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On uniserial modules in the Auslander–Reiten quiver

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Abstract

This article begins the study of irreducible maps involving finite-dimensional uniserial modules over finite-dimensional associative algebras. We work on the classification of irreducible maps between two uniserials over triangular algebras, and give estimates for the number of middle terms of an almost split sequence with a uniserial end term.

Keywords: Auslander–Reiten quiver; Uniserial modules; Irreducible maps; Almost split sequences

1. Introduction

The study of finite-dimensional uniserial modules over finite-dimensional associative algebras was begun in earnest by Huisgen-Zimmermann in [8]; Huisgen-Zimmermann and Bongartz achieved a description of uniserial modules in terms of certain varieties in [5] (see also [4,6,9]). In the present article, which is based on the authors' theses [3,12], certain questions regarding the position of uniserial modules in the Auslander–Reiten quiver of finite-dimensional algebras are investigated; most of the work applies to basic split triangular algebras only.

The article is organized as follows. In Section 2 we fix our notation and conventions and recall the basic description of uniserials via varieties. In Section 3 we present a general result that motivates much of the following work: any irreducible map between two uniserials is either the radical embedding or the socle factor projection of a uniserial module. The two cases being dual,

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we go on to state a conjecture giving a concrete necessary and sufficient condition for a uniserial over a triangular algebra to have an irreducible radical embedding. The criterion is combinatorial in nature—as a consequence, while slightly technical when phrased in full generality, it is readily checkable for a given quiver with relations. The sufficiency of this condition is proved using the technique of quiver representations. The necessity of one part of the condition is then proved in a slightly more general context.

We have not yet managed to prove the necessity of the full condition for all triangular algebras. In Section 4 we prove it under an additional assumption, which includes the case of all triangular multiserial algebras. In Section 5 we prove it for all monomial algebras; the condition takes on a very simple form in this situation.

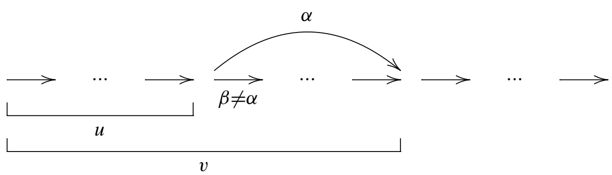
In Section 6 we study a general finite-dimensional algebra and focus on a different circle of questions: almost split sequences with a uniserial end term. First we give a simple general result: any short exact sequence with uniserial end terms has a middle term which is either indecomposable or a direct sum of two uniserials. Then we study the number of indecomposable middle terms in an almost split sequence ending in a uniserial module; an upper bound is given for multiserial algebras.

2. Notation and preliminaries

We will use the notation and terminology of [1]. Throughout, $\mathfrak{K}$ will be a field and Γ will be a finite quiver with vertex set Γ_0 and arrow set Γ_1. We compose arrows, paths and maps from right to left: if $p: e \to f$ and $q: f \to g$ then $qp: e \to g$. The starting point of the path p is denoted by $\mathrm{s}(p)$ and its end point by $\mathrm{t}(p)$. $\Lambda = \mathfrak{K}\Gamma/I$ will be a finite-dimensional $\mathfrak{K}$-algebra presented as the quotient of the path algebra of Γ by an admissible ideal I. Λ is called *triangular* if Γ does not contain any directed cycles. Whenever useful, we identify elements of Γ_0 and paths in Γ with their corresponding classes in Λ.

The category of finitely generated left Λ-modules is denoted by Λ-mod. The direct sum of two modules M and N is denoted by $M \sqcup N$. A module is called *uniserial* if it has only one composition series with simple factors. If $U \in \Lambda$-mod is uniserial with length n, then there exists a path p in Γ of length $n-1$ and an element $x \in U$ such that $px \neq 0$. Any such path is called a *mast* of U and any such element x is called a *top element* of U. The terminology is that of [8].

Let p be a path in Γ. A path u is a *right subpath* of p if there exists a path r with $p = ru$. Following [8], a *detour* on the path p is a pair (α, u) with α an arrow and u a right subpath of p, where αu is a path in Γ which is not a right subpath of p, but there exists a right subpath v of p with $\mathrm{length}(v) \geqslant \mathrm{length}(u) + 1$ such that the endpoint of v coincides with the endpoint of α.



We will abbreviate the statement "(α, u) is a detour on p" by $(\alpha, u) \wr\wr p$. Given any detour on p, let $V(\alpha, u) = \{v_i(\alpha, u) \mid i \in I(\alpha, u)\}$ be the family of right subpaths of p in $\mathfrak{K}\Gamma$ which are longer than u and have the same endpoint as α.

Now suppose p has length l and passes consecutively through the vertices $e(1), \ldots, e(l+1)$ (which need not be distinct). A *route* on p is any path in Γ which starts in $e(1)$ and passes through a subsequence of the sequence $(e(1), \ldots, e(l+1))$ in this order and through no other vertices. A *non-route* on p is any path in Γ which starts in $e(1)$ and is not a route on p.

Given any uniserial module with mast p and top element x, if $(\alpha, u) \wr\wr p$, then $\alpha u x = \sum_{i \in I(\alpha,u)} k_i(\alpha,u) v_i(\alpha,u) x$ for unique scalars $k_i(\alpha,u)$. By [8], the points $(k_i(\alpha,u))_{i \in I(\alpha,u),(\alpha,u)\wr\wr p}$ corresponding to uniserials with mast p form an affine variety, called V_p, which lives in $\mathbb{A}^N$, where $N = \sum_{(\alpha,u)\wr\wr p} |I(\alpha,u)|$. Moreover, there is a surjective map Φ_p from V_p onto the set of isomorphism types of uniserial Λ-modules with mast p. It assigns to each point $k = (k_i(\alpha,u))_{i \in I(\alpha,u),(\alpha,u)\wr\wr p}$ in V_p the isomorphism type of the module $\Lambda e(1)/U_k$, where

$$U_k = \sum_{(\alpha,u)\wr\wr p} \Lambda\left(\alpha u - \sum_{i \in I(\alpha,u)} k_i(\alpha,u) v_i(\alpha,u)\right) + \sum_{q \text{ non-route on } p} \Lambda q.$$

3. Irreducible radical embeddings of uniserials

In this section, we first show that the only irreducible maps between uniserial modules are certain radical embeddings $JU \hookrightarrow U$ and socle factor projections $U \to U/\operatorname{soc} U$. Then for a triangular algebra $\Lambda = \mathfrak{K}\Gamma/I$, we propose necessary and sufficient combinatorial conditions for the radical embedding $JU \hookrightarrow U$ of a uniserial module U to be irreducible.

Proposition 3.1. *Let R be a left artinian ring with Jacobson radical J.*

(1) *If $f : M \to U$ is an irreducible injective map from the module $M \in R$-mod to the uniserial $U \in R$-mod, then there exists an isomorphism $\varphi : JU \to M$ so that $f\varphi$ is the natural radical embedding $JU \hookrightarrow U$.*
(2) *If $g : U \to M$ is an irreducible surjective map from the uniserial $U \in R$-mod to the module $M \in R$-mod, then there exists an isomorphism $\psi : M \to U/\operatorname{soc} U$ so that ψg is the natural socle factor projection $U \to U/\operatorname{soc} U$.*

Proof. (1) Since $\operatorname{im}(f)$ is a proper submodule of U, $\operatorname{im}(f) = J^l U$ with $l \geqslant 1$ and $M \cong J^l U$ via f. However, if $l > 1$, then $J^l U \to J^{l-1} U \to U$ would be a non-trivial factorization of $J^l U \to U$, giving us a factorization of f, which is impossible. The proof of (2) is similar to that of (1). □

Since every irreducible morphism is either injective or surjective, the only irreducible maps between two uniserial modules are among radical embeddings $JU \hookrightarrow U$ and socle factor projections $U \to U/\operatorname{soc} U$. Since the two cases are clearly dual, we will focus on radical embeddings in the sequel.

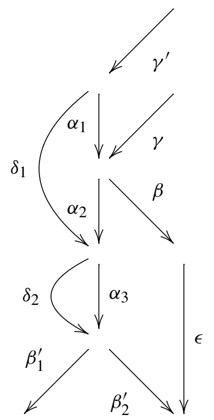
Now assume that $\Lambda = \mathfrak{K}\Gamma/I$ is a triangular algebra. To prepare for our analysis in this section, we fix a finitely generated uniserial left Λ-module U with mast

$$p = 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} n.$$

On several occasions, we will refer to certain subpaths $\alpha_i \cdots \alpha_j$ of p; whenever $i < j$, this expression will simply stand for 1. We now name all the arrows in Γ that touch p, classifying them according to the type of contact with p:

$$B := \{\beta \in \Gamma_1 \mid \mathrm{s}(\beta) \in \{1, \dots, n-1\} \text{ and } \mathrm{t}(\beta) \notin \{1, \dots, n\}\},$$
$$B' := \{\beta' \in \Gamma_1 \mid \mathrm{s}(\beta') = n\},$$
$$C := \{\gamma \in \Gamma_1 \mid \mathrm{s}(\gamma) \notin \{1, \dots, n\} \text{ and } \mathrm{t}(\gamma) \in \{2, \dots, n\}\},$$
$$C' := \{\gamma' \in \Gamma_1 \mid \mathrm{t}(\gamma') = 1\},$$
$$D := \{\delta \in \Gamma_1 \mid \{\mathrm{s}(\delta), \mathrm{t}(\delta)\} \subset \{1, \dots, n\} \text{ and } \delta \notin \{\alpha_1, \dots, \alpha_{n-1}\}\}.$$

For an illustration of these definitions with an example, consider the following quiver Γ, together with the path $p = \alpha_3\alpha_2\alpha_1$:



We then have

$$B = \{\beta\}, \quad B' = \{\beta'_1, \beta'_2\}, \quad C = \{\gamma\}, \quad C' = \{\gamma'\}, \quad D = \{\delta_1, \delta_2\}.$$

Observe that, in general, our uniserial module U may be identified with a representation $U = ((U_x), (f_\alpha))$ of Γ, where

$$U_x = \begin{cases} \mathfrak{K}, & \text{if } x \in \{1, \dots, n\}; \\ 0, & \text{otherwise} \end{cases}$$

and

$$f_{\alpha_i} = \mathrm{id} \quad \text{for every } i \in \{1, \dots, n-1\}.$$

The module U is then completely determined by the choice of the mast p and the scalars $f_\delta(1)$ for $\delta \in D$, different sets of scalars corresponding to non-isomorphic modules. Unlike the hereditary case, not every path is a mast, however, and not every set of scalars appears in this fashion, since the relations in I impose restrictions.

