by the Editors

Received April 13, 1987

Theorems on the localization of the conditions of G. A. Dirac (*Proc. London Math. Soc. (3)* **2**, 1952, 69–81), O. Ore (*Amer. Math. Monthly* **67**, 1960, 55), and Geng-hua Fan (*J. Combin. Theory Ser. B* **37**, 1984, 221–227) for a graph to be hamiltonian are obtained. It is proved, in particular, that a connected graph G on $p \geqslant 3$ vertices is hamiltonian if $d(u) \geqslant |M^3(u)|/2$ for each vertex u in G, where $M^3(u)$ is the set of vertices v in G that are a distance at most three from u. © 1990 Academic Press, Inc.

1. INTRODUCTION

Our notation and terminology follows Harary [4]. Let k be a positive integer. For each vertex u of a graph $G = (V, X)$ we will denote by $M^k(u)$ and $N(u)$ the sets of all $v \in V$ with $d(u, v) \leqslant k$ and $d(u, v) = 1$, respectively. The subgraph of G induced by $M^k(u)$ is denoted by $G_k(u)$. The degree in $G_k(u)$ of a vertex $v \in M^k(u)$ is denoted by $d_{G_k(u)}(v)$.

The closure $C(G)$ of G is the graph obtained from G by recursively joining pairs of nonadjacent vertices whose degree-sum is at least $|V|$, until no such pair remains.

The following results are known. A graph $G = (V, X)$ on $p \geqslant 3$ vertices is hamiltonian if:

$$d(v) \geqslant p/2 \quad \text{for each} \quad v \in V \quad \text{(Dirac [2]).} \tag{1.1}$$

$$uv \notin X \Rightarrow d(u) + d(v) \geqslant p \quad \text{(Ore [6]).} \tag{1.2}$$

$$d(u) = k < (p-1)/2 \Rightarrow |\{v \in V/d(v) \leqslant k\}| < k \quad \text{(Posa [7]).} \tag{1.3}$$

0095-8956/90 $3.00

and

$$d(u) = (p-1)/2 \Rightarrow |\{v \in V/d(v) \leqslant (p-1)/2\}| \leqslant (p-1)/2.$$

$$C(G) \text{ is a complete graph} \qquad \text{(Bondy and Chvátal [1]).} \tag{1.4}$$

$$G \text{ is 2-connected and } d(v) < p/2,\ d(u, v) = 2 \Rightarrow d(u) \geqslant p/2 \quad \text{(Geng-hua Fan [3]).} \tag{1.5}$$

In [5] the following theorem on a localization of condition (1.3) is proved:

THEOREM. *A connected graph G on $p \geqslant 3$ vertices is hamiltonian if*

$$d(u) = k < (p-1)/2 \Rightarrow |\{v \in M^2(u)/d(v) \leqslant k\}| < k$$

and

$$d(u) = (p-1)/2 \Rightarrow |\{v \in M^2(u)/d(v) \leqslant (p-1)/2\}| \leqslant (p-1)/2.$$

In this paper we obtain the theorems on localizations of conditions (1.1), (1.2), and (1.5).

2. RESULTS

LEMMA. *Let G be a graph with $d(u, v) = 2$, $w \in N(u) \cap N(v)$, and $d(u) + d(v) \geqslant |N(u) \cup N(v) \cup N(w)|$. Then $|N(w) \setminus (N(u) \cup N(v))| \leqslant |N(u) \cap N(v)|$.*

Proof.

$$\begin{aligned}
&|N(w) \setminus (N(u) \cup N(v))| \\
&\quad = |N(w)| - |N(w) \cap (N(u) \cup N(v))| \\
&\quad = |N(w)| - (|N(w)| + |N(u) \cup N(v)| - |N(u) \cup N(v) \cup N(w)|) \\
&\quad = |N(u) \cup N(v) \cup N(w)| - (|N(u)| + |N(v)| - |N(u) \cap N(v)|) \\
&\quad = |N(u) \cap N(v)| - (d(u) + d(v) - |N(u) \cup N(v) \cup N(w)|) \\
&\quad \leqslant |N(u) \cap N(v)|.
\end{aligned}$$

THEOREM 1. *Let $G = (V, X)$ be a connected graph with at least three vertices. If*

$$d(u) + d(v) \geqslant |N(u) \cup N(v) \cup N(w)|$$

for each triple of vertices u, v, w with $d(u, v) = 2$ and $w \in N(u) \cap N(v)$, then G is hamiltonian.

