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# Chamber behavior of double Hurwitz numbers in genus 0

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**Abstract**

We study double Hurwitz numbers in genus zero counting the number of covers $\mathbb{C}\mathbf{P}^1 \to \mathbb{C}\mathbf{P}^1$ with two branching points with a given branching behavior. By the recent result due to Goulden, Jackson and Vakil, these numbers are piecewise polynomials in the multiplicities of the preimages of the branching points. We describe the partition of the parameter space into polynomiality domains, called chambers, and provide an expression for the difference of two such polynomials for two neighboring chambers. Besides, we provide an explicit formula for the polynomial in a certain chamber called totally negative, which enables us to calculate double Hurwitz numbers in any given chamber as the polynomial for the totally negative chamber plus the sum of the differences between the neighboring polynomials along a path connecting the totally negative chamber with the given one.


*Keywords:* Double Hurwitz numbers; Piecewise polynomiality; Chambers; Wall crossing

## 1. Introduction and results

Let $\mu = (\mu_1, \dots, \mu_m)$ be a partition of an integer $d$. The *Hurwitz number* $h^g_\mu$ is the number of genus $g$ branched covers of $\mathbb{C}\mathbf{P}^1$ with branching corresponding to $\mu$ over a fixed point (usually

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identified with $\infty$) and an appropriate number of fixed simple branching points. Recall that each cover is counted with the weight $c^{-1}$, where $c$ is the number of automorphisms of the cover.

Hurwitz numbers possess a rich structure explored by many authors in different fields, such as algebraic geometry, representation theory, integrable systems, combinatorics, and mathematical physics. We mention here only the so-called ELSV-formula (see [1,2,6])

$$h^g_\mu = \frac{(m+d+2g-2)!}{|\mathrm{Aut}\,\mu|} \prod_{i=i}^{m} \frac{\mu_i^{\mu_i}}{\mu_i!} \int\limits_{\overline{\mathcal{M}}_{g,m}} \frac{c(\Lambda^\vee)}{(1-\mu_1\psi_1)\cdots(1-\mu_m\psi_m)},$$

where $\mathrm{Aut}\,\mu$ is the group of symmetries of the set $\mu$, $\overline{\mathcal{M}}_{g,m}$ is the Deligne–Mumford compactification of the moduli space of genus $g$ curves with $m$ marked points, $\psi_i$ is the first Chern class of the cotangent bundle at the $i$th marked point, and $c(\Lambda^\vee)$ is the total Chern class of the dual to the Hodge bundle (see the above mentioned papers for exact definitions).

Let $\nu = (\nu_1, \dots, \nu_n)$ be another partition of the same integer $d$. The *double Hurwitz number* $h^g_{\mu;\nu}$ is the number of genus $g$ branched covers of $\mathbb{C}\mathbf{P}^1$ with branchings corresponding to $\mu$ and $\nu$ over two fixed points (in what follows we identify them with $\infty$ and 0, respectively) and an appropriate number of fixed simple branching points. We denote the latter number $r^g_{\mu;\nu}$; by the Riemann–Hurwitz theorem, $r^g_{\mu;\nu} = m+n+2g-2$. To simplify the exposition, we assume that the points mapped to $\infty$ and 0 are labelled, so the double Hurwitz numbers under this convention are $|\mathrm{Aut}\,\mu||\mathrm{Aut}\,\nu|$ larger than they would be under the usual convention.

Most of the known results concerning double Hurwitz numbers treat only the so-called one-part (or polynomial) case, when $m = 1$ and $n$ is arbitrary. One-part double Hurwitz numbers in genus zero where studied in [21] (see also [19] for an earlier version of the same result). It is proved there that

$$h^0_{(d);\nu} = (n-1)!d^{n-2}; \tag{1.1}$$

recall that $n - 1 = r^0_{(d);\nu}$. In fact, a combinatorial result much more general than (1.1) was obtained already in [4] using Lagrange inversion; in [21] the same formula was reproved by methods of algebraic geometry, and in this way its algebro–geometric meaning was clarified.

Another way to get (1.1) is suggested in [14]. The aim of the paper is to derive an ELSV-type formula for $h^g_{(d);\nu}$. Two such formulas are derived; the first of them (see [14,16]) represents $h^g_{(d);\nu}$ as

$$h^g_{(d);\nu} = r!d^{r-1} \int\limits_{\Delta} \psi_1^{r-1},$$

where $r = r^g_{(d);\nu} = n+2g-1$, $\Delta$ is a certain cycle in the moduli space $\overline{\mathcal{M}}_{g,n+1}$ and $\psi_1$ has the same meaning as in the ELSV-formula. In the case $g = 0$ the cycle $\Delta$ coincides with the whole space $\overline{\mathcal{M}}_{0,n+1}$, and since the integral is known to be equal to 1, we immediately get (1.1). The second formula (see [14,15]) represents $h^0_{(d);\nu}$ as

$$h^0_{(d);\nu} = d^{n-2} \int\limits_{\overline{\mathcal{M}}_{0,d(n-1)+2}} \psi_1^{n-2}\Psi_2\cdots\Psi_{n+1}.$$

The definition of the classes $\Psi_i$ is recursive and rather involved. The only example considered in [14] is $n = 2$, in which case, after some effort, one gets the correct answer.

A different ELSV-type formula for the one-part double Hurwitz numbers is conjectured in [5]. It is proved there for the case $g = 0$. The answer looks as follows:

$$h^0_{(d);\nu} = (n-1)!d \int\limits_{\overline{\mathcal{M}}_{0,n}} \frac{1}{(1-\nu_1\psi_1)\cdots(1-\nu_n\psi_n)},$$

where $\psi_i$ are the same as in the ELSV formula. For higher genera, the conjectured formula looks very similar, however, it is yet not clear over which moduli space should the integral be taken. On the computational side, [5] suggests the following result, which is a common generalization of (1.1) and [17, Theorem 6]:

$$h^g_{(d);\nu} = r!d^{r-1}\bigl[t^{2g}\bigr]\prod_{k\geqslant 1}\left(\frac{\sinh kt/2}{kt/2}\right)^{c_k},$$

where $r = r^g_{(d);\nu}$, $\{c_k\}$ is a finite sequence defined solely by $\nu$, and $[A]B$ is the coefficient of $A$ in $B$. This result extends previous computations carried out in [11] for $n = 1, 2$.

Much less is known for double Hurwitz numbers with arbitrary $\mu$ and $\nu$. In fact, there are only two general results. First, it is proved in [12] that the exponent of the generating function for the double Hurwitz numbers is a $\tau$-function for the Toda hierarchy of Ueno and Takasaki. Second, Theorem 2.1 in [5] states that for fixed $g$, $m$, and $n$, double Hurwitz numbers are piecewise polynomial in variables $\mu_1, \dots, \mu_m, \nu_1, \dots, \nu_n$, and that the highest degree of this piecewise polynomial is constant and equal to $m + n + 4g - 3$. Moreover, it is proved in [5] that for genus zero case this piecewise polynomial is homogeneous.

In a different direction, Theorem 4.1 of the same paper treats $h^g_{\mu;\nu}$ as a function of $g$ for fixed partitions $\mu$ and $\nu$. In principle, any such function can be obtained by recursive computation; the only case done explicitly is $m = n = 2$ and $\mu_1 > \nu_1 > \nu_2 > \mu_2$. Similar results are also obtained in [11] for all double Hurwitz numbers of degree at most 5.