We know from 3.1 that, in order to understand irreducible maps between uniserial modules, it is sufficient to study radical embeddings (and their duals, socle factor projections). The following conjecture covers this situation; we manage to prove "(2) ⇒ (1)" and a generalization of

"(1) ⇒ (2)(a)" in the sequel. We will also prove "(1) ⇒ (2)(b)" for monomial and for multiserial algebras.

Conjecture 3.2. *(See [3, Conjecture 1.2.1].) Suppose Λ is a triangular algebra and U is a uniserial Λ-module with mast p. Then the following statements are equivalent*:

(1) *The embedding $JU \to U$ is irreducible.*
(2) *U is not simple and satisfies both* (a) *and* (b) *below*:
   (a) *For every $\beta \in B$,*

$$\beta\alpha_{\mathrm{s}(\beta)-1}\cdots\alpha_1 \in Jp,$$

   and for every $\delta \in D$,

$$\delta\alpha_{\mathrm{s}(\delta)-1}\cdots\alpha_1 \in \mathfrak{K}\alpha_{\mathrm{t}(\delta)-1}\cdots\alpha_1.$$

   (b) *There exists a subset $R \subset J$ such that $\{rp + J^2p \mid r \in R\}$ forms a $\mathfrak{K}$-basis for Jp/J^2p and* (i) *and* (ii) *both hold*:
      (i) *For every $\gamma \in C$ there exists $w \in pJ$ such that, for every $r \in R$,*

$$r\alpha_{n-1}\cdots\alpha_{\mathrm{t}(\gamma)}\gamma = rw.$$

      (ii) *For every $\delta \in D$ and every $r \in R$,*

$$r\alpha_{n-1}\cdots\alpha_{\mathrm{t}(\delta)}\delta \in \mathfrak{K}r\alpha_{n-1}\cdots\alpha_{\mathrm{s}(\delta)}.$$

Proof of "(2) ⇒ (1)." Let $V = ((V_x), (g_\alpha)) \in \Lambda$-mod and suppose there exist Λ-linear maps

$$JU \xrightarrow{\Phi=(\Phi_x)} V \xrightarrow{\Psi=(\Psi_x)} U$$

such that $\Psi\Phi$ is the embedding $JU \hookrightarrow U$.

Observe that we can assume without loss of generality that the elements of the set R arising from condition (2) are normed in the following fashion: $r = e_{u(r)}re_n$ for certain vertices $u(r) \in \Gamma_0$. We can thus denote by g_r the $\mathfrak{K}$-linear map $V_n \to V_{u(r)}$ induced by left multiplication by r.

Note furthermore that we can strengthen the conditions on $\delta \in D$ in the following manner:

$$\delta\alpha_{\mathrm{s}(\delta)-1}\cdots\alpha_1 = f_\delta(1)\alpha_{\mathrm{t}(\delta)-1}\cdots\alpha_1$$

and for every $r \in R$

$$r\alpha_{n-1}\cdots\alpha_{\mathrm{t}(\delta)}\delta = f_\delta(1)r\alpha_{n-1}\cdots\alpha_{\mathrm{s}(\delta)}.$$

The first equation is clear, and the second one follows then from

$$r\alpha_{n-1}\cdots\alpha_{\mathrm{t}(\delta)}\delta\alpha_{\mathrm{s}(\delta)-1}\cdots\alpha_1 = f_\delta(1)r\alpha_{n-1}\cdots\alpha_1$$

since $rp \neq 0$ for $r \in R$.

Case 1. There exists $v \in V_1$ with $\Psi_1(v) = 1$ and $(g_r g_{\alpha_{n-1}} \cdots g_{\alpha_1})(v) = 0$ for all $r \in R$.

Our goal is to construct a section χ for Ψ in this case. Define $\chi = (\chi_x) : U \to V$ by

$$\begin{aligned} \chi_1(1) &:= v, \\ \chi_i(1) &:= (g_{\alpha_{i-1}} \cdots g_{\alpha_1})(v) \quad &&\text{for } i \in \{2, \dots, n\} \text{ and} \\ \chi_x &:= 0 &&\text{for } x \notin \{1, \dots, n\}. \end{aligned}$$

Once we have checked that $\chi \in \mathrm{Hom}_\Lambda(U, V)$, the equality $\Psi_1 \chi_1(1) = 1$ will clearly imply $\Psi \chi = \mathrm{id}$, completing the treatment of the first case.

So let us check that χ is Λ-linear. That $g_{\alpha_i} \chi_i = \chi_{i+1} = \chi_{i+1} f_{\alpha_i}$ for $i \in \{1, \dots, n-1\}$ is clear; moreover, we compute

$$\begin{aligned} g_\delta \chi_{\mathrm{s}(\delta)}(1) &= (g_\delta g_{\alpha_{\mathrm{s}(\delta)-1}} \cdots g_{\alpha_1})(v) \\ &= f_\delta(1)(g_{\alpha_{\mathrm{t}(\delta)-1}} \cdots g_{\alpha_1})(v) \\ &= \chi_{\mathrm{t}(\delta)} f_\delta(1) \end{aligned}$$

for $\delta \in D$.

Next observe that $Jpv \subset \sum_r \mathfrak{K} rpv + J^2 pv = J^2 pv$ (because R generates $Jp/J^2 p$ and because of our assumption in Case 1). If follows $Jpv = 0$.

Now let $\beta \in B \cup B'$. Then $(g_\beta g_{\alpha_{\mathrm{s}(\beta)-1}} \cdots g_{\alpha_1})(v) \in Jpv = 0$, and again $g_\beta \chi_{\mathrm{s}(\beta)} = 0 = \chi_{\mathrm{t}(\beta)} f_\beta$.

Case 2. For every $v \in V_1$ with $\Psi_1(v) = 1$, there exists $r \in R$ with $(g_r g_{\alpha_{n-1}} \cdots g_{\alpha_1})(v) \neq 0$.

In this case, we will construct a retraction χ for Φ. We may clearly assume that R is finite, and then the condition of Case 2 immediately implies that there exist linear maps $\omega_r : V_{u(r)} \to \mathfrak{K}$ for $r \in R$ such that

$$\Psi_1 = \sum_{r \in R} \omega_r g_r g_{\alpha_{n-1}} \cdots g_{\alpha_1}.$$

Define $\chi = (\chi_x) : V \to JU$ by

$$\begin{aligned} \chi_i &:= \Psi_i - \textstyle\sum_{r \in R} \omega_r g_r g_{\alpha_{n-1}} \cdots g_{\alpha_i} \quad &&\text{for } i \in \{1, \dots, n\} \text{ and} \\ \chi_x &:= 0 &&\text{for } x \notin \{1, \dots, n\}. \end{aligned}$$

Again we need to check that χ is Λ-linear. For that purpose, we compute $\chi_1 = 0$,

$$\begin{aligned} f_{\alpha_i} \chi_i &= \Psi_{i+1} g_{\alpha_i} - \left(\sum_{r \in R} \omega_r g_r g_{\alpha_{n-1}} \cdots g_{\alpha_{i+1}} \right) g_{\alpha_i} \\ &= \chi_{i+1} g_{\alpha_i} \end{aligned}$$

for $i \in \{1, \dots, n-1\}$, and

$$
\begin{aligned}
f_\delta \chi_{\mathrm{s}(\delta)} &= \Psi_{\mathrm{t}(\delta)} g_\delta - f_\delta(1) \sum_{r\in R} \omega_r g_r g_{\alpha_{n-1}} \cdots g_{\alpha_{\mathrm{s}(\delta)}} \\
&= \Psi_{\mathrm{t}(\delta)} g_\delta - \sum_{r\in R} \omega_r g_r g_{\alpha_{n-1}} \cdots g_{\alpha_{\mathrm{t}(\delta)}} g_\delta \\
&= \chi_{\mathrm{t}(\delta)} g_\delta
\end{aligned}
$$

for $\delta \in D$. In addition, we obtain $\chi_1 g_{\gamma'} = 0 = f_{\gamma'} \chi_{\mathrm{s}(\gamma')}$ for $\gamma' \in C'$. If $\gamma \in C$, then we can clearly assume that the corresponding element $w \in pJ$ from condition (2)(b)(i) has the form $w = pw'$ with $w' \in e_1 J e_{\mathrm{s}(\gamma)}$, and it follows

$$
\begin{aligned}
\chi_{\mathrm{t}(\gamma)} g_\gamma &= f_\gamma \Psi_{\mathrm{s}(\gamma)} - \sum_{r\in R} \omega_r g_r g_{\alpha_{n-1}} \cdots g_{\alpha_{\mathrm{t}(\gamma)}} g_\gamma \\
&= 0 - \sum_{r\in R} \omega_r g_r g_{\alpha_{n-1}} \cdots g_{\alpha_1} g_{w'} \\
&= -\Psi_1 g_{w'} \\
&= -f_{w'} \Psi_{\mathrm{s}(\gamma)} \\
&= 0 = f_\gamma \chi_{\mathrm{s}(\gamma)}.
\end{aligned}
$$

Hence χ belongs indeed to $\mathrm{Hom}_\Lambda(V, JU)$. That $\chi\Phi = \mathrm{id}_{JU}$ is a consequence of the following computation:

$$
\begin{aligned}
\chi_2 \Phi_2(1) &= \Psi_2 \Phi_2(1) - \sum_{r\in R} \omega_r g_r g_{\alpha_{n-1}\cdots\alpha_2} \Phi_2(1) \\
&= 1 - \sum_{r\in R} \omega_r \Phi_{u(r)} f_r f_{\alpha_{n-1}\cdots\alpha_2}(1) \\
&= 1.
\end{aligned}
$$

Thus Φ is a split monomorphism in the second case, which shows that the inclusion $JU \hookrightarrow U$ cannot be factored non-trivially. □

The implication (1) ⇒ (2)(a) is proved in [3] using representations of algebras. In the sequel, we will generalize (1) ⇒ (2)(a), by weakening the assumption that the quiver has no oriented cycle, and use the language of modules. The following result (which does not assume that Λ is triangular) gives a first necessary condition for $JU \hookrightarrow U$ to be irreducible.

