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Theoretical Computer Science 400 (2008) 113–136

Theoretical Computer Science

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Matrices of 3-iet preserving morphisms

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Received 12 February 2007; received in revised form 4 February 2008; accepted 24 February 2008

Communicated by B. Durand

Abstract

We study matrices of morphisms preserving the family of words coding 3-interval exchange transformations. It is well known that matrices of morphisms preserving Sturmian words (i.e. words coding 2-interval exchange transformations with the maximal possible factor complexity) form the monoid $\{\boldsymbol{M} \in \mathbb{N}^{2\times 2} \mid \det \boldsymbol{M} = \pm 1\} = \{\boldsymbol{M} \in \mathbb{N}^{2\times 2} \mid \boldsymbol{M}\boldsymbol{E}\boldsymbol{M}^T = \pm \boldsymbol{E}\}$, where $\boldsymbol{E} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

We prove that in the case of exchange of three intervals, the matrices preserving words coding these transformations and having the maximal possible subword complexity belong to the monoid $\{\boldsymbol{M} \in \mathbb{N}^{3\times 3} \mid \boldsymbol{M}\boldsymbol{E}\boldsymbol{M}^T = \pm \boldsymbol{E}, \det \boldsymbol{M} = \pm 1\}$, where $\boldsymbol{E} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$.

Keywords: Interval exchange transformation; Sturmian words; Morphism; Incidence matrix

1. Introduction

Sturmian words are the most studied class of infinite aperiodic words. By their nature, they are defined purely over a binary alphabet. There exist several equivalent definitions of Sturmian words [6], which give rise to several different generalizations of Sturmian words over larger alphabets. For example, the generalization of Sturmian words to Arnoux–Rauzy words of order r uses the characterization of Sturmian words by means of the so-called left and right special factors [3].

Another natural generalization can be derived from the definition of a Sturmian word as an aperiodic word coding a transformation of exchange of two intervals. The r-interval exchange transformation has been introduced by Katok and Stepin [16]: An exchange T of r intervals is defined by a vector of r lengths and by a permutation of r letters; the unit interval is then partitioned according to the vector of lengths, and T interchanges these intervals according to the given permutation. According to the Marseille mathematical folklore, Rauzy was the first one to observe that interval exchange transformation can be used for the generalization of Sturmian words [11].

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0304-3975/$ - see front matter
doi:10.1016/j.tcs.2008.02.044

In contrast to ergodic properties of these transformations, which were studied by many authors [17,26,28,29], combinatorial properties of associated words have been so far explored only a little. Some results, analogical to the properties known for Sturmian words, have been derived for the most simple case, namely for 3-interval exchange transformations. Note that for the exchange of three intervals, the most interesting permutation[1] is (321), since it gives infinite words with reversal closed languages as in the case of Sturmian words. All the results cited below apply to transformations with this permutation. Words coding 3-interval exchange transformation can be periodic or aperiodic, depending on the choice of parameters. In accordance with the terminology introduced by [10], infinite words which code 3-interval exchange transformations and are aperiodic, are called 3-iet words. The factor complexity $\mathcal{C}_u(n)$ of a 3-iet word u, i.e., the number of different factors of length n occurring in u, is known to satisfy $\mathcal{C}_u(n) \leq 2n+1$ for all $n \in \mathbb{N}$. Words for which $\mathcal{C}_u(n) = 2n+1$, for all $n \in \mathbb{N}$, are called non-degenerate (or regular) 3-iet words.

In paper [11], minimal sequences coding 3-interval exchange transformations are fully characterized. The structure of palindromes of these words was described in [10,5], whereas paper [11] deals with their return words. Here we study morphisms which map the set of 3-iet words to itself.

Morphisms preserving Sturmian words were completely described by Berstel, Mignosi and Séébold [7,20,27]. Recall that there are two ways to define such a morphism:

- A morphism φ over the binary alphabet $\{0, 1\}$ is said to be *locally Sturmian* if there is a Sturmian word u such that $\varphi(u)$ is also Sturmian.
- A morphism φ over the binary alphabet $\{0, 1\}$ is said to be *Sturmian* if $\varphi(u)$ is Sturmian for all Sturmian words u.

Berstel, Mignosi and Séébold showed that the families of Sturmian and locally Sturmian morphisms coincide and that they form a monoid generated by three morphisms, ψ_1, ψ_2 and ψ_3, given by

$$\psi_1 : \begin{array}{l} 0 \mapsto 01 \\ 1 \mapsto 1 \end{array}, \qquad \psi_2 : \begin{array}{l} 0 \mapsto 10 \\ 1 \mapsto 1 \end{array}, \qquad \psi_3 : \begin{array}{l} 0 \mapsto 1 \\ 1 \mapsto 0 \end{array}. \tag{1}$$

To each morphism φ over a k-letter alphabet $\{a_1, \ldots, a_k\}$ one can assign its incidence matrix $\boldsymbol{M}_\varphi \in \mathbb{N}^{k\times k}$ by putting

$$(\boldsymbol{M}_\varphi)_{ij} = \text{number of letters } a_j \text{ in the word } \varphi(a_i). \tag{2}$$

As a simple consequence of the fact that the monoid of Sturmian morphisms is generated by ψ_1, ψ_2 and ψ_3 from (1), one has the following fact: *A matrix* $\boldsymbol{M} \in \mathbb{N}^{2\times 2}$ *is the incidence matrix of a Sturmian morphism if and only if* $\det \boldsymbol{M} = \pm 1$. By an easy calculation we can derive that for matrices of order 2×2

$$\det \boldsymbol{M} = \pm 1 \iff \boldsymbol{M}\boldsymbol{E}\boldsymbol{M}^T = \pm \boldsymbol{E}, \quad \text{where } \boldsymbol{E} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

In the theory of Lie groups, one can formulate this claim by stating that the group $\mathrm{SL}(2, \mathbb{Z})$ is isomorphic to the group $\mathrm{Sp}(2, \mathbb{Z})$, see [15].

The aim of this paper is to derive similar properties for matrices of morphisms preserving the family of 3-iet words, which we call here 3-iet preserving morphisms. We will prove the following theorems.

Theorem A. *Let φ be a 3-iet preserving morphism and let $\boldsymbol{M}$ be its incidence matrix. Then*

$$\boldsymbol{M}\boldsymbol{E}\boldsymbol{M}^T = \pm \boldsymbol{E}, \text{ where } \boldsymbol{E} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}.$$

Theorem B. *Let φ be a 3-iet preserving morphism and let $\boldsymbol{M}$ be its incidence matrix. Then one of the following holds*

- $\det \boldsymbol{M} = \pm 1$ *and $\varphi(u)$ is non-degenerate for every non-degenerate 3-iet word u,*
- $\det \boldsymbol{M} = 0$ *and $\varphi(u)$ is degenerate for every 3-iet word u.*

[1] For a permutation $\pi \in S_n$ we write $(\pi(1)\pi(2)\cdots\pi(n))$.

In the proof of Theorem A we use the description of matrices of Sturmian morphisms given above, while the main tool employed in the proof of Theorem B is the connection between words coding 3-interval exchange transformations and cut-and-project sets.

Note that Theorems A and B give only necessary conditions for $\boldsymbol{M}$ to be the incidence matrix of a 3-iet preserving morphism. In Section 9 we provide an example which shows that the conditions are not sufficient.

2. Preliminaries

In this paper we deal with finite and infinite words over a finite alphabet $\mathcal{A}$, whose elements are called letters. The set of all finite words over $\mathcal{A}$ is denoted by $\mathcal{A}^*$. This set, equipped with the concatenation as a binary operation, is a free monoid having the empty word as its identity. The length of a word $w = w_1 w_2 \cdots w_n$ is denoted by $|w| = n$, the number of letters a in the word w is denoted by $|w|_a$.

2.1. *Infinite words*

The set of two-sided infinite words over an alphabet $\mathcal{A}$, i.e., of two-sided infinite sequences of letters of $\mathcal{A}$, is denoted by $\mathcal{A}^{\mathbb{Z}}$, its elements are words $u = (u_n)_{n\in\mathbb{Z}}$, where $u_n \in \mathcal{A}$. Note that in all our considerations we will not identify infinite words $(u_{n+k})_{n\in\mathbb{Z}}$ and $(u_n)_{n\in\mathbb{Z}}$, and therefore we will mark the position corresponding to the index 0, usually using | as the delimiter, e.g. for $u \in \mathcal{A}^{\mathbb{Z}}$,

$$u = \cdots u_{-3}u_{-2}u_{-1}|\, u_0u_1u_2 \cdots .$$

The words of this form are sometimes called pointed biinfinite words. Naturally, one can define a metric on the set $\mathcal{A}^{\mathbb{Z}}$.

Definition 1. Let $u = (u_n)_{n\in\mathbb{Z}}$ and $v = (v_n)_{n\in\mathbb{Z}}$ be two biinfinite words over $\mathcal{A}$. We define the *distance* $\mathrm{d}(u, v)$ between u and v by setting

$$\mathrm{d}(u, v) := \frac{1}{1 + j}, \tag{3}$$

where $j \in \mathbb{N}$ is the minimal index such that either $u_j \neq v_j$ or $u_{-j} \neq v_{-j}$.

It can be easily verified that the above defined distance $\mathrm{d}(u, v)$ is a metric and that the set $\mathcal{A}^{\mathbb{Z}}$ with this metric is a compact metric space.

We consider also one-sided infinite words $u = (u_n)_{n\in\mathbb{N}}$, either right-sided $u = u_0u_1u_2 \cdots$ or left-sided $u = \cdots u_2u_1u_0$.

The degree of diversity of an infinite word u is expressed by the complexity function, which counts the number of factors of length n in the word u. Formally, a word w of length n is said to be a *factor* of a word $u = (u_n)_{n\in\mathbb{Z}}$ if there is an index $i \in \mathbb{Z}$ such that $w = u_iu_{i+1} \cdots u_{i+n-1}$. The set of all factors of u of length n is denoted by $\mathcal{L}_n(u)$. The *language* $\mathcal{L}(u)$ of an infinite word u is the set of all its factors, that is,

$$\mathcal{L}(u) = \bigcup_{n\in\mathbb{N}} \mathcal{L}_n(u).$$

The *(factor) complexity* $\mathcal{C}_u$ of an infinite word u is the function $\mathcal{C}_u : \mathbb{N} \to \mathbb{N}$ defined as

$$\mathcal{C}_u(n) := \#\mathcal{L}_n(u).$$

Clearly, $\mathcal{C}_u(n)$ is a non-decreasing function. Recall that if there exists $n_0 \in \mathbb{N}$ such that $\mathcal{C}_u(n_0) \leq n_0$, then the word u is eventually periodic (if $u = (u_n)_{n\in\mathbb{N}}$), or periodic (if $u = (u_n)_{n\in\mathbb{Z}}$), see [22]. Hence for an aperiodic[2] word u, one has $\mathcal{C}_u(n) \geq n + 1$, for all $n \in \mathbb{N}$.

A one-sided Sturmian word $(u_n)_{n\in\mathbb{N}}$ is often defined as an aperiodic word with complexity $C_u(n) = n + 1$, for all $n \in \mathbb{N}$. However, for biinfinite words, the condition $\mathcal{C}_u(n) \geq n + 1$ is not enough for u to be aperiodic. For example, the word $\cdots 111|000 \cdots$ has the complexity $\mathcal{C}(n) = n + 1$ for all $n \in \mathbb{N}$. In order to define a biinfinite Sturmian word

[2] A one-sided word $(u_n)_{n\in\mathbb{N}}$ is called aperiodic if it is not eventually periodic. A biinfinite word $(u_n)_{n\in\mathbb{Z}}$ is called aperiodic if neither $u_0u_1u_2\cdots$ nor $\cdots u_{-3}u_{-2}u_{-1}$ is eventually periodic.

$(u_n)_{n\in\mathbb{Z}}$ by means of complexity, we need to add another condition. We introduce the notion of the density of letters, representing the frequency of occurrence of a given letter in an infinite word.

The *density of a letter* $a \in \mathcal{A}$ in a word $u \in \mathcal{A}^{\mathbb{Z}}$ is defined as

$$\rho(a) := \lim_{n\to\infty} \frac{\#\{i \mid -n \le i \le n,\ u_i = a\}}{2n+1},$$

if the limit exists.

A biinfinite word $u = (u_n)_{n\in\mathbb{Z}}$ is called Sturmian, if $\mathcal{C}_u(n) = n+1$ for each $n \in \mathbb{N}$ and the densities of letters are irrational.

Another equivalent definition of Sturmian words uses the balance property. We say that an infinite word u over the alphabet $\{0,1\}$ is *balanced*, if for every pair of factors $v, w \in \mathcal{L}_n(u)$ we have $\big||v|_0 - |w|_0\big| \le 1$. A one-sided infinite word over the alphabet $\{0,1\}$ is Sturmian, if and only if it is aperiodic and balanced [23]. A biinfinite word over $\{0,1\}$ is Sturmian, if and only if it is balanced and has irrational densities of letters. For other properties of one-sided and two-sided infinite Sturmian words the reader is referred to [19,24].

