Liouville theorem and coupling on negatively curved Riemannian manifolds

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Received 5 March 2001; received in revised form 26 February 2002; accepted 12 March 2002

Abstract

By using probabilistic approaches, Liouville theorems are proved for a class of Riemannian manifolds with Ricci curvatures bounded below by a negative function. Indeed, for these manifolds we prove that all harmonic functions (maps) with certain growth are constant. In particular, the well-known Liouville theorem due to Cheng for sublinear harmonic functions (maps) is generalized. Moreover, our results imply the Brownian coupling property for a class of negatively curved Riemannian manifolds. This leads to a negative answer to a question of Kendall concerning the Brownian coupling property. © 2002 Elsevier Science B.V. All rights reserved.

MSC: 35C21; 58B05

Keywords: Liouville theorem; Harmonic function; Diffusion process; Semigroup; Coupling

1. Introduction

Let \((M, \langle \cdot, \cdot \rangle)\) be a connected, noncompact, complete Riemannian manifold of dimension \(d\). The classical Liouville theorem says that all bounded harmonic functions are constant for \(M = \mathbb{R}^2\). This result was improved by Yau (1975) as follows: all positive harmonic functions are constant provided the Ricci curvature is nonnegative. According to Grigor’yan (1992) and Salo-Coste (1992), the curvature condition in Yau’s result can be replaced by the doubling property together with a local Poincaré inequality. Moreover, Cheng (1980) proved Yau’s result for sublinear harmonic functions (more generally, harmonic maps) in place of positive harmonic functions, see also Stafford (1990) for a probabilistic proof.
In this paper we aim to prove the Liouville theorem for the case that the Ricci curvature is allowed to be negative. Let

\[ k(x) = \sup \{ -\text{Ric}(X,X): X \in T_xM, |X| = 1 \}, \quad x \in M. \tag{1.1} \]

Let \( o \in M \) be fixed, and let \( \rho(x) \) denote the Riemannian distance between \( x \) and \( o \). In the sequel, \( \lim_{\rho \to \infty} f = 0 \) for a function \( f \) means that \( \lim_{r \to \infty} \sup_{v \leq r} f = 0 \). To understand our results in the compact manifold case (which is a trivial setting for Liouville type theorems), one puts \( \sup_{\emptyset} = 0 \) as usual so that \( \lim_{\rho \to \infty} f = 0 \) for all \( f \).

**Theorem 1.1.** Let \( d = 2 \) and

\[ \gamma(r) = \sup \{ k(x): \rho(x) = r \}, \quad r \geq 0, \]

\[ G(R) = \int_1^R \frac{1}{s} \exp \left[ - \int_1^s \frac{1}{r^2} \int_0^r r^2 \gamma(r) \, dr \right] \, ds, \quad R \geq 1. \]

If \( G(\infty) = \infty \) then a harmonic function \( u \) is constant provided \( |u|/G(\rho) \to 0 \) as \( \rho \to \infty \). In particular, if there exists \( r_0 > 1 \) such that \( k \leq [\rho^2 \log(\rho)]^{-1} \) for \( \rho \geq r_0 \), one has \( G(R) \geq c \log \log(1 + R) \) for some \( c > 0 \).

We stress that Theorem 1.1 (also Theorem 1.2 below) is presented for the negative curvature case, and for nonnegatively curved manifolds it is weaker than some known results. For instance, when \( \text{Ric} \geq 0 \) all sublinear harmonic functions have to be constant (see Cheng, 1980), but Theorem 1.1 only applies to (by taking \( \gamma = 0 \)) harmonic functions with sub-log growth. Nevertheless, this known result for sublinear harmonic functions is generalized in the paper by other means (see Corollary 3.4 below).

**Theorem 1.2.** Assume that \( o \) is a pole and \( \Delta \rho^2 \geq 2\delta \), for some \( \delta > 0 \).

1. If \( \delta \geq 6 \) and there exists \( \epsilon > 0 \) such that \( k \leq (\delta - 4)/(\epsilon + \rho^2) \), then \( u \) is constant provided \( \Delta u = 0 \) and \( |u|/\sqrt{\log \rho} \to 0 \) as \( \rho \to \infty \).

2. If \( 2 < \delta < 6 \) and there exists \( \epsilon > 0 \) such that \( k \leq (\delta - 2)/[8(\rho^2 + \epsilon)] \), then \( u \) is constant provided \( \Delta u = 0 \) and \( |u|\rho^{(\delta - 6)/4} \to 0 \) as \( \rho \to \infty \).

Obviously, Theorem 1.2 applies to Cartan–Hadamard manifolds for which one has \( \Delta \rho^2 \geq 2d \).