Proof. Let G satisfy the hypothesis of Theorem 1. Clearly, G contains a circuit; let C be the largest one. If G has no hamiltonian circuit, then there is a vertex u outside of C that is adjacent to at least one vertex in C. Let $\{w_1, ..., w_n\}$ be the set of vertices in C that are adjacent to u, and for each $i = 1, ..., n$ let v_i be the successor of w_i in a fixed cyclic ordering of C.

Note, that if $1 \leqslant i < j \leqslant n$ then $v_i v_j \notin X$. Otherwise delete the edges $w_i v_i$, $w_j v_j$ from C and add the edges $v_i v_j$, $w_i u$, $u w_j$. In this way we obtain a circuit longer than C, which is a contradiction.

For each $i = 1, ..., n$ let F_i and T_i denote the sets $N(u) \cap N(v_i)$ and $N(w_i) \backslash (N(u) \cup N(v_i))$, respectively. Since C is the longest circuit, $uv_i \notin X$, $i = 1, ..., n$. Then $d(u, v_i) = 2$ and from the Lemma we have $|T_i| \leqslant |F_i|$ for each $i = 1, ..., n$.

We shall show that there is a vertex u' such that $u' \notin C$ and $u' \in F_i$ for some i, $1 \leqslant i \leqslant n$.

Consider the following iterated algorithm.

Step 1. $k := 1$, $m := 1$, and $Z_i^1 := \{u, v_i\}$, $Y_i^1 := \{w_i\}$ for each $i = 1, ..., n$.

Step 2. If the set $F_m \backslash Y_m^k$ contains a vertex $u' \notin C$, then stop. Otherwise, choose an arbitrary vertex w in $F_m \backslash Y_m^k$. Clearly, $w = w_r$ for some r, $1 \leqslant r \leqslant n$.

Set

$$Y_m^{k+1} := Y_m^k \cup \{w_r\}; Z_m^{k+1} := Z_m^k;$$

$$Y_r^{k+1} := Y_r^k; Z_r^{k+1} := Z_r^k \cup \{v_m\};$$

$$Y_i^{k+1} := Y_i^k; Z_i^{k+1} := Z_i^k \quad \text{for} \quad i \neq m, r \text{ and } 1 \leqslant i \leqslant n.$$

Step 3. $k := k + 1$, $m := r$ and go to Step 2.

It is not difficult to see that before the kth iteration of the algorithm we have

(a) $Z_i^k \subseteq T_i$, $Y_i^k \subseteq F_i$, $|Z_i^k| \geqslant |Y_i^k|$ for each i, $1 \leqslant i \leqslant n$.

(b) $F_m \backslash Y_m^k \neq \varnothing$, because $|T_m| \leqslant |F_m|$ and $|Z_m^k| > |Y_m^k|$.

(c) $Y_i^k \subseteq \{w_1, ..., w_n\}$ for each i, $1 \leqslant i \leqslant n$.

(d) $|Y_m^k| = |Y_m^{k-1}| + 1$ if $k \geqslant 2$.

From (b), (c), (d) it follows that if $k \geqslant 2$ then $\sum_{i=1}^n |Y_i^{k-1}| < \sum_{i=1}^n |Y_i^k| \leqslant n^2$. Hence there exists k such that $1 \leqslant k \leqslant n^2$ and the set $F_m \backslash Y_m^k$ contains a vertex $u' \notin C$. Delete the edge $w_m v_m$ from C and add the edges $w_m u$, uu', $u'v_m$. In this way we obtain a circuit longer than C, which is a contradiction. The proof is complete.