Still another approach suggested in [5] provides exact formulas for double Hurwitz numbers in genus zero when $m = 2$ or $m = 3$. In particular,

$$h^0_{\mu_1,\mu_2;\nu} = \frac{n!|\mathrm{Aut}\,\nu|}{d}\sum\frac{l(\rho)!\prod_{j\geqslant 1}\rho_j}{|\mathrm{Aut}\,\rho||\mathrm{Aut}\,\sigma||\mathrm{Aut}\,\tau|}\bigl(\mu_1-|\sigma|\bigr)\bigl(\mu_2-|\tau|\bigr)\mu_1^{l(\sigma)-1}\mu_2^{l(\tau)-1}, \qquad (1.2)$$

where the summation is over partitions $\rho = (\rho_1, \dots, \rho_{l(\rho)})$, $\sigma = (\sigma_1, \dots, \sigma_{l(\sigma)})$, $\tau = (\tau_1, \dots, \tau_{l(\tau)})$ with $\rho \cup \sigma \cup \tau = \nu$ and

$$|\sigma| = \sum_{i=1}^{l(\sigma)}\sigma_i < \mu_1, \qquad |\tau| = \sum_{i=1}^{l(\tau)}\tau_i < \mu_2,$$

see [5, Corollary 5.11].

Finally, the approach developed in [10,20] allows in principle to obtain formulas for $h^0_{\mu;\nu}$ as functions of the degree $d$ for a very special choice of the partitions $\mu$ and $\nu$: $\mu = (\alpha, 1, \dots, 1)$, $\nu = (\beta, 1, \dots, 1)$, $\alpha$ and $\beta$ being fixed partitions. For example, for $\alpha = \beta = (3)$ one gets

$$h^0_{3^1 1^{d-3}; 3^1 1^{d-3}} = \frac{3}{4}\left(27d^2 - 137d + 180\right)\frac{d^{d-6}(2d-6)!}{(d-3)!}.$$

In this note we study the homogeneous piecewise polynomial of degree $m+n-3$ defining double Hurwitz numbers $h^0_{\mu;\nu}$. In what follows we always assume that $m+n-3 \geqslant 0$. To formulate the results we need to introduce some notation. We consider $x_1 = \mu_1, \dots, x_m = \mu_m$, $y_1 = \nu_1, \dots, y_n = \nu_n$ as coordinates of a point in $\mathbb{R}^{m+n}$. The *parameter space* is the cone in $\mathbb{R}^{m+n}$ given by the inequalities $x_1 \geqslant \cdots \geqslant x_m \geqslant 0$, $y_1 \geqslant \cdots \geqslant y_n \geqslant 0$ and the equality $\sum_{i=1}^m x_i = \sum_{j=1}^n y_j$. A *resonance* is a hyperplane $x_I = y_J$, where $I \subset [1,m]$ and $J \subset [1,n]$ are proper subsets and $a_K$ stands for $\sum_{i \in K} a_i$ for any sequence $a_1, \dots, a_k$ and any subset $K \subseteq [1,k]$. The connected components of the complement to the union of all resonances are called *chambers*.

**Theorem 1.1.** *Let $(\mu, \nu)$ vary within a closure of a chamber, then double Hurwitz numbers $h^0_{\mu;\nu}$ are given by a homogeneous polynomial of degree $m+n-3$.*

Consider a chamber $C$, and let $P_C$ be the corresponding homogeneous polynomial. The most convenient way to identify $C$ is to pick up a reference point $(\alpha, \beta) \in C$; in this case we write $P_{\alpha;\beta}$ instead of $P_C$. Observe that the only role of the reference point $(\alpha, \beta)$ is to indicate the choice of the chamber, and that its coordinates are not necessary integers. It is clear that $P_{\mu;\nu}(\mu, \nu) = h^0_{\mu;\nu}$; however, $P_C(\mu', \nu')$ may differ from $h^0_{\mu';\nu'}$ if $(\mu', \nu') \notin C$.

The total number of resonances is equal to $2(2^{m-1} - 1)(2^{n-1} - 1)$, since each resonance $x_I = y_J$ can be also written as $x_{\bar{I}} = y_{\bar{J}}$, where bar stands for the complement. In what follows we always assume that $1 \notin I$. Therefore, each chamber is defined by a sequence of $w = 2(2^{m-1} - 1)(2^{n-1} - 1)$ signs of the expressions $x_I - y_J$; observe that the total number of chambers is less than $2^w$, since certain combinations of signs are impossible. We say that two chambers are *neighboring* along the resonance $x_I = y_J$ if the corresponding sign sequences differ only in the position corresponding to this resonance.

Let $C$ and $C'$ be two chambers neighboring along the resonance $x_I = y_J$; without loss of generality we assume that $x_I - y_J > 0$ in $C$.

**Theorem 1.2.** *Let $(\mu, \nu)$ be an arbitrary point in $C$. Then*

$$P_C - P_{C'} = \binom{m+n-2}{|I|+|J|-1}(x_I - y_J) P_{\mu(\bar{I}), \mu_I - \nu_J; \nu(\bar{J})} P_{\mu(I); \nu(J), \mu_I - \nu_J},$$

*where $a(K)$ is the subsequence of $a_1, \dots, a_k$ consisting of terms $a_i$, $i \in K$.*

**Remark.** Here and in what follows we omit the arguments of polynomials whenever this does not lead to a confusion. Usually, the arguments are formed from the components of $x$ and $y$ according to the same rules as the coordinates of the reference point are formed from the parts of $\mu$ and $\nu$. For example, $P_{\mu(I);\nu(J),\mu_I - \nu_J}$ has $|I| + |J| + 1$ arguments. The first $|I|$ of them are $x_i$, $i \in I$, then follow $y_j$, $j \in J$, and the last argument is $x_I - y_J$.

Consider the *totally negative* chamber, that is, the one for which all the signs in the corresponding sequence are negative.

**Theorem 1.3.** *The polynomial corresponding to the totally negative chamber is given by* $(m+n-2)!x_1^{n-1}(x_1+\cdots+x_m)^{m-2}$.

Theorems 1.1–1.3 give rise to recurrence relations expressing double Hurwitz numbers of degree $d$ via double Hurwitz numbers of lesser degrees. In general, these relations are rather cumbersome, however, for $m=2$ one gets a very simple explicit formula, which is easier than (1.2). The details of the corresponding computations are presented in Section 2.

## 2. Computations

### *2.1. General recurrence*

To find a recurrence relation for the double Hurwitz number $h^0_{\mu;\nu}$ via Theorems 1.1–1.3, one has to pick a path connecting the totally negative chamber with the chamber containing the point $(\mu,\nu)$ in the parameter space. If $(\mu,\nu)$ lies on one or more resonances, one can choose any of the adjacent chambers in an arbitrary way. By multiplying all coordinates by a sufficiently big integer $t>d^{m+n}$ and perturbing slightly the resulting point within the same chamber, one can ensure that the obtained point $(t\mu,\nu')$ is in general position, that is, $\nu'_I=\nu'_J$ if and only if $I=J$. Pick the point $(td-m(m-1)/2, m-1, m-2, \dots, 1, \nu')$ in the totally negative chamber and connect it with $(t\mu,\nu')$ by the following path consisting of $m-1$ segments. The first segment is of the form $(td-m(m-1)/2-s, m-1+s, m-2, \dots, 1, \nu')$, $s=0,1,\dots,t\mu_2-m+1$, the second segment is of the form $(td-t\mu_2-(m-1)(m-2)/2-s, t\mu_2, m-2+s, m-3, \dots, 1, \nu')$, $s=0,1,\dots,t\mu_3-m+2$, and so on. It is easy to see that each point on this path belongs to at most one resonance. To formulate the recurrence relation, pick arbitrary numbers $\varepsilon_3,\varepsilon_4,\dots,\varepsilon_m$ and $\delta$ satisfying inequalities $0<\delta<\varepsilon_m<\cdots<\varepsilon_3<1/m$ and denote $\varepsilon=(\varepsilon_3,\dots,\varepsilon_m)$; the exact values of $\varepsilon_i$ and $\delta$ do not have any meaning, since these numbers will be only used to indicate the corresponding chamber. Clearly, a resonance $x_I=y_J$ is intersected by the above path, and hence contributes to $h^0_{\mu;\nu}$, if and only if $\mu_I>\nu_J$. If this is the case, we consider $I=\{i_1,\dots,i_{|I|}\}$ and define $k=\min\{j\in[1,|I|]\colon\ \mu_{i_1}+\cdots+\mu_{i_j}>\nu_J\}$. Any $i\in[1,m]$, $i\neq 1, i_k$, can be related to one of the four subsets: $I_1=\{i\in I\colon\ i<i_k\}$, $\bar I_1=\{i\notin I\colon\ 1<i<i_k\}$, $I_2=\{i\in I\colon\ i>i_k\}$, $\bar I_2=\{i\notin I\colon\ i>i_k\}$. We thus get the following result.