Theorems 1.1 and 1.2 can be considered as somehow inverse results of Greene-Wu’s conjecture (Greene and Wu, 1979): If \( M \) is a Cartan–Hadamard manifold with sectional curvatures bounded above by \( -c/\rho^2 \) for some \( c > 0 \) and large \( \rho \), then there exist nonconstant bounded harmonic functions. This conjecture has been proved by Hsu and Kendall (1992) for \( d = 2 \) and almost confirmed by Le (1996) for \( d \geq 3 \). Actually, for \( d \geq 3 \) Le proved that the angular part of the Brownian motion converges (as the time goes to infinity) to a random variable on \( \mathbb{S}^{d-1} \) with full support provided there exist \( c, c' > 0 \) (where \( c > 3/4 \) when \( d = 3 \)) such that

\[ -c' \rho^2 \leq \text{sectional curvatures} \leq -c \rho^{-2} \tag{1.2} \]
outside a compact set. Therefore, according to Hsu and March (see Section 3 in Hsu and March, 1985), (1.2) implies the existence of nonconstant bounded harmonic functions. Our results, however, cannot be compared with this conjecture.

Moreover, we would like to mention a result by March (1986) which is related to Theorem 1.1. Let $M$ be a rotationally symmetric manifold with metric given by

$$ds^2 = dr^2 + g(r)^2 d\theta^2,$$

where $g(0) = 0$, $g'(0) = 1$, $g(r) > 0$ for $r > 0$. It was proved in March (1986) that all bounded harmonic functions are constant if and only if

$$\int_1^\infty g^{d-3}(r) dr \int_r^\infty g^{1-d}(s) ds = \infty. \quad (1.3)$$

In particular, (1.3) holds provided the radial curvature is bounded below by $-c/\rho^2 \log \rho$ outside a compact set, where $c = 1$ when $d = 2$ and $c = \frac{1}{2}$ otherwise. In this case one has $G(\infty) = \infty$ and hence Theorem 1.1 applies. But Theorem 1.1 works also for unbounded harmonic functions and the manifold therein is not necessarily rotationally symmetric.

Theorem 1.1 will be proved in the next section by using Brownian motion and an upper bound for $\Delta f$. Indeed, we prove a more general result Theorem 2.1. To prove Theorem 1.2, we take a conformal change of the metric $\langle \cdot, \cdot \rangle' = f^{-2} \langle \cdot, \cdot \rangle$, where $f$ is a positive smooth function such that $\langle \cdot, \cdot \rangle'$ is complete, i.e. $(M, \langle \cdot, \cdot \rangle')$ is a complete Riemannian manifold. Let $\Lambda'$ denote the Laplace–Beltrami operator w.r.t. $\langle \cdot, \cdot \rangle'$. We have (see Thalmaier and Wang, 1998)

$$f^2 \Lambda = \Lambda' + (d - 2) f \nabla f := L'.$$

Then, it suffices to study the $L'$-harmonic functions on the Riemannian manifold $(M, \langle \cdot, \cdot \rangle')$. The key point in the proof of Theorem 1.2 is to choose a suitable $f$ such that the curvature of $L'$ is nonnegative.

It turns out that we need to establish the Liouville theorem for the Laplacian with a drift. Certainly, one may modify Stafford’s argument to treat such an operator with nonnegative curvature (cf. Section 5). But we intend to use a derivative formula which provides more information. We present in Section 3 a derivative formula for the diffusion semigroup of (unbounded) functions. Such a formula is already available in Elworthy and Li (1994) and Thalmaier (1997) for bounded smooth functions, and can be proved for some unbounded functions in a similar manner as in these references. We present a brief proof of the formula for completeness. This formula leads to some general sufficient conditions for the Liouville theorem (see Corollaries 3.3 and 3.4). In particular, Cheng’s result for harmonic functions is recovered. Combining results obtained in Section 3 and a conformal change of the metric, we prove Theorem 1.2 in Section 4. Actually, we obtain a more general result Theorem 4.1.

In Section 5.1 we extend the main results to harmonic maps from $M$ to a Cartan–Hadamard manifold. We will use Stafford’s argument instead of the derivative formula, because gradient estimates obtained in Section 3 are not available for harmonic maps so far. We refer to Arnaudon and Thalmaier (1998) for a different type of gradient estimates. Applications of derivative formulae to harmonic maps will be addressed in a
forthcoming paper joint with Anton Thalmaier. Finally, in terms of Theorems 1.1 and 1.2 and a recent result in Cranston and Wang (2000), we see that the Brownian motion has a successful coupling for a class of manifolds with negative Ricci curvature. This improves a result by Kendall (1986) for nonnegative Ricci curvature. See Section 5.2 for details.

2. Proof of Theorem 1.1

Proof of Theorem 1.1. For \( p \notin \text{cut}(o) \cup \{o\} \), where \( \text{cut}(o) \) denotes the cut locus of \( o \), let \( \{J_i\}_{i=1}^{d-1} \) be the Jacobi fields along the minimal geodesic \( \{z_s: s \in [0, \rho(p)]\} \) from \( o \) to \( p \) such that \( J_i(o) = 0 \) and \( \{U_iJ_i(p): 1 \leq i \leq d-1\} \in \Omega_p(M) \), where \( U(z_s) := dz_s/ds \) is the unit tangent vector field along this geodesic. We have, by the second variation formula of the distance, that

\[
\Delta \rho(p) = \sum_{i=1}^{d-1} \int_0^{\rho(p)} \left( |\nabla U_i J_i|^2 - \langle R(J_i, U_i)U_i, J_i \rangle \right) ds.
\]