Proposition 3.3. *Let U be a uniserial Λ-module with mast p. Then $JU \hookrightarrow U$ is not irreducible if there is an arrow leaving $e := s(p)$ besides the first arrow of p.*

Proof. Suppose $p = p'\beta$ with $\beta \in \Gamma_1$ and $U = \Lambda e/K$ where

$$
K = \sum_{(\delta,u) \wr\wr p} \Lambda\left(\delta u - \sum_{i\in I(\delta,u)} k_i(\delta,u) v_i(\delta,u)\right) + \sum_{q \text{ non-route on } p} \Lambda q.
$$

Suppose there is an arrow α leaving e besides β. Then either $(\alpha, e) \wr\wr p$ or α is a non-route on p. Here, we assume $(\alpha, e) \wr\wr p$ and we will prove that $JU \hookrightarrow U$ is not irreducible. The proof for the case where α is a non-route on p is similar. Let $V = \Lambda e/L$ with

$$L = \sum_{(\delta,u)\wr\wr p,\ (\delta,u)\neq(\alpha,e)} \Lambda\Big(\delta u - \sum_{i\in I(\delta,u)} k_i(\delta,u)v_i(\delta,u)\Big) + J\Big(\alpha - \sum_{i\in I(\alpha,e)} k_i(\alpha,e)v_i(\alpha,e)\Big) + \sum_{q \text{ non-route on } p} \Lambda q.$$

We will prove that $JU \hookrightarrow U$ factors non-trivially through V. Let φ and ψ be the unique Λ-homomorphisms

$$JU \xrightarrow{\varphi} V \xrightarrow{\psi} U,$$

defined by $\varphi(\beta + K) = \beta + L$ and $\psi(e + L) = e + K$. First, we show that φ is well defined. Suppose $\lambda\beta \in K$, for some $\lambda \in \Lambda$. We will need to show that $\lambda\beta \in L$. We have

$$\lambda\beta = l\Big(\alpha - \sum_{i\in I(\alpha,e)} k_i(\alpha,e)v_i(\alpha,e)\Big) + w,$$

where $l \in \mathfrak{K}$ and $w \in L$. On the other hand, $\lambda\beta = k\beta + w'\beta$ for some $k \in \mathfrak{K}$ and $w' \in J$. Hence,

$$l\Big(\alpha - \sum_{i\in I(\alpha,e)} k_i(\alpha,e)v_i(\alpha,e)\Big) - k\beta = w - w'\beta. \tag{1}$$

Since α does not appear in any terms in the right-hand side of (1), we have $l = 0$. Therefore $\lambda\beta = w \in L$. It is clear that ψ is well defined and $\psi\varphi$ equals the radical embedding $JU \hookrightarrow U$.

Claim 1. *φ is not a split monomorphism. Otherwise, suppose $\chi : V \to JU$ is a splitting of φ. Then $\chi(e + L) \in JU$, thus $\chi(\beta + L) \in J^2U$, so $\beta + K = \chi\varphi(\beta + K) \in J^2U$, which is a contradiction since $p = p'\beta$ is a mast for U.*

Claim 2. *ψ is not a split epimorphism. Since $L \subset J\Lambda e$, the module V has simple top and is thus indecomposable. Therefore, all we have to show is that ψ is not an isomorphism. This is the case because L is properly contained in K and the dimension of V is larger than the dimension of U.* □

Definition 3.4. (See [10].) A detour (α, u) on a path p is called *inessential* if

$$\alpha u = s + \sum_{i\in I(\alpha,u)} k_i v_i(\alpha,u)$$

in Λ, where s is a $\mathfrak{K}$-linear combination of paths, none of which is a route on p, and $k_i \in \mathfrak{K}$ for all $i \in I(\alpha, u)$. A detour is *essential* if it is not inessential.

The following result establishes (1) ⇒ (2)(a) of Conjecture 3.2, and indeed it is somewhat stronger since the quiver is allowed to have oriented cycles here.

Theorem 3.5. *Let U be a non-simple uniserial module with mast p, where p does not start with an oriented cycle. If $JU \hookrightarrow U$ is irreducible, then*

(i) *All detours on p are inessential.*
(ii) *All non-routes on p are in Jp.*

In particular, $U = \Lambda e/Jp$ with $e = \mathrm{s}(p)$.

Proof. Let $p = \alpha_n \cdots \alpha_1$ and suppose $(\delta_j, u_j) \wr\wr p$ for $0 \leqslant j \leqslant m$. Let $N_R = \sum_{q \text{ non-route on } p} \Lambda q$ and

$$\Delta_j = \delta_j u_j - \sum_{i \in I(\delta_j, u_j)} k_i(\delta_j, u_j) v_i(\delta_j, u_j).$$

Proof of (i). Suppose $U = \Lambda e/K$, where $K = \sum_{j=1}^m \Lambda\Delta_j + N_R$, with m minimal. We have to show that $m = 0$. If $m > 0$, let $U' = \Lambda e/L$ where $L = \sum_{j=2}^m \Lambda\Delta_j + J\Delta_1 + N_R$. Notice that $eJe \subseteq N_R$ by our assumption on p; hence $eU' = (\mathfrak{K}e + L)/L$. We first assume that $\mathfrak{K} \neq \mathbb{Z}_2$. Let

$$V = \frac{U' \sqcup JU'}{H},$$

where $H = \Lambda(p + L, kp + L) + \Lambda(\Delta_1 + L, \Delta_1 + L)$ with $0, 1 \neq k \in \mathfrak{K}$. Recall from Proposition 3.3 that α_1 is the only arrow leaving $e = \mathrm{s}(p)$. Let φ and ψ be the unique Λ-homomorphisms

$$JU \xrightarrow{\varphi} V \xrightarrow{\psi} U,$$

defined by $\varphi(\alpha_1 + K) = (\alpha_1 + L, \alpha_1 + L) + H$ and $\psi((e + L, 0 + L) + H) = se + K$ and $\psi((0 + L, \alpha_1 + L) + H) = l\alpha_1 + K$, with $s, l \in \mathfrak{K}$ such that $s + l = 1$ and $s + lk = 0$. Note that such elements exist, since $\mathfrak{K} \neq \mathbb{Z}_2$.

1. φ is well defined:

$$\varphi(\Delta_1 + K) = (\Delta_1 + L, \Delta_1 + L) + H = H.$$

2. ψ is well defined: We have $\psi((p + L, kp + L) + H) = sp + lkp + K = K$, and $\psi((\Delta_1 + L, \Delta_1 + L) + H) = s\Delta_1 + l\Delta_1 + K = K$.

3. $\psi\varphi = \mathrm{id}_{JU}$:

$$\psi\varphi(\alpha_1 + K) = \psi\big((\alpha_1 + L, \alpha_1 + L) + H\big) = s\alpha_1 + l\alpha_1 + K = \alpha_1 + K.$$

4. φ is not a split monomorphism: Otherwise there would exist $\chi \in \mathrm{Hom}_\Lambda(V, JU)$ such that $\chi\varphi = \mathrm{id}_{JU}$. Then $\chi((e + L, L) + H) = K$, since $eJe \subseteq K$. Hence,

$$\alpha_1 + K = \chi\varphi(\alpha_1 + K) = \chi\big((\alpha_1 + L, \alpha_1 + L) + H\big) = \chi\big((L, \alpha_1 + L) + H\big).$$

Then $\chi((L, \alpha_1 + L) + H) = \alpha_1 + K$. Therefore,

$$\chi(H) = \chi\big((p+L, kp+L)+H\big) = \chi\big((p+L, L)+H\big) + \chi\big((L, kp+L)+H\big)$$
$$= kp + K \neq K,$$

which is a contradiction.

Therefore, ψ splits; i.e., there exists $\chi_1 \in \operatorname{Hom}_\Lambda(U, V)$ such that $\psi\chi_1 = \mathrm{id}_U$. Hence $\chi_1(e+K) = (s^{-1}e + L, L) + H$ because of the assumption that p does not start with an oriented cycle. Then,

$$\chi_1(K) = \chi_1(\Delta_1 + K) = \big(s^{-1}\Delta_1 + L, L\big) + H = H.$$

Then, $(s^{-1}\Delta_1 + L, L) \in H$. Hence,

$$\big(s^{-1}\Delta_1 + L, L\big) = z(p+L, kp+L) + z'(\Delta_1 + L, \Delta_1 + L),$$

with $z, z' \in \Lambda$. Therefore we have

$$s^{-1}\Delta_1 + L = zp + z'\Delta_1 + L,$$
$$L = kzp + z'\Delta_1 + L.$$

Then, $s^{-1}\Delta_1 + L = (1-k)zp + L$. Hence $\Delta_1 - s(1-k)zp \in L$. Thus $s(1-k)zp \in K$, since $\Delta_1 \in K$ and $L \subseteq K$. This implies $zp \in Jp$, since $pU \neq 0$. Hence $\Delta_1 \in L$. This contradicts the minimality of m, finishing the proof of (i) for these base fields. □

Now suppose $\mathfrak{K} = \mathbb{Z}_2$. With the same notation, let

$$V = \frac{U' \sqcup JU' \sqcup JU'}{H},$$

where $H = \Lambda(L, p+L, p+L) + \Lambda(\Delta_1 + L, \Delta_1 + L, \Delta_1 + L)$. Then as in the previous case,

$$JU \xrightarrow{\varphi} V \xrightarrow{\psi} U,$$

is a non-trivial factorization of the radical embedding $JU \hookrightarrow U$ through V, where φ and ψ are the (unique) Λ-homomorphisms defined by $\varphi(\alpha_1 + K) = (\alpha_1 + L, \alpha_1 + L, \alpha_1 + L) + H$ and $\psi((e+L, L, L)+H) = e + K$, $\psi((L, \alpha_1 + L, L)+H) = \alpha_1 + K$ and $\psi((L, L, \alpha_1 + L)+H) = \alpha_1 + K$.