Unlike the metric space $\mathcal{A}^{\mathbb{Z}}$, the set of all Sturmian words equipped with the same metric (3) is not compact, however we have the following result.

Lemma 2. *Let $u \in \{0,1\}^{\mathbb{Z}}$ be a limit of a sequence of Sturmian words $u^{(m)}$. Then u is either Sturmian or the densities of letters in u are rational.*

Proof. Let $w, \widehat{w} \in \mathcal{L}(u)$ be factors of the same length in u. Since $u = \lim_{m\to\infty} u^{(m)}$ there exists $m_0 \in \mathbb{N}$ such that $w, \widehat{w}$ are factors of $u^{(m_0)}$, which is Sturmian. Therefore $\big||w|_0 - |\widehat{w}|_0\big| \le 1$ and u is balanced. If, moreover, the densities are irrational, then u is Sturmian. The statement follows. □

2.2. Morphisms and incidence matrices

A mapping $\varphi : \mathcal{A}^* \to \mathcal{A}^*$ is said to be a *morphism* over $\mathcal{A}$ if $\varphi(w\widehat{w}) = \varphi(w)\varphi(\widehat{w})$ holds for any pair of finite words $w, \widehat{w} \in \mathcal{A}^*$. Obviously, a morphism is uniquely determined by the images $\varphi(a)$ for all letters $a \in \mathcal{A}$. A morphism φ is called non-erasing if $\varphi(a)$ is non-empty for all $a \in \mathcal{A}$.

The action of a non-erasing[3] morphism φ can be naturally extended to biinfinite words by the prescription

$$\varphi(u) = \varphi(\cdots u_{-2}u_{-1}|\, u_0u_1\cdots) := \cdots \varphi(u_{-2})\varphi(u_{-1})|\, \varphi(u_0)\varphi(u_1)\cdots.$$

The mapping $\varphi : u \mapsto \varphi(u)$ is continuous on $\mathcal{A}^{\mathbb{Z}}$; a word $u \in \mathcal{A}^{\mathbb{Z}}$ is said to be a *fixed point* of φ if $\varphi(u) = u$.

Recall that the incidence matrix of a morphism φ over the alphabet $\mathcal{A}$ is defined by (2). A morphism φ is called primitive if there exist an integer k such that the matrix $\boldsymbol{M}_\varphi^k$ is positive.

Morphisms over $\mathcal{A}$ form a monoid, whose neutral element is the identity morphism. Let φ and ψ be morphisms over $\mathcal{A}$, then the matrix of their composition, that is, of the morphism $u \mapsto (\varphi\circ\psi)(u) = \varphi\big(\psi(u)\big)$ is obtained by

$$\boldsymbol{M}_{\varphi\circ\psi} = \boldsymbol{M}_\psi \boldsymbol{M}_\varphi. \tag{4}$$

Let us now explain the importance of the incidence matrix of a morphism φ for the properties of infinite words on which the morphism φ acts. For details we refer to [25]. Assume that an infinite word u over the alphabet $\mathcal{A} = \{a_1, \dots, a_k\}$ has well-defined densities of letters, given by the vector

$$\vec{\rho}_u = \big(\rho(a_1), \dots, \rho(a_k)\big).$$

It is easy to see that the densities of letters in the infinite word $\varphi(u)$ are also well defined and it holds that

$$\vec{\rho}_{\varphi(u)} = \frac{\vec{\rho}_u \boldsymbol{M}_\varphi}{\vec{\rho}_u \boldsymbol{M}_\varphi \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}}, \tag{5}$$

where $\boldsymbol{M}_\varphi$ is the incidence matrix of φ.

[3] In general, a morphism can map an infinite word to a finite word. However, we will see in Corollary 18 that any 3-iet preserving morphism is non-erasing.

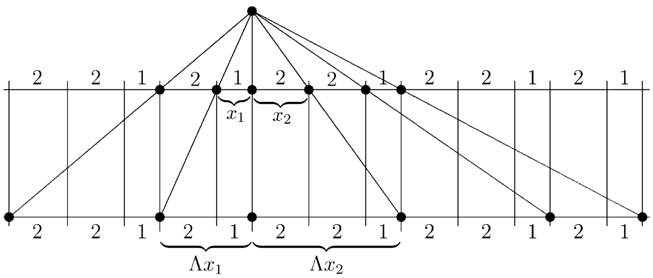

Fig. 1. Action of the morphism $1 \mapsto 21$, $2 \mapsto 221$ on the geometric representation of its fixed point $u = \lim_{n\to\infty} \varphi^n(1)|\varphi^n(2)$.

Now assume that the infinite word u is a fixed point of a morphism φ. Then from (5), we obtain that the vector of densities $\vec{\rho}_u$ is a left eigenvector of the incidence $\boldsymbol{M}_\varphi$, i.e., $\vec{\rho}_u \boldsymbol{M}_\varphi = \Lambda \vec{\rho}_u$. Since $\boldsymbol{M}_\varphi$ is a non-negative integral matrix, we can use the Perron–Frobenius Theorem stating that Λ is the dominant eigenvalue of $\boldsymbol{M}_\varphi$. Moreover, all eigenvalues of $\boldsymbol{M}_\varphi$ are algebraic integers.

The right eigenvector of the incidence matrix corresponding to the dominant eigenvalue has also a nice interpretation. It plays an important role for the *geometric representation* of a fixed point of a morphism. Let u be a fixed point of a morphism φ over a k-letter alphabet $\{a_1, \ldots, a_k\}$ and let $\boldsymbol{M}_\varphi$ have a positive right eigenvector $\vec{x}$. The infinite word u can be geometrically represented by a self-similar set Σ as follows.

Let us denote by $x_1, x_2, \ldots, x_k$ the positive components of $\vec{x}$, and let Λ be the corresponding eigenvalue, i.e., $\boldsymbol{M}_\varphi \vec{x} = \Lambda \vec{x}$. Since $\boldsymbol{M}_\varphi$ is non-negative and $\vec{x}$ is positive, the eigenvalue Λ is equal to the spectral radius of the matrix $\boldsymbol{M}_\varphi$. Moreover, $\boldsymbol{M}_\varphi$ being an integral matrix implies $\Lambda \geq 1$.

For a biinfinite word $u = \cdots u_{-3}u_{-2}u_{-1}|u_0u_1u_2\cdots$ we denote

$$\Sigma = \left\{ \sum_{i=1}^{k} |w|_{a_i} x_i \;\middle|\; w \text{ is an arbitrary prefix of } u_0u_1u_2\cdots \right\}$$
$$\cup \left\{ -\sum_{i=1}^{k} |w|_{a_i} x_i \;\middle|\; w \text{ is an arbitrary suffix of } \cdots u_{-3}u_{-2}u_{-1} \right\}.$$

The set Σ can be equivalently defined as

$$\Sigma = \{t_n \mid n \in \mathbb{Z}\}, \quad \text{where} \quad t_0 = 0 \text{ and } t_{n+1} - t_n = x_i \Leftrightarrow u_n = a_i.$$

Since u is a fixed point of a morphism, the construction of Σ implies that $\Lambda\Sigma \subset \Sigma$. A set having this property is called self-similar.

Moreover, if $u_n = a_i$ then the number of points of the set Σ belonging to $(\Lambda t_n, \Lambda t_{n+1}]$ is equal to the length of $\varphi(a_i)$. Formally, we have

$$\#\big((\Lambda t_n, \Lambda t_{n+1}] \cap \Sigma\big) = |\varphi(a_i)|. \tag{6}$$

In Fig. 1, one can see the geometric representation of the fixed point of the morphism $1 \mapsto 21$, $2 \mapsto 221$. The matrix of this morphism, $\boldsymbol{M} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, has the dominant eigenvalue $\Lambda = \tau^2$, where $\tau = \frac{1+\sqrt{5}}{2}$ is the golden ratio. The corresponding right eigenvector of M is $\begin{pmatrix} 1 \\ \tau \end{pmatrix}$. Hence the lengths assigned to letters 1 and 2 are $x_1 = 1$ and $x_2 = \tau$, respectively.

3. Interval exchange words

Before we define infinite words coding a 3-interval exchange transformation, we will show the definition of Sturmian words using a 2-interval exchange transformation. It is well known (see e.g. [23,19]) that every Sturmian word $u = (u_n)_{n\in\mathbb{Z}}$ over the alphabet $\{0, 1\}$ satisfies

$$u_n = \lfloor (n+1)\alpha + x_0 \rfloor - \lfloor n\alpha + x_0 \rfloor \quad \text{for all } n \in \mathbb{Z}, \tag{7}$$

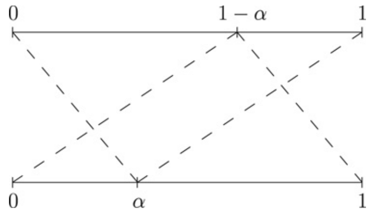

Fig. 2. Graph of a 2-interval exchange transformation.

or

$$u_n = \lceil (n+1)\alpha + x_0 \rceil - \lceil n\alpha + x_0 \rceil \quad \text{for all } n \in \mathbb{Z}, \tag{8}$$

where $\alpha \in (0, 1)$ is an irrational number called the slope, and $x_0 \in [0, 1)$ is called the intercept of u. In the former case, $(u_n)_{n\in\mathbb{Z}}$ is the so-called upper mechanical word, in the latter case the lower mechanical word, with slope α and intercept x_0.

If $(u_n)_{n\in\mathbb{Z}}$ is of the form (7) then, obviously,

$$u_n = \begin{cases} 0 & \text{if } \{n\alpha + x_0\} \in [0, 1-\alpha), \\ 1 & \text{if } \{n\alpha + x_0\} \in [1-\alpha, 1), \end{cases} \tag{9}$$

where $\{x\}$ denotes the fractional part of x, i.e., $\{x\} = x - \lfloor x \rfloor$. We can define a transformation $T : [0, 1) \to [0, 1)$ by the prescription

$$T(x) = \begin{cases} x + \alpha & \text{if } \{n\alpha + x_0\} \in [0, 1-\alpha) =: I_0, \\ x + \alpha - 1 & \text{if } \{n\alpha + x_0\} \in [1-\alpha, 1) =: I_1, \end{cases} \tag{10}$$

which satisfies $T(x) = \{x + \alpha\}$. It follows easily that the nth iteration of T is given as

$$T^n(x) = \{x + n\alpha\} \quad \text{for all } n \in \mathbb{Z}. \tag{11}$$

Putting (9) and (11) together, we see that a Sturmian word $(u_n)_{n\in\mathbb{Z}}$ can be defined using the transformation T by

$$u_n = \begin{cases} 0 & \text{if } T^n(x_0) \in I_0, \\ 1 & \text{if } T^n(x_0) \in I_1. \end{cases}$$

Hence a Sturmian word is given by iterations of the intercept x_0 under the mapping T, that is, by the orbit of x_0 under T.

The action of the mapping T from (10) is illustrated on Fig. 2.

We see that T is in fact an exchange of two intervals $I_0 = [0, 1-\alpha)$ and $I_1 = [1-\alpha, 1)$. It is therefore called a 2-interval exchange transformation.

Let us mention that if $(u_n)_{n\in\mathbb{Z}}$ is an upper mechanical word, the corresponding 2-interval exchange transformation is given by $T : (0, 1] \mapsto (0, 1]$, with $I_0 = (0, 1-\alpha]$ and $I_1 = (1-\alpha, 1]$. Note also that it was not necessary that T was acting on a unit interval. We could choose an arbitrary interval divided into two parts, ratio of whose lengths would be irrational.

Analogously to the case of exchange of two intervals, we can define a 3-interval exchange transformation.

Definition 3. Let α, β, γ be three positive real numbers. Denote

$$\begin{array}{lcl} I_A := [0, \alpha) & & I_A := (0, \alpha] \\ I_B := [\alpha, \alpha+\beta) & \text{or} & I_B := (\alpha, \alpha+\beta] \\ I_C := [\alpha+\beta, \alpha+\beta+\gamma) & & I_C := (\alpha+\beta, \alpha+\beta+\gamma] \end{array}$$

respectively, and $I := I_A \cup I_B \cup I_C$. A mapping $T : I \to I$, given by

$$T(x) = \begin{cases} x + \beta + \gamma & \text{if } x \in I_A, \\ x - \alpha + \gamma & \text{if } x \in I_B, \\ x - \alpha - \beta & \text{if } x \in I_C, \end{cases} \tag{12}$$

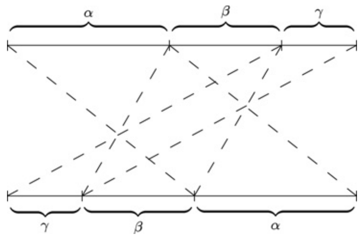

Fig. 3. Graph of a 3-interval exchange transformation.

is called a 3-*interval exchange transformation* (3-iet)[4] with parameters α, β, γ.

The graph of a 3-interval exchange is given in Fig. 3.