**Theorem 2.1.** *Double Hurwitz numbers are given by*

$$\begin{aligned}h^0_{\mu;\nu}={}&(m+n-2)!d^{m-2}\mu_1^{n-1}+\sum_{\mu_I>\nu_J}\binom{m+n-2}{|I|+|J|-1}(\mu_I-\nu_J)\\&\times P_{\xi_1,\mu(\bar I_1),\varepsilon(\bar I_2),\delta;\nu(\bar J)}\big(\mu_1,\mu(\bar I_1\cup\bar I_2),\mu_I-\nu_J,\nu(\bar J)\big)\\&\times P_{\mu(I_1),\xi_2,\varepsilon(I_2);\nu(J),\delta}\big(\mu(I_1),\mu_{i_k},\mu(I_2),\nu(J),\mu_I-\nu_J\big),\end{aligned}$$

*where* $\xi_1=\nu_{\bar J}-\mu_{\bar I_1}-\varepsilon_{\bar I_2}-\delta$, $\xi_2=\nu_J-\mu_{I_1}-\varepsilon_{I_2}+\delta$.

### 2.2. *Two-part double Hurwitz numbers*

The general expression in Theorem 2.1 looks very cumbersome. However, in the case of two-part double Hurwitz numbers, when $m = 2$, it can be written in a very simple way. Indeed, in this case all the resonances are of the form $x_2 = y_J$, therefore $I = \{2\}$, $i_k = 2$, $I_1 = \bar{I}_1 = I_2 = \bar{I}_2 = \emptyset$. Therefore Theorem 2.1 yields

$$h^0_{\mu_1,\mu_2;\nu} = n!\mu_1^{n-1} + \sum_{\mu_2>\nu_J} \binom{n}{|J|}(\mu_2 - \nu_J) \\ \times P_{\nu_{\bar{J}}-\delta,\delta;\nu(\bar{J})}\big(\mu_1, \mu_2 - \nu_J, \nu(\bar{J})\big) P_{\nu_J+\delta;\nu(J),\delta}\big(\mu_2, \nu(J), \mu_2 - \nu_J\big).$$

The second polynomial in the right-hand side corresponds to one-part double Hurwitz numbers of total degree $\mu_2$ with $|J| + 1$ zeros, hence, by (1.1),

$$P_{\nu_J+\delta;\nu(J),\delta}\big(\mu_2, \nu(J), \mu_2 - \nu_J\big) = |J|!\mu_2^{|J|-1}.$$

The first polynomial in the right-hand side corresponds to the totally negative chamber for two-part double Hurwitz numbers of total degree $d - \nu_J$ with $n - |J|$ zeros, hence, by Theorem 1.3,

$$P_{\nu_{\bar{J}}-\delta,\delta;\nu(\bar{J})}\big(\mu_1, \mu_2 - \nu_J, \nu(\bar{J})\big) = \big(n - |J|\big)!\mu_1^{n-|J|-1}.$$

Observe that the first summand in the right-hand side of the above formula can be also included in the regular part of the sum for $J = \emptyset$. In what follows we indicate this by writing $\sum^{\emptyset}$ instead of $\sum$.

We thus obtain the following explicit formula for the two-part double Hurwitz numbers in genus zero, which is simpler than (1.2).

**Corollary 2.2.** *The two-part double Hurwitz numbers are given by*

$$h^0_{\mu_1,\mu_2;\nu} = n! \sum_{\mu_2>\nu_J}^{\emptyset} (\mu_2 - \nu_J)\mu_1^{n-|J|-1}\mu_2^{|J|-1}.$$

### 2.3. *Three-part double Hurwitz numbers*

Consider now the case of three-part double Hurwitz numbers, when $m = 3$. Then the first segment of the path intersects resonances of the form $x_2 = y_J$ and $x_2 + x_3 = y_J$, while the second segment of the path intersects resonances of the form $x_3 = y_J$ and $x_2 + x_3 = y_J$. Therefore, we have the following four types of intersections.

**Type 1.** $I = \{2\}$, $i_k = 2$, $I_1 = \bar{I}_1 = I_2 = \emptyset$, $\bar{I}_2 = \{3\}$.

By Theorem 2.1, the contribution of such an intersection equals

$$\binom{n+1}{|J|}(\mu_2 - \nu_J) P_{\nu_{\bar{J}}-\varepsilon_3-\delta,\varepsilon_3,\delta;\nu(\bar{J})}\big(\mu_1, \mu_3, \mu_2 - \nu_J, \nu(\bar{J})\big) \\ \times P_{\nu_J+\delta;\nu(J),\delta}\big(\mu_2, \nu(J), \mu_2 - \nu_J\big).$$

The second polynomial in the above expression is the same as in the case of two-part double Hurwitz numbers, and its value is equal to $|J|!\mu_2^{|J|-1}$. The first polynomial corresponds to the totally negative chamber for three-part double Hurwitz numbers of total degree $d-\nu_J$ with $n-|J|$ zeros, hence, by Theorem 1.3,

$$P_{\nu_{\bar J}-\varepsilon_3-\delta,\varepsilon_3,\delta;\nu(\bar J)}\bigl(\mu_1,\mu_3,\mu_2-\nu_J,\nu(\bar J)\bigr)=\bigl(n-|J|+1\bigr)!\mu_1^{n-|J|-1}(d-\nu_J).$$

Therefore, the total contribution of all intersections of Type 1 equals

$$(n+1)!\sum_{\mu_2>\nu_J}(\mu_2-\nu_J)\mu_1^{n-|J|-1}\mu_2^{|J|-1}(d-\nu_J).$$

**Type 2.** $I=\{2,3\}$, $i_k=2$, $I_1=\bar I_1=\bar I_2=\emptyset$, $I_2=\{3\}$.

By Theorem 2.1, the contribution of such an intersection equals

$$\binom{n+1}{|J|+1}(\mu_2+\mu_3-\nu_J)P_{\nu_{\bar J}-\delta,\delta;\nu(\bar J)}\bigl(\mu_1,\mu_2+\mu_3-\nu_J,\nu(\bar J)\bigr)$$
$$\times P_{\nu_J-\varepsilon_3+\delta,\varepsilon_3;\nu(J),\delta}\bigl(\mu_2,\mu_3,\nu(J),\mu_2+\mu_3-\nu_J\bigr).$$

The first polynomial in the above expression we have already encountered in the case of two-part double Hurwitz numbers, and its value is equal to $(n-|J|)!\mu_1^{n-|J|-1}$. The second polynomial corresponds to the chamber neighboring with the totally negative chamber for two-part double Hurwitz numbers of total degree $\mu_2+\mu_3$ with $|J|+1$ zeros, hence, by Corollary 2.2,

$$P_{\nu_J-\varepsilon_3+\delta,\varepsilon_3;\nu(J),\delta}\bigl(\mu_2,\mu_3,\nu(J),\mu_2+\mu_3-\nu_J\bigr)$$
$$=\bigl(|J|+1\bigr)!\bigl(\mu_2^{|J|}+\bigl(\mu_3-(\mu_2+\mu_3-\nu_J)\bigr)\mu_2^{|J|-1}\bigr)=\bigl(|J|+1\bigr)!\mu_2^{|J|-1}\nu_J.$$

Therefore, the total contribution of all intersections of Type 2 equals

$$(n+1)!\sum_{\mu_2>\nu_J}(\mu_2+\mu_3-\nu_J)\mu_1^{n-|J|-1}\mu_2^{|J|-1}\nu_J.$$