Proof of (ii). Again first assume that $\mathfrak{K} \neq \mathbb{Z}_2$. By part (i), $U = \Lambda e/K$ where $K = \sum_{i=1}^m \Lambda\beta_i u_i + Jp$, and each $\beta_i u_i$ is a non-route on p with u_i a right subpath of p, $\beta_i \in \Gamma_1$. Assume m is minimum. If $m > 0$, then let $U' = \Lambda e/L$ where $L = (\sum_{i=2}^m \Lambda\beta_i u_i + Jp)$ and

$$V = \frac{U' \sqcup JU'}{H},$$

where $H = \Lambda(p+L, kp+L) + \Lambda(\beta_1 u_1 + L, \beta_1 u_1 + L)$ for some $k \in \mathfrak{K}$, $k \neq 0, 1$. Let φ and ψ be the Λ-homomorphisms

$$JU \xrightarrow{\varphi} V \xrightarrow{\psi} U,$$

defined by $\varphi(\alpha_1 + K) = (\alpha_1 + L, \alpha_1 + L)$ and $\psi((e + L, L) + H) = se + K$, and $\psi((L, \alpha_1 + L) + H) = l\alpha_1 + K$ with $s, l \in \mathfrak{K}$ such that $s + l = 1$ and $s + kl = 0$. As in (i) we can see that φ, ψ are well defined, $\psi\varphi$ equals the radical embedding $JU \hookrightarrow U$, and φ is not a split monomorphism. Therefore ψ is split; i.e., there is a $\chi \in \mathrm{Hom}_\Lambda(U, V)$ such that $\psi\chi = \mathrm{id}_U$. Then $\chi(e + K) = (s^{-1}e + L, L) + H$ since no cycles start at p. It follows

$$\chi(K) = \chi(\beta_1 u_1 + K) = \left(s^{-1}\beta_1 u_1 + L, L\right).$$

Therefore, $(s^{-1}\beta_1 u_1 + L, L) = w(p + L, kp + L) + w'(\beta_1 u_1 + L, \beta_1 u_1 + L)$ where $w, w' \in \Lambda$. Hence,

$$\begin{aligned} s^{-1}\beta_1 u_1 + L &= wp + w'\beta_1 u_1 + L, \\ L &= kwp + w'\beta_1 u_1 + L. \end{aligned}$$

Therefore $s^{-1}\beta_1 u_1 + L = (1 - k)wp + L$. Hence

$$s^{-1}\beta_1 u_1 + (k - 1)wp = vp + \sum_{i=2}^{m} w_i \beta_i u_i, \tag{2}$$

where $v \in J$ and $w_i \in \Lambda$. If we multiply Eq. (2) by $\mathrm{t}(\beta_1)$ from the left, we get that $\mathrm{t}(\beta_1)wp$ is zero or a non-route on p, since $\mathrm{t}(\beta_1) \neq \mathrm{t}(p)$. Then Eq. (2) contradicts the minimality of m since it expresses $\beta_1 u_1$ as an element of L. □

Now suppose that $\mathfrak{K} = \mathbb{Z}_2$. With the same notation, let

$$V = \frac{U' \sqcup JU' \sqcup JU'}{H},$$

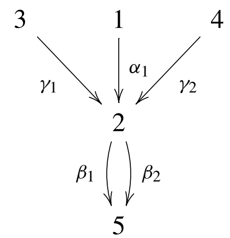
where $H = \Lambda(L, p + L, p + L) + \Lambda(\beta_1 u_1 + L, \beta_1 u_1 + L, \beta_1 u_1 + L)$. Let φ and ψ be the Λ-homomorphisms

$$JU \xrightarrow{\varphi} V \xrightarrow{\psi} U,$$

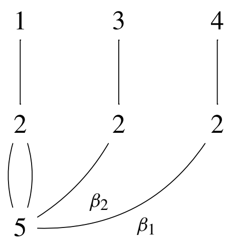
defined by $\varphi(\alpha_1 + K) = (\alpha_1 + L, \alpha_1 + L, \alpha_1 + L)$ and $\psi((e + L, L, L) + H) = e + K$, $\psi((L, \alpha_1 + L, L) + H) = \alpha_1 + K$ and $\psi((L, L, \alpha_1 + L) + H) = \alpha_1 + K$. Similarly, this is a non-trivial factorization of the radical embedding $JU \hookrightarrow U$ through V. □

Example 3.6. In order to provide a better understanding of the different cases that would have to be dealt with in a proof of "(1) ⇒ (2)(b)," we include here a series of examples where condition (2)(b) of Conjecture 3.2 is violated. A non-trivial factorization of the radical embedding is given in each of these cases.

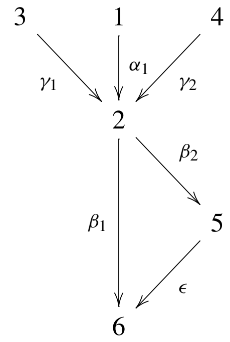
(a) Suppose Γ is given by



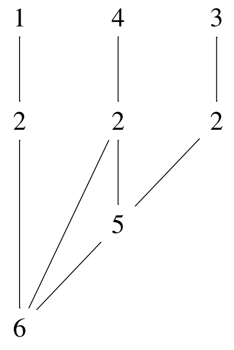
with relations $\beta_1\alpha_1 = \beta_2\alpha_1$ and $\beta_1\gamma_1 = 0 = \beta_2\gamma_2$. Here U is the unique uniserial with mast α_1. The embedding $JU \hookrightarrow U$ can then be factored non-trivially through a module with graph



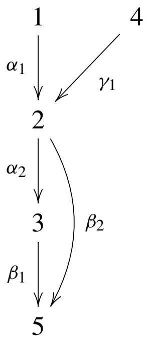
(b) Now Γ is given by



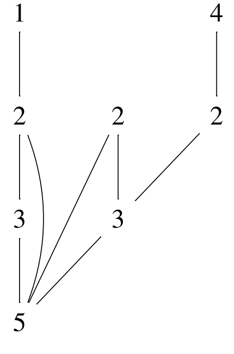
with relations $\epsilon\beta_2\gamma_2 = \beta_1\gamma_2$ and $\beta_1\gamma_1 = 0 = \beta_2\alpha_1$. Again, U is the unique uniserial with mast α_1. In this case, the radical embedding can be factored through the indecomposable with graph



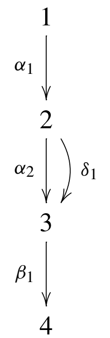
(c) Consider the quiver Γ



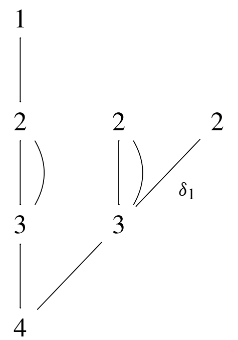
with relations $\beta_2\alpha_1 = \beta_1\alpha_2\alpha_1$ and $\beta_2\gamma_1 = 0$. The radical embedding of the uniserial with mast $\alpha_2\alpha_1$ can be factored through the following indecomposable module:



(d) In our final example, let Γ be given by



and consider the relation $\delta_1\alpha_1 = \alpha_2\alpha_1$. We can factor the radical embedding of the uniserial with mast $\alpha_2\alpha_1$ through the module



Remark 3.7. In order to tackle the remaining implication "(1) ⇒ (2)(b)" of Conjecture 3.2, it is convenient to have the following reformulation of condition (2)(b) at hand:

(2)(b′) *There exists a family* $(w_\gamma) \in (pJ)^C$, *such that for every* $x \in \Gamma_0$ *and* $\mu \in e_x Jp/e_x J^2 p$, *we can find* $r \in e_x J e_n$ *with* $\mu = rp + e_x J^2 p$ *and* $r\alpha_{n-1}\cdots\alpha_{\mathrm{t}(\gamma)}\gamma = rw_\gamma$ *for all* $\gamma \in C$ *and* $r\alpha_{n-1}\cdots\alpha_{\mathrm{t}(\delta)}\delta \in \mathfrak{K}r\alpha_{n-1}\cdots\alpha_{\mathrm{s}(\delta)}$ *for all* $\delta \in D$.

Assume that condition (1) holds, i.e., that the canonical embedding $JU \to U$ is irreducible, and that (2)(b′) is violated. We then get, for every family (w_γ), a special vertex x and an element $\mu \in e_x Jp/e_x J^2 p$ from the negation of this statement. Since (2)(a) holds, this allows us to "lengthen" U to a uniserial module $\hat{U}$ in such a fashion that U is an epimorphic image of $\hat{U}$ and $\operatorname{soc}\hat{U} \simeq \Lambda e_x/Je_x$ (note however that there is a choice involved: $\hat{U}$ is not uniquely determined by U and μ). Here are two potential approaches to the construction of a module M through which the radical embedding of U factors non-trivially:

(a) Let M be the module obtained from gluing the socles of $\hat{U}$ and $\mathrm{D}(e(x)\Lambda)$ (where $\mathrm{D} = \mathrm{Hom}_{\mathfrak{K}}(-, \mathfrak{K})$ denotes the usual duality). The problem then is to find a "good" map from JU to M.
(b) This time, we begin by gluing the socles of $\hat{U}$ and $J\hat{U}$ together to obtain $\check{M}$; this allows for a natural embedding of JU. Of course, this particular embedding splits, and we have to

extend $\check{M}$ to a module M having $\check{M}$ as an epimorphic image in order to prevent this from happening.

4. The case of left multiserial triangular algebras

Throughout this section we assume that the algebra Λ is a triangular algebra. In this section, using approach (b) from above, we will show that Conjecture 3.2 is true whenever the mast p has the following additional property:

$$\dim_{\mathfrak{K}}\big(J\alpha_{n-1}/J^2\alpha_{n-1}\big) \leqslant 1.$$

As a consequence, Conjecture 3.2 is valid for multiserial algebras (see Corollary 4.6).

Lemma 4.1. *Let U be a uniserial module with mast p. If the radical embedding $JU \hookrightarrow U$ is irreducible, and β' is an arrow with $\beta' p \neq 0$, then there is a uniserial module V with mast $q := \beta' p$.*

Specifically, if $\{\beta_i' p + J^2 p \mid 1 \leqslant i \leqslant m\}$ is a basis for $Jp/J^2 p$, with $\beta_i' \in \Gamma_1$ and $\beta_1' = \beta'$, then such a uniserial module V can be constructed as $V = \Lambda e/L$ with

$$L := Jq + \sum_{i=2}^{m} \Lambda\beta_i' p + \sum_{(\delta,u) \wr\wr q,\, \mathrm{t}(\delta)=\mathrm{t}(q)} \Lambda\big(\delta u - l(\delta,u)q\big),$$

where $l(\delta,u) \in \mathfrak{K}$ is a suitable scalar for every $(\delta,u) \wr\wr q$ with $\mathrm{t}(\delta) = \mathrm{t}(q)$.