With a 3-interval exchange transformation T, one can naturally associate a ternary biinfinite word $u_T(x_0) = (u_n)_{n\in\mathbb{Z}}$, which codes the orbit of a point x_0 from the domain of T, as

$$u_n = \begin{cases} A & \text{if } T^n(x_0) \in I_A, \\ B & \text{if } T^n(x_0) \in I_B, \\ C & \text{if } T^n(x_0) \in I_C. \end{cases} \tag{13}$$

Similarly as in the case of a 2-interval exchange transformation, the infinite word coding a 3-iet can be periodic or aperiodic, according to the choice of parameters α, β, γ. We will focus only on aperiodic words.

Definition 4. An aperiodic word $u_T(x_0)$ coding the orbit of the point x_0 under the 3-iet T defined above is called a *3-iet word* with parameters α, β, γ and x_0.

The following lemma shows a close relation between words coding 3-interval exchange and 2-interval exchange transformations.

Lemma 5. *Let $u = (u_n)_{n\in\mathbb{Z}}$ be a word coding a* 3*-interval exchange transformation and let $\sigma : \{A, B, C\}^* \to \{0, 1\}^*$ be a morphism given by*

$$A \mapsto 0, \qquad B \mapsto 01, \qquad C \mapsto 1. \tag{14}$$

Then $\sigma(u)$ codes a 2-interval exchange transformation.

Proof. Let u be the coding of x_0 under the 3-interval exchange transformation T with intervals $[0, \alpha)$, $[\alpha, \alpha + \beta)$ and $[\alpha + \beta, \alpha + \beta + \gamma)$.

Let S be the 2-interval exchange transformation of the intervals $I_0 = [0, \alpha + \beta)$ and $I_1 = [\alpha + \beta, \alpha + 2\beta + \gamma)$, i.e.,

$$S(x) = \begin{cases} x + \beta + \gamma & \text{if } x \in I_0, \\ x - \alpha - \beta & \text{if } x \in I_1. \end{cases}$$

One can easily see that

$$\begin{aligned} &x \in [0, \alpha) && \Rightarrow && x \in I_0 \text{ and } T(x) = S(x), \\ &x \in [\alpha, \alpha + \beta) && \Rightarrow && x \in I_0,\ S(x) \in I_1 \text{ and } S^2(x) = T(x), \\ &x \in [\alpha + \beta, \alpha + \beta + \gamma) && \Rightarrow && x \in I_1, \text{ and } S(x) = T(x). \end{aligned}$$

This proves that $\sigma(u)$ is the coding of x_0 under S. □

[4] Note that the above defined mapping T should be more precisely called 3-interval exchange with the permutation (321), since the initial arrangement of intervals $I_A < I_B < I_C$ is changed to $T(C) < T(B) < T(A)$. Indeed, one can define also 3-iet with a different permutation of intervals, e.g. (312). We will not consider such transformations.

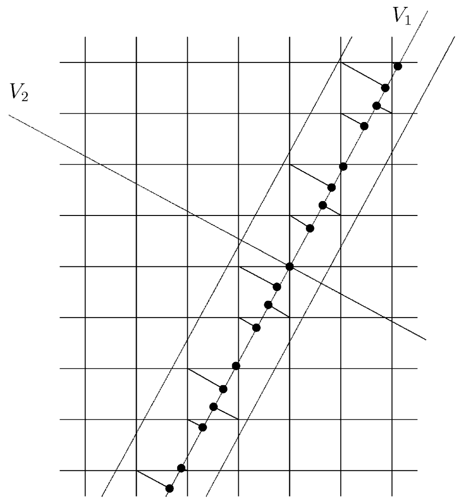

Fig. 4. Construction of a cut-and-project set.

4. Periodic and aperiodic words coding 3-iet

In order to clarify the relation between the parameters of a 3-iet and the complexity of the corresponding infinite words, we recast the definition of these words in a new formalism. We show that every 3-iet word codes distances in a discrete set arising as a projection of points of the lattice $\mathbb{Z}^2$. This construction is known as the cut-and-project method [21].

Let ε, η be real numbers, $\varepsilon \neq -\eta$. Every point $(a, b) \in \mathbb{Z}^2$ can be written in the form

$$(a, b) = (a + b\eta)\vec{x}_1 + (a - b\varepsilon)\vec{x}_2,$$

where

$$\vec{x}_1 = \frac{1}{\varepsilon + \eta}(\varepsilon, 1) \quad \text{and} \quad \vec{x}_2 = \frac{1}{\varepsilon + \eta}(\eta, -1).$$

Let V_1 and V_2 denote the lines in $\mathbb{R}^2$ spanned by $\vec{x}_1$ and $\vec{x}_2$, respectively. Then $(a + b\eta)\vec{x}_1$ is the projection of the lattice point (a, b) on V_1, whereas $(a - b\varepsilon)\vec{x}_2$ is its projection on V_2. Let Ω be a bounded interval. Then the set

$$\Sigma_{\varepsilon,\eta}(\Omega) := \{a + b\eta \mid a, b \in \mathbb{Z}, a - b\varepsilon \in \Omega\} \tag{15}$$

is called the *Cut-and-project (C&P) set* with parameters $\varepsilon, \eta, \Omega$. Thus C&P sets arise by projection on the line V_1 of points of $\mathbb{Z}^2$ having their second projection in a chosen segment on V_2, see Fig. 4.

Proposition 6. *Let α, β, γ be positive real numbers, and let $T : [0, \alpha + \beta + \gamma) \mapsto [0, \alpha + \beta + \gamma)$ be a 3-iet defined by (12). Let $x_0 \in [0, \alpha + \beta + \gamma)$ and let $u_T(x_0) = (u_n)_{n\in\mathbb{Z}}$ be the biinfinite word given by (13). Put*

$$\varepsilon := \frac{\beta + \gamma}{\alpha + 2\beta + \gamma}, \quad l := \frac{\alpha + \beta + \gamma}{\alpha + 2\beta + \gamma}, \quad c := \frac{x_0}{\alpha + 2\beta + \gamma} \quad \text{and} \quad \Omega = (c - l, c], \tag{16}$$

and choose arbitrary $\eta > 0$. Then the C&P set $\Sigma_{\varepsilon,\eta}(\Omega)$ is a discrete set with the following properties:

(1) $0 \in \Sigma_{\varepsilon,\eta}(\Omega)$;
(2) *the distances between adjacent elements of $\Sigma_{\varepsilon,\eta}(\Omega)$ take values $\mu_A = \eta$, $\mu_B = 1 + 2\eta$, and $\mu_C = 1 + \eta$;*
(3) *the ordering of the distances with respect to the origin is coded by the word $u_T(x_0)$;*
(4) $\Sigma_{\varepsilon,\eta}(\Omega) = \big\{\lfloor c + n\varepsilon\rfloor + n\eta \mid n \in \mathbb{Z},\ \{c + n\varepsilon\} \in [0, l)\big\}$.

Proof. The parameters ε, l, and c satisfy clearly

$$\varepsilon \in (0, 1), \quad \max\{\varepsilon, 1 - \varepsilon\} < l < 1, \quad 0 \in (c - l, c]. \tag{17}$$

The condition in (15) determining whether a given point $a + b\eta$ belongs to the C&P set $\Sigma_{\varepsilon,\eta}(\Omega)$ can be rewritten

$$a - b\varepsilon \in \Omega \quad \Leftrightarrow \quad c + b\varepsilon - l < a \le c + b\varepsilon \quad \Leftrightarrow \quad a = \lfloor c + b\varepsilon \rfloor \text{ and } \{c + b\varepsilon\} \in [0, l).$$

Therefore, the C&P set $\Sigma_{\varepsilon,\eta}(\Omega)$ can be expressed as

$$\Sigma_{\varepsilon,\eta}(\Omega) = \big\{\lfloor c + n\varepsilon \rfloor + n\eta \mid n \in \mathbb{Z},\ \{c + n\varepsilon\} \in [0, l)\big\}. \tag{18}$$

Let us denote $y_n := \lfloor c + n\varepsilon \rfloor + n\eta$ and $y_n^* := \{c + n\varepsilon\}$. From the choice of the parameters ε and η we can derive that the sequence $(y_n)_{n\in\mathbb{Z}}$ is strictly increasing. Since $\Sigma_{\varepsilon,\eta}(\Omega) \subset \{y_n \mid n \in \mathbb{Z}\}$, to every element $y \in \Sigma_{\varepsilon,\eta}(\Omega)$ corresponds a point $y^* \in [0, l)$. We show that the distance of y and its right neighbour depends on the position of y^* in the interval $[0, l)$. Moreover, if z is the right neighbour of y in $\Sigma_{\varepsilon,\eta}(\Omega)$, then $z^* = \widetilde{T}(y^*)$, where $\widetilde{T} : [0, l) \to [0, l)$ is a 3-iet given by the prescription

$$\widetilde{T}(x) = \begin{cases} x + \varepsilon & \text{if } x \in [0, l - \varepsilon), \\ x + 2\varepsilon - 1 & \text{if } x \in [l - \varepsilon, 1 - \varepsilon), \\ x + \varepsilon - 1 & \text{if } x \in [1 - \varepsilon, l). \end{cases} \tag{19}$$

Let us determine the right neighbour of a point $y \in \Sigma_{\varepsilon,\eta}(c - l, c]$. Let $y = y_n$, $n \in \mathbb{Z}$, i.e., $y_n^* = \{c + n\varepsilon\} \in [0, l)$. We discuss three separate cases, all the time using the fact that $\max\{\varepsilon, 1 - \varepsilon\} < l \le 1$.

(i) If $y_n^* \in [0, l - \varepsilon)$ then $y_{n+1}^* = \{c + (n + 1)\varepsilon\} = y_n^* + \varepsilon \in [0, l)$ and $\lfloor c + n\varepsilon \rfloor = \lfloor c + (n + 1)\varepsilon \rfloor$. Hence the distance between y_n and its right neighbour is $y_{n+1} - y_n = \eta$.
(ii) If $y_n^* \in [l - \varepsilon, 1 - \varepsilon)$ then $y_{n+1}^* = \{c + (n + 1)\varepsilon\} = y_n^* + \varepsilon \in [l, 1)$, hence y_{n+1} does not belong to the set $\Sigma_{\varepsilon,\eta}(c - l, c]$. However, $y_{n+2}^* = \{c + (n + 2)\varepsilon\} = y_n^* + 2\varepsilon - 1 \in [0, l)$ and $\lfloor c + (n + 2)\varepsilon \rfloor = 1 + \lfloor c + n\varepsilon \rfloor$. Therefore the right neighbour of y_n is y_{n+2} and we have $y_{n+2} - y_n = 1 + 2\eta$.
(iii) If $y_n^* \in [1 - \varepsilon, l)$ then $y_{n+1}^* = \{c + (n + 1)\varepsilon\} = y_n^* + \varepsilon - 1 \in [0, l)$, y_{n+1} is the right neighbour of y_n and $y_{n+1} - y_n = 1 + \eta$.

As $y_0 = 0 \in \Sigma_{\varepsilon,\eta}(c - l, c]$ and $y_0^* = \{c\} = c$, the distances between consecutive elements of the C&P set $\Sigma_{\varepsilon,\eta}(c - l, c]$ are coded by the infinite word $u_{\widetilde{T}}(c)$. It is easy to see that with our choice of l, ε, and c, the lengths of the partial intervals in the definition of the 3-iet $\widetilde{T}$ and the starting point c are only $(\alpha+2\beta+\gamma)$-multiples of the partial intervals of the 3-iet T and its starting point x_0, ($\widetilde{T}$ and T are homothetic 3-iets). Therefore $u_{\widetilde{T}}(c) = u_T(x_0)$. □

Let us mention that a 3-iet T with the domain $(0, \alpha + \beta + \gamma]$ corresponds also to a C&P set with parameters similar to (16).

It is known that a word coding an r-interval exchange transformation with arbitrary permutation of intervals has complexity $\mathcal{C}(n) \le (r - 1)n + 1$ for all $n \in \mathbb{N}$, see [17]. It is useful to distinguish the words with full complexity and the others.

Definition 7. A 3-iet word is called *non-degenerate*, if $\mathcal{C}(n) = 2n + 1$ for all $n \in \mathbb{N}$. Otherwise it is called *degenerate*.

The following proposition allows one to classify the words coding 3-iet according to the parameters into periodic, 3-iet degenerate, and 3-iet non-degenerate infinite words.

Proposition 8. *Let T be a 3-iet transformation of the interval I with parameters α, β, γ, and let $x_0 \in I$.*

- *The infinite word $u_T(x_0)$ defined by (13) is aperiodic if and only if*

$$\alpha + \beta \text{ and } \beta + \gamma \text{ are linearly independent over } \mathbb{Q}.$$

- *If the word $u_T(x_0)$ is aperiodic then it is degenerate if and only if*

$$\alpha + \beta + \gamma \in (\alpha + \beta)\mathbb{Z} + (\beta + \gamma)\mathbb{Z}.$$

Proof. Item 4 in Proposition 6 for the C&P set $\Sigma_{\varepsilon,\eta}(c - l, c]$ implies easily that if ε is rational, then the set $\Sigma_{\varepsilon,\eta}(\Omega)$ is periodic, i.e., the orbit of every point under the 3-iet $\widetilde{T}$ is periodic. On the other hand, if ε is irrational, the sequence $\{c + n\varepsilon\}$ is uniformly distributed, and thus also the orbit of every point under $\widetilde{T}$ is dense in $[0, l)$. The relation (16) between the parameters ε and α, β, γ implies the statement about periodicity of $u_T(x_0)$.