Let \( X_i \) be the parallel vector field along \( z_s \) such that \( X_i(p) = J_i(p) \). Let \( h \in C^1[0, \rho(p)] \) be such that \( h(0) = 0 \) and \( h(\rho(p)) = 1 \). By the index lemma we obtain

\[
\Delta \rho(p) \leq \sum_{i=1}^{d-1} \int_0^{\rho(p)} \left[ ds \left( h(s)X_i(z_s)\right)^2 - \langle R(h(s)X_i(z_s), U_i(z_s))U(z_s), h(s)X_i(z_s) \rangle \right] ds
\]

\[
= (d - 1) \int_0^{\rho(p)} (h'(s))^2 ds - \int_0^{\rho(p)} h(s)^2 \text{Ric}(U_i, U_i)(z_s) ds.
\]

(2.1)

Taking \( h(s) = s/\rho \) we obtain for \( d = 2 \) that

\[
\Delta \rho \leq \frac{1}{\rho} + \frac{1}{\rho^2} \int_0^\rho s^2 \gamma(s) ds := \psi(\rho)
\]

outside \( \text{cut}(o) \cup \{o\} \). Then the proof is completed by the following result. \( \square \)

Theorem 2.1. Let \( L = \Delta + Z \) for some \( C^1 \)-vector field \( Z \). Let \( \psi \in C(0, \infty) \) be such that \( L \psi \leq \psi(\rho) \) outside \( \text{cut}(o) \cup \{o\} \). Define

\[
G(r) = \int_1^r \exp \left[ - \int_s^r \psi(t) dt \right] ds.
\]

If \( G(\infty) = \infty \) then an \( L \)-harmonic function \( u \) (i.e. \( Lu = 0 \)) is constant provided \( |u|/G(\rho) \to 0 \) as \( \rho \to \infty \).

Proof. For \( x \neq o \), let \( x_t \) be the diffusion process generated by \( \Delta \) with \( x_0 = x \). For any \( R > \rho(x) > \varepsilon \in (0,1) \), let

\[
S_\varepsilon = \inf \{t \geq 0: \rho(x_t) \leq \varepsilon\}, \quad T_R = \inf \{t \geq 0: \rho(x_t) \geq R\}.
\]
Since $G'(\rho) \geq 0$, by the Itô’s formula for the radial part obtained by Kendall (1987),
\[
dG(\rho)(x_t) = \sqrt{2}G'(\rho)(x_t) \, dB_t + 1_{\{x_t \not\in \text{cut}(\rho)\}}LG(\rho)(x_t) \, dt - dL_t
\]
before the stopping time $S_\varepsilon$, where $b_t$ is a Brownian motion on $\mathbb{R}$ and $L_t$ is an increasing process. Noting that $LG(\rho) \leq 0$ outside cut(\rho) \cup \{o\}$, $S_\varepsilon \land T_R < \infty$, and $G$ is bounded on $[\varepsilon, R]$, we obtain
\[
G(\rho)(x) \geq \mathbb{E}G(\rho)(x_{S_\varepsilon \land T_R}) \geq G(R)\mathbb{P}(S_\varepsilon > T_R) + G(\varepsilon)\mathbb{P}(T_R \geq S_\varepsilon).
\]
Hence $\mathbb{P}(S_\varepsilon > T_R) \leq [G(\rho)(x) - G(\varepsilon)]/[G(R) - G(\varepsilon)]$. Assume that $G(\infty) = \infty$. Let $u$ be a harmonic function with $|u|/G(\rho) \to 0$ as $\rho \to \infty$, one has sup\{\{u\}: \rho = R\}G(R)^{-1} \to 0 as $R \to \infty$. Then
\[
|u(x) - u(o)| = |\mathbb{E}u(x_{S_\varepsilon \land T_R}) - u(o)| \leq \sup_{\rho \leq \varepsilon} \left| u - u(o) \right| + \left( |u(o)| + \sup_{\rho = R} |u| \right) \mathbb{P}(S_\varepsilon > T_R)
\]
\[
\leq \sup_{\rho \leq \varepsilon} |u - u(o)| + \left( |u(o)| + \sup_{\rho = R} |u| \right) \frac{G(\rho)(x) - G(\varepsilon)}{G(R) - G(\varepsilon)}.
\]
By letting first $R \uparrow \infty$ then $\varepsilon \downarrow 0$, we obtain $u(x) = u(o)$. \hfill \square