**Type 3.** $I=\{3\}$, $i_k=3$, $I_1=I_2=\bar I_2=\emptyset$, $\bar I_1=\{2\}$.

By Theorem 2.1, the contribution of such an intersection equals

$$\binom{n+1}{|J|}(\mu_3-\nu_J)P_{\nu_{\bar J}-\mu_2-\delta,\mu_2,\delta;\nu(\bar J)}\bigl(\mu_1,\mu_2,\mu_3-\nu_J,\nu(\bar J)\bigr)$$
$$\times P_{\nu_J+\delta;\nu(J),\delta}\bigl(\mu_3,\nu(J),\mu_3-\nu_J\bigr).$$

The second polynomial in the above expression is again the same as in the case of two-part double Hurwitz numbers, and its value is equal to $|J|!\mu_3^{|J|-1}$. The first polynomial corresponds to three-part double Hurwitz numbers of total degree $d-\nu_J$ with $n-|J|$ zeros, computed in some chamber intersected by the first segment of our path. To compute these numbers we have to take into account only intersections of Types 1 and 2. By the above reasoning, we get

$$\begin{aligned}
&P_{\nu_{\bar J}-\mu_2-\delta,\mu_2,\delta;\nu(\bar J)}\bigl(\mu_1,\mu_2,\mu_3-\nu_J,\nu(\bar J)\bigr)\\
&\quad=\bigl(n-|J|+1\bigr)!\sum_{\substack{K\subseteq\bar J\\ \mu_2>\nu_K}}^{\emptyset}(\mu_2-\nu_K)(d-\nu_J-\nu_K)\mu_1^{n-|J|-|K|-1}\mu_2^{|K|-1}\\
&\qquad+\sum_{\substack{K\subseteq\bar J\\ \mu_2>\nu_K}}^{\emptyset}\bigl(\mu_2+(\mu_3-\nu_J)-\nu_K\bigr)\mu_1^{n-|J|-|K|-1}\mu_2^{|K|-1}\nu_K.
\end{aligned}$$

Therefore, the total contribution of all intersections of Type 3 equals

$$(n+1)!\sum_{\mu_3>\nu_J}(\mu_3-\nu_J)\mu_3^{|J|-1}\sum_{\substack{K\subseteq\bar J\\ \mu_2>\nu_K}}^{\emptyset}\mu_1^{n-|J|-|K|-1}\mu_2^{|K|-1}\bigl(\mu_2(d-\nu_J)-\nu_K(\mu_1+\mu_2)\bigr).$$

**Type 4.** $I=\{2,3\}$, $i_k=3$, $\bar I_1=I_2=\bar I_2=\emptyset$, $I_1=\{2\}$.

By Theorem 2.1, the contribution of such an intersection equals

$$\begin{aligned}
&\binom{n+1}{|J|+1}(\mu_2+\mu_3-\nu_J)P_{\nu_{\bar J}-\delta,\delta;\nu(\bar J)}\bigl(\mu_1,\mu_2+\mu_3-\nu_J,\nu(\bar J)\bigr)\\
&\quad\times P_{\mu_2,\nu_J-\mu_2+\delta;\nu(J),\delta}\bigl(\mu_2,\mu_3,\nu(J),\mu_2+\mu_3-\nu_J\bigr).
\end{aligned}$$

The first polynomial in the above expression we have already encountered in the case of two-part double Hurwitz numbers, and its value is equal to $(n-|J|)!\mu_1^{n-|J|-1}$. The second polynomial corresponds to arbitrary two-part double Hurwitz numbers of total degree $\mu_2+\mu_3$ with $|J|+1$ zeros. By Corollary 2.2, such numbers are given by

$$\begin{aligned}
&P_{\mu_2,\nu_J-\mu_2+\delta;\nu(J),\delta}\bigl(\mu_2,\mu_3,\nu(J),\mu_2+\mu_3-\nu_J\bigr)\\
&\quad=\bigl(|J|+1\bigr)!\sum_{\substack{K\subseteq J\\ \nu_J-\mu_2>\nu_K}}^{\emptyset}(\mu_3-\nu_K)\mu_2^{|J|-|K|}\mu_3^{|K|-1}\\
&\qquad+\bigl(|J|+1\bigr)!\sum_{\substack{K\subseteq J\\ \nu_J-\mu_2>\nu_K}}^{\emptyset}(\mu_3-\nu_K)\bigl(\mu_3-\nu_K-(\mu_2+\mu_3-\nu_J)\bigr)\mu_2^{|J|-|K|-1}\mu_3^{|K|-1};
\end{aligned}$$

the first sum in the right-hand side corresponds to resonances not involving the last $y$-coordinate, and the second one to those involving this coordinate. Therefore, the total contribution of all intersections of Type 4 equals

$$(n+1)! \sum_{\mu_2<\nu_J<\mu_2+\mu_3} (\mu_2+\mu_3-\nu_J)\mu_1^{n-|J|-1}$$
$$\times \sum_{\substack{K\subseteq J\\ \nu_J-\mu_2>\nu_K}}^{\emptyset} (\mu_3-\nu_K)\mu_2^{|J|-|K|-1}\mu_3^{|K|-1}\big(\nu_J\mu_3-\nu_K(\mu_2+\mu_3)\big).$$

Collecting all the summands and taking into account the contribution of the totally negative chamber, we get the following result.

**Corollary 2.3.** *The three-part double Hurwitz numbers are given by*

$$\frac{h^0_{\mu_1,\mu_2,\mu_3;\nu}}{(n+1)!}$$
$$= \sum_{\mu_2>\nu_J}^{\emptyset} (\mu_2-\nu_J)\mu_1^{n-|J|-1}\mu_2^{|J|-1}A_J$$
$$+ \sum_{\mu_3>\nu_J} (\mu_3-\nu_J)\mu_3^{|J|-1} \sum_{\substack{K\subseteq \bar J\\ \mu_2>\nu_K}}^{\emptyset} \mu_1^{n-|J|-|K|-1}\mu_2^{|K|-1}B_{JK}$$
$$+ \sum_{\mu_2<\nu_J<\mu_2+\mu_3} (\mu_2+\mu_3-\nu_J)\mu_1^{n-|J|-1} \sum_{\substack{K\subseteq J\\ \nu_J-\mu_2>\nu_K}}^{\emptyset} (\mu_3-\nu_K)\mu_2^{|J|-|K|-1}\mu_3^{|K|-1}C_{JK},$$

*where* $A_J = d\mu_2-\nu_J(\mu_1+\mu_2)$, $B_{JK}=(d-\nu_J)\mu_2-\nu_K(\mu_1+\mu_2)$, *and* $C_{JK}=\nu_J\mu_3-\nu_K(\mu_2+\mu_3)$.