Proof. Let

$$p = 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \cdots \xrightarrow{\alpha_{n-1}} n,$$

and $n+1 := \mathrm{t}(\beta')$. Suppose $(\delta,u) \wr\wr q$. If $\mathrm{t}(\delta) \in \{1,2,\ldots,n\}$, then $(\delta,u) \wr\wr p$ and so by Theorem 3.5, $\delta u \in \mathfrak{K}\alpha_{\mathrm{t}(\delta)-1}\cdots\alpha_1 + s$, where $s \in Jp$. Since there are no oriented cycles, we get $s = 0$. If $\mathrm{t}(\delta) = n+1$, then by Theorem 3.5(ii), $\delta u \in Jp$. Hence,

$$\delta u = l_1\beta' p + l_2\beta_2' p + \cdots + l_m\beta_m' p + wp, \tag{3}$$

with $w \in J^2$, $l_i \in \mathfrak{K}$. Set $l(\delta,u) := l_1$. If for some $\beta \in \Gamma_1$, βu is a non-route on q, then it is a non-route on p as well and so $\beta u \in Jp$ and $\mathrm{t}(\beta) \notin \{1,\ldots,n+1\}$. Hence, in this case, $\beta u \in \sum_{i=2}^{m} \mathfrak{K}\beta_i' p + J^2 p$. Define $V = \Lambda e/L$, where

$$L := Jq + \sum_{i=2}^{m} \Lambda\beta_i' p + \sum_{(\delta,u) \wr\wr q,\, \mathrm{t}(\delta)=n+1} \Lambda\big(\delta u - l(\delta,u)q\big). \tag{4}$$

Thus, V is a uniserial module. We only need to show that $qV \neq 0$. Suppose $qV = 0$. Then, $q \in L$ and by Eqs. (3) and (4), we get $q \in Jq + \sum_{i=2}^{m} \Lambda\beta_i' p + J^2 p$. Then,

$$q = vq + \sum_{i=2}^{m} \lambda_i\beta_i' p + w' p, \tag{5}$$

with $v \in J$, $\lambda_i \in \Lambda$ and $w' \in J^2$. Multiply Eq. (5) by $\mathrm{t}(\beta')$ from the left. Since the quiver does not have oriented cycles, $vq = 0$, which contradicts the choice of the basis of Jp/J^2p. □

Lemma 4.2. *Suppose* $\dim_{\mathfrak{K}} J\alpha_{n-1}/J^2\alpha_{n-1} = 1$. *Then there exists an arrow* β' *such that* $\mathfrak{K}\beta'\alpha_{n-1} + J\beta'\alpha_{n-1} = J\alpha_{n-1}$.

Proof. By the hypothesis there is some $\beta' \in \Gamma_1$ with $\beta'\alpha_{n-1} \notin J^2\alpha_{n-1}$. We will show that $J^2\alpha_{n-1} = J\beta'\alpha_{n-1}$. For this we only need to show that any path in $J^2\alpha_{n-1}$ is in $J\beta'\alpha_{n-1}$. If not, let q be a longest path in $J^2\alpha_{n-1} \backslash J\beta'\alpha_{n-1}$. Then $q = \gamma_r \cdots \gamma_1\alpha_{n-1}$, where $\gamma_i \in \Gamma_1$ and $\gamma_1\alpha_{n-1} \notin J^2\alpha_{n-1}$, otherwise q could be replaced by a longer path. Hence $\gamma_1\alpha_{n-1} = k\beta'\alpha_{n-1} + w\alpha_{n-1}$, where $0 \neq k \in \mathfrak{K}$ and $w \in J^2$. Therefore,

$$q = \gamma_r \cdots \gamma_1\alpha_{n-1} = k\gamma_r \cdots \gamma_2\beta'\alpha_{n-1} + \gamma_r \cdots \gamma_2 w\alpha_{n-1}. \tag{6}$$

Since $\gamma_r \cdots \gamma_2 w\alpha_{n-1}$ is a linear combination of paths in $J^2\alpha_{n-1}$ longer than q, we get $\gamma_r \cdots \gamma_2 w\alpha_{n-1} \in J\beta'\alpha_{n-1}$. Then, by Eq. (6), $q \in J\beta'\alpha_{n-1}$. This is a contradiction. □

Theorem 4.3. *Let* Λ *be a triangular algebra and* U *be a uniserial* Λ*-module with mast* $p = \alpha_{n-1} \cdots \alpha_1$. *If* $\dim_{\mathfrak{K}} J\alpha_{n-1}/J^2\alpha_{n-1} \leqslant 1$, *then the following statements are equivalent*:

(1) *The embedding* $JU \hookrightarrow U$ *is irreducible.*
(2) U *is not simple and satisfies both* (a) *and* (b) *below*:
   (a) *For every* $\beta \in B$,

$$\beta\alpha_{\mathrm{s}(\beta)-1} \cdots \alpha_1 \in Jp,$$

   and for every $\delta \in D$,

$$\delta\alpha_{\mathrm{s}(\delta)-1} \cdots \alpha_1 \in \mathfrak{K}\alpha_{\mathrm{t}(\delta)-1} \cdots \alpha_1.$$

   (b) $Jp = 0$ *or there is an arrow* β' *such that* $\{\beta' p + J^2 p\}$ *forms a* $\mathfrak{K}$*-basis for* Jp/J^2p *and* (i) *and* (ii) *both hold*:
      (i) *For every* $\gamma \in C$ *there exists* $w \in pJ$ *such that*

$$\beta'\alpha_{n-1} \cdots \alpha_{\mathrm{t}(\gamma)}\gamma = \beta' w.$$

      (ii) *For every* $\delta \in D$

$$\beta'\alpha_{n-1} \cdots \alpha_{\mathrm{t}(\delta)}\delta \in \mathfrak{K}\beta'\alpha_{n-1} \cdots \alpha_{\mathrm{s}(\delta)}.$$

Proof. Note first that, under the present hypotheses, the conditions (2) are equivalent to those in Conjecture 3.2. The conditions (2)(a) are identical. We have that $\dim_{\mathfrak{K}} J\alpha_{n-1}/J^2\alpha_{n-1} \leqslant 1$ so that, by Lemma 4.2, we can take the set R of Conjecture 3.2(2)(b) to be $\{\beta' p + J^2 p\}$ or $\emptyset$. Then Conjecture 3.2(2)(b)(i) and (ii) reduce to the corresponding parts of this theorem.

(1) ⇒ (2)(b)(i):

Suppose $Jp \neq 0$. Then $\dim_{\mathfrak{K}} J\alpha_{n-1}/J^2\alpha_{n-1} = 1$. By Lemma 4.2, there exists an arrow $\beta' \in \Gamma_1$ such that $\mathfrak{K}\beta'\alpha_{n-1} + J\beta'\alpha_{n-1} = J\alpha_{n-1}$. Then $\{\beta'\alpha_{n-1} + J^2\alpha_{n-1}\}$ is a basis for $J\alpha_{n-1}/J^2\alpha_{n-1}$ and $\{\beta' p + J^2 p\}$ is a basis for Jp/J^2p. We will show that for $\gamma \in C$,

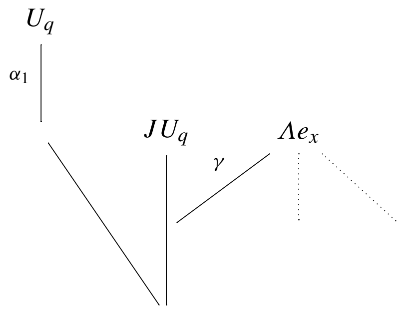
$\beta'\alpha_{n-1}\cdots\alpha_{t(\gamma)}\gamma \in \beta' pJ$. By Theorem 3.5, we know that $U = \Lambda e_1/Jp$ where $e_1 = s(p)$. Let $q = \beta' p$ and $K = Jp$. By Lemma 4.1, there exists a uniserial module $U_q = \Lambda e_1/L$ with mast q, where

$$L = Jq + \sum_{(\delta,u)\wr\wr q,\, t(\delta)=t(q)} \Lambda\big(\delta u - l(\delta,u)q\big).$$

Let

$$V = \frac{U_q \sqcup JU_q \sqcup \Lambda e_x}{H},$$

where $H = \Lambda(q + L, q + L, 0) + \Lambda(L, \alpha_{n-1}\cdots\alpha_1 + L, \alpha_{n-1}\cdots\alpha_{t(\gamma)}\gamma)$ with $e_x = s(\gamma)$.



Notice that for $v \in V$, $e_1 v \in \mathfrak{K}(e_1 + L, 0, z) + H$, where z is a linear combination of paths from e_x to e_1. Let φ and ψ be the Λ-homomorphisms

$$JU \xrightarrow{\varphi} V \xrightarrow{\psi} U,$$

defined by $\varphi(\alpha_1 + K) = (\alpha_1 + L, \alpha_1 + L, 0) + H$ and $\psi((e_1 + L, L, 0) + H) = e_1 + K$, $\psi((L, \alpha_1 + L, 0) + H) = K$ and $\psi((L, L, e_x) + H) = K$. Then, using Lemma 4.2 we can prove that φ and ψ are well defined. Clearly, $\psi\varphi$ is the radical embedding $JU \hookrightarrow U$.

Claim. *φ is not a split monomorphism; otherwise there would exist $\chi : V \to JU$ such that $\chi\varphi = \mathrm{id}_{JU}$. We have $\alpha_1 + K = \chi\varphi(\alpha_1 + K) = \chi((\alpha_1 + L, \alpha_1 + L, 0) + H) = \chi((\alpha_1 + L, L, 0) + H) + \chi((L, \alpha_1 + L, 0) + H) = \chi((L, \alpha_1 + L, 0) + H)$, because $\chi((e_1 + L, L, 0) + H) = K$. Also we have $\chi((L, L, e_x) + H) = K$. But*

$$\begin{aligned}\chi(H) &= \chi\big((L, \alpha_{n-1}\cdots\alpha_1 + L, \alpha_{n-1}\cdots\alpha_{t(\gamma)}\gamma) + H\big)\\ &= \alpha_{n-1}\cdots\alpha_2\alpha_1 + K \neq K,\end{aligned}$$

which is a contradiction. Therefore ψ splits, i.e., there exists $\chi_1 : U \to V$ with $\psi\chi_1 = \mathrm{id}_U$. We have $\chi_1(e_1 + K) = ((e_1 + L, L, \sum_{i=1}^m k_i w_i) + H)$, where w_i are the paths from e_x to e_1 and $k_i \in \mathfrak{K}$. But $q \in K$ and so

$$\chi_1(K) = \chi_1(q + K) = \left(q + L, L, \sum_{i=1}^m k_i q w_i\right) + H.$$

Hence,

$$\left(q+L, L, \sum_{i=1}^{m} k_i q w_i\right) \in \Lambda(q+L, q+L, 0) + \Lambda(L, \alpha_{n-1}\cdots\alpha_1 + L, \alpha_{n-1}\cdots\alpha_{\mathrm{t}(\gamma)}\gamma).$$