### 3. Liouville theorems via gradient estimates

Let $L = A + Z$ for some $C^1$-vector field $Z$ such that
\[
\text{Ric}(X,X) - \langle \nabla_X Z, Z \rangle \geq -K_Z |X|^2
\] (3.1)
for some $K_Z \geq 0$ and all $X \in TM$. Define
\[
k_Z(x) = \sup\{-\text{Ric}(X,X) + \langle \nabla_X Z, X \rangle\colon X \in T_xM, \ |X| = 1\}, \quad x \in M.
\] (3.2)
Let $O(M)$ be the frame bundle over $M$, and $\pi: O(M) \to M$ the natural projection. Denote by $d$ and $d_\pi$ respectively the Itô differential and the Stratonovich differential. Let $\{(x_t, \Phi_t) \in M \times O(M)\colon t \geq 0\}$ solve the following stochastic differential equations:
\[
d_\pi x_t = \sqrt{2} \Phi_t d_\pi B_t + Z(x_t) \, dt,
\]
\[
d_\pi \Phi_t = H_\Phi d_\pi x_t, \quad \Phi_0 \in O(M), \quad x_0 = \pi \Phi_0,
\]
where $B_t$ is the Brownian motion on $\mathbb{R}^d$ and $H_\Phi : T_{\pi \Phi}M \to T_{\Phi}O(M)$ is the horizontal lift. It is well-known that $x_t(-\pi \Phi_t)$ is the diffusion process generated by $L$, and $\Phi_t$ is its horizontal lift to $O(M)$. By Itô’s formula, for $f \in C^2(M)$ one has
\[
d f(x_t) = \sqrt{2} \langle \nabla f(x_t), \Phi_t dB_t \rangle + Lf(x_t) \, dt.
\] (3.3)
For any \( X \in T_xM \), let \( \text{Ric}^\#(X) \in T_xM \) be defined by
\[
\langle \text{Ric}^\#(X), Y \rangle = \text{Ric}(X, Y), \quad Y \in T_xM.
\]

Let \( v_t \in T_{x_t}M \) be the process on \( TM \) solving the following differential equation:
\[
\frac{Dv_t}{dt} = -\text{Ric}^\#(v_t) + \nabla v_t Z.
\] (3.4)

Recall that \( Dv_t/dt \in T_{x_t}M \) is defined by \( //_{0,t}(d/dt)//_{0,0}v_t \), where \( //_{0,t} := \Phi_0^{-1} \Phi_t^{-1} \) are parallel translations along the path of \( x_t \). Therefore, one obtains from (3.3) that
\[
d|v_t|^2 = -2[\text{Ric}(v_t, v_t) - \langle \nabla v_t Z, v_t \rangle] dt \leq 2k_Z(x_t)|v_t|^2 dt.
\]

This then implies
\[
|v_t| \leq |v_0| \exp \left[ \int_0^t k_Z(x_s) ds \right].
\] (3.5)

For \( f \in C^2(M) \) and \( t > 0 \), let \( P_t f(x) = \mathbb{E}^t f(x_t) \) if the right-hand side is well defined. We have
\[
d\langle v_s, \nabla P_{t-s} f(x_s) \rangle = \left\langle \frac{Dv_s}{ds}, \nabla P_{t-s} f(x_s) \right\rangle ds + \sqrt{2} \langle v_s, \nabla \Phi_s dB_s \nabla P_{t-s} f(x_s) \rangle
\]
\[
+ \langle v_s, (\Box + \nabla Z) \nabla P_{t-s} f(x_s) - \nabla L P_{t-s} f(x_s) \rangle ds,
\]

where \( \Box(x) = \sum_i \nabla e_i \nabla e_i \) for any \( x \in M \) and a normal orthonormal frame \( \{e_i\} \) at \( x \). By combining this with (3.4) and the identity \( \Box \nabla f = \nabla \Delta f + \text{Ric}^\#(\nabla f) \), we obtain
\[
d\langle v_s, \nabla P_{t-s} f(x_s) \rangle = \sqrt{2} \langle v_s, \nabla \Phi_s dB_s \nabla P_{t-s} f(x_s) \rangle.
\] (3.6)

We are now ready to prove the following derivative formula for \( P_t \).

**Theorem 3.1.** Let \( T > 0 \) and \( x_0 = x \in M \). Let \( f \in C^2(M) \) be such that
\[
\mathbb{E}[f(x_t)^2 + |\nabla P_{T-t} f(x_t)|^2] \leq C,
\] (3.7)

for some \( C > 0 \) and all \( t \in [0, T] \). For any \( l \in C^1[0, T] \) such that \( l(0) = 1 \) and \( l(T) = 0 \), and any \( v_0 \in T_{x_t}M \), one has
\[
\langle \nabla P_T f(x), v_0 \rangle = -\frac{1}{\sqrt{2}} \mathbb{E} f(x_T) \int_0^T \langle v_s l'(s), \Phi_s dB_s \rangle.
\] (3.8)