The complexity of an infinite word coding a C&P set with irrational parameters ε, η has been described in [13]. It is shown that such a word has the complexity $\mathcal{C}(n) = 2n + 1$ for all n if and only if the length l of the interval Ω from (15) satisfies $l \notin \mathbb{Z} + \mathbb{Z}\varepsilon$. The relation (16) implies the necessary and sufficient conditions for the degeneracy of the corresponding infinite word. $\square$

We will use the following reformulation of the above statements.

Corollary 9. *The infinite word $u_T(x_0)$, defined by (13), with parameters $\alpha, \beta, \gamma > 0$ is*

- *periodic if there exist $K, L \in \mathbb{Z}$, $K, L \neq 0$ such that*

$$(\alpha, \beta, \gamma) \begin{pmatrix} K \\ K + L \\ L \end{pmatrix} = 0, \tag{20}$$

- *aperiodic degenerate if there exist unique $K, L \in \mathbb{Z}$ such that*

$$(\alpha, \beta, \gamma) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = (\alpha, \beta, \gamma) \begin{pmatrix} K \\ K + L \\ L \end{pmatrix}. \tag{21}$$

Note that the sequence $\{c + n\varepsilon\}$ being uniformly distributed for ε irrational implies not only the aperiodicity of the infinite word, but also that the densities of letters are well defined.

Corollary 10. *All letters in a 3-iet word u with parameters α, β, γ have a well-defined density and the vector of densities of u, denoted by $\vec{\rho}_u := \big(\rho(A), \rho(B), \rho(C)\big)$, is proportional to the vector (α, β, γ).*

For the transformation T of exchange of r intervals, it is generally difficult to describe the conditions under which the corresponding dynamical system is minimal, i.e., under which condition the orbit $\{T^n(x_0) \mid n \in \mathbb{Z}\}$ of any point x_0 is dense in the domain of T. Keane provides in [17] two sufficient conditions for the minimality of T: one of them is the linear independence of parameters α, β and γ over $\mathbb{Q}$; second, weaker condition is that the orbits of all discontinuity points of T are disjoint. This condition is called i.d.o.c. In [11] it is shown that the parameters α, β, γ fulfill i.d.o.c. if and only if they satisfy neither (20) nor (21). Nevertheless, even the weaker condition i.d.o.c. is only sufficient, but not necessary for the minimality of the dynamical system of T. The geometric representation of 3-iet T using a cut-and-project set allows us to provide a simple characterization of minimal dynamical systems among 3-iet.

Corollary 11. *The dynamical system given by a 3-interval exchange transformation T with parameters α, β, γ is minimal if and only if the numbers $\alpha + \beta$ and $\beta + \gamma$ are linearly independent over $\mathbb{Q}$.*

Remark 12. It can be shown, (see [3,11,13]), that a 3-iet word is degenerate if and only if the orbits of the two discontinuity points of the corresponding 3-iet T have a non-empty intersection, formally, $\{T^n(\alpha) \mid n \in \mathbb{Z}\} \cap \{T^n(\alpha + \beta) \mid n \in \mathbb{Z}\} \neq \emptyset$. The complexity of a degenerate 3-iet word is $\mathcal{C}(n) = n + \text{const}$ for sufficiently large n. Cassaigne [9] calls one-sided infinite words with such complexity quasi-Sturmian words. By a slight modification of his proof one can show that for any biinfinite word u with complexity $\mathcal{C}_u(n) \leq n + \text{const}$ there exists a Sturmian word $(v_n)_{n \in \mathbb{Z}}$ over $\{0, 1\}$ and finite words $w_1, w_2 \in \{A, B, C\}^*$ such that

$$u = \cdots w_{v_{-2}} w_{v_{-1}} \mid w_{v_0} w_{v_1} w_{v_2} \cdots,$$

that is, u is obtained from v by applying the morphism $0 \mapsto w_0$ and $1 \mapsto w_1$.

5. Morphisms preserving 3-iet words

Definition 13. A morphism on the alphabet $\{A, B, C\}$ is said to be *3-iet preserving* if $\varphi(u)$ is a 3-iet word for every 3-iet word u.

Let us recall that 3-iet words are defined as those words coding 3-interval exchange transformations, which are aperiodic. Similarly, Sturmian words are aperiodic words coding 2-interval exchange transformations.

In the rest of this section we give several useful examples of 3-iet preserving morphisms.

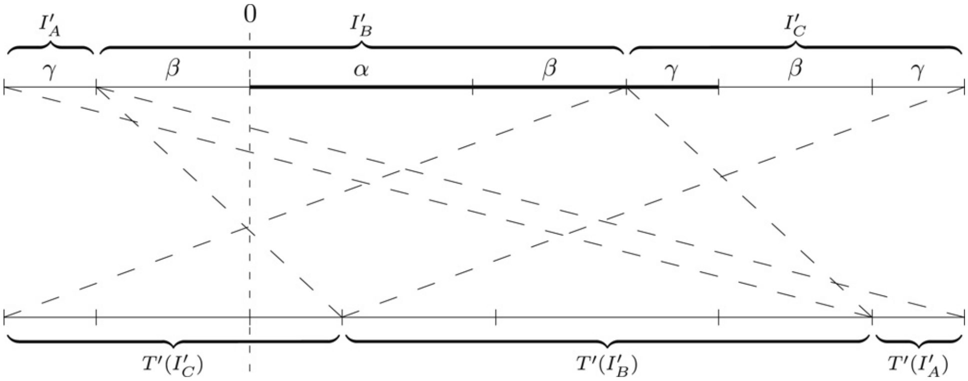

Fig. 5. Action of the transformation T'.

Example 14. We will prove that the morphism ψ over $\{A, B, C\}$ given by prescriptions

$$A \mapsto B, \qquad B \mapsto BCB, \qquad C \mapsto CAC, \tag{22}$$

is 3-iet preserving. Let us consider an arbitrary 3-iet word u with arbitrary parameters α, β, γ and x_0. The corresponding transformation T is given by (12).

Let us define the transformation T' (see Fig. 5) by

$$T'(x) = \begin{cases} x + \alpha + 3\beta + 2\gamma & \text{if } x \in [-\beta - \gamma, -\beta) =: I'_A, \\ x + \beta + \gamma & \text{if } x \in [-\beta, \alpha + \beta) =: I'_B, \\ x - \alpha - 2\beta - \gamma & \text{if } x \in [\alpha + \beta, \alpha + 2\beta + 2\gamma) =: I'_C. \end{cases} \tag{23}$$

Indeed, T' is a 3-iet with parameters $\alpha' = \gamma$, $\beta' = \beta + \alpha + \beta$, and $\gamma' = \gamma + \beta + \gamma$, up to shift of the domain. We show that the infinite word $\psi(u)$ is a coding of the point x_0 under the transformation $T' : [-\gamma - \beta, \alpha + 2\beta + 2\gamma] \to [-\gamma - \beta, \alpha + 2\beta + 2\gamma]$. This will be sufficient for claiming that ψ is 3-iet preserving.

Obviously, (12) and (23) imply for a point $x \in I_A \subsetneqq I'_B$ that

$$T'(x) = x + \beta + \gamma = T(x).$$

Hence any point $x \in I_A$ belongs in the new 3-iet to the interval I'_B. The transformation T' sends such a point to the same place as the original transformation T. Therefore we substitute $A \mapsto B$. Similarly, for a point $x \in I_B \subsetneqq I'_B$ we have

$$\begin{aligned} T'(x) &= x + \beta + \gamma \in I'_C, \\ (T')^2(x) &= x - \alpha - \beta \in I'_B, \\ (T')^3(x) &= x - \alpha + \gamma = T(x). \end{aligned}$$

Such point $x \in I_B$ belongs to the interval I'_B. Its first iteration is $T'(x) \in I'_C$, the second iteration is $(T')^2(x) \in I'_B$ and the third iteration $(T')^3$ sends x to the same place as the first iteration of the original transformation T. Therefore we substitute $B \mapsto BCB$. Finally, for $x \in I_C \subsetneqq I'_C$ we get

$$\begin{aligned} T'(x) &= x - \alpha - 2\beta - \gamma \in I'_A, \\ (T')^2(x) &= x + \beta + \gamma \in I'_C, \\ (T')^3(x) &= x - \alpha - \beta = T(x), \end{aligned}$$

and we substitute $C \mapsto CAC$. Thus we see that the 3-iet word coding x_0 under T' coincides with the word $\psi(u)$. Note that ψ is a primitive morphism such that $\det \boldsymbol{M}_\psi = 1$ and $\boldsymbol{M}_\psi^3 > 0$.

Example 15. It is easy to see that the morphism ξ over $\{A, B, C\}$ given by prescriptions

$$A \mapsto C, \qquad B \mapsto B, \qquad C \mapsto A, \tag{24}$$

is a 3-iet preserving morphism. To a 3-iet word, which codes the orbit of x_0 under the transformation T with intervals $[0, \alpha) \cup [\alpha, \alpha+\beta) \cup [\alpha+\beta, \alpha+\beta+\gamma)$, it assigns a 3-iet word, which codes the orbit of $\alpha+\beta+\gamma-x_0$ under the transformation $\tilde{T}$ with intervals $(0, \gamma] \cup (\gamma, \gamma+\beta] \cup (\gamma+\beta, \gamma+\beta+\alpha]$.

Example 16. Let us consider the morphism φ on $\{A, B, C\}$ given by $A \mapsto AC$, $B \mapsto BC$ and $C \mapsto C$. Let u be an arbitrary 3-iet word with parameters α, β, γ. Using the same technique as in Example 14 one can show that φ is 3-iet preserving; the 3-iet word coinciding with $\varphi(u)$ has parameters $\alpha' = \alpha$, $\beta' = \beta$, $\gamma' = \alpha+\beta+\gamma$, and $x_0' = x_0$. Contrary to ψ from Example 14, the morphism φ is non-primitive.

6. Proof of Theorem A

The aim of this section is to prove that the matrix $\boldsymbol{M}$ of a 3-iet preserving morphism fulfills the following condition

$$\boldsymbol{M}\boldsymbol{E}\boldsymbol{M}^T = \pm\boldsymbol{E}, \quad \text{where } \boldsymbol{E} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}. \tag{25}$$

The main tool used in the proof of this property of $\boldsymbol{M}$ is the fact that the matrix of a Sturmian morphism has determinant ± 1 and some auxiliary statements formulated as Lemmas 17 and 20.

Lemma 17. *Let φ be a 3-iet preserving morphism and $\boldsymbol{M}$ its incidence matrix. Let $\mathcal{P}$ be a subspace of $\mathbb{R}^3$ spanned by the vectors $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. Then $\boldsymbol{M}\mathcal{P} = \mathcal{P}$.*

Proof. If u is a 3-iet word with parameters (α, β, γ), then according to (5), $\varphi(u)$ is a 3-iet word with parameters $(\alpha, \beta, \gamma)\boldsymbol{M}$.

Let us first show that $\boldsymbol{M}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\boldsymbol{M}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ are linearly independent over $\mathbb{R}$. Suppose the opposite, i.e., that there exist K, L such that

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = K\boldsymbol{M}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + L\boldsymbol{M}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \boldsymbol{M}\begin{pmatrix} K \\ K+L \\ L \end{pmatrix}.$$

Since $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ are integer vectors, one can choose $K, L \in \mathbb{Z} \setminus \{0\}$.

This, however, implies that for arbitrary parameters (α, β, γ), we have

$$(\alpha, \beta, \gamma)\boldsymbol{M}\begin{pmatrix} K \\ K+L \\ L \end{pmatrix} = 0,$$

i.e., according to Corollary 9, the infinite word $\varphi(u)$ is periodic for arbitrary 3-iet word u, which is a contradiction with the assumption that φ is a 3-iet preserving morphism.

We now show that $\boldsymbol{M}\mathcal{P} \subseteq \mathcal{P}$. Together with the fact that $\boldsymbol{M}\mathcal{P}$ has dimension two, it will imply $\boldsymbol{M}\mathcal{P} = \mathcal{P}$, which is to be shown.