## 3. Proofs

### 3.1. Integral representation for double Hurwitz numbers

Define the *Hurwitz space* $\mathcal{H}^0_{\mu;\nu}$ as the space of degree $d$ meromorphic functions on genus 0 curves having $m+n-2$ simple critical values $z_1,\dots,z_{m+n-2}$ and monodromies given by $\mu$ and $\nu$ over two other points $x$ and $y$; the functions are considered modulo $SL(2,\mathbb{C})$-action in the image. We assume that all preimages of the points $x,y,z_1,\dots,z_{m+n-2}$ are labelled. The Lyashko–Looijenga map $\ell\ell$ that takes a function $f\in\mathcal{H}^0_{\mu;\nu}$ to the points $x,y,z_1,\dots,z_{m+n-2}$ can be viewed as an unramified covering of degree $h^0_{\mu;\nu}(d-2)!^{m+n-2}$ over the moduli space $\mathcal{M}_{0,m+n}$ (recall that $m+n\geqslant 3$, and hence the special case of $\mathcal{M}_{0,2}$ does not occur). Note that $\ell\ell$ extends continuously to the mapping $\ell\ell:\overline{\mathcal{H}}^0_{\mu;\nu}\to\overline{\mathcal{M}}_{0,m+n}$, where $\overline{\mathcal{H}}^0_{\mu;\nu}$ is the compactification of $\mathcal{H}^0_{\mu;\nu}$ by admissible covers studied by Ionel (see [9, Definition 1.7]). On the other hand, let $\mathrm{st}:\mathcal{H}^0_{\mu;\nu}\to\mathcal{M}_{0,m+n+(m+n-2)(d-1)}$ be the mapping that takes a function $f$ to the set of all preimages of the points $x,y,z_1,\dots,z_{m+n-2}$ and let $\pi:\mathcal{M}_{0,m+n+(m+n-2)(d-1)}\to\mathcal{M}_{0,m+n}$ be the projection that forgets the preimages of the points $z_1,\dots,z_{m+n-2}$. Both these mappings extend continuously to the mappings between the compactified spaces $\mathrm{st}:\overline{\mathcal{H}}^0_{\mu;\nu}\to\overline{\mathcal{M}}_{0,m+n+(m+n-2)(d-1)}$ and $\pi:\overline{\mathcal{M}}_{0,m+n+(m+n-2)(d-1)}\to\overline{\mathcal{M}}_{0,m+n}$. Denote by $x_1,\dots,x_m$ the preimages of $x$ having multiplicities $\mu_1, \dots, \mu_m$, and by $y_1, \dots, y_n$ the preimages of $y$ having multiplicities $\nu_1, \dots, \nu_n$. Finally, denote by $D$ the divisor on $\overline{\mathcal{M}}_{0,m+n+(m+n-2)(d-1)}$ whose generic point is a two-component curve such that $x_1$ lies on one component and $x_2, \dots, x_m, y_1, \dots, y_n$ lie on the other component.

**Lemma 3.1.** *One has*

$$h^0_{\mu;\nu} = (m+n-2)!\mu_1^{m+n-3} + (d-2)!^{2-m-n} \sum_{u+v=m+n-4} \mu_1 \int_{\mathrm{st}(\overline{\mathcal{H}}^0_{\mu;\nu})} D\big(\mu_1 \pi^* \psi(x_1)\big)^u \big(\mathrm{st}_* \, \ell\!\ell^* \psi(x)\big)^v. \quad (3.1)$$

**Proof.** Indeed, the degree of $\ell\!\ell$ equals $\int_{\overline{\mathcal{H}}^0_{\mu;\nu}} \ell\!\ell^* \omega / \int_{\overline{\mathcal{M}}_{0,m+n}} \omega$ for any top class $\omega$ on $\overline{\mathcal{M}}_{0,m+n}$ such that $\int_{\overline{\mathcal{M}}_{0,m+n}} \omega \neq 0$. It is well known that $\psi(x)^{m+n-3}$ is a top class on $\overline{\mathcal{M}}_{0,m+n}$, and that $\int_{\overline{\mathcal{M}}_{0,m+n}} \psi(x)^{m+n-3} = 1$. Therefore,

$$h^0_{\mu;\nu}(d-2)!^{m+n-2} = \int_{\overline{\mathcal{H}}^0_{\mu;\nu}} \big(\ell\!\ell^* \psi(x)\big)^{m+n-3}.$$

Besides, by the Ionel lemma [9, Lemma 1.17], $\ell\!\ell^* \psi(x) = \mu_1 \, \mathrm{st}^* \, \tilde{\psi}(x_1)$, where tilde stands for the classes on $\overline{\mathcal{M}}_{0,m+n+(m+n-2)(d-1)}$, and by the standard pullback formula, $\tilde{\psi}(x_1) = \pi^* \psi(x_1) + D$. Therefore, on $\mathrm{st}(\overline{\mathcal{H}}^0_{\mu;\nu})$ one can write

$$\big(\mathrm{st}_* \, \ell\!\ell^* \psi(x)\big)^{m+n-3} = \big(\mu_1 \pi^* \psi(x_1)\big)^{m+n-3} + \mu_1 D \sum_{u+v=m+n-4} \big(\mu_1 \pi^* \psi(x_1)\big)^u \big(\mathrm{st}_* \, \ell\!\ell^* \psi(x)\big)^v.$$

Finally, $\pi$ is a covering of degree $(m+n-2)!(d-2)!^{m+n-2}$, hence

$$\int_{\mathrm{st}(\overline{\mathcal{H}}^0_{\mu;\nu})} \big(\mu_1 \pi^* \psi(x_1)\big)^{m+n-3} = (m+n-2)!(d-2)!^{m+n-2} \mu_1^{m+n-3} \int_{\overline{\mathcal{M}}_{0,m+n}} \psi(x_1)^{m+n-3}$$
$$= (m+n-2)!(d-2)!^{m+n-2} \mu_1^{m+n-3},$$

and the result follows. □

### *3.2. Encoding irreducible components of* $\mathrm{st}(\overline{\mathcal{H}}^0_{\mu;\nu}) \cap D$

In view of Lemma 3.1, we will be interested in the description of the irreducible components of $\mathrm{st}(\overline{\mathcal{H}}^0_{\mu;\nu}) \cap D$. Let $f$ be a function whose image belongs to this intersection. Points $x$ and $y$ on the target curve of $f$ belong to different components. Moreover, the number of components is exactly two, for dimensionality reasons. Therefore, the source curve of $f$ has a number of double points, which are all mapped to the unique double point on the target curve. The components of the source curve are of two types: those covering the component of the target curve containing $x$, and those covering the component of the target curve containing $y$. Each component of the first type contains one or more preimages of $x$, and each component of the second type contains one or more preimages of $y$. Finally, the component of the source curve containing $x_1$ does not contain any other preimages of $x$.

The irreducible components of $\mathrm{st}(\overline{\mathcal{H}}^0_{\mu;\nu}) \cap D$ can be encoded by geometric trees in the following way. Consider two arbitrary partitions

$$[1,m] = \bigsqcup_{i=1}^{k} I_i, \quad [1,n] = \bigsqcup_{j=1}^{l} J_j,$$

such that all parts $I_i$ and $J_j$ are nonempty and $I_1 = \{1\}$. Let $T$ be a tree viewed as a bipartite graph with the vertices $I_1, \dots, I_k$ in one part and $J_1, \dots, J_l$ in the other part and let $\gamma_e$ be a weight assigned to an edge $e$ of $T$ in such a way that the sum of $\gamma_e$ over all edges incident to an arbitrary vertex $I_i$ equals $\mu_{I_i}$, and the sum of $\gamma_e$ over all edges incident to an arbitrary vertex $J_j$ equals $\nu_{J_j}$. Evidently, $\gamma_e$ are defined by the above condition in a unique way, and each $\gamma_e$ is an integer. We say that $T$ is a *geometric tree* if the following three conditions are satisfied:

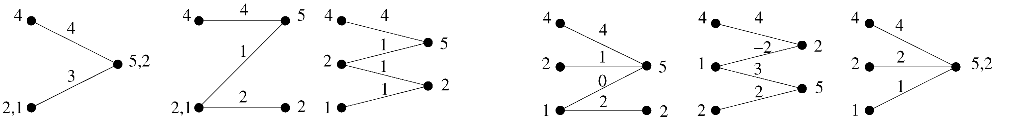
(i) at least one among $I_1, \dots, I_k$ is not a singleton;
(ii) all $\gamma_e$ are positive;
(iii) $I_1$ is a leaf of $T$.

The first condition follows from the fact that the target curve of $f$ is stable. A similar condition for $J_1, \dots, J_l$ follows from (iii) and hence is omitted. The second condition follows from the fact that edges of $T$ correspond to double points of the source curve and weights are the multiplicities at these double points. The third condition is the definition of $D$. The set of all geometric trees is denoted by $\mathcal{T}_{\mu;\nu}$, the irreducible component of $\mathrm{st}(\overline{\mathcal{H}}^0_{\mu;\nu}) \cap D$ encoded by a geometric tree $T \in \mathcal{T}_{\mu;\nu}$ is denoted by $D_T$.

**Example.** Let $\mu = (4,2,1)$, $\nu = (5,2)$. This data defines three geometric trees, presented on Fig. 1a. Three examples of non-geometric trees defined by the same data are given on Fig. 1b. The vertices are labelled by the corresponding subsets of $\mu$ and $\nu$, and the edges are labelled by the weights $\gamma_e$.