**Proof.** By (3.3)
\[
dP_{T-t} f(x_t) = \sqrt{2} \langle \nabla P_{T-t} f(x_t), \Phi_t dB_t \rangle.
\]
Then $P_{T-t}f(x_t)$ is a square integrable martingale on $[0, T]$ according to (3.7). Hence
\[
\frac{P_{T-t}f(x_t)}{\sqrt{2}} = M_t + \int_0^t \langle \nabla P_{T-t}f(x_s), v_s l'(s) \rangle \, ds, \quad t \in [0, T]
\] (3.9)
for some martingale $M_t$. On the other hand, by (3.6) one has
\[
\langle v_t, \nabla P_{T-t}f(x_t) \rangle l(t) = 2 \int_0^t \langle v_s l(s), \nabla \phi_s \nabla P_{T-s}f(x_s) \rangle \\
+ \int_0^t \langle v_s l'(s), \nabla P_{T-s}f(x_s) \rangle \, ds, \quad t \in [0, T].
\] (3.10)
Combining this with (3.9), we obtain
\[
\langle v_t, \nabla P_{T-t}f(x_t) \rangle l(t) - \frac{P_{T-t}f(x_t)}{\sqrt{2}} = \int_0^t \langle v_s l'(s), \Phi_s dB_s \rangle
\]
\[
= \sqrt{2} \int_0^t \langle v_s l(s), \nabla \phi_s dB_s \nabla P_{T-s}f(x_s) \rangle - M_t,
\] (3.10)
which is a local martingale on $[0, T]$. Let $\tau_n = \inf \{ t \geq 0; \, \rho(x_t) \geq n \}$. By (3.1) we have $\tau_n \uparrow \infty$ as $n \uparrow \infty$. Then (3.10) implies that (recall $l(0) = 1$)
\[
\langle v, \nabla P_T f(x) \rangle = \mathbb{E} \langle v_{T \wedge \tau_n}, \nabla P_{T-T \wedge \tau_n} f(x_{T \wedge \tau_n}) \rangle l(T \wedge \tau_n)
\]
\[
- \frac{1}{\sqrt{2}} \mathbb{E}P_{T-T \wedge \tau_n} f(x_{T \wedge \tau_n}) \int_0^{T \wedge \tau_n} \langle v_s l'(s), \Phi_s dB_s \rangle.
\] (3.11)
Noting that $l(T) = 0$, $\int_0^T \langle v_s l'(s), \Phi_s dB_s \rangle$ is square integrable and
\[
\mathbb{E}(P_{T-T \wedge \tau_n} f(x_{T \wedge \tau_n}))^2 \leq \mathbb{E}(\mathbb{E}^T \wedge \tau_n f(x_{T \wedge \tau_n}))^2 \leq \mathbb{E}(f(x_T)^2) < \infty
\] according to (3.7), by letting $n \uparrow \infty$ in (3.11) we obtain (3.8). □

**Theorem 3.2.** In the situation of Theorem 2.1, let $u$ be an $L$-harmonic function. If
\[
u^2 \leq \exp[c(1 + \rho)]
\] (3.12)
for some $c > 0$ (recall that $k = k_0$), then
\[
|\nabla u(x)|^2 \leq \frac{\mathbb{E}^T[u(x_T)^2]}{2} \int_0^T l'(t)^2 \mathbb{E}^T \exp \left[ 2 \int_0^t k_z(x_s) \, ds \right] \, dt, \quad T > 0.
\] (3.13)

**Proof.** For $p \notin \text{cut}(o) \cup \{0\}$, let $z_s$ and $h$ be as in (2.1). Noting that
\[
Z \rho(p) := \langle Z, U \rangle(p) = \int_0^{\rho(p)} \frac{d}{ds} \langle h(s)^2 U(z_s), Z(z_s) \rangle \, ds
\]
\[
= \int_0^{\rho(p)} [(h^2)'(s) \langle U, Z \rangle(z_s) + h(s)^2 \langle \nabla U, Z \rangle(z_s)] \, ds,
\]

it follows from (2.1) that
\[ L \rho \leq \int_0^{\rho(p)} [(d - 1)(h'(s))^2 + K_Z h(s)^2 + (h^2)'(s)|Z|(z_s)] \, ds. \]

Letting \( h \in C^1[0, \rho(p)] \) be such that \( h(0) = 0, h(s) = 1 \) for \( s \geq \rho(p) \wedge 1 \), and \( |h'| \leq 2/\rho(p) \wedge 1 \), we obtain \( L \rho \leq c_1(1 + \rho^{-1}) \) outside \( \text{cut}(o) \cup \{o\} \) for some \( c_1 > 0 \). Therefore, by the Itô’s formula for the radial process (see Kendall, 1987), we obtain, for any \( n \geq 1 \),
\[
d \exp[n\sqrt{1 + \rho^2}(x_t)] \leq \frac{\sqrt{2}n\rho(x_t)}{\sqrt{1 + \rho^2(x_t)}} \exp[n\sqrt{1 + \rho^2}(x_t)] \, db_t + c(n) \exp[n\sqrt{1 + \rho^2(x_t)}] \, dt
\]
for some \( c(n) > 0 \), where \( b_t \) is a Brownian motion on \( \mathbb{R} \). This implies, by a standard argument (see, e.g. Thalmaier and Wang, 1998, p. 118), that
\[
\mathbb{E}^x \exp[n\sqrt{1 + \rho^2}(x_t)] \leq \exp[c(n)t + n\sqrt{1 + \rho^2(x)}]. \tag{3.14}
\]

Next, by Bochner’s formula we have
\[
\frac{1}{2} L |\nabla u|^2 = \langle \nabla Lu, u \rangle + (\text{Ric} - \langle \nabla Z, \cdot \rangle)(\nabla u, \nabla u) + \|\text{Hess} u\|^2 \geq -K_Z |\nabla u|^2.
\]
Then \( \mathbb{E}^x |\nabla u|^2(x_t) \geq \exp[-2K_Z t]|\nabla u|^2(x) \). Since \( Lu^2 = 2|\nabla u|^2 \), we obtain
\[
\mathbb{E}^x u^2(x_t) \geq 2 \int_0^t \mathbb{E}^x |\nabla u|^2(x_s) \, ds \geq \frac{|\nabla u|^2(x)}{K_Z} (1 - \exp[-2K_Z t]).
\]
Combining this with (3.12) and (3.14), we obtain (3.7) for \( f = u \). Therefore (3.8) holds for \( f = u \) and hence the proof is completed. \( \square \)

The following is a direct consequence of Theorem 3.2 by taking \( l(s) = (T - s)/T \).