Note that for every $t \geqslant 5$ there exists a graph $G_t = (V_t, X_t)$ with $V_t = \{v_1, v_2, ..., v_{2t}\}$ and

$$X_t = \bigcup_{k=0}^{t-2} \{v_i v_j / 2k+1 \leqslant i < j \leqslant 2k+4\},$$

which fulfills the condition in Theorem 1 and does not fulfill conditions (1.1)–(1.5). Clearly, $C(G_t) = G_t$, $|V_t| = 2t$, and $|X_t| = 5t - 4 = (5/2) \cdot |V_t| - 4$.

Corollary 1. *Let G be a connected graph on $p \geqslant 3$ vertices. If $d(u) + d_{G_2(u)}(v) \geqslant |M^2(u)|$ for each pair of vertices u, v with $d(u, v) = 2$, then G is hamiltonian.*

Proof. Clearly,

$$d(u) + d(v) - |N(v) \backslash M^2(u)| = d(u) + d_{G_2(u)}(v) \geqslant |M^2(u)|.$$

Then

$$d(u) + d(v) \geqslant |M^2(u)| + |N(v) \backslash M^2(u)| \geqslant |N(u) \cup N(v) \cup N(w)|$$

for each vertex $w \in N(u) \cap N(v)$. Hence, Corollary 1 follows from Theorem 1.

Corollary 2. *Let G be a connected graph on $p \geqslant 3$ vertices. If $d(u) + d(v) \geqslant |M^3(u)|$ for each pair of vertices u, v with $d(u, v) = 2$, then G is hamiltonian.*

Corollary 2 follows from Theorem 1 because $|M^3(u)| \geqslant |N(u) \cup N(v) \cup N(w)|$ for each vertex $w \in N(u) \cap N(v)$.

Corollary 3. *Let G be a connected graph on $p \geqslant 3$ vertices. If $d(u) \geqslant |M^3(u)|/2$ for every vertex u in G then G is hamiltonian.*

Proof. Let $G \neq K_p$, $d(u, v) = 2$, and $d(u) \leqslant d(v)$. Since $d(u) \geqslant |M^3(u)|/2$, then $d(u) + d(v) \geqslant |M^3(u)| \geqslant |N(u) \cup N(v) \cup N(w)|$ for each vertex $w \in N(u) \cap N(v)$. Therefore Corollary 3 follows from Theorem 1.

Corollary 4. *Let G be a connected graph on $p \geqslant 3$ vertices. If*

$$d_{G_1(w)}(u) + d_{G_1(w)}(v) \geqslant |M^1(w)|$$

or

$$d(u) + d(v) \geqslant |M^2(w)|$$

for each triple of vertices u, v, w with $d(u, v) = 2$ and $w \in N(u) \cap N(v)$, then G is hamiltonian.

Proof. Let $d(u, v)=2$ and $w \in N(u) \cap N(v)$.

If $d(u) + d(v) \geqslant |M^2(w)|$, then $d(u) + d(v) \geqslant |N(u) \cup N(v) \cup N(w)|$ because $|M^2(w)| \geqslant |N(u) \cup N(v) \cup N(w)|$.

Suppose that $d_{G_1(w)}(u) + d_{G_1(w)}(v) \geqslant |M^1(w)|$. Clearly, $d_{G_1(w)}(u) = d(u) - |N(u)\backslash M^1(w)|$ and $d_{G_1(w)}(v) = d(v) - |N(v)\backslash M^1(w)|$. Hence

$$\begin{aligned} d(u)+d(v) &\geqslant |M^1(w)| + |N(u)\backslash M^1(w)| + |N(v)\backslash M^1(w)| \\ &\geqslant |N(u) \cup N(v) \cup N(w)| \end{aligned}$$

and Corollary 4 follows from Theorem 1.