Then, by Lemma 4.2*,*

$$\left(q+L, L, \sum_{i=1}^{m} k_i q w_i\right) = k(q+L, q+L, 0) + l\beta'\big(L, \alpha_{n-1}\cdots\alpha_1 + L, \alpha_{n-1}\cdots\alpha_{\mathrm{t}(\gamma)}\gamma\big)$$
$$+ \sum_{l(u_i)\geqslant 1} l_i u_i \beta'(L, \alpha_{n-1}\cdots\alpha_1 + L, \alpha_{n-1}\cdots\alpha_{\mathrm{t}(\gamma)}\gamma),$$

where $k, l, l_i \in \mathfrak{K}$*. It follows* $k = 1$ *and* $l = -1$*. Hence,*

$$\beta'\alpha_{n-1}\cdots\alpha_{\mathrm{t}(\gamma)}\gamma = -\sum k_i q w_i + \sum_{l(u_i)\geqslant 1} l_i u_i \beta'\alpha_{n-1}\cdots\alpha_{\mathrm{t}(\gamma)}\gamma.$$

If we multiply the above equation from the left by $\mathrm{t}(\beta')$*, using the fact that the quiver does not have any oriented cycles and therefore* $\mathrm{t}(\beta')u_i = 0$*, we obtain*

$$\beta'\alpha_{n-1}\cdots\alpha_{\mathrm{t}(\gamma)}\gamma = -\sum k_i q w_i \in \beta' p J.$$

(1) $\Rightarrow$ (2)(b)(ii):

Suppose $\delta \in D$. We will show that $\beta'\alpha_{n-1}\cdots\alpha_{\mathrm{t}(\delta)}\delta \in \mathfrak{K}\beta'\alpha_{n-1}\cdots\alpha_{\mathrm{s}(\delta)}$. Let $\delta : i \to j$ and $q := \beta'\alpha_{n-1}\cdots\alpha_1$. Again let $U_q = \Lambda e/L$ be the uniserial with mast q, with L as above. Let

$$V = \frac{U_q \sqcup JU_q \sqcup \Lambda e_i}{H},$$

where

$$H = \Lambda(q+L, q+L, 0) + \Lambda(L, \alpha_{n-1}\cdots\alpha_1 + L, \alpha_{n-1}\cdots\alpha_j\delta) + \Lambda(L, L, \alpha_{n-1}\cdots\alpha_i).$$

Let φ and ψ be the Λ-homomorphisms

$$JU \xrightarrow{\varphi} V \xrightarrow{\psi} U,$$

defined by $\varphi(\alpha_1 + K) = (\alpha_1 + L, \alpha_1 + L, 0) + H$ and $\psi((e_1 + L, L, 0) + H) = e_1 + K$, $\psi((L, \alpha_1 + L, 0) + H) = K$ and $\psi((L, L, e_i) + H) = K$. Then, φ and ψ are well defined and $\psi\varphi$ is the radical embedding $JU \hookrightarrow U$.

Claim. *φ is not split monomorphism; otherwise there would exist $\chi : V \to JU$ such that $\chi\varphi = \mathrm{id}_{JU}$. Then we would have $\alpha_1 + K = \chi\varphi(\alpha_1 + K) = \chi((\alpha_1 + L, \alpha_1 + L, 0) + H) = \chi((\alpha_1 + L, L, 0) + H) + \chi((L, \alpha_1 + L, 0) + H) = \chi((L, \alpha_1 + L, 0) + H)$, because $\chi((e_1 + L, L, 0) + H) = K$. By Theorem* 3.5(ii)*, $\chi((L, L, e_i) + H) = k\alpha_{i-1}\cdots\alpha_1 + K$, where $k \in \mathfrak{K}$.*

Thus, $\chi(H) = \chi((L, L, \alpha_{n-1} \cdots \alpha_i) + H) = k\alpha_{n-1} \cdots \alpha_1 + K$. *Therefore,* $k = 0$. *But*

$$\chi(H) = \chi\big((L, \alpha_{n-1} \cdots \alpha_1 + L, \alpha_{n-1} \cdots \alpha_j \delta) + H\big) = \alpha_{n-1} \cdots \alpha_2 \alpha_1 + K \neq K,$$

which is a contradiction.

Therefore, ψ splits, i.e., there exists $\chi_1 : U \to V$ with $\psi\chi_1 = \mathrm{id}_U$. We have $\chi_1(e_1 + K) = (e_1 + L, L, 0) + H$. Hence $\chi_1(K) = \chi_1(q + K) = (q_1 + L, L, 0) + H$. Therefore,

$$\begin{aligned}(q + L, L, 0) \in \Lambda(q + L, q + L, 0) &+ \Lambda(L, \alpha_{n-1} \cdots \alpha_1 + L, \alpha_{n-1} \cdots \alpha_j \delta) \\ &+ \Lambda(L, L, \alpha_{n-1} \cdots \alpha_i).\end{aligned}$$

Then by Lemma 4.2

$$\begin{aligned}(q + L, L, 0) = k(q + L, q + L, 0) &+ l\beta'(L, \alpha_{n-1} \cdots \alpha_1 + L, \alpha_{n-1} \cdots \alpha_j \delta) \\ &+ \sum_{l(u_s) \geqslant 1} l_s u_s \beta'(L, \alpha_{n-1} \cdots \alpha_1 + L, \alpha_{n-1} \cdots \alpha_j \delta) \\ &+ v(L, L, \alpha_{n-1} \cdots \alpha_i),\end{aligned}$$

where $l, l_s \in \mathfrak{K}$, $u_s \in J$ and $v \in \Lambda$. Hence $k = 1$ and $l = -1$. Therefore,

$$\beta'\alpha_{n-1} \cdots \alpha_j \delta = \sum_{l(u_s) \geqslant 1} l_s u_s \beta' \alpha_{n-1} \cdots \alpha_j \delta + v\alpha_{n-1} \cdots \alpha_i, \tag{7}$$

in Λe_i. If we multiply Eq. (7) from the left by $\mathrm{t}(\beta')$, using the fact that there are no oriented cycles, $\mathrm{t}(\beta')u_s = 0$, we get

$$\beta'\alpha_{n-1} \cdots \alpha_j \delta = \mathrm{t}(\beta') v \alpha_{n-1} \cdots \alpha_i.$$

Since there are no oriented cycles, $\mathrm{t}(\beta') \neq \mathrm{t}(\alpha_{n-1})$ and so $\mathrm{t}(\beta') v \alpha_{n-1} \in J\alpha_{n-1}$. But $J\alpha_{n-1} = \mathfrak{K}\beta'\alpha_{n-1} + J\beta'\alpha_{n-1}$ by Lemma 4.2. Therefore,

$$\beta'\alpha_{n-1} \cdots \alpha_j \delta = k\beta'\alpha_{n-1} \cdots \alpha_i + w\beta'\alpha_{n-1} \cdots \alpha_i,$$

where $w \in J$. But, $\mathrm{t}(\beta')w = 0$, since there are no oriented cycles. Therefore, $\beta'\alpha_{n-1} \cdots \alpha_j \delta = k\beta'\alpha_{n-1} \cdots \alpha_i$. □

By the work above, Conjecture 3.2 is true for all triangular algebras with a presentation so that for each $\alpha \in \Gamma_1$, $\Lambda\alpha$ is uniserial.

Definition 4.4. An algebra Λ with Jacobson radical J is called *left multiserial* (*m-multiserial*) if, for each primitive idempotent e of Λ, the left ideal Je is a sum of uniserial (m uniserial) Λ-modules.

For the convenience of the reader, we provide here the following theorem from [10, Remark 2.3].

Theorem 4.5. *(See [10, Remark 2.3].) Every left multiserial algebra is isomorphic to one with a presentation so that for each $\alpha \in \Gamma_1$, $\Lambda\alpha$ is uniserial.*

Corollary 4.6. *Conjecture* 3.2 *is true for all left triangular multiserial algebras.*

5. The case of monomial algebras

Throughout this section we assume that the algebra Λ is a triangular algebra. We will prove that the conjecture is true for monomial algebras. Recall that for any path p, non-zero in Λ, there is an affine variety V_p and a map Φ_p from V_p onto the set of isomorphism types of uniserial Λ-modules with mast p (see page 3).

Theorem 5.1. *Suppose Λ is a triangular monomial algebra and U is a uniserial Λ-module with mast p. Then the following statements are equivalent*:

(1) *The embedding $JU \hookrightarrow U$ is irreducible.*
(2) *U is not simple and satisfies both* (a) *and* (b) *below*:
  (a)
    (i) *For every $\beta \in B$, $\beta\alpha_{s(\beta)-1}\cdots\alpha_1 = 0$, and*
    (ii) *For every $\delta \in D$, $\delta\alpha_{s(\delta)-1}\cdots\alpha_1 = 0$.*
  (b) *For every $\beta' \in B'$ such that $\beta' p \neq 0$ we have*:
    (i) *For every $\gamma \in C$, $\beta'\alpha_{n-1}\cdots\alpha_{t(\gamma)}\gamma = 0$, and*
    (ii) *For every $\delta \in D$, $\beta'\alpha_{n-1}\cdots\alpha_{t(\delta)}\delta = 0$.*

Proof. Note first that, since the algebra is monomial, the conditions (2) are equivalent to the ones in Conjecture 3.2.

(1) $\Rightarrow$ (2)(b)(i):

Let $p = \alpha_{n-1}\cdots\alpha_1$ and $U = \Lambda e_1/K$. Suppose that there is $\beta' \in B'$ such that $\beta' p \neq 0$ and $\beta'\alpha_{n-1}\cdots\alpha_i\gamma \neq 0$ for some $\gamma \in C$, where $x \xrightarrow{\gamma} i$, with $x \notin \{1, 2, \ldots, n\}$. By condition (2)(a), $V_p = \{\underline{0}\}$. Let

$$q_1 := \beta'\alpha_{n-1}\cdots\alpha_1, \qquad q_2 := \beta'\alpha_{n-1}\cdots\alpha_i\gamma.$$

Since Λ is a monomial algebra and $q_i \neq 0$; by [11, Proposition II.3], $\underline{0} \in V_{q_i}$ for $i = 1$ and 2. Let $U_{q_1} := \Phi_{q_1}(\underline{0}) = \Lambda e_1/L$ and $U_{q_2} := \Phi_{q_2}(\underline{0}) = \Lambda e_x/F$, where $e_x = s(\gamma)$. Let

$$V = \frac{U_{q_1} \sqcup JU_{q_1} \sqcup U_{q_2}}{H},$$

where

$$H = \Lambda(q_1 + L, q_1 + L, F) + \Lambda(L, \alpha_{n-1}\cdots\alpha_1 + L, \alpha_{n-1}\cdots\alpha_i\gamma + F).$$

Once again, for $v \in V$, $e_1 v = (ke_1 + L, L, z + F) + H$, where z is a linear combination of paths from $s(\gamma)$ to e_1. However, such a path goes through e_1 and so is a non-route on q_2, i.e., $z \in F$. Let φ and ψ be the Λ-homomorphisms

$$JU \xrightarrow{\varphi} V \xrightarrow{\psi} U,$$

defined by $\varphi(\alpha_1 + K) = (\alpha_1 + L, \alpha_1 + L, F) + H$ and $\psi((e_1 + L, L, F) + H) = e_1 + K$, $\psi((L, \alpha_1 + L, F) + H) = K$ and $\psi((L, L, e_x + F) + H) = K$. We will first show that φ is well defined. Note that $K = L + \Lambda q_1$. Suppose $\lambda\alpha_1 \in K$, for some $\lambda \in \Lambda$. Then $\lambda\alpha_1 = w + \gamma q_1$, for some $w \in L$ and $\gamma \in \Lambda$. Thus, $(\lambda\alpha_1 + L, \lambda\alpha_1 + L, F) = (\gamma q_1 + L, \gamma q_1 + L, F) \in H$.