**Corollary 3.3.** If
\[
\lim_{T \to \infty} \frac{1}{T^2} \int_0^T \mathbb{E}^x \exp \left[ 2 \int_0^t k_Z(x_s) \, ds \right] \, dt = 0, \quad x \in M,
\]
then all bounded \( L \)-harmonic functions are constant.

**Corollary 3.4.** Assume that
\[
\nu(x) := \sup_{r \geq 0} \mathbb{E}^x \exp \left[ 2 \int_0^r k_Z(x_s) \, ds \right] < \infty, \quad x \in M.
\]
Let \( \psi \in C(0, \infty) \) be such that \( L \rho \leq \psi(\rho) \) outside \( \text{cut}(o) \cup \{o\} \). Define
\[
\Psi(r) = \int_0^r \exp \left[ - \int_s^r \psi(t) \, dt \right] \, ds \int_0^s \exp \left[ \int_1^t \psi(a) \, da \right] \, dt, \quad r \geq 0.
\]
We have $\Psi(\infty) = \infty$. If $u$ is a $L$-harmonic function satisfying (3.12) for some $c > 0$ and $|u|^2/\Psi(\rho) \to 0$ as $\rho \to \infty$, then $u$ is constant.

**Proof.** It is easy to check that $L\Psi(\rho) \leq 1$ outside cut$(o)$. Therefore $\mathbb{E}^x \Psi(\rho)(x) \leq t + \Psi(\rho)(x)$. Let $u$ be such that $Lu = 0$ and $|u|^2/\Psi(\rho) \to 0$ as $\rho \to \infty$. For any $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that $|u|^2 \leq c_\varepsilon + \varepsilon \Psi(\rho)$. Then $\mathbb{E}^x u_\varepsilon^2(x_T) \leq c(\varepsilon, x) + \varepsilon t$ for some $c(\varepsilon, x) > 0$. By taking $t(s)=(T-s)/T$ in (3.13), we obtain $|\nabla u(x)|^2 \leq (\alpha(x)/T)(c(\varepsilon, x) + \varepsilon T)$. By letting first $T \to \infty$ then $\varepsilon \to 0$, we obtain $\nabla u(x) = 0$.

We note that Corollary 3.4 implies Cheng’s Liouville theorem for harmonic functions, since when $k := k_0 \leq 0$ one has $\alpha(x) \leq 1$ and $\Delta \rho \leq (d-1)/\rho$.

4. Proof of Theorem 1.2

Let $L = \Delta + Z$ be as in Section 3. Let $f$ be a positive smooth function such that $\langle \cdot, \cdot \rangle' := f^{-2}\langle \cdot, \cdot \rangle$ is complete. In particular, this new metric is complete provided $f \leq h \circ \rho$ for some positive $h \in C(\mathbb{R}_+)$ such that $\int_0^{\infty} h(r)^{-1} \, dr = \infty$. Indeed, let $\rho'$ be the distance function from $o$ induced by the new metric, then $\rho' \to \infty$ as $\rho \to \infty$. Hence, for a Cauchy sequence $\{x_i\}$ w.r.t. the new metric, $\{\rho(x_i)\}$ is bounded and hence $\{x_i\}$ is also a Cauchy sequence w.r.t. the original metric since these two metrics are locally equivalent.

**Theorem 4.1.** If there exists $f$ such that $\langle \cdot, \cdot \rangle'$ is complete and

$$f^2k_Z + 3|\nabla f|^2 + f|\nabla f| \cdot |Z| - f\Delta f \leq 0,$$

then $u$ is constant provided $Lu = 0$ and $|u|/\sqrt{f} \to 0$ as $\rho' \to \infty$.

**Proof.** Let $L' = f^2 L$. We have (see Thalmaier and Wang, 1998)

$$L' = \Lambda' + f^2 Z + (d-2)f\nabla f := \Lambda' + Z'. $$

Let $\text{Ric}'$ and $\nabla'$ denote, respectively, the Ricci curvature tensor and the Levi–Civita connection induced by $\langle \cdot, \cdot \rangle'$. Then for any $X \in TM$ one has (see Thalmaier and Wang, 1998)

$$\text{Ric}'(f X, f X) - \langle \nabla' f X', f X \rangle'$$

$$\geq f^2[\text{Ric}(X, X) - \langle \nabla X Z, X \rangle] + [3|\nabla f|^2 + f|\nabla f| \cdot |Z| - f\Delta f]\|X\|^2.$$

Therefore $k'_{Z'} \leq 0$ according to (4.1), where $k'_{Z'}$ is defined as $k_Z$ for $L'$ on $(M, \langle \cdot, \cdot \rangle')$ in place of $L$ on $(M, \langle \cdot, \cdot \rangle)$. Hence, as has been shown in the proof of Theorem 3.2, there exists $c > 0$ such that $L' \rho' \leq c(1 + (\rho')^{-1})$ outside cut$(o)$, the cut locus of $o$ under the new metric. Then the proof is completed by applying Corollary 3.4 to $L'$ on $(M, \langle \cdot, \cdot \rangle')$ and $\psi(r) = c(1 + r^{-1})$.  \[\square\]
Proof of Theorem 1.2. (a) For \( \delta \geq 6 \), let \( f = \sqrt{\rho^2 + \varepsilon} \). We have

\[
f \Delta f - 3|\nabla f|^2 \geq \delta - \frac{4\rho^2}{\rho^2 + \varepsilon} \geq \delta - 4.
\]

Then (4.1) holds for \( Z = 0 \) since \( k_0 := k \leq (\delta - 4)/f^2 = (\delta - 4)/f^2 \). Hence the first assertion follows from Theorem 4.1 by noting that \( \rho' \leq c \log(1 + \rho) \) for some \( c > 0 \).