COROLLARY 5. *Let G be a connected graph on $p \geqslant 3$ vertices. If for each vertex u in G at least one of the graphs $G_1(u)$ or $G_2(u)$ satisfies Ore's condition, then G is hamiltonian.*

Corollary 5 follows from Corollary 4.

THEOREM 2. *Let $G=(V, X)$ be a 2-connected graph on $p \geqslant 3$ vertices and let v and u be distinct vertices of G. If*

$$d(u) < p/2,\ d(u, v) = 2 \Rightarrow d(v) \geqslant |M^3(u)|/2, \tag{2.1}$$

then G is hamiltonian.

Proof. Let $A = \{P^1, ..., P^h\}$ be the set of all longest paths in G. For each $i=1, ..., h$ let $P^i = v_0^i v_1^i \cdots v_m^i$ and $f(P^i)$ be the smallest r from $\{0, 1, ..., m-1\}$ such that $v_m^i v_r^i \in X$. We denote by A_1 the set of all $P^i \in A$ with $d(v_0^i) = \max_{1 \leqslant j \leqslant h} d(v_0^j)$.

Suppose that G is a graph satisfying the condition of Theorem 2 and that G has no hamiltonian circuit. We shall arrive at a contradiction.

Let $P = v_0 v_1 \cdots v_m$ be some longest path in G of length m, chosen so that $f(P) = \min_{P^i \in A_1} f(P^i)$. Clearly, $d(v_0) \geqslant d(v_m)$. If $d(v_0) + d(v_m) \geqslant p$ then there are at least two consecutive vertices on P, v_i, and v_{i+1}, such that $v_i v_m \in X$ and $v_{i+1} v_0 \in X$, and so we obtain a circuit of length $m+1$. By the connectedness of G, we have either a hamiltonian circuit or a path of length $m+1$. Each leads to contradictions. Consequently $d(v_0) + d(v_m) < p$. Since $d(v_0) \geqslant d(v_m)$, $d(v_m) < p/2$.

From the proof above we can also suppose that

(a) G has no circuit of length $m+1$.

Since G is 2-connected, $d(v_m) \geqslant 2$. Let $N(v_m) = \{v_{j_1}, ..., v_{j_t}\}$ and $j_1 < \cdots < j_t$. Clearly, $j_1 \geqslant 1$, otherwise G has a circuit of length $m+1$, which is contrary to (a). We show now that

(b) if $v_m v_i \in X$ and $j_1 \leqslant i \leqslant m-1$ then $N(v_{1+i}) \subseteq \{v_{j_1}, ..., v_m\}$,

(c) $v_m v_i \notin X$ for some i, $j_1 < i < m$,

(d) $v_{j_1-1} v_{j_i+1} \notin X$ for every i, $1 \leqslant i \leqslant t$.

Proof. (b) Clearly, we have $N(v_{1+i}) \subseteq \{v_0, v_1, ..., v_m\}$ otherwise G has a path of length $m+1$. Suppose that there is s such that $1 \leqslant s < j_1$ and $v_s v_{1+i} \in X$. Then

$$P' = v_0 v_1 \cdots v_s v_{1+i} v_{2+i} \cdots v_m v_i v_{i-1} \cdots v_{s+1}$$

is the longest path in G with $f(P') < f(P)$. This contradicts the choice of P. Therefore $N(v_{1+i}) \subseteq \{v_{j_i}, v_{1+j_i}, ..., v_m\}$.

(c) If $v_m v_i \in X$ for every i, $j_1 \leqslant i < m$, then $\bigcup_{i=j_1}^{m-1} N(v_{1+i}) \subseteq \{v_{j_1}, v_{1+j_1}, ..., v_m\}$. This contradicts the 2-connectedness of G.

(d) It is obvious that (d) follows from (b).