**Lemma 3.2.** *The set of geometric trees is an invariant of a chamber.*

**Proof.** It is enough to check that as $(\mu, \nu)$ varies within a chamber, the weights of the edges of a given graph remain positive. Any edge $e$ of the initial tree $T$ defines two subtrees of $T$:




(a) (b)

Fig. 1. (a) Geometric trees; (b) several non-geometric trees.

the connected components of $T \setminus e$. Denote by $I$, $\bar{I}$ the unions of their $\mu$-vertices, and by $J$, $\bar{J}$ the unions of their $\nu$-vertices. It follows immediately from the definition of weights that the weight of $e$ can be written as $\mu_I - \nu_J$. Clearly, this quantity does not change its sign inside a chamber. □

The integral featuring in (3.1) can be rewritten as

$$\int\limits_{\mathrm{st}(\overline{\mathcal{H}}^0_{\mu;\nu})} D\big(\mu_1\pi^*\psi(x_1)\big)^u\big(\mathrm{st}_*\,\ell^*\psi(x)\big)^v = \sum_{T\in T_{\mu;\nu}} \delta_T \int\limits_{D_T} \big(\mu_1\pi^*\psi(x_1)\big)^u\big(\mathrm{st}_*\,\ell^*\psi(x)\big)^v, \tag{3.2}$$

where $\delta_T$ is the multiplicity arising from the non-transversal intersection of $\mathrm{st}(\overline{\mathcal{H}}^0_{\mu;\nu})$ and $D$. This multiplicity can be calculated as follows.

**Lemma 3.3.** *The multiplicity $\delta_T$ is given by*

$$\delta_T = \frac{1}{\mu_1}\prod_{e\in T}\gamma_e.$$

**Proof.** Let $f \in \overline{\mathcal{H}}^0_{\mu;\nu}$ be a function such that $\mathrm{st}(f) \in D_T$. To find $\delta_T$, we have to parametrize the local universal deformation space in a small neighborhood of $C = \mathrm{st}(f)$. Recall that $C_0 = \ell(f)$ is a two-component curve. Following [7, pp. 61, 62], we choose coordinates $u, v$ at the double point of $C_0$ so that $C_0$ is given locally by $uv = 0$. Then, for $t$ suitably chosen, the universal deformation of $C_0$ is given locally near the double point by the equation $uv = t$. Similarly, at each double point $p_e$, $e \in T$, of $C$ the universal deformation of $\mathrm{st}(\overline{\mathcal{H}}^0_{\mu;\nu})$ is defined by $x_e y_e = t_e$ with $u = x_e^{\gamma_e}$, $v = y_e^{\gamma_e}$. We get therefore $t_e^{\gamma_e} = t$, $e \in T$. Recall that $D_T$ locally is given by the equation $t_{e_1} = 0$, where $e_1$ is the edge incident to the leaf $I_1$. Therefore, $\delta_T$ is equal to the multiplicity of the point $\{t_e = 0, e \in T, e \neq e_1\}$ in the curve $\{t_e^{\gamma_e} = t, e \in T, e \neq e_1\}$. The latter is equal to $\frac{1}{\mu_1}\prod_{e\in T}\gamma_e$, since $\gamma_{e_1} = \mu_1$. □

*3.3. Essential geometric trees*

We say that a geometric tree $T \in \mathcal{T}_{\mu;\nu}$ is *essential* if all vertices $J_j$ except for the one connected to $I_1$ are singletons. The set of all essential geometric trees for a given pair $(\mu, \nu)$ is denoted $\mathcal{E}_{\mu;\nu}$. As follows immediately from Lemma 3.2, $\mathcal{E}_{\mu;\nu}$ is an invariant of a chamber. The importance of this notion is revealed in the following statement. Denote by $\mathcal{I}(T, u, v)$ the integral in the right-hand side of (3.2).

**Lemma 3.4.** *For any inessential tree $T \in \mathcal{T}_{\mu;\nu} \setminus \mathcal{E}_{\mu;\nu}$ one has $\mathcal{I}(T, u, v) = 0$.*

**Proof.** For any vertex $I_i$ of $T$ denote by $\gamma(I_i)$ the partition of $\mu_{I_i}$ formed by the weights of the edges incident to $I_i$, and by $\deg I_i$ the number of such edges; $\gamma(J_j)$ and $\deg J_j$ will have a similar meaning for the vertex $J_j$. Let $\overline{\mathcal{H}}^0_{I_1,\dots,I_k}$ denote the compactified Hurwitz space of degree $d$ meromorphic functions on disconnected genus 0 curves with $k$ connected components such that on the $i$th component the function has $|I_i| + \deg I_i - 2$ simple critical values and monodromies given by $\mu(I_i)$ and $\gamma(I_i)$ over the points $x$ and $y$. Note that this space does not coincide with the direct product of the spaces $\overline{\mathcal{H}}^0_{\mu(I_i);\gamma(I_i)}$; they both are obtained from the same space by taking quotient by two different actions: the first one by the action of $\mathbb{C}^*$, and the second one by the action of the direct product of several copies of $\mathbb{C}^*$. Consider a natural mapping

$$\rho_T : D_T \to \prod_{j=1}^{l} \overline{\mathcal{H}}^0_{\nu(J_j);\gamma(J_j)} \times \overline{\mathcal{H}}^0_{I_1,\dots,I_k}. \tag{3.3}$$

Denote by $\overline{\mathcal{H}}$ the space in the right-hand side of the above formula. It is easy to see that the class to be integrated over $D_T$ is a pullback under $\rho_T$ of a certain class on $\overline{\mathcal{H}}$. Therefore, $\mathcal{I}(T,u,v)$ will vanish whenever the dimension of $\overline{\mathcal{H}}$ differs from that of $D_T$.

Let us calculate the dimension of $\overline{\mathcal{H}}$. Clearly, $\dim \overline{\mathcal{H}}^0_{\nu(J_j);\gamma(J_j)} = \max\{|J_j| + \deg J_j - 3, 0\}$. Since $|J_j| + \deg J_j - 3$ equals $-1$ if $J_j$ is a singleton and is nonnegative otherwise, we see that the total dimension of the direct product of all $\overline{\mathcal{H}}^0_{\nu(J_j);\gamma(J_j)}$ equals

$$n + (k + l - 1) - 3l + l^* = n + k - 2l + l^* - 1,$$

where $l^*$ is the number of singletons among $J_1, \dots, J_l$. Further,

$$\dim \overline{\mathcal{H}}^0_{I_1,\dots,I_k} = \sum_{i=1}^{k} \bigl(|I_i| + \deg I_i - 2\bigr) - 1 = m + (k + l - 1) - 2k - 1 = m + l - k - 2.$$

Finally, $\dim D_T = m + n - 4$. Equating the two dimensions we get $l - l^* = 1$, which means that $T$ is essential. □

The value of the integral $\mathcal{I}(T,u,v) = 0$ for essential trees is calculated in the following proposition.

**Lemma 3.5.** *For any essential tree $T \in \mathcal{E}_{\mu;\nu}$ one has*

$$\mathcal{I}(T,u,v) = \begin{cases} (d-2)!^r r! \mu_1^u \prod_{i=1}^k \frac{h^0_{\mu(I_i);\gamma(I_i)}}{r_i!} \prod_{j \in \bar{J}_1} \nu_j^{-1} & \text{if } u = |J_1| + \deg J_1 - 3, \\ 0 & \text{otherwise,} \end{cases}$$

*where $r = m + n - 2$ and $r_i = |I_i| + \deg I_i - 2$.*

**Proof.** Define $\rho_T$ as in (3.3). Since $T$ is essential, $\rho_T$ is a finite map, and its degree is equal to $\binom{m+n-2}{u+1}[(d-2)!/(\nu_{J_1} - 2)!]^{u+1}$. Using a Fubini-type argument we get

$$\mathcal{I}(T,u,v) = \binom{m+n-2}{u+1}\left(\frac{(d-2)!}{(\nu_{J_1} - 2)!}\right)^{u+1} \times \int\limits_{\overline{\mathcal{H}}^0_{\nu(J_1);\gamma(J_1)}} \bigl(\mu_1 \pi^* \psi(x_1)\bigr)^u \prod_{j=2}^{l} \int\limits_{\overline{\mathcal{H}}^0_{\nu(J_j);\gamma(J_j)}} 1 \int\limits_{\overline{\mathcal{H}}^0_{I_1,\dots,I_k}} \bigl(\tilde{\ell\ell}^* \psi(x)\bigr)^v, \tag{3.4}$$

where $\tilde{\mathcal{L}}$ is the Lyashko–Looijenga map for the space $\overline{\mathcal{H}}^0_{I_1,\dots,I_k}$. The first integral in (3.4) vanishes whenever $u$ is distinct from the dimension of $\overline{\mathcal{H}}^0_{\nu(J_1);\gamma(J_1)}$, which is equal to $|J_1| + \deg J_1 - 3$.