(b) For \( 2 < \delta < 6 \), let \( f = (\rho^2 + \varepsilon)^r \), where \( r \in (0, 1/2) \) is to be determined. We have

\[
f \Delta f - 3|\nabla f|^2 \geq 2r[\delta - 2 - 4r](\rho^2 + \varepsilon)^{2r-1}.
\]

Taking \( r = (\delta - 2)/8 \), we obtain (4.1) for \( Z = 0 \) since \( k_6 = f^2 = (\delta - 2)/2 \). By applying Theorem 3.2 to \( L' \) on the manifold with the new metric and letting \( l(t) = (T - t)/T \), it suffices to show that

\[
\lim_{T \to \infty} \frac{1}{T} \mathbb{E} \langle u(x'_t) \rangle^2 = 0, \quad x \in M
\]

for any \( L \)-harmonic \( u \) with \( |u|\rho^{(\delta-6)/4} \to 0 \) as \( \rho \to 0 \), where \( x'_t \) is the diffusion process generated by \( L' \). To prove this, taking \( h(s) = s/\rho(p) \) in (2.1) we obtain

\[
\Delta \rho \leq \frac{d - 1}{\rho} + \int_0^\rho \frac{s^2(\delta - 2)^2}{8\rho^2(s^2 + \varepsilon)} \, ds \leq \frac{c}{\rho}
\]

for some \( c > 0 \) since \( k \leq (\delta - 2)^2/[8(\rho^2 + \varepsilon)] \). Then

\[
L'(\rho^2 + 1)^{(6-\delta)/4} := f^2 \Delta (\rho^2 + 1)^{(6-\delta)/4} \leq \frac{6 - \delta}{2} (1 + c) := c_1.
\]

It then follows that

\[
\mathbb{E}(1 + \rho^2)^{(6-\delta)/4}(x'_t) \leq [1 + \rho^2(x)]^{(6-\delta)/4} + c_1 t.
\]

Since \( |u|\rho^{(\delta-6)/4} \to 0 \) as \( \rho \to \infty \), we obtain (4.2) as did in the proof of Corollary 3.4.

5. Applications to harmonic maps and coupling

5.1. Harmonic maps

In this section we aim to extend our results to harmonic maps. Let \( u : M \to N \) be a harmonic map, where \( N \) is a Cartan–Hadamard manifold. We say that \( u \) is constant if its image contains only a single point. We add a subscript to an operator or function to denote the corresponding one on \( N \). For instance \( \rho_N \) is the Riemannian distance function on \( N \) from a fixed point.

**Theorem 5.1.** Let \( G \) be defined in Theorem 2.1 for \( Z = 0 \). \( u \) is constant provided \( \rho_N \circ u/G(\rho) \to 0 \) as \( \rho \to \infty \). Consequently, the result holds for \( G \) given in Theorem 1.1 for \( d = 2 \).
**Proof.** Without loss of generality, let $\rho_N$ be the distance function from $u(o)$. Let $S_\epsilon$ and $T_R$ be in the proof of Theorem 2.1 for $Z = 0$. Since $u$ is harmonic and $N$ is a Cartan–Hadamard manifold, one has

$$\Delta \rho_N \circ u = \text{Hess}_N(\rho_N)(du, du) := \sum_{i=1}^{d} \text{Hess}_N(\rho_N)(du(e_i), du(e_i)) \geq 0,$$

where $du : TM \rightarrow T_u(N)$ is induced by $u$ and $\{e_i\}$ is a normal frame on $M$. Then

$$\rho_N \circ u(x) \leq \mathbb{E} \rho_N \circ u(x_{S_\epsilon \wedge T_R}) \leq \sup_{\rho \leq \epsilon} \rho_N \circ u + \mathbb{P}(S_\epsilon > T_R) \sup_{\rho = R} \rho_N \circ u.$$

This implies $\rho_N \circ u = 0$ (i.e. $u(x) = u(o)$) as in the proof of Theorem 2.1. \qed

To extend Theorem 1.2, we adopt Stafford’s argument (Stafford, 1990) as explained in the first section.

**Theorem 5.2.** If there exists $f$ such that $\langle \cdot, \cdot \rangle' := f^{-2}\langle \cdot, \cdot \rangle$ is complete and $f^2 k + 3|\nabla f|^2 - f \Delta f \leq 0$, then $u$ is constant provided $\rho_N \circ u / \rho' \rightarrow 0$ as $\rho' \rightarrow \infty$. Consequently, Theorem 1.2 holds for the present case with $|u|$ replaced by $\rho_N \circ u$.