From (c) it follows that there is a k such that $j_1 < k < m-1$, $v_m v_k \notin X$, and $v_m v_i \in X$ for every i, $j_1 \leqslant i \leqslant k-1$. Thus we have

(e) there is no i such that $j_1 < i \leqslant m-1$, $v_{j_1-1} v_i \in X$, and $v_k v_{1+i} \in X$.

Indeed, if $v_i v_{j_1-1} \in X$ and $v_k v_{1+i} \in X$ then from (d) it follows that $i > k$. Then G has the longest path P',

$$P' = v_0 v_1 \cdots v_{j_1-1} v_i v_{i-1} \cdots v_k v_{1+i} \cdots v_m v_{k-1} \cdots v_{j_1}$$

with $f(P') < f(P)$. This contradicts the choice of P.

Clearly, $d(v_k, v_m) = 2$ and $d(v_{j_1-1}, v_m) = 2$. Since $d(v_m) < p/2$, it follows from (2.1) that $d(v_{j_1-1}) \geqslant |M^3(v_m)|/2$ and $d(v_k) \geqslant |M^3(v_m)|/2$.

Since $v_k v_{j_1-1} \notin X$ and the degree-sum of vertices v_k and v_{j_1-1} in $G_3(v_m)$ is at least $|M^3(v_m)|$, $d(v_k, v_{j_1-1}) = 2$.

From (d) it follows that $d(v_{j_1-1}) < |M^3(v_m)| - d(v_m)$. Since $d(v_{j_1-1}) \geqslant |M^3(v_m)|/2$, then $d(v_m) < |M^3(v_m)|/2$. Therefore $d(v_{j_1-1}) > d(v_m)$ and $d(v_k) > d(v_m)$.

Case 1. $d(v_k) < p/2$. Since $d(v_k, v_m) = d(v_k, v_{j_1-1}) = 2$, it follows from (2.1) that $d(v_m) \geqslant |M^3(v_k)|/2$ and $d(v_{j_1-1}) \geqslant |M^3(v_k)|/2$. Together with $d(v_k) > d(v_m)$ this implies that

$$d(v_k) + d(v_{j_1-1}) \geqslant |M^3(v_k)|. \tag{2.2}$$

From (d) it follows that $v_i v_{j_1-1} \notin X$ for each i, $1 + j_1 \leqslant i < k$. From (e) it follows that $v_i v_{j_1-1} \notin X$ for every i such that $i > j_1$ and $v_k v_{i+1} \in X$. Besides, $v_m v_{j_1-1} \notin X$ and $v_m, v_{j_1-1} \in M^3(v_k)$. Thus $d(v_{j_1-1}) \leqslant |M^3(v_k)| - d(v_k) - 1$. This contradicts (2.2).

Case 2. $d(v_k) \geqslant p/2$. If $d(v_{j_1-1}) < p/2$ then (2.1) and $d(v_{j_1-1}, v_m) =$

$d(v_{j_1-1}, v_k)=2$ imply that $d(v_m) \geqslant |M^3(v_{j_1-1})|/2$ and $d(v_k) \geqslant |M^3(v_{j_1-1})|/2$. Since $d(v_{j_1-1}) > d(v_m)$, we have

$$d(v_{j_1-1}) + d(v_k) \geqslant |M^3(v_{j_1-1})|. \tag{2.3}$$

If $d(v_{j_1-1}) \geqslant p/2$ then $d(v_{j_1-1}) + d(v_k) \geqslant p \geqslant |M^3(v_{j_1-1})|$, so (2.3) holds again.

From (b) it follows that $v_k v_{j_1-1} \notin X$ and v_k is not adjacent to every vertex $v \in N(v_{j_1-1}) \backslash \{v_{j_1}, v_{1+j_1}, ..., v_m\}$.

