Hypermaps and indecomposable permutations

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1. Introduction

Hypermaps have been introduced as an algebraic tool for the representation of the embedding of graphs in an orientable surface. A hypermap of size $n$ is an ordered pair of permutations generating a transitive subgroup of $\Sigma_n$. A survey of the main results on hypermaps can be found in [2].

A natural question is to ask what is the probability that two random permutations of $\Sigma_n$ yield a hypermap. The answer to this question was given by Dixon in ([4] Theorem 2). More recently, a refinement of this result was given in ([5] Theorem 1). The purpose of the present paper is to give an elementary combinatorial presentation of these results.

2

Let $a_n$ be the number of ordered pair of permutations $(\alpha, \beta)$ generating a transitive subgroup of $\Sigma_n$.

Lemma 1.

$$n! n! = \sum_{i=1}^{n} \binom{n-1}{i-1} a_i (n-i)! (n-i)! .$$

Proof. Let $(\alpha, \beta)$ be an ordered pair of permutations of $\Sigma_n$ and let $\Omega_1$ be the orbit of the group they generate containing 1. Denote by $i$ the number of elements of $\Omega_1$. Then $\alpha, \beta$, can be written as:

$$\alpha = \alpha_1 \alpha_2 \quad \beta = \beta_1 \beta_2$$

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where \((\alpha_1, \beta_1)\) acts transitively on \(\Omega_1\) and \((\alpha_2, \beta_2)\) is an ordered pair of permutations acting on \(\{1, 2, \ldots, n\} \setminus \Omega_1\). Now observe:

- \(nn!\) is the number of ordered pairs of permutations of \(\delta_n\)
- \(\binom{n-1}{i-1}\) is the number of subsets with \(i\) elements containing 1 of \(\{1, 2, \ldots, n\}\) (i.e. the number of possible orbits \(\Omega_1\)),
- \(a_i\) is the number of possible transitive pairs on \(\Omega_1\)
- \((n-i)!(n-i)!\) is the number of possible pairs \((\alpha_2, \beta_2)\)

The result follows. \(\square\)

A permutation \(\theta = a_1, a_2, \ldots, a_n\) is said to be indecomposable, if for any \(p < n\) the left factor subsequence \(a_1, a_2, \ldots, a_p\) contains at least one \(a_j > p\). Equivalently for \(p < n\), there is no initial interval \([1, 2, \ldots, p]\) fixed by \(\theta\). Indecomposable permutations were probably considered for the first time by Lentin [6] in his thesis. An asymptotics for the number of indecomposable permutations \(c_n\) in \(\delta_n\) was found by Comtet [1] using the following simple formula relating the number of indecomposable permutations to the indecomposable ones of smaller length:

\[
c_n = n! - \sum_{i=1}^{n-1} c_i(n-i)!.
\]

**Lemma 2.** \(a_i = c_{i+1}(i-1)\!\)!

**Proof.** From the above equality,

\[
\sum_{i=1}^{n} c_i(n-i)! = n!.
\]

Replacing \(n\) by \(n + 1\) and \(i\) by \(k + 1\) we get:

\[
\sum_{k=1}^{n} c_{k+1}(n-k)! = (n+1)! - n!c_1 = nn!.
\]

Moreover, from Lemma 1,

\[
\sum_{i=1}^{n} \frac{a_i}{(i-1)!}(n-i)! = nn!.
\]

The result follows from the fact that \(\frac{a_i}{(i-1)!}\) and \(c_{i+1}\) satisfy the same recursion formula and that \(a_1 = c_2\). \(\square\)

**Proposition.** The probability \(t_n\) that an ordered pair of permutations generates a transitive subgroup of \(\delta_n\) and the probability \(g_n\) that a permutation of \(S_n\) be indecomposable are related by:

\[
t_n = \frac{n+1}{n} g_{n+1}.
\]

**Proof.** Use Lemma 2 and \(t_n = \frac{a_n}{nn!}, g_n = \frac{c_n}{nn!}\). \(\square\)

**Remark.** P. Ossona de Mendez and P. Rosenstiehl in [3] gave a bijection between rooted hypermaps and indecomposable permutations. Our proposition can also be proved using their result.

**References**