From now on we assume that $u = |J_1| + \deg J_1 - 3$. Than the first integral in (3.4) equals $\mu_1^u \int_{\overline{\mathcal{M}}_{0,|J_1|+\deg J_1}} \psi(x_1)^u$ times the transversal multiplicity of the projection $\pi$ at a point of $D_T$. The latter is equal to $(u+1)!(\nu_{J_1} - 2)!^{u+1}$, and the integral over the moduli space equals 1, so the first integral in (3.4) equals $\mu_1^u (u+1)!(\nu_{J_1} - 2)!^{u+1}$.

The dimension of the space $\overline{\mathcal{H}}^0_{I_1,\dots,I_k}$ equals $v$, and hence the last integral in (3.4) gives the degree of $\tilde{\mathcal{L}}$, similarly to the argument in Section 3.1. The degree of $\tilde{\mathcal{L}}$ equals $(d-2)!^{v+1}\binom{v+1}{r_1\cdots r_k}$ times the corresponding double Hurwitz number of disconnected coverings. The latter is equal to the product of double Hurwitz numbers over the components of the covering. Finally, since $T$ is essential, all $J_j$ for $j = 2,\dots,l$ are singletons, hence the $j$th among the $l-1$ intermediate integrals in (3.4) equals $1/\nu_{J_j}$, and their product is $\prod_{j\in\bar{J}_1} \nu_j^{-1}$. Multiplying all the above factors and taking into account that $u + v = m + n - 4$, we arrive to the desired result. □

Let us take a more precise look at essential geometric trees. The structure of such trees is very simple, as shown on Fig. 2. We denote $\mu$-vertices by circles and $\nu$-vertices by squares. Singletons are white and non-singletons are black. An essential geometric tree has a unique black square vertex denoted $J_1$. All the other black vertices are circles; we denote them $\widehat{I_i}$, $i = 1,\dots,\widehat{k}$. Note that the collection of all $\widehat{I_i}$'s forms a proper subset of the initial collection of all $I_i$'s. For any black circle vertex $\widehat{I_i}$ we denote by $\widehat{J_i}$ the (possibly empty) union of all white squares incident to it. Note that in general $\widehat{J_i}$ does not coincide with any of the initial $J_j$.

We can now represent double Huritz numbers as a sum over the set of essential geometric trees of products of double Hurwitz numbers of a smaller size. For unification purposes, we extend the set of essential geometric trees by adding the tree with $m$ white circle vertices corresponding to $\mu_i$, and no black circle or white square vertices. The extended set is denoted $\mathcal{E}^*_{\mu;\nu}$.

**Theorem 3.6.** *Double Hurwitz numbers are given by*

$$h^0_{\mu;\nu} = (m+n-2)! \sum_{T\in\mathcal{E}^*_{\mu;\nu}} \mu_1^{u_T} \prod_{i=1}^{\widehat{k}} (\mu_{\widehat{I_i}} - \nu_{\widehat{J_i}}) \frac{h^0_{\mu(\widehat{I_i});\nu^*(\widehat{J_i})}}{(|\widehat{I_i}| + |\widehat{J_i}| - 1)!}, \tag{3.5}$$

*where* $u_T = |J_1| + \deg J_1 - 3$ *and* $\nu^*(\widehat{J_i})$ *is obtained from* $\nu(\widehat{J_i})$ *by insertion of* $\mu_{\widehat{I_i}} - \nu_{\widehat{J_i}}$ *at the proper place.*

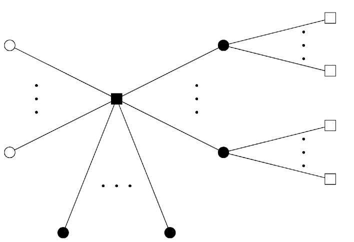


Fig. 2. Essential geometric trees.

**Proof.** Follows from Lemmas 3.1, 3.3–3.5 once we note that edges connecting $J_1$ with white circle vertices contribute $\mu_i$ to $\delta_T$ and $\mu_i^{-1}$ to the product of double Hurwitz numbers in the formula for $\mathcal{I}(T, u, v)$ in Lemma 3.5, so that their contributions are cancelled. Further, edges connecting $J_1$ with black circle vertices contribute $\mu_{\widehat{I_i}} - \nu_{\widehat{J_i}}$ to $\delta_T$. Finally, edges connecting black circle vertices to white square vertices contribute $\nu_j$ to $\delta_T$, and so their contribution is cancelled with the last factor in the formula for $\mathcal{I}(T, u, v)$ in Lemma 3.5. The added tree accounts for the first term in the right-hand side of (3.1), since in this case $|J_1| = n$ and $\deg J_1 = m$. □

*3.4. Proofs of Theorems 1.1–1.3*

Let us start from the following observation.

**Proposition 3.7.** *Resonances for the double Hurwitz numbers $h^0_{\mu(\widehat{I_i});\nu^*(\widehat{J_i})}$ correspond bijectively to resonances $x_I = y_J$ for $h^0_{\mu;\nu}$ with $I \subset \widehat{I_i}$, $J \subset \widehat{J_i}$.*

The proof of Theorem 1.1 follows immediately from Lemma 3.2, Theorem 3.6 and Proposition 3.7 by induction over $m + n$. The base of induction is formed by the cases when either $m$ or $n$ equals 1, and there are no resonances.

To prove Theorem 1.2, observe that by Proposition 3.7, (3.5) can be rewritten as

$$P_{\mu;\nu} = (m + n - 2)! \sum_{T \in \mathcal{E}^*_{\mu;\nu}} x_1^{u_T} \prod_{i=1}^{\widehat{k}} (x_{\widehat{I_i}} - y_{\widehat{J_i}}) \frac{P_{\mu(\widehat{I_i});\nu^*(\widehat{J_i})}}{(|\widehat{I_i}| + |\widehat{J_i}| - 1)!}. \tag{3.6}$$

We denote by $Q^T_{\mu;\nu}$ the contribution of a tree $T$ to the right-hand side of (3.6).

Assume that $(\mu, \nu) \in C$ and $(\mu', \nu') \in C'$. Clearly, $\mathcal{E}^*_{\mu';\nu'} \subset \mathcal{E}^*_{\mu;\nu}$. Therefore, the expression for $P_{\mu;\nu}$ may change for two reasons:

(1) birth of new essential geometric trees, and
(2) changes in the expressions for $Q^T_{\mu;\nu}$.

Let $\mathcal{E}_{\mathrm{res}}$ be the set of essential geometric trees $T \in \mathcal{E}_{\mu';\nu'}$ such that $Q^T_{\mu;\nu} \neq Q^T_{\mu';\nu'}$, and let $\mathcal{E}_{\mathrm{new}} = \mathcal{E}_{\mu;\nu} \setminus \mathcal{E}_{\mu';\nu'}$.