Again, ψ is well defined, $\psi\varphi$ is $JU \hookrightarrow U$ and φ is not split monomorphism. We will prove that ψ also is not a split epimorphism, which contradicts the irreducibility of $JU \hookrightarrow U$. Suppose ψ is a split epimorphism. Then, there exists $\chi : U \to V$ with $\psi\chi = \mathrm{id}_U$. We have $\chi(e_1 + K) = (e_1 + L, L, F) + H$. But $q_1 = \beta'\alpha_{n-1} \cdots \alpha_1 \in K$. Hence $\chi(K) = \chi(q_1 + K) = (q_1 + L, L, F) + H$ is zero in V. Then,

$$(q_1 + L, L, F) = k(q_1 + L, q_1 + L, F) + w(L, \alpha_{n-1} \cdots \alpha_1 + L, \alpha_{n-1} \cdots \alpha_{\mathrm{t}(\gamma)}\gamma + F), \tag{8}$$

where $k \in \mathfrak{K}$ and $w \in \Lambda$. Note that if $\beta'_1 \neq \beta'$, then either $\beta'_1\alpha_{n-1} \cdots \alpha_1$ is non-route on q_1 or $(\beta'_1, \alpha_{n-1} \cdots \alpha_1) \wr\wr q_1$. Hence, $\beta'_1\alpha_{n-1} \cdots \alpha_1 \in L$. Similarly, $\beta'_1\alpha_{n-1} \cdots \alpha_{\mathrm{t}(\gamma)}\gamma \in F$. Therefore Eq. (8) becomes $(q_1 + L, L, F) = k(q_1 + L, q_1 + L, F) + l\beta'(L, \alpha_{n-1} \cdots \alpha_1 + L, \alpha_{n-1} \cdots \alpha_{\mathrm{t}(\gamma)}\gamma + F)$, where $k, l \in \mathfrak{K}$. Therefore $k = 1$, $k + l = 0$, $l = 0$, which is a contradiction.

(1) $\Rightarrow$ (2)(b)(ii):

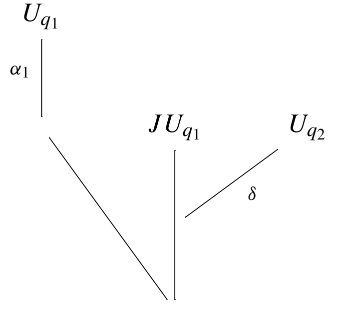
Suppose there is $\beta' \in B'$ such that $\beta' p \neq 0$ and $\beta'\alpha_{n-1} \cdots \alpha_{\mathrm{t}(\delta)}\delta \neq 0$ for some $\delta \in D$. Let $\delta : i \to j$. By (a)(ii), $\mathrm{s}(\delta) = i \neq 1$. Let

$$q_1 := \beta'\alpha_{n-1} \cdots \alpha_1, \qquad q_2 := \beta'\alpha_{n-1} \cdots \alpha_j\delta$$

and let $U_{q_1} = \Lambda e_1/L$ and $U_{q_2} = \Lambda e_2/F$ be the uniserial modules corresponding to $0 \in V_{q_1}$ and $0 \in V_{q_2}$ respectively. Let

$$V = \frac{U_{q_1} \sqcup JU_{q_1} \sqcup U_{q_2}}{H},$$

where $H = \Lambda(q_1 + L, q_1 + L, F) + \Lambda(L, \alpha_{n-1} \cdots \alpha_1 + L, \alpha_{n-1} \cdots \alpha_j\delta + F)$.



Let φ and ψ be the Λ-homomorphisms

$$JU \xrightarrow{\varphi} V \xrightarrow{\psi} U,$$

defined by $\varphi(\alpha_1 + K) = (\alpha_1 + L, \alpha_1 + L, F) + H$ and $\psi((e_1 + L, L, F) + H) = e_1 + K$, $\psi((L, \alpha_1 + L, F) + H) = K$ and $\psi((L, L, e_i + F) + H) = K$. Again, φ and ψ are well defined and $\psi\varphi$ is the radical embedding $JU \hookrightarrow U$.

Claim. *φ is not a split monomorphism*:

Suppose there exists $\chi : V \to JU$ such that $\chi\varphi = \mathrm{id}_{JU}$. Then, we have $\alpha_1 + K = \chi\psi(\alpha_1 + K) = \chi((\alpha_1 + L, \alpha_1 + L, F) + H) = \chi((\alpha_1 + L, L, F) + H) + \chi((L, \alpha_1 + L, F) + H) = \chi((L, \alpha_1 + L, F) + H)$. Also we know that $\chi((L, L, e_i + F) + H) = k\alpha_{i-1} \cdots \alpha_1 + K$, where $k \in \mathfrak{K}$. Then $\chi((L, L, \delta e_i + F) + H) = k\delta\alpha_{i-1} \cdots \alpha_1 + K = K$, by (a)(ii), and

$$\chi(H) = \chi(L, \alpha_{n-1} \cdots \alpha_1 + L, \alpha_{n-1} \cdots \alpha_j\delta + F) = \alpha_{n-1} \cdots \alpha_2\alpha_1 + K \neq K,$$

which is a contradiction.

Claim. *ψ is not a split epimorphism.*

Suppose there exists $\chi_1 : U \to V$ with $\psi\chi_1 = \mathrm{id}_U$. We have $\chi_1(e_1 + K) = (e_1 + L, L, F) + H$. Hence $\chi_1(K) = \chi_1(q_1 + K) = (q_1 + L, L, F) + H$. Therefore $(q_1 + L, L, F) + H = H$, and so

$$(q_1 + L, L, F) \in \Lambda(q_1 + L, q_1 + L, F) + \Lambda\beta'(L, \alpha_{n-1} \cdots \alpha_1 + L, \alpha_{n-1} \cdots \alpha_j\delta + F).$$

Then $(q_1 + L, L, F) = k(q_1 + L, q_1 + L, F) + l\beta'(L, \alpha_{n-1} \cdots \alpha_1 + L, \alpha_{n-1} \cdots \alpha_j\delta + F)$, with $k, l \in \mathfrak{K}$. Therefore $k = 1$, $k + l = 0$, $l = 0$, which is a contradiction. □

6. Almost split sequences with uniserial end terms

In this section, we first show that if we have an arbitrary exact sequence with uniserial end terms, then the middle term is either indecomposable or a direct sum of two uniserials. Then we study $\alpha(U)$, the number of indecomposable summands of the middle term of an almost split sequence ending in U, where U is a uniserial non-projective Λ-module, and give a global upper bound for it in the case that Λ is a multiserial algebra. See [7] for related work.

Proposition 6.1. *Let R be a left artinian ring and consider a short exact sequence*

$$0 \to U_1 \xrightarrow{f} M \xrightarrow{g} U_2 \to 0$$

*in R-*mod *with uniserial modules U_1 and U_2. Then M is either indecomposable or a direct sum of two uniserial modules.*

Proof. We will again denote the Jacobson radical of R by J. Assume we have a decomposition $M = M_1 \oplus M_2$ with both M_1 and M_2 non-zero. Decompose f and g accordingly, i.e., write $f = \binom{f_1}{f_2}$ and $g = (g_1, g_2)$, and let

$$\bar{\ }: R\text{-mod} \to (R/J)\text{-mod}$$

be the functor $R/J \otimes_R -$. We then get the right exact sequence

$$\bar{U}_1 \xrightarrow{\binom{\bar{f}_1}{\bar{f}_2}} \bar{M}_1 \oplus \bar{M}_2 \xrightarrow{(\bar{g}_1,\bar{g}_2)} \bar{U}_2 \to 0$$

where $\bar{U}_1$ and $\bar{U}_2$ are simple and $\bar{M}_1$, $\bar{M}_2$ non-zero semisimple. Comparing the lengths of the involved modules, we see that both $\bar{M}_1$ and $\bar{M}_2$ must be simple and $\bar{f} \neq 0$. Without loss of generality, we may assume $\bar{f}_1(\bar{U}_1) = \bar{M}_1$.

Pick $u_1 \in U_1 \setminus JU_1$. Then $f_1(u_1) \in M_1 \setminus JM_1$ generates M_1. Hence f_1 is surjective and M_1 is uniserial. If $f_2(u_1) = 0$, then $f_2 = 0$ and g_2 is injective, and consequently M_2 is uniserial. If $f_2(u_1) \neq 0$, we can find $l \geqslant 0$ with $f_2(u_1) \in J^l M_2 \setminus J^{l+1} M_2$. If $l = 0$, then $f_2(u_1)$ generates M_2 and M_2 is therefore uniserial. We will assume $l > 0$ from now on.

Claim 1. $\mathrm{im}(g_1) \subset J^l U_2$.

Let $m_1 \in M_1$; write $m_1 = \alpha f_1(u_1) = f_1(\alpha u_1)$ with $\alpha \in \Lambda$. Then $g_1(m_1) = g(m_1) = gf_1(\alpha u_1) - gf(\alpha u_1) = -gf_2(\alpha u_1) \subset g(J^l M_2) \subset J^l U_2$. Hence we have $g_1(M_1) \subset J^l U_2$.

Claim 2. *g_2 is surjective and the map $M_2/J^l M_2 \to U_2/J^l U_2$ induced by g_2 is an isomorphism.*

Let $m_2 \in M_2 \setminus JM_2$. Then $u_2 := g_2(m_2) \in U_2 \setminus JU_2$ (since $g_2(m_2) \in JU_2$ would imply $\mathrm{im}(g) = \mathrm{im}(g_1) + \mathrm{im}(g_2) \subset J^l U_2 + JU_2 \subsetneq U_2$, a contradiction). Since u_2 generates U_2, g_2 is surjective. Now let $x \in M_2 \setminus J^l M_2$ and assume $g_2(x) \in J^l U_2$, say $g_2(x) = \alpha u_2 = g_2(\alpha m_2)$ with $\alpha \in J^l$. Then $x - \alpha m_2 \in \ker(g_2) \setminus J^l M_2 \subset \mathrm{im}(f_2) \setminus J^l M_2 = \emptyset$, again a contradiction.