Since φ is a 3-iet preserving morphism, it means that $\varphi(u)$ is non-periodic, whenever u is non-periodic. With the help of Corollary 9, it implies that for every triple of positive numbers (α, β, γ), we have

$$\exists K, L \in \mathbb{Z} \setminus \{0\} \text{ such that } (\alpha, \beta, \gamma)\boldsymbol{M}\begin{pmatrix} K \\ K+L \\ L \end{pmatrix} = 0 \quad \Longrightarrow$$

$$\Longrightarrow \quad \exists H, S \in \mathbb{Z} \setminus \{0\} \text{ such that } (\alpha, \beta, \gamma)\begin{pmatrix} H \\ H+S \\ S \end{pmatrix} = 0. \tag{26}$$

Suppose there exists a pair $K_0, L_0 \in \mathbb{Z} \setminus \{0\}$ such that $\boldsymbol{M}\begin{pmatrix} K_0 \\ K_0+L_0 \\ L_0 \end{pmatrix}$ has both positive and negative components. Then one can find a triple $\alpha_0, \beta_0, \gamma_0 > 0$ such that $(\alpha_0, \beta_0, \gamma_0)\boldsymbol{M}\begin{pmatrix} K_0 \\ K_0+L_0 \\ L_0 \end{pmatrix} = 0$. The set of vectors in $\mathbb{R}^3$ orthogonal to $\boldsymbol{M}\begin{pmatrix} K_0 \\ K_0+L_0 \\ L_0 \end{pmatrix}$ is a plane in $\mathbb{R}^2$ with non-empty intersection with the first octant. Therefore there exist uncountably many triples (α, β, γ) such that $\alpha, \beta, \gamma > 0, \alpha+\beta+\gamma = 1$ and $(\alpha, \beta, \gamma)\boldsymbol{M}\begin{pmatrix} K_0 \\ K_0+L_0 \\ L_0 \end{pmatrix} = 0$. For every such triple there exists, due to (26), a pair $H, S \in \mathbb{Z} \setminus \{0\}$ satisfying

$$(\alpha, \beta, \gamma)\begin{pmatrix} H \\ H+S \\ S \end{pmatrix} = 0.$$

As there are only countably many pairs $H, S \in \mathbb{Z}$, there must be a pair $H_0, S_0 \in \mathbb{Z} \setminus \{0\}$ such that

$$(\alpha_i, \beta_i, \gamma_i)\begin{pmatrix} H_0 \\ H_0+S_0 \\ S_0 \end{pmatrix} = 0$$

for two distinct triples $(\alpha_i, \beta_i, \gamma_i)$ such that $\alpha_i, \beta_i, \gamma_i > 0$, $\alpha_i + \beta_i + \gamma_i = 1$, $i = 1, 2$. The vectors $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ are obviously linearly independent. Both the vector $\boldsymbol{M}\begin{pmatrix} K_0 \\ K_0+L_0 \\ L_0 \end{pmatrix}$ and the vector $\begin{pmatrix} H_0 \\ H_0+S_0 \\ S_0 \end{pmatrix}$ are orthogonal to them, which implies

$$\boldsymbol{M}\begin{pmatrix} K_0 \\ K_0+L_0 \\ L_0 \end{pmatrix} = \text{const.}\begin{pmatrix} H_0 \\ H_0+S_0 \\ S_0 \end{pmatrix}, \tag{27}$$

and thus $\boldsymbol{M}\begin{pmatrix} K_0 \\ K_0+L_0 \\ L_0 \end{pmatrix} \in \mathcal{P}$.

Now since $\dim \boldsymbol{M}\mathcal{P} = 2$, we have that $\left\{\boldsymbol{M}\begin{pmatrix} K \\ K+L \\ L \end{pmatrix} \;\middle|\; K, L \in \mathbb{Z}\right\}$ is a two-dimensional lattice. Therefore we can choose two linearly independent pairs K_1, L_1, and K_2, L_2 with the property of K_0, L_0, and thus the proof is completed. $\square$

It follows from Lemma 17 that the incidence matrix of a 3-iet preserving morphism cannot have a row full of zeroes. Therefore we have the following corollary.

Corollary 18. *A 3-iet preserving morphism is non-erasing.*

Remark 19. Denote

$$\vec{x}_1 := \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \vec{x}_2 := \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \qquad \vec{x}_3 := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

The triplet of vectors $\vec{x}_1, \vec{x}_2, \vec{x}_3$ forms a basis of $\mathbb{R}^3$. Denoting $\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, we have $\det \boldsymbol{P} = -1$, and thus $\vec{x}_1$, $\vec{x}_2, \vec{x}_3$ is also a basis of the integer lattice $\mathbb{Z}^3$. At the same time, the pair $\vec{x}_1, \vec{x}_2$ is a basis of the invariant subspace $\mathcal{P}$ of the matrix $\boldsymbol{M}$. We have

$$\boldsymbol{P}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & -1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{P}^{-1}\boldsymbol{M}\boldsymbol{P} = \begin{pmatrix} m_{11}+m_{12} & m_{12}+m_{13} & m_{12} \\ m_{31}+m_{32} & m_{32}+m_{33} & m_{32} \\ 0 & 0 & -m_{12}+m_{22}-m_{32} \end{pmatrix},$$

where the 0's in the third row correspond to the fact that $\boldsymbol{P}^{-1}\boldsymbol{M}\boldsymbol{P}$ can be seen as the matrix $\boldsymbol{M}$ written in the basis $\vec{x}_1$, $\vec{x}_2, \vec{x}_3$, where the first two vectors form a basis of the invariant subspace $\mathcal{P}$. Since $\boldsymbol{M}\mathcal{P} = \mathcal{P}$, we have

$$\det \begin{pmatrix} m_{11}+m_{12} & m_{12}+m_{13} \\ m_{31}+m_{32} & m_{32}+m_{33} \end{pmatrix} \neq 0.$$

Lemma 20. *Let $\boldsymbol{M} = (m_{ij})$ be the incidence matrix of a 3-iet preserving morphism φ. Then*

$$\det \begin{pmatrix} m_{11}+m_{12} & m_{12}+m_{13} \\ m_{31}+m_{32} & m_{32}+m_{33} \end{pmatrix} = \delta \in \{1, -1\}. \tag{28}$$

Proof. Let us choose a Sturmian word $u \in \{A, C\}^{\mathbb{Z}}$ and a sequence $(u^{(m)})_{m\in\mathbb{N}}$ of 3-iet words such that $u = \lim_{m\to\infty} u^{(m)}$. For example, let u be the coding of $x_0 = 0$ under the 2-interval exchange transformation T with $I_0 = [0, 1-\alpha)$ and $I_1 = [1-\alpha, 1)$, where α is an arbitrary irrational number. Then we can choose $u^{(m)}$ to be the 3-iet word that codes $x_0 = 0$ under the 3-interval exchange transformation with intervals $I_A = [0, 1-\alpha-\frac{1}{m})$, $I_B = [1-\alpha-\frac{1}{m}, 1-\alpha)$ and $I_C = [1-\alpha, 1)$.

Let σ be a morphism given by

$$A \mapsto A, \quad B \mapsto AC, \quad C \mapsto C.$$

Since any morphism on $\{A, B, C\}^{\mathbb{Z}}$ is a continuous mapping, we have

$$(\sigma \circ \varphi)(u^{(m)}) \to (\sigma \circ \varphi)(u).$$

According to the assumption, the morphism φ is 3-iet preserving, hence $\varphi(u^{(m)})$ are 3-iet words. By Lemma 5, the words $(\sigma \circ \varphi)(u^{(m)})$, $m \in \mathbb{N}$, code 2-interval exchange transformations, and by Lemma 2, the limit of these words, that is the word $(\sigma \circ \varphi)(u)$, is either Sturmian or the densities of its letters are rational.

The matrix of σ is $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, which implies by (4) that the matrix of $\sigma \circ \varphi$ is

$$\boldsymbol{M}_{\sigma\circ\varphi} = \begin{pmatrix} m_{11}+m_{12} & 0 & m_{12}+m_{13} \\ m_{21}+m_{22} & 0 & m_{22}+m_{23} \\ m_{31}+m_{32} & 0 & m_{32}+m_{33} \end{pmatrix}.$$

Since $\sigma \circ \varphi$ maps a Sturmian word u over $\{A, C\}$ to a word over the same alphabet, we are interested only in the matrix of this morphism over $\{A, C\}$, that is,

$$\widetilde{\boldsymbol{M}} = \begin{pmatrix} m_{11}+m_{12} & m_{12}+m_{13} \\ m_{31}+m_{32} & m_{32}+m_{33} \end{pmatrix}. \tag{29}$$

Let us suppose that the densities of A and C in u are $1-\alpha$ and α, respectively. Using (5) we find the density of A in $(\sigma \circ \varphi)(u)$ to be

$$\rho(A) = \frac{(1-\alpha, \alpha)\, \widetilde{\boldsymbol{M}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{(1-\alpha, \alpha)\, \widetilde{\boldsymbol{M}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}. \tag{30}$$

If $\rho(A)$ is irrational, the word $(\sigma \circ \varphi)(u)$ is Sturmian and hence the morphism $\sigma \circ \varphi$ is Sturmian. This implies $\det \widetilde{\boldsymbol{M}} = \pm 1$.

The irrational number α, i.e., the density of A in the Sturmian word u, was chosen arbitrarily. Therefore $\rho(A)$, given by (30), will be rational for any irrational α only in the case when

$$p\widetilde{\boldsymbol{M}}\begin{pmatrix}1\\0\end{pmatrix} = q\widetilde{\boldsymbol{M}}\begin{pmatrix}1\\1\end{pmatrix}, \quad \text{for some } \ p, q \in \mathbb{Z} \setminus \{0\}. \tag{31}$$

This however implies that the matrix $\widetilde{\boldsymbol{M}}$ is singular, which contradicts Remark 19. $\square$

We are now in position to finish the proof of Theorem A.

Theorem A. *Let $\boldsymbol{M}$ be the incidence matrix of a 3-iet preserving morphism. Then*

$$\boldsymbol{M}\boldsymbol{E}\boldsymbol{M}^T = \pm \boldsymbol{E}, \quad \textit{where } \boldsymbol{E} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}. \tag{32}$$

Proof. Using the notation of Remark 19 for the matrix $\boldsymbol{P}$, we obviously see that the matrix $\boldsymbol{P}^{-1}\boldsymbol{M}\boldsymbol{P}$ has $(0, 0, -1)$ for its left eigenvector corresponding to the eigenvalue $-m_{12}+m_{22}-m_{32}$. It is then trivial to verify that $(0, 0, -1)\boldsymbol{P}^{-1} = (1, -1, 1)$ is a left eigenvector of the matrix $\boldsymbol{M}$ corresponding to the same eigenvalue. Since

$$\det \boldsymbol{M} = \det(\boldsymbol{P}^{-1}\boldsymbol{M}\boldsymbol{P}) = \delta(-m_{12} + m_{22} - m_{32}), \tag{33}$$

where $\delta \in \{-1, 1\}$ is given by (28), we derive that $(1, -1, 1)$ is a left eigenvector of the matrix $\boldsymbol{M}$ corresponding to the eigenvalue $\delta \det \boldsymbol{M}$. Denoting $\Delta := \det \boldsymbol{M}$, we can write

$$(1, -1, 1)\boldsymbol{M} = \delta\Delta(1, -1, 1). \tag{34}$$

This implies that the matrix $\boldsymbol{M}$ can be written in the following form,

$$\boldsymbol{M} = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{11} + m_{31} - \delta\Delta & m_{12} + m_{32} + \delta\Delta & m_{13} + m_{33} - \delta\Delta \\ m_{31} & m_{32} & m_{33} \end{pmatrix}. \tag{35}$$

With this, one can verify by inspection, that $\boldsymbol{M}\boldsymbol{E}\boldsymbol{M}^T = \delta\boldsymbol{E}$, using Lemma 20 for simplification of algebraic expressions. $\square$

As a partial result, we have shown in the above proof the following interesting statement.

Corollary 21. *Let $\boldsymbol{M}$ be the matrix of a 3-iet preserving morphism φ. Then the vector $(1, -1, 1)$ is a left eigenvector of $\boldsymbol{M}$, associated with the eigenvalue $\det \boldsymbol{M}$ or $-\det \boldsymbol{M}$, i.e.,*

$$(1, -1, 1)\boldsymbol{M} = \pm \det \boldsymbol{M}(1, -1, 1). \tag{36}$$

The other eigenvalues λ_1 and λ_2 of the matrix $\boldsymbol{M}$ are either quadratic mutually conjugate algebraic units, or $\lambda_1, \lambda_2 \in \{1, -1\}$.

From form (35) of the matrix $\boldsymbol{M}$ we derive the following corollary.

Corollary 22. *Let $\boldsymbol{M}$ be a matrix of a 3-iet preserving morphism. Then the sum of its first and the third rows differs from the sum of its second row by $\pm \det \boldsymbol{M}$. Formally,*

$$(1, 0, 1)\boldsymbol{M}\begin{pmatrix}1\\1\\1\end{pmatrix} - (0, 1, 0)\boldsymbol{M}\begin{pmatrix}1\\1\\1\end{pmatrix} = \pm \det \boldsymbol{M}.$$

7. 3-iet preserving morphisms versus fixed points

The proof of Theorem B, which is performed in Section 8, is based on the properties of 3-iet words which are fixed points of morphisms. In this section we therefore inspect which 3-iet preserving morphisms have a fixed point.