**Proof.** By the proofs of Theorems 1.2 and 4.1, it suffices to show the following Proposition 5.3. \qed

**Proposition 5.3.** Let $L = \Delta + Z$ for some $C^1$-vector field $Z$ such that $K_Z \leq 0$. If $Lu = 0$ and $\mathbb{E}^3 \rho_N^2 \circ u(x_t)/t \rightarrow 0$ as $t \rightarrow \infty$ for each $x$, where $x_t$ is generated by $L$, then $u$ is constant.

**Proof.** Since $N$ is a Cartan–Hadamard manifold, $\text{Hess}_N(\rho_N^2)(X, X) \geq 2|X|^2$. One has

$$\frac{1}{2} L \rho_N^2 \circ u = \sum_{i=1}^{d} \left[ \langle \nabla e_i, du(e_i) \rangle N + \text{Hess}_N(\rho_N^2)(du(e_i), du(e_i)) \right]$$

$$+ \langle du(Z), \nabla_N(\rho_N^2) \circ u \rangle N$$

$$\geq \langle Lu, (\nabla_N \rho_N^2) \circ u \rangle N + \|du\|^2 = \|du\|^2,$$

where $\nabla e_i X$ is the covariant differentiation of $X$ in the direction of $e_i$ for a $C^1$-vector field $X$ along $u$. Then

$$\mathbb{E}^3 \rho_N^2 \circ u(x_t) - \rho_N^2 \circ u(x) \geq 2 \int_0^t \mathbb{E}^3 \|du\|^2(x_s) \, ds.$$  \hspace{1cm} (5.1)

On the other hand, one has

$$\frac{1}{2} Z \|du\|^2 = \sum_{i=1}^{d} \langle \nabla e_i, du(Z), du(e_i) \rangle N - \sum_{i=1}^{d} \langle du(\nabla e_i Z), du(e_i) \rangle N$$

$$= \sum_{i=1}^{d} \langle \nabla e_i, du(Z), du(e_i) \rangle N - \sum_{i,j=1}^{d} \langle du(e_i), du(e_j) \rangle N \langle \nabla e_i Z, e_j \rangle.$$
Combining this with the Bochner’s formula (cf. Cheng, 1980; Stafford, 1990), we obtain

\[ L\|du\|^2 \geq 2 \sum_{i,j=1}^{d} (\text{Ric}(e_i, e_j) - \langle \nabla e_i Z, e_j \rangle) (\langle du(e_i), du(e_j) \rangle_N) \geq 0 \]

since \( Lu = 0 \), the sectional curvatures on \( N \) are nonpositive, and both \((\text{Ric}(e_i, e_j) - \langle \nabla e_i Z, e_j \rangle)_{d\times d}\) and \((\langle du(e_i), du(e_j) \rangle_N)_{d\times d}\) are nonnegatively definite. Hence \( \|du\|^2(x) \leq \mathbb{E}^x\|du\|^2(x_s) \). The proof is completed by combining this with (5.1).

### 5.2. Coupling

The coupling method has become a powerful tool in probability theory and related fields, we refer to, e.g. Chen (1997), Kendall (1998) and Lindvall (1992) for backgrounds and applications of the coupling theory. We say that the Brownian motion has a successful coupling, or the manifold has the Brownian coupling property (Kendall, 1986), if for any \( x, y \in M \) there exist two Brownian motions \( x_t \) and \( y_t \) with \( x_0 = x \) and \( y_0 = y \) such that

\[
\tau := \inf\{t \geq 0 : x_t = y_t\} < \infty \quad \text{a.s.}
\]

Combining Theorems 1.1 and 1.2 with the following result implied by results in Cranston and Wang (2000), we see that there exist a class of negatively curved manifolds which possess the Brownian coupling property.

**Theorem 5.4.** Assume that the Ricci curvature is bounded below. Then all bounded harmonic functions are constant if and only if the Brownian motion has a successful coupling.

**Proof.** Obviously (see Kendall, 1986), the Brownian coupling property implies that all bounded harmonic functions are constant. It suffices to prove the converse. According to Cranston and Greven (1995), if all bounded harmonic functions are constant, then there exists a successful shift coupling, i.e. for any \( x, y \in M \) there exist two Brownian motions \( x_t \) and \( y_t \) with \( x_0 = x \) and \( y_0 = y \), and two finite stopping times \( \tau_1 \) and \( \tau_2 \) such that \( x_{\tau_1} = y_{\tau_2} \). Since the Ricci curvature is bounded below, the heat semigroup satisfies a parabolic Harnack inequality according to Li and Yau (1986). Then there also exists a successful coupling for the Brownian motion by Theorem 2.1 in Cranston and Wang (2000), which says that under a parabolic Harnack inequality, the existence of successful coupling is equivalent to that of successful shift coupling.

Recall that Kendall asked in Kendall (1986): is there any Liouville manifold which does not possess the Brownian coupling property? Theorem 5.4 provides a negative answer to this question if the Ricci curvature is bounded below.

### Acknowledgements

The author would like to thank Prof. Anton Thalmaier for useful conversations and a referee for careful comments.
References