**Lemma 3.8.** *There exists a bijection between $\mathcal{E}_{\mathrm{res}} \cup \mathcal{E}_{\mathrm{new}}$ and $\mathcal{E}^*_{\mu(\bar{I}),\mu_I - \nu_J;\nu(\bar{J})}$.*

**Proof.** The bijection $\Phi$ is presented on Fig. 3. The upper part of the figure corresponds to the trees in $\mathcal{E}_{\mathrm{new}}$, the lower part corresponds to the trees in $\mathcal{E}_{\mathrm{res}}$. The asterisk stands for the additional index corresponding to $\mu_I - \nu_J$ in $\mathcal{E}^*_{\mu(\bar{I}),\mu_I - \nu_J;\nu(\bar{J})}$. □

For a tree $T \in \mathcal{E}_{\mathrm{new}}$ we have

$$Q^T_{\mu;\nu} = (m + n - 2)! x_1 \frac{Q^{T_1}_{\mu(\bar{I});\nu(\bar{J})}}{(m - |I| + n - |J| - 2)!} (x_I - y_J) \frac{P_{\mu(I);\nu(J),\mu_I - \nu_J}}{(|I| + |J| - 1)!}.$$

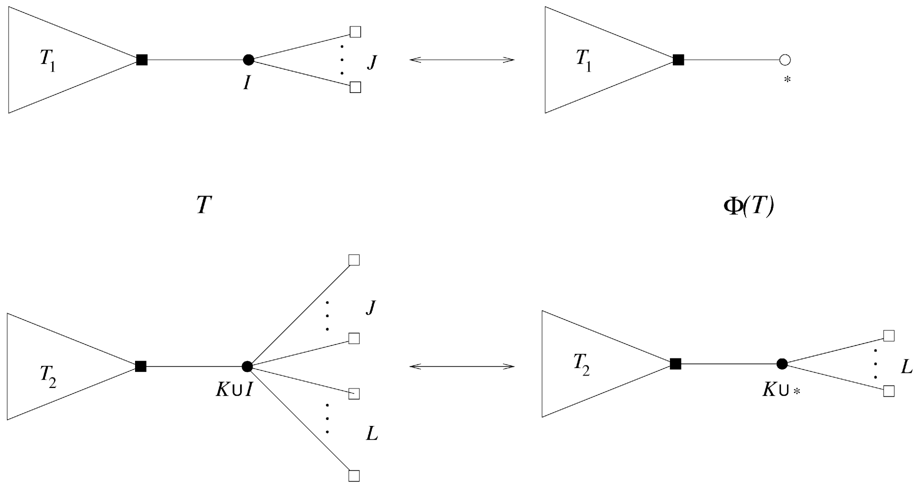



Fig. 3. The definition of the bijection $\Phi$.

Besides, it follows from the definition of $\Phi$ that

$$\frac{x_1 Q^{T_1}_{\mu(\bar I);\nu(\bar J)}}{(m-|I|+n-|J|-2)!} = \frac{Q^{\Phi(T)}_{\mu(\bar I),\mu_I-\nu_J;\nu(\bar J)}}{(m-|I|+n-|J|-1)!},$$

and hence

$$Q^T_{\mu;\nu} = \binom{m+n-2}{|I|+|J|-1} Q^{\Phi(T)}_{\mu(\bar I),\mu_I-\nu_J;\nu(\bar J)}(x_I-y_J)P_{\mu(I);\nu(J),\mu_I-\nu_J}.$$

For a tree $T\in\mathcal{E}_{\mathrm{res}}$ we have

$$Q^T_{\mu;\nu} - Q^T_{\mu';\nu'} = (m+n-2)!x_1\frac{Q^{T_2}_{\mu(\overline{I\cup K});\nu(\overline{J\cup L})}}{(m-|I|-|K|+n-|J|-|L|-2)!}$$
$$\times (x_I+x_K-y_J-y_L)\big(P_{\mu(I\cup K);\nu^*(J\cup L)} - P_{\mu'(I\cup K);\nu'^*(J\cup L)}\big).$$

By induction over $m+n$, the latter difference equals

$$\binom{|I|+|K|+|J|+|L|-1}{|I|+|J|-1}(x_I-y_J)P_{\mu(K),\mu_I-\nu_J;\nu^*(L)}P_{\mu(I);\nu(J),\mu_I-\nu_J}.$$

Besides, it follows from the definition of $\Phi$ that

$$\frac{x_1 Q^{T_2}_{\mu(\overline{I\cup K});\nu(\overline{J\cup L})}}{(m-|I|-|K|+n-|J|-|L|-2)!}(x_I+x_K-y_J-y_L)\frac{P_{\mu(K),\mu_I-\nu_J;\nu^*(L)}}{(|K|+|L|)!}$$
$$= \frac{Q^{\Phi(T)}_{\mu(\bar I),\mu_I-\nu_J;\nu(\bar J)}}{(m-|I|+n-|J|-1)!},$$

and hence

$$Q^T_{\mu;\nu} - Q^T_{\mu';\nu'} = \binom{m+n-2}{|I|+|J|-1} Q^{\Phi(T)}_{\mu(\bar{I}),\mu_I-\nu_J;\nu(\bar{J})}(x_I - y_J) P_{\mu(I);\nu(J),\mu_I-\nu_J}.$$

Summing up the contributions of all trees in $\mathcal{E}_{\mathrm{new}} \cup \mathcal{E}_{\mathrm{res}}$ and using Lemma 3.8 we get

$$\begin{aligned} P_C - P_{C'} &= \binom{m+n-2}{|I|+|J|-1}(x_I - y_J) P_{\mu(I);\nu(J),\mu_I-\nu_J} \sum_{T \in \mathcal{E}^*_{\mu(\bar{I}),\mu_I-\nu_J,\nu(\bar{J})}} Q^{\Phi(T)}_{\mu(\bar{I}),\mu_I-\nu_J;\nu(\bar{J})} \\ &= \binom{m+n-2}{|I|+|J|-1}(x_I - y_J) P_{\mu(I);\nu(J),\mu_I-\nu_J} P_{\mu(\bar{I}),\mu_I-\nu_J;\nu(\bar{J})}, \end{aligned}$$

as required.

It remains to prove Theorem 1.3. We start from expression (3.5). Note that essential geometric trees for the totally negative chamber have a very simple structure: they do not have white square vertices. Therefore,

$$\mu_{\widehat{I_i}} - \nu_{\widehat{J_i}} = \mu_{\widehat{I_i}} \quad \text{and} \quad h^0_{\mu(\widehat{I_i});\nu^*(\widehat{J_i})} = \big(|\widehat{I_i}| - 1\big)!\,\mu_{\widehat{I_i}}^{|\widehat{I_i}|-2}.$$

Finally, $|J_1| = n$, and hence the double Hurwitz numbers for the totally negative chamber are given by

$$(m+n-2)!\,\mu_1^{n-1} \sum_{k=1}^{m-1} \mu_1^{k-1} \sum_{I_1,\ldots,I_k} \prod_{i=1}^{k} \mu_{I_i}^{|I_i|-1},$$

where the inner sum is taken over all unordered partitions of $[2,m]$ into $k$ nonempty parts $I_1,\ldots,I_k$. It is easy to see that

$$\sum_{k=1}^{m-1} x_1^k \sum_{I_1,\ldots,I_k} \prod_{i=1}^{k} x_{I_i}^{|I_i|-1} = x_1(x_1 + \cdots + x_m)^{m-2}, \tag{3.7}$$

since both parts of the above formula enumerate trees on $[1,n]$ rooted at 1 classified according to the degrees of the vertices (see e.g. [13, Cayley's tree volume formula]. Therefore, the double Hurwitz numbers in question equal $(m+n-2)!\,\mu_1^{n-1}(\mu_1 + \cdots + \mu_m)^{m-2}$, as required.

**Remark.** Identity (3.7) is a Hurwitz-type multinomial identity, see [13] and references therein. Identities of this kind were discovered by Hurwitz in [8] and, apparently, used by him in his studies of Hurwitz numbers (see [18] for a conjectural reconstruction of the original Hurwitz derivation for $h^0_\mu$). It is interesting to note that such identities arose again recently in connection with Gromov–Witten invariants in [3].

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