Similarly to the case of Sturmian words, the set of 3-iet words is not compact, and therefore in general the accumulation point u of a sequence $(u^{(m)})_{m\in\mathbb{N}}$ of 3-iet words is not necessarily a 3-iet word. The special case when the accumulation point belongs to the set of 3-iet words is treated by the following Lemma.

Lemma 23. *Let α, β, γ be positive real numbers such that $\alpha+\beta$ and $\beta+\gamma$ are linearly independent over $\mathbb{Q}$. Let T_1, T_2 be the 3-iet transformations with parameters α, β, γ and domain $[0, \alpha+\beta+\gamma)$, $(0, \alpha+\beta+\gamma]$, respectively. Let $(u^{(n)})_{n\in\mathbb{N}}$ be a sequence of 3-iet words and $(x^{(n)})_{n\in\mathbb{N}}$ a sequence of points in $[0, \alpha+\beta+\gamma]$ such that*

- *$u^{(n)} = u_{T_1}(x^{(n)})$ or $u^{(n)} = u_{T_2}(x^{(n)})$ for all $n \in \mathbb{N}$;*
- *$x^{(n)}$ is a monotonous sequence with the limit x.*

Then $\lim_{n\to\infty} u^{(n)}$ exists and is equal to the 3-iet word $u_{T_1}(x)$ or $u_{T_2}(x)$.

Proof. We use a statement from [13]. For a given m put

$$D_m := \{T_1^i(\alpha), T_1^i(\alpha+\beta), T_2^i(\alpha), T_2^i(\alpha+\beta) \mid -m \le i \le m\}.$$

Let $a < b$ and let $(a, b) \cap D_m = \emptyset$. It follows from the definition of D_m that one has $T_1^i(z) = T_2^i(z)$ for all $z \in (a, b)$, and for all $i = -m, \ldots, m$. Moreover, if we denote by $w(z)$ the word $w_{-m}w_{-m+1}\cdots w_{-1}w_0w_1\cdots w_m$ defined by the prescription

$$w_i = \begin{cases} A & \text{if } T_1^i(z) \in I_A, \\ B & \text{if } T_1^i(z) \in I_B, \\ C & \text{if } T_1^i(z) \in I_C, \end{cases}$$

then for all $z \in (a, b)$ the words $w(z)$ coincide.

Since the transformation T_1 is right-continuous and the transformation T_2 is left-continuous, for all $z \in (a, b)$ we have

$$\mathrm{d}\big(u_{T_1}(a), u_{T_1}(z)\big) < \frac{1}{1+m}, \qquad \mathrm{d}\big(u_{T_1}(a), u_{T_2}(z)\big) < \frac{1}{1+m}, \tag{37}$$

$$\mathrm{d}\big(u_{T_2}(b), u_{T_1}(z)\big) < \frac{1}{1+m}, \qquad \mathrm{d}\big(u_{T_2}(b), u_{T_2}(z)\big) < \frac{1}{1+m}. \tag{38}$$

Assume that the sequence $(x^{(n)})_{n\in\mathbb{N}}$ is decreasing. For $\varepsilon > 0$, we find $m \in \mathbb{N}$ such that $\varepsilon > \frac{1}{m+1}$ and we put $\delta_m := \sup\{y > x \mid y \notin D_m\}$. Since $x^{(n)} \searrow x$, there exists n_0 such that for all $n > n_0$ we have $x \le x^{(n)} < x + \delta_m$. Since $u^{(n)} = u_{T_1}(x^{(n)})$ or $u^{(n)} = u_{T_2}(x^{(n)})$, we obtain, using (37) for the interval $(a, b) = (x, x+\delta_m)$, that $\mathrm{d}\big(u_{T_1}(x), u^{(n)}\big) < \varepsilon$, which implies $\lim_{n\to\infty} u^{(n)} = u_{T_1}(x)$. Similarly we use (38) in the case $x^{(n)} \nearrow x$. $\square$

The following proposition deals with the original aim of this section, namely with the search for 3-iet preserving morphisms having 3-iet words as their fixed points.

Proposition 24. *Let φ be a primitive 3-iet preserving morphism. Then there exists $p \in \mathbb{N}$, $p \ge 1$, such that φ^p has a fixed point, and this fixed point is a 3-iet word.*

Proof. Without loss of generality, we may assume that the incidence matrix $\boldsymbol{M}$ of the morphism φ is positive. Otherwise, we show the validity of the statement for $\psi = \varphi^k$ for some k, which implies the validity of the statement for φ.

Let (α, β, γ) be a positive left eigenvector of $\boldsymbol{M}$. First we show that an infinite word coding a 3-iet with such parameters is not periodic. For contradiction, assume that (α, β, γ) satisfy (20), that is,

$$(\alpha, \beta, \gamma)\begin{pmatrix} K \\ K+L \\ L \end{pmatrix} = 0, \quad \text{for some } K, L \in \mathbb{Z} \setminus \{0\}. \tag{39}$$

If the Perron eigenvalue λ_1 of $\boldsymbol{M}$ is a quadratic irrational number, one can assume without loss of generality that the components of the vector (α, β, γ) belong to the quadratic field $\mathbb{Q}(\lambda_1)$. For any $x \in \mathbb{Q}(\lambda_1)$, denote by x' the image of x under the Galois automorphism of $\mathbb{Q}(\lambda_1)$. Since the matrix $\boldsymbol{M}$ and the vector $\begin{pmatrix} K \\ K+L \\ L \end{pmatrix}$ have integer components, the vector $(\alpha', \beta', \gamma')$ is an eigenvector to the eigenvalue $\lambda_1' = \lambda_2$ and satisfies

$$(\alpha', \beta', \gamma') \begin{pmatrix} K \\ K+L \\ L \end{pmatrix} = 0.$$

Using Corollary 21, the vector $(1, -1, 1)$ is a left eigenvector of $\boldsymbol{M}$ corresponding to the eigenvalue $\pm \det \boldsymbol{M}$. Therefore vectors $(\alpha', \beta', \gamma')$, (α, β, γ) and $(1, -1, 1)$ are eigenvectors of $\boldsymbol{M}$ corresponding to different eigenvalues, which means that they are linearly independent. All of them are orthogonal to the vector $\begin{pmatrix} K \\ K+L \\ L \end{pmatrix}$, which implies $K = L = 0$. This contradicts (39).

By Corollary 21, it remains to discuss the case when the Perron eigenvalue of $\boldsymbol{M}$ is $\lambda_1 = 1$. This is impossible due to the fact that a positive integral matrix $\boldsymbol{M}$ cannot have 1 as its eigenvalue corresponding to a positive eigenvector. Thus we have shown that the infinite word coding a 3-iet with parameters α, β, γ is not periodic.

Denote T_1, T_2 the 3-iet transformations with parameters α, β, γ and domain $[0, \alpha + \beta + \gamma)$, $(0, \alpha + \beta + \gamma]$, respectively.

Let $u^{(0)}$ be an arbitrary 3-iet word coding the orbit of a point by T_1. Put

$$u^{(n)} := \varphi^n(u^{(0)}), \quad \text{for } n \geq 1.$$

Since the vector of densities of $u^{(0)}$ is a left eigenvector of the incidence matrix of the morphism φ, every word $u^{(n)}$, $n \in \mathbb{N}$, has the same density of letters, see (5). As φ is a 3-iet preserving morphism, the word $u^{(n)}$ is a 3-iet word coding the orbit of a point under T_1 or T_2, for every $n \in \mathbb{N}$.

The space of infinite words over the alphabet $\{A, B, C\}$ is compact, and thus there exists a Cauchy subsequence of the sequence $(u^{(n)})_{n \in \mathbb{N}}$. Therefore there exist $m_0, n_0 \in \mathbb{N}$, $n_0 > m_0$, such that

$$\mathrm{d}\big(u^{(n_0)}, u^{(m_0)}\big) < \frac{1}{2}. \tag{40}$$

Set $p := n_0 - m_0$ and $v = \cdots v_{-2} v_{-1} | v_0 v_1 \cdots := u^{(m_0)}$. Since $u^{(n_0)} = \varphi^{n_0 - m_0}(u^{(m_0)}) = \varphi^p(v)$, inequality (40) can be rewritten as

$$\mathrm{d}\big(\varphi^p(v), v\big) < \frac{1}{2}. \tag{41}$$

The latter, together with the primitivity of the morphism φ, implies

$$\varphi^p(v_0) = v_0 w_0 \quad \text{and} \quad \varphi^p(v_{-1}) = w_{-1} v_{-1}$$

for some non-empty words $w_0, w_{-1} \in \{A, B, C\}^*$. Therefore the morphism φ^p has the fixed point

$$\lim_{n \to \infty} \varphi^{np}(v).$$

Since $\varphi^{np}(v)$ is a 3-iet word given by T_1, or T_2, there exists for every n a number $x^{(n)} \in [0, \alpha + \beta + \gamma]$, such that

$$\varphi^{np}(v) = u_{T_1}(x^{(n)}) \text{ or } u_{T_2}(x^{(n)}).$$

Denote by x the limit of some monotonous subsequence of $(x^{(n)})_{n \in \mathbb{N}}$, i.e., $x = \lim_{n \to \infty} x^{(k_n)}$. According to Lemma 23,

$$\lim_{n \to \infty} \varphi^{np}(v) = u_{T_1}(x) \text{ or } u_{T_2}(x),$$

which means that φ^p has as its fixed point a 3-iet word, namely $u_{T_1}(x)$ or $u_{T_2}(x)$, respectively. $\square$

Remark 25. The assumption of primitivity of the morphism φ is essential in the above statement. For example, the morphism φ defined by (22) is 3-iet preserving, yet the only fixed points of an arbitrary power φ^p, $p \in \mathbb{N}$, $p \geq 1$, are

$$\cdots CCC|ACCC\cdots, \qquad \cdots CCC|BCCC\cdots, \qquad \cdots CCC|CCC\cdots$$

8. Proof of Theorem B

In the proof of Theorem B we use certain properties of discrete sets associated with 3-iet words. Every 3-iet word can be geometrically represented using a C&P set. On the other hand, every fixed point of a primitive morphism can be represented by a self-similar set, which is constructed using a right eigenvector of the matrix of the morphism. The crucial point in the proof of Theorem B is the fact that for a 3-iet word being a fixed point of a primitive morphism these two geometric representations coincide. In what follows the Galois automorphism on a quadratic algebraic field $\mathbb{Q}(\varepsilon)$ plays an important role. Let us recall that it is the mapping $' : \mathbb{Q}(\varepsilon) \to \mathbb{Q}(\varepsilon)$, which to any element $x \in \mathbb{Q}(\varepsilon)$ assigns its algebraic conjugate $x' \in \mathbb{Q}(\varepsilon)$. This mapping satisfies $(x+y)' = x' + y'$ and $(xy)' = x'y'$.

We first show that the determinant of the incidence matrix of a 3-iet preserving morphism is in modulus smaller than or equal to 1. For that we use the following technical lemma.

Lemma 26. *Let $\varepsilon \in (0,1)$ be a quadratic irrational number with conjugate $\varepsilon' < 0$. Let $\lambda \in (0,1)$ be a quadratic unit such that its conjugate satisfies $\lambda' > 1$ and $\lambda'\mathbb{Z}[\varepsilon'] = \mathbb{Z}[\varepsilon'] := \mathbb{Z} + \varepsilon'\mathbb{Z}$. Let us denote $\Lambda := \lambda'$, $\eta := -\varepsilon'$ and*

$$P_n(x) := \#\big(x, x + (1+2\eta)\Lambda^n\big] \cap \Sigma_{\varepsilon,\eta}(\Omega),$$

where Ω is a bounded interval. Then there is a constant R such that

$$|P_n(x) - P_n(y)| \leq R,$$

for any $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$.

The proof of the Lemma exploits some simple properties of C&P sets, which are however not related to infinite words. Therefore we postpone it to the Appendix.

Proposition 27. *The incidence matrix $\boldsymbol{M}$ of a primitive 3-iet preserving morphism φ satisfies $|\det \boldsymbol{M}| \leq 1$.*

Proof. Without loss of generality we assume that φ has a 3-iet fixed point u, and, moreover, that both the matrix $\boldsymbol{M}$ and its spectrum are positive. This is possible since according to Proposition 24 for any primitive 3-iet preserving morphism there exists $p \in \mathbb{N}$ such that φ^p has a fixed point and $|\det \boldsymbol{M}| \leq 1 \Leftrightarrow |\det \boldsymbol{M}^p| \leq 1$.

Let us denote by Λ the dominant (Perron) eigenvalue of $\boldsymbol{M}$. Its second eigenvalue is by Corollary 21 equal to $\pm\det \boldsymbol{M}$, the third one is denoted by λ. A positive integer matrix cannot have 1 as its dominant eigenvalue, hence by Corollary 21, $\Lambda > 1$ is a quadratic algebraic unit such that its conjugate $\Lambda' = \lambda$.