From (e) it follows that $v_k v_{1+i} \notin X$ for every i such that $v_i \in N(v_{j_1-1}) \cap \{v_{1+j_1}, v_{2+j_1}, ..., v_m\}$. Besides, we have $v_{j_1-1}, v_k \in M^3(v_{j_1-1})$. Therefore $d(v_k) \leqslant |M^3(v_{j_1-1})| - d(v_{j_1-1}) - 1$.

This contradicts (2.3). The proof is complete.

Note that for every $r \geqslant 2$ there exists a graph $G_r = (V_r, X_r)$ with $V_r = \{w_1, w_2\} \cup \{v_1, ..., v_{3r-1}\} \cup \{u_1, ..., u_{3r-1}\}$ and $X_r = \{w_1 v_i, w_2 v_i / i = 1, ..., 2r\} \cup \{v_i v_j, u_i u_j / 1 \leqslant i < j \leqslant 3r-1\} \cup \{v_i u_j / 1 + 2r \leqslant i,\ j \leqslant 3r-1\}$ that satisfies the condition of Theorem 2 and does not satisfy the condition (1.5).

Besides, for every $n \geqslant 5$ there exists a graph $G_n = (V_n, X_n)$ with $V_n = \{v_1, ..., v_n\}$ and $X_n = \{v_i v_j / 1 \leqslant i < j \leqslant n-2\} \cup \{v_{n-1} v_1, v_{n-1} v_2\} \cup \{v_n v_i / i = 2, 3, ..., n-2\}$ that satisfies the condition of Theorem 2 and does not satisfy the condition of Theorem 1.

Let $G = (V, X)$. It is shown in [1] (by paraphrasing Ore's proof [6]) that if $G + uv$ is hamiltonian and $d(u) + d(v) \geqslant |V|$ then G itself is hamiltonian.

THEOREM 3. *If $G + uv$ is hamiltonian, $d(u, v) = 2$, and*

$$d(u) + d_{G_2(u)}(v) \geqslant |M^2(u)|, \tag{2.4}$$

then G itself is hamiltonian.

Proof. Suppose $G + uv$ is hamiltonian but G is not. Then G has a hamiltonian path $u_1, u_2, ..., u_p$ with $u_1 = v$ and $u_p = u$. Let $N(u) = \{u_{i_1}, ..., u_{i_t}\}$. If $vu_{1+i_j} \notin X$ for every j, $1 \leqslant j \leqslant t$, then $d_{G_2(u)}(v) < |M^2(u)| - d(u)$. This contradicts (2.4). Hence there is m such that $1 \leqslant m \leqslant t-1$, $vu_{1+i_m} \in X$, and $uu_{i_m} \in X$.

But then G has the hamiltonian circuit

$$u_1 u_{1+i_m} u_{2+i_m} \cdots u_p u_{i_m} u_{i_m-1} \cdots u_1.$$

This contradicts the hypothesis.

COROLLARY 6. *If $G + uv$ is hamiltonian, $d(u, v) = 2$, and $d(u) + d(v) \geqslant |M^3(v)|$, then G itself is hamiltonian.*

REFERENCES

1. J. A. BONDY AND V. CHVÁTAL, A method in graph theory, *Discrete Math.* **15** (1976), 111–135.
2. G. A. DIRAC, Some theorems on abstract graphs, *Proc. London Math. Soc. (3)* **2** (1952), 69–81.
3. GENG-HUA FAN, New sufficient conditions for cycles in graphs, *J. Combin. Theory Ser. B* **37** (1984), 221–227.
4. F. HARARY, "Graph Theory," Addison–Wesley, Reading, MA, 1969.
5. A. S. HASRATIAN AND N. K. KHACHATRIAN, A two theorems on a hamiltonian graphs (Russian), *Math. Zametki* **35**, No. 1 (1984), 55–61.
6. O. ORE, Note on Hamiltonian circuits, *Amer. Math. Monthly* **67** (1960), 55.
7. L. POSA, A theorem concerning Hamilton lines, *Magyar Tud. Acad. Mat. Kutató Int. Közl.* **7** (1962), 225–226.