Claim 3. *$J^l M_2$ is uniserial.*

By restricting our maps f and g, we obtain the following short exact sequence:

$$0 \to U_1 \to M_1 \oplus J^l M_2 \to J^l U_2 \to 0$$

and we see as above that $J^l M_2 / J^{l+1} M_2$ is simple, hence $J^l M_2$ is generated by $f_2(u_1)$ and $f_2 : U_1 \to J^l M_2$ is therefore surjective.

Claim 4. *M_2 is uniserial.*

We know that $J^k M_2 / J^{k+1} M_2$ is simple or 0 for all $k \in \mathbb{N}$. □

In the sequel, Λ will be a finite-dimensional algebra over $\mathfrak{K}$.

The following proposition gives a general upper bound for the number $\alpha(U)$ for a uniserial module U:

Proposition 6.2. *If $U \in \Lambda$-*mod *is a non-projective uniserial module, then*

$$\alpha(U) \leqslant \mathrm{length}(\mathrm{soc}\, DTrU) + 1.$$

Proof. Let $0 \to DTrU \to B \to U \to 0$ be an almost split sequence. Then $0 \to \operatorname{soc} DTrU \to \operatorname{soc} B \to \operatorname{soc} U$ is left exact. Therefore,

$$\begin{aligned}\alpha(U) &\leqslant \operatorname{length}(\operatorname{soc} B)\\ &\leqslant \operatorname{length}(\operatorname{soc} DTrU) + \operatorname{length}(\operatorname{soc} U)\\ &= \operatorname{length}(\operatorname{soc} DTrU) + 1. \qquad \square\end{aligned}$$

The following proposition gives more precise information.

Proposition 6.3. *Let $0 \to DTrU \xrightarrow{f} \bigsqcup_{i\in I} B_i \xrightarrow{g} U \to 0$ be an almost split sequence where U is a uniserial module and the B_i are indecomposable.*

(i) *At most one of the induced maps $g_i : B_i \to U$ is a monomorphism.*
(ii) *If $B_i \xrightarrow{g_i} U$ is an epimorphism and* $\operatorname{soc} B_i$ *is simple then* $\operatorname{soc} B_i \subseteq f(\operatorname{soc} DTrU)$.
(iii) *Let $I' = \{i \in I \mid g_i : B_i \to U$ is an epimorphism$\}$. Then* $|I'| \leqslant \operatorname{length}(\operatorname{soc} DTrU)$.

Proof. (i) Suppose g_1 and g_2 are monomorphisms. Using Proposition 3.1 again, we have $B_1 \cong JU$ and $B_2 \cong JU$. The induced irreducible morphism $B_1 \sqcup B_2 \to U$ cannot be an epimorphism and therefore is a monomorphism and $B_1 \sqcup B_2 \cong JU$, which is impossible.

(ii) We have $\operatorname{soc} B_i \cap \ker(g_i) \neq 0$ since $\ker(g_i) \neq 0$ and $\operatorname{soc} B_i$ is essential in B_i. But $\operatorname{soc} B_i$ is simple, so $\operatorname{soc} B_i \subseteq \ker(g_i)$. We know that $0 \to \operatorname{soc} DTrU \xrightarrow{\bar f} \bigsqcup_{i\in I} \operatorname{soc} B_i \xrightarrow{\bar g} \operatorname{soc} U$ is exact. Hence $\operatorname{soc} B_i \subseteq \ker \bar g = \operatorname{im} \bar f$. Therefore, $\operatorname{soc} B_i \subseteq f(\operatorname{soc} DTrU)$.

(iii) We distinguish two cases:

Case 1. There is an i such that g_i is a monomorphism. Then

$$|I'| \leqslant \alpha(U) - 1 \leqslant \operatorname{length}(\operatorname{soc} DTrU)$$

by Proposition 6.2.

Case 2. For each $i \in I$, the map g_i is an epimorphism. We consider the exact sequence $0 \to \operatorname{soc} DTrU \xrightarrow{\bar f} \bigsqcup_{i\in I} \operatorname{soc} B_i \xrightarrow{\bar g} \operatorname{soc} U$ and we use (ii): if $\operatorname{soc} B_i$ is simple for all i, then $\bar f$ is an isomorphism and we get

$$|I'| = \alpha(U) = \operatorname{length}\left(\operatorname{soc} \bigsqcup_{i\in I} B_i\right) = \operatorname{length}(\operatorname{soc} DTrU).$$

If however at least one $\operatorname{soc} B_i$ is not simple, then the same exact sequence gives

$$|I'| = \alpha(U) \leqslant \operatorname{length}\left(\operatorname{soc} \bigsqcup_{i\in I} B_i\right) - 1 \leqslant \operatorname{length}(\operatorname{soc} DTrU). \qquad \square$$

Let e, f be primitive idempotents in Λ. For a non-zero element $a \in fJe$, the Λ-module $\Lambda e/\Lambda a$ is indecomposable and non-projective. We are interested in the case where this module is a uniserial module and consider the almost split sequence ending in $\Lambda e/\Lambda a$.

Proposition 6.4. *If $U = \Lambda e/\Lambda a$ is a uniserial module, then $\alpha(U) \leqslant 2$.*

Proof. $\Lambda f \xrightarrow{.a} \Lambda e \to \Lambda e/\Lambda a \to 0$ (where $.a$ denotes the right multiplication by a) is exact and is the start of a minimal projective presentation of $\Lambda e/\Lambda a$. From [2, Proposition V.6.1] we have that the middle term B in the almost split sequence $\delta : 0 \to DTrU \to B \to U \to 0$ has a decomposition $B = B' \sqcup B''$ with B' indecomposable and such that if $B'' \neq 0$, the induced morphism $g'' : B'' \to U$ is an irreducible monomorphism. But, by Proposition 3.1, $B'' \cong JU$ is indecomposable and therefore $\alpha(U) \leqslant 2$. □

Uniserial representations of left multiserial algebras are studied in [10]. Here we find an upper bound for $\alpha(U)$ where U is a uniserial module over a left m-multiserial algebra.

Theorem 6.5. *Let U be a non-projective uniserial module over a left m-multiserial algebra Λ with $m \geqslant 2$. Then $\alpha(U) \leqslant m$.*

Proof. By [10, Remark 2.3], we can assume that $\Lambda = \mathfrak{K}\Gamma/I$ such that $\Lambda\alpha$ is uniserial for every arrow α in Γ_1. Suppose p is a mast for U and let α_1 be the first arrow of p. Let $\mathcal{A} = \{\Lambda\gamma p \mid \gamma \in \Gamma_1\}$. Any two members of $\mathcal{A}$ are comparable; i.e., for $\gamma_1, \gamma_2 \in \Gamma_1$, either $\Lambda\gamma_1 p \subseteq \Lambda\gamma_2 p$ or $\Lambda\gamma_2 p \subseteq \Lambda\gamma_1 p$, since $\Lambda\alpha_1$ is uniserial. Hence there exists a greatest element in $\mathcal{A}$, say $\Lambda\gamma p$. Notice that $\Lambda\gamma p$ can be zero. This happens when $Jp = 0$.

Case 1. There is no arrow leaving $e := \mathrm{s}(p)$ except α_1. Here Λe is uniserial, we have $U = \Lambda e/\Lambda\gamma p$ and since U is not projective, $\gamma p \neq 0$. Therefore $\alpha(U) \leqslant 2 \leqslant m$ by Proposition 6.4.

Case 2. There are arrows $\beta_1, \ldots, \beta_l, \delta_{l+1}, \ldots, \delta_n$ leaving e except α_1. Assume $(\beta_j, e) \wr\wr p$ $(1 \leqslant j \leqslant l)$ and $\delta_t e$ $(l+1 \leqslant t \leqslant n)$ are non-routes on p. Note that $n < m$, since Λ is m-multiserial. Let $b_j = \beta_j - \sum_{i \in I(\beta_j, e)} k_i(\beta_j, e) v_i(\beta_j, e)$ and $b_t = \delta_t$. If $\Lambda\gamma p = 0$, then $U = \Lambda e/\sum_{i=1}^{n} \Lambda b_i$. Otherwise $U = \Lambda e/(\sum_{i=1}^{n} \Lambda b_i + \Lambda\gamma p)$. Let $0 \to DTrU \xrightarrow{f} \bigsqcup_{i \in I} B_i \xrightarrow{g} U \to 0$ be an almost split sequence. By Proposition 3.3, all the induced irreducible maps $g_i : B_i \to U$ are epimorphisms. By Proposition 6.3(iii), $\alpha(U) \leqslant \operatorname{length} \operatorname{soc} DTrU$. But by [2, Proposition IV.1.11], we know that $\operatorname{soc} DTrU \cong P_1/JP_1$ where $P_1 \to \Lambda e \to U \to 0$ is a minimal projective presentation of U. Therefore $\alpha(U) \leqslant m$. □

The following proposition gives more precise information in certain situations; it follows from the proof of the above theorem.

Proposition 6.6. *Suppose U is a non-projective uniserial module with mast p over a left m-multiserial algebra Λ. Then*

(i) *If $m = 1$ (i.e. if Λ is left serial) then $\alpha(U) \leq 2$.*
(ii) *If there is only one arrow leaving $\mathrm{s}(p)$, then $\alpha(U) \leqslant 2$.*
(iii) *If $m = 2$ (for example, if Λ is a left biserial algebra), and $Jp = 0$, then $\alpha(U) = 1$.*

Proof. The parts (i) and (ii) follow directly from the proof of the above theorem. As to part (iii), let α_1 be the first arrow of p. Then there is an arrow $\beta \neq \alpha_1$ starting at $e = \mathrm{s}(p)$ (otherwise, following the above proof again, $U = \Lambda e$ would be projective). Thus, either $(\beta, e) \wr\wr p$ or β is a non-route on p. If $(\beta, e) \wr\wr p$, then $U = \Lambda e/\Lambda b$, where $b := \beta - \sum_{i \in I(\beta,e)} k_i(\beta, e)v_i(\beta, e)$. If β is a non-route, then $U = \Lambda e/\Lambda b$, where $b := \beta$. In both cases then, $\alpha(U) = 1$ by [2, Proposition V.6.3], because the image of $\Lambda f \xrightarrow{\cdot b} \Lambda e$ is not in $J^2 e$, where $f = \mathrm{t}(\beta)$. □

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