Without loss of generality we assume that a positive right eigenvector associated with the Perron eigenvalue Λ is such that the modulus of its third component is greater than the modulus of the first one. Otherwise, we use $\xi \circ \varphi \circ \xi$ instead of φ, where ξ is defined as in Example 15. Matrices corresponding to φ and $\xi \circ \varphi \circ \xi$ have the same spectrum, the first and the last component of eigenvectors being interchanged.

The fixed point u of φ is the coding of a 3-iet with parameters α, β, γ, with a starting point x_0. By (5) and Corollary 10 the vector (α, β, γ) is a left eigenvector of $\boldsymbol{M}$ corresponding to Λ.

In the proof we use properties of a C&P set; we construct it in such a way that it coincides with the geometric representation of the fixed point u. Let us define parameters ε, l, c and the interval Ω by (16). Note that $(l-\varepsilon, 1-l, l-1+\varepsilon)$ is also an eigenvector to Λ, because it is just scalar multiple of (α, β, γ), and, moreover, since Λ is a quadratic irrational number, the parameters ε, l belong to the same quadratic algebraic field $\mathbb{Q}(\Lambda)$. By x' we denote the image of $x \in \mathbb{Q}(\Lambda)$ under the Galois automorphism on $\mathbb{Q}(\Lambda)$.

Let us denote $\vec{F} = \begin{pmatrix} -\varepsilon \\ 1-2\varepsilon \\ 1-\varepsilon \end{pmatrix}$. The vector $\vec{F}$ is orthogonal to two left eigenvectors $(1,-1,1)$ and $(l-\varepsilon, 1-l, l-1+\varepsilon)$ associated with eigenvalues $\pm\det \boldsymbol{M}$ and Λ, respectively. The matrix $\boldsymbol{M}$ has three different eigenvalues, therefore $\vec{F}$ is a right eigenvector[5] to the third eigenvalue λ.

[5] We used the following simple fact: let $\vec{x}$ and $\vec{y}$ be a left and a right eigenvector, respectively. If they correspond to different eigenvalues then $xy = 0$, i.e., x and y are orthogonal provided that they are real [12].

Since the matrix $\boldsymbol{M}$ is integral, the vector $\vec{F}' := \begin{pmatrix} -\varepsilon' \\ 1-2\varepsilon' \\ 1-\varepsilon' \end{pmatrix}$ is a right eigenvector corresponding to the dominant eigenvalue $\lambda' = \Lambda$, that is,

$$\boldsymbol{M}\begin{pmatrix} -\varepsilon' \\ 1-2\varepsilon' \\ 1-\varepsilon' \end{pmatrix} = \lambda'\begin{pmatrix} -\varepsilon' \\ 1-2\varepsilon' \\ 1-\varepsilon' \end{pmatrix} = \Lambda\begin{pmatrix} -\varepsilon' \\ 1-2\varepsilon' \\ 1-\varepsilon' \end{pmatrix}. \tag{42}$$

Therefore by Perron–Frobenius theorem (see e.g. [12]) the components of $\vec{F}'$ are either all positive or all negative. By assumption, the modulus of the third component of a right dominant eigenvector is greater than the modulus of the first one, which implies that all components of $\vec{F}'$ are positive, i.e., $-\varepsilon' > 0$.

We define a C&P set with parameters $\varepsilon, \eta, \Omega$, where ε and Ω are as above and we put $\eta := -\varepsilon'$. By Proposition 6, the distances between adjacent elements of $\Sigma_{\varepsilon,\eta}(\Omega)$ take values $\mu_A = \eta$, $\mu_B = 1+2\eta$, and $\mu_C = 1+\eta$ and their ordering with respect to the origin is coded by the word u. Let $(t_n)_{n\in\mathbb{Z}}$ denote a strictly increasing sequence such that $\Sigma_{\varepsilon,\eta}(\Omega) = \{t_n \mid n \in \mathbb{Z}\}$. According to Section 2.2, this C&P set is also the geometric representation of the fixed point u of φ and

$$\#\big((\Lambda t_n, \Lambda t_{n+1}] \cap \Sigma_{\varepsilon,\eta}(\Omega)\big) = \begin{cases} |\varphi(A)| & \text{if } t_{n+1}-t_n = \mu_A = \eta, \\ |\varphi(B)| & \text{if } t_{n+1}-t_n = \mu_B = 1+2\eta. \\ |\varphi(B)| & \text{if } t_{n+1}-t_n = \mu_C = 1+\eta. \end{cases}$$

As the fixed point of a morphism is also the fixed point of an arbitrary power of this morphism, the geometric representations of φ and φ^n are the same for any $n \in \mathbb{N}$. Since AC is a factor of any 3-iet word,[6] there exist $k, m \in \mathbb{N}$ such that

$$|\varphi^n(AC)| = \#\big((\Lambda^n t_k, \Lambda^n t_{k+2}] \cap \Sigma_{\varepsilon,\eta}(\Omega)\big), \tag{43}$$

$$|\varphi^n(B)| = \#\big((\Lambda^n t_m, \Lambda^n t_{m+1}] \cap \Sigma_{\varepsilon,\eta}(\Omega)\big). \tag{44}$$

By definition of the matrix of a morphism and by Corollary 22, we have $|\varphi^n(AC)| - |\varphi^n(B)| = \pm(\det \boldsymbol{M})^n$. Observe that intervals $(\Lambda^n t_k, \Lambda^n t_{k+2}]$ and $(\Lambda^n t_m, \Lambda^n t_{m+1}]$ have the same length, namely $\Lambda^n(1+2\eta)$, and that the equality $\lambda'\mathbb{Z}[\varepsilon'] = \mathbb{Z}[\varepsilon']$ holds due to (42). We can therefore use Lemma 26, which states that the difference between the right-hand sides of (43) and (44) is bounded by a constant R independent of n. Putting both facts together one obtains

$$|\det \boldsymbol{M}^n| \leq R \quad \text{for any } n \in \mathbb{N}.$$

The statement follows from the fact that $\det \boldsymbol{M}$ is an integer. □

Corollary 28. *The incidence matrix of a 3-iet preserving morphism satisfies* $|\det \boldsymbol{M}| \leq 1$.

Proof. Consider the primitive morphism ψ, defined in Example 14, and let us denote by $\boldsymbol{M}_0$ a power of its incidence matrix (say kth power), which is positive. Let φ be a non-primitive 3-iet preserving morphism and let $\boldsymbol{M}$ be its matrix. It follows from Theorem A that in any row of $\boldsymbol{M}$ there is at least one non-zero element. Therefore the matrix $\boldsymbol{M}\boldsymbol{M}_0$ is positive, and by (4) it is the incidence matrix of 3-iet preserving morphism $\psi^k \circ \varphi$, which is thus primitive.

By Proposition 27 we have

$$1 \geq |\det(\boldsymbol{M}\boldsymbol{M}_0)| = |\det \boldsymbol{M}| \underbrace{|\det \boldsymbol{M}_0|}_{=1} = |\det \boldsymbol{M}|. \quad \square$$

Theorem B. *Let φ be a 3-iet preserving morphism and let $\boldsymbol{M}$ be its incidence matrix. Then one of the following holds*

- $\det \boldsymbol{M} = 0$ *and $\varphi(u)$ is degenerate for every 3-iet word u,*
- $\det \boldsymbol{M} = \pm 1$ *and $\varphi(u)$ is non-degenerate for every non-degenerate 3-iet word u.*

[6] This fact follows directly from the definition of the three interval exchange transformation with the permutation (321).

Proof. Since $\boldsymbol{M}$ is an integer matrix its determinant is an integer as well, and by Corollary 28 one has $\det \boldsymbol{M} \in \{-1, 0, 1\}$.

We use notation from Lemma 17 and Remark 19. In particular, recall that

$$\vec{x}_1 := \begin{pmatrix}1\\1\\0\end{pmatrix}, \qquad \vec{x}_2 := \begin{pmatrix}0\\1\\1\end{pmatrix}, \qquad \vec{x}_3 := \begin{pmatrix}0\\1\\0\end{pmatrix}.$$

Lemma 17 states that $\boldsymbol{M}\mathcal{P} = \mathcal{P}$, where $\mathcal{P}$ is a subspace of $\mathbb{R}^3$ spanned by vectors $\vec{x}_1, \vec{x}_2$. Let us denote by $\mathcal{S}$ the lattice $\mathcal{S} = \mathbb{Z}\vec{x}_1 + \mathbb{Z}\vec{x}_2$. The action of the matrix $\boldsymbol{M}$ on the two-dimensional subspace $\mathcal{P}$ has the matrix $\widetilde{M}$ from (29), which has by Lemma 20 determinant $\delta \in \{1, -1\}$. Therefore the vectors $\boldsymbol{M}\vec{x}_1$ and $\boldsymbol{M}\vec{x}_2$ form a basis of $\mathcal{S}$ as well and thus

$$\boldsymbol{M}\mathcal{S} = \mathcal{S}. \tag{45}$$

By an easy computation, $\boldsymbol{M}\vec{x}_3 = m_{12}\vec{x}_1 + m_{32}\vec{x}_2 + (-m_{12} - m_{32} + m_{22})\vec{x}_3$, hence by (33) we have $\boldsymbol{M}\vec{x}_3 \in \delta\Delta\vec{x}_3 + \mathcal{S}$, where $\Delta = \det \boldsymbol{M}$ as before. Moreover,

$$\begin{pmatrix}1\\1\\1\end{pmatrix} = -\vec{x}_3 + \vec{x}_1 + \vec{x}_2 \quad \Longrightarrow \quad \boldsymbol{M}\begin{pmatrix}1\\1\\1\end{pmatrix} \in -\delta\Delta\vec{x}_3 + \mathcal{S},$$

and if we replace $\vec{x}_3$ on the right-hand side using $-\vec{x}_3 = \begin{pmatrix}1\\1\\1\end{pmatrix} - \vec{x}_1 - \vec{x}_2$ we obtain

$$\boldsymbol{M}\begin{pmatrix}1\\1\\1\end{pmatrix} \in \delta\Delta\begin{pmatrix}1\\1\\1\end{pmatrix} + \mathcal{S}. \tag{46}$$

Case 1: Let $\Delta = \det \boldsymbol{M} = 0$. Then by (46) and (45) there exist $K_1, L_1 \in \mathbb{Z}$ such that

$$\boldsymbol{M}\begin{pmatrix}1\\1\\1\end{pmatrix} = \boldsymbol{M}\begin{pmatrix}K_1\\K_1+L_1\\L_1\end{pmatrix} \quad \text{which implies} \quad (\alpha,\beta,\gamma)\boldsymbol{M}\begin{pmatrix}1\\1\\1\end{pmatrix} = (\alpha,\beta,\gamma)\boldsymbol{M}\begin{pmatrix}K_1\\K_1+L_1\\L_1\end{pmatrix},$$

for arbitrary parameters (α, β, γ). By Corollary 9 it means that $(\alpha, \beta, \gamma)\boldsymbol{M}$ are parameters of a degenerate 3-iet word.

Case 2: Let $\Delta = \det \boldsymbol{M} = \pm 1$. Again, by (46) there exist $K_2, L_2 \in \mathbb{Z}$ such that

$$\boldsymbol{M}\begin{pmatrix}1\\1\\1\end{pmatrix} = \pm\begin{pmatrix}1\\1\\1\end{pmatrix} + \begin{pmatrix}K_2\\K_2+L_2\\L_2\end{pmatrix}. \tag{47}$$

We show that parameters $(\alpha, \beta, \gamma)\boldsymbol{M}$ correspond to a degenerate 3-iet word only if the original parameters (α, β, γ) correspond to a degenerate 3-iet word.

Let (α, β, γ) be such that $(\alpha, \beta, \gamma)\boldsymbol{M}$ are parameters of a degenerate 3-iet word, i.e., by Corollary 9 there exist $K_3, L_3, H, S \in \mathbb{Z}$

$$(\alpha,\beta,\gamma)\boldsymbol{M}\begin{pmatrix}1\\1\\1\end{pmatrix} = (\alpha,\beta,\gamma)\boldsymbol{M}\begin{pmatrix}K_3\\K_3+L_3\\L_3\end{pmatrix} = (\alpha,\beta,\gamma)\begin{pmatrix}H\\H+S\\S\end{pmatrix}, \tag{48}$$

where the last equality comes from (45). Multiplying equation (47) by (α, β, γ) from the left one obtains

$$(\alpha,\beta,\gamma)\boldsymbol{M}\begin{pmatrix}1\\1\\1\end{pmatrix} = \pm(\alpha,\beta,\gamma)\begin{pmatrix}1\\1\\1\end{pmatrix} + (\alpha,\beta,\gamma)\begin{pmatrix}K_2\\K_2+L_2\\L_2\end{pmatrix}. \tag{49}$$

Finally, comparing the right-hand sides of (48) and (49) we have

$$(\alpha, \beta, \gamma) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \pm(\alpha, \beta, \gamma) \begin{pmatrix} K_2 - H \\ K_2 + L_2 - H - S \\ L_2 - S \end{pmatrix},$$

which means that (α, β, γ) are parameters of a degenerate 3-iet word. □

9. Comments and open problems

(1) We have derived that matrices of 3-iet preserving morphisms belong to the monoid $\mathrm{E}(3, \mathbb{N}) := \{\boldsymbol{M} \in \mathbb{N}^{3\times3} \mid \boldsymbol{MEM}^T = \pm\boldsymbol{E} \text{ and } \det\boldsymbol{M} = \pm1\}$ where $\boldsymbol{E} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$. Unfortunately – in contrast to the Sturmian case – the opposite is not true. In fact, the monoid $\mathrm{E}(3, \mathbb{N})$ contains matrices such that no associated morphism is 3-iet preserving.

As an example we can consider the matrix $\boldsymbol{M} = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 3 & 5 \\ 3 & 0 & 5 \end{pmatrix}$. We will use the following two simple facts (see [11]): For any $X \in \{A, B, C\}$ there exists a 3-iet word u such that XX is a factor of u. For any 3-iet word u there exists a unique $Y \in \{A, B, C\}$ such that YY is a factor of u.

Let φ be a morphism such that $\boldsymbol{M}$ its incidence matrix. One can easily see that any morphism φ with $\boldsymbol{M}$ as its incidence matrix has CC as the factor of $\varphi(C)$. Hence $\varphi(A) = BCB$. Now take a 3-iet word u which contains AA and C. Its image under φ contains both CC and BB, thus it is not a 3-iet word. By definition φ is not 3-iet preserving.

(2) The mapping $\varphi \to \boldsymbol{M}_\varphi$, where φ a is 3-iet preserving morphism and $\boldsymbol{M}_\varphi$ is its incidence matrix is not one-to-one. One can show that for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{N}^{2\times2}$ with $ad - bc = \pm1$ there exist $a + b + c + d - 1$ different Sturmian morphisms. The same question for matrices of 3-iet preserving morphisms is not solved.

(3) Unlike the free monoid $\mathrm{SL}(2, \mathbb{N}) = \{\boldsymbol{M} \in \mathbb{N}^{2\times2} \mid \det\boldsymbol{M} = 1\}$, which is generated by two matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, the monoid $\mathrm{SL}(3, \mathbb{N})$ is not free, and, moreover, it is not finitely generated [24, Appendix A]. It would be interesting to derive similar results for the monoid $\mathrm{E}(3, \mathbb{N})$.

(4) Even though the aim of this paper is not to investigate explicit prescriptions of 3-iet preserving morphisms, we can still provide some information about it, based on our results. It follows from the proof of Lemma 20 that for every 3-iet preserving morphism $\varphi : \{A, B, C\}^* \to \{A, B, C\}^*$ the morphism given by $A \mapsto \sigma_{A,B} \circ \varphi(A)$ and $B \mapsto \sigma_{A,B} \circ \varphi(B)$ is Sturmian, where $\sigma_{A,B}$: $A \mapsto A$, $B \mapsto AB$, $C \mapsto B$; analogously for morphisms $\sigma_{A,C}$ and $\sigma_{B,C}$.

(5) In this paper we were not at all interested in the characterization of 3-iet words, which are fixed points of primitive morphisms, that is, of 3-iet words u such that there exists a primitive morphisms φ for which $\varphi(u) = u$. This question is completely solved for Sturmian words [30,4], partial results can be found in [2,18]. Adamczewski [1] studied for 3-iet words a weaker property, the so-called primitive substitutivity. An infinite word u over an alphabet $\mathcal{A}$ is said to be primitively substitutive if there exists a word v over an alphabet $\mathcal{B}$, which is a fixed point of a primitive morphism, and a morphism $\psi : \mathcal{B}^* \to \mathcal{A}^*$ such that $\psi(v) = u$. Adamczewski, using results of Boshernitzan and Carroll [8], proved that a non-degenerate 3-iet word is primitively substitutive if and only if normalized parameters ε, l, c (see (16)) of the corresponding transformation belong to the same quadratic field. Similar study can be found in [14].

Acknowledgements

The authors acknowledge the financial support by the Czech Science Foundation GA ČR 201/05/0169, and by the grant LC06002 of the Ministry of Education, Youth, and Sports of the Czech Republic.

Appendix. Proof of Lemma 26

In this Appendix we prove Lemma 26, which is rather technical. The proof uses the following four claims.

Claim 29. *Let ε, η be irrational numbers, $\varepsilon \neq -\eta$, and let Ω_1, Ω_2 be arbitrary bounded intervals. Then $\#\big(\Omega_1 \cap \Sigma_{\varepsilon,\eta}(\Omega_2)\big) = \#\big(\Omega_2 \cap \Sigma_{\eta,\varepsilon}(\Omega_1)\big)$.*

Proof.

$$\begin{aligned}\#\big(\Omega_1 \cap \Sigma_{\varepsilon,\eta}(\Omega_2)\big) &= \#\{a + b\eta \mid a, b \in \mathbb{Z},\ a + b\eta \in \Omega_1,\ a - b\varepsilon \in \Omega_2\}\\ &= \#\{a + c\varepsilon \mid a, c \in \mathbb{Z},\ a + c\varepsilon \in \Omega_2,\ a - c\eta \in \Omega_1\}\\ &= \#\big(\Omega_2 \cap \Sigma_{\eta,\varepsilon}(\Omega_1)\big). \quad \square\end{aligned}$$

Claim 30. *Let ε be a quadratic irrational number with conjugate ε'. Let λ be a quadratic unit whose conjugate λ' satisfies $\lambda'\mathbb{Z}[\varepsilon'] = \mathbb{Z}[\varepsilon']$. Then $\lambda'\Sigma_{\varepsilon,-\varepsilon'}(\Omega) = \Sigma_{\varepsilon,-\varepsilon'}(\lambda\Omega)$.*

Proof. Let us consider $x = a - b\varepsilon \in \mathbb{Z}[\varepsilon]$. If we denote $\eta = -\varepsilon'$, the number $a + b\eta = a - b\varepsilon' \in \mathbb{Z}[\varepsilon']$ is the image of x under the Galois automorphism and therefore we denote it by $x' = a - b\varepsilon'$.

Note that the condition $\lambda'\mathbb{Z}[\varepsilon'] = \mathbb{Z}[\varepsilon']$ is equivalent to the condition $\lambda\mathbb{Z}[\varepsilon] = \mathbb{Z}[\varepsilon]$, and that these two equalities imply that the mappings $x' \mapsto \lambda'x'$ and $x \mapsto \lambda x$ are bijections on $\mathbb{Z}[\varepsilon']$ and $\mathbb{Z}[\varepsilon]$, respectively.

By definition of a C&P set we have

$$\Sigma_{\varepsilon,-\varepsilon'}(\Omega) = \{x' \in \mathbb{Z}[\varepsilon'] \mid x \in \Omega\}.$$

We derive

$$\begin{aligned}\lambda'\Sigma_{\varepsilon,-\varepsilon'}(\Omega) &= \lambda'\{x' \in \mathbb{Z}[\varepsilon'] \mid x \in \Omega\} = \{\lambda'x' \in \mathbb{Z}[\varepsilon'] \mid \lambda x \in \lambda\Omega\}\\ &= \{y' \in \mathbb{Z}[\varepsilon'] \mid y \in \lambda\Omega\} = \Sigma_{\varepsilon,-\varepsilon'}(\lambda\Omega). \quad \square\end{aligned}$$

The following claim is given without the proof, since it is just a special case of Proposition 6.2 in [13].

Claim 31. *Let $\hat{\varepsilon}, \hat{\eta}$ be irrational numbers, $\hat{\varepsilon} \neq -\hat{\eta}$, and let $\hat{\Omega}$ be an arbitrary bounded interval. Then*

$$\Sigma_{\hat{\varepsilon},\hat{\eta}}((1 + 2\hat{\varepsilon})\hat{\Omega}) = (1 - 2\hat{\eta})\Sigma_{\frac{\hat{\varepsilon}}{1+2\hat{\varepsilon}},\frac{\hat{\eta}}{1-2\hat{\eta}}}(\hat{\Omega}).$$

Claim 32. *Let $\tilde{\varepsilon}, \tilde{\eta}$ be irrational numbers such that $\tilde{\varepsilon} \neq -\tilde{\eta}$. Let $z \in \mathbb{R}$ and let J be a bounded interval. We denote $Q(J, z) := \#(J \cap \Sigma_{\tilde{\varepsilon},\tilde{\eta}}(z - 1, z])$. Then there is a constant R such that $|Q(J, z) - Q(J, t)| \leq R$ for every $z, t \in \mathbb{R}$ and for every interval J.*

Proof. The condition $a - b\tilde{\varepsilon} \in (z - 1, z]$, where $a, b \in \mathbb{Z}$, can be equivalently rewritten as $a = \lfloor z + b\tilde{\varepsilon}\rfloor = z + b\tilde{\varepsilon} - \{z + b\tilde{\varepsilon}\}$. Hence

$$\Sigma_{\tilde{\varepsilon},\tilde{\eta}}(z - 1, z] = \{b(\tilde{\varepsilon} + \tilde{\eta}) + z - \{z + b\tilde{\varepsilon}\} \mid b \in \mathbb{Z}\}.$$

We consider the interval J with boundary points $c, c+l$, where $c, l \in \mathbb{R}$ and $l > 0$. If the point $b(\tilde{\varepsilon}+\tilde{\eta})+z-\{z+b\tilde{\varepsilon}\}$ belongs to the set $J \cap \Sigma_{\tilde{\varepsilon},\tilde{\eta}}(z - 1, z]$, then $c - z \leq b(\tilde{\varepsilon} + \tilde{\eta}) \leq c + l - z + 1$. On the other hand, if $c - z + 1 < b(\tilde{\varepsilon} + \tilde{\eta}) < c + l - z$ then the point $b(\tilde{\varepsilon} + \tilde{\eta}) + z - \{z + b\tilde{\varepsilon}\}$ belongs to $J \cap \Sigma_{\tilde{\varepsilon},\tilde{\eta}}(z - 1, z]$. It means that the number of points in the set $J \cap \Sigma_{\tilde{\varepsilon},\tilde{\eta}}(z - 1, z]$ is at least $\lfloor \frac{l-1}{\tilde{\varepsilon}+\tilde{\eta}} \rfloor$ and at most $\lceil \frac{l+1}{\tilde{\varepsilon}+\tilde{\eta}} \rceil$, and hence

$$\left\lfloor \frac{l - 1}{\tilde{\varepsilon} + \tilde{\eta}} \right\rfloor \leq Q(J, z) \leq \left\lceil \frac{l + 1}{\tilde{\varepsilon} + \tilde{\eta}} \right\rceil.$$

Note that the bounds on $Q(J, z)$ do not depend on z, and thus the same estimate holds for $Q(J, t)$. Therefore

$$|Q(J, z) - Q(J, t)| \leq \left\lceil \frac{l + 1}{\tilde{\varepsilon} + \tilde{\eta}} \right\rceil - \left\lfloor \frac{l - 1}{\tilde{\varepsilon} + \tilde{\eta}} \right\rfloor \leq 2\left(1 + \frac{1}{\tilde{\varepsilon} + \tilde{\eta}}\right) =: R. \quad \square$$

Now we are in the position to conclude the proof of Lemma 26.

Proof of Lemma 26. Let us recall the definition of $P_n(x) = \#\Big(\big(x, x + (1+2\eta)\Lambda^n\big] \cap \Sigma_{\varepsilon,\eta}(\Omega)\Big)$. By Claim 29, we have

$$P_n(x) = \#\Big(\Omega \cap \Sigma_{\eta,\varepsilon}(x, x + (1+2\eta)\Lambda^n]\Big) = \#\Big(\Lambda^n\Omega \cap \Lambda^n\Sigma_{\eta,\varepsilon}(x, x + (1+2\eta)\Lambda^n]\Big).$$

As $\eta = -\varepsilon'$ and $\lambda' = \Lambda$, we have, by Claim 30,

$$P_n(x) = \#\Big(\Lambda^n\Omega \cap \Sigma_{\eta,\varepsilon}\big(\lambda^n x, \lambda^n x + (1+2\eta)\big]\Big),$$

where we used $\lambda\lambda' = \lambda\Lambda = 1$. Claim 31 further implies

$$P_n(x) = \#\left(\Lambda^n\Omega \cap (1-2\varepsilon)\Sigma_{\frac{\eta}{1+2\eta},\frac{\varepsilon}{1-2\varepsilon}}\left(\frac{\lambda^n x}{1+2\eta}, \frac{\lambda^n x}{1+2\eta} + 1\right]\right).$$

Thus $P_n(x) = Q\big(J, \frac{\lambda^n x}{1+2\eta} + 1\big)$ as defined in Claim 32, where $J = \frac{1}{1-2\varepsilon}\Lambda^n\Omega$, $\tilde{\varepsilon} = \frac{\eta}{1+2\eta}$ and $\tilde{\eta} = \frac{\varepsilon}{1-2\varepsilon}$. The statement of the lemma follows by application of Claim 32. $\square$

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