Codiameters of 3-domination critical graphs with toughness more than one

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Abstract

A graph $G$ is 3-domination-critical (3-critical, for short), if its domination number $\gamma$ is 3 and the addition of any edge decreases $\gamma$ by 1. In this paper, we show that every 3-critical graph with independence number 4 and minimum degree 3 is Hamilton-connected. Combining the result with those in [Y.J. Chen, F. Tian, B. Wei, Hamilton-connectivity of 3-domination critical graphs with $\alpha \leq \delta$, Discrete Mathematics 271 (2003) 1–12; Y.J. Chen, F. Tian, Y.Q. Zhang, Hamilton-connectivity of 3-domination critical graphs with $\alpha = \delta + 2$, European Journal of Combinatorics 23 (2002) 777–784; Y.J. Chen, T.C.E. Cheng, C.T. Ng, Hamilton-connectivity of 3-domination critical graphs with $\alpha = \delta + 1 \geq 5$, Discrete Mathematics 308 (2008) (in press)], we solve the following conjecture: a connected 3-critical graph $G$ is Hamilton-connected if and only if $\tau(G) > 1$, where $\tau(G)$ is the toughness of $G$.

Keywords: Domination-critical graph; Hamilton-connectivity

1. Introduction

Let $G = (V(G), E(G))$ be a graph. For the notations that are not defined here, we follow [2]. A graph $G$ is said to be $t$-tough if for every cutset $S \subseteq V(G)$, $|S| \geq t \omega(G - S)$, where $\omega(G - S)$ is the number of components of $G - S$. The toughness of $G$, denoted by $\tau(G)$, is defined to be $\min\{|S|/\omega(G - S) \mid S$ is a cutset of $G\}$. Let $u, v \in V(G)$ be any two distinct vertices. We denote by $p(u, v)$ the length of a longest path connecting $u$ and $v$. The codiameter of $G$, denoted by $d^*(G)$, is defined to be $\min\{p(u, v) \mid u, v \in V(G)\}$. A graph $G$ of order $n$ is said to be Hamilton-connected if $d^*(G) = n - 1$, i.e., every two distinct vertices are joined by a hamiltonian path. A graph $G$ is called $k$-domination critical, abbreviated as $k$-critical, if $\gamma(G) = k$ and $\gamma(G + e) = k - 1$ holds for any $e \in E(\overline{G})$, where $\overline{G}$ is the complement of $G$. The concept of domination-critical graphs was introduced by Sumner [7]. Given three vertices $u, v$ and $x$ such that $\{u, x\}$ dominates $V(G) - \{v\}$ but not $v$, we will write $[u, x] \rightarrow v$. It was observed in [7] that if $u, v$ are any two nonadjacent vertices of a 3-critical graph $G$, then since $\gamma(G + uv) = 2$, there exists a vertex $x$ such that either $[u, x] \rightarrow v$ or $[v, x] \rightarrow u$. In [2], Chen et al. posed the following.

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Conjecture 1 (Chen et al. [2]). A connected 3-critical graph $G$ is Hamilton-connected if and only if $\tau(G) > 1$.

In the same paper, they proved that the conjecture is true when $\alpha(G) \leq \delta(G)$.

Theorem 1 (Chen et al. [2]). Let $G$ be a connected 3-critical graph with $\alpha(G) \leq \delta(G)$. Then $G$ is Hamilton-connected if and only if $\tau(G) > 1$.

Let $G$ be a 3-connected 3-critical graph. It is shown in [3] that $\tau(G) \geq 1$ and $\tau(G) = 1$ if and only if $G$ belongs to a special infinite family $\mathcal{G}$ described in [3]. Since $\alpha(G) = \delta(G) = 3$ for each $G \in \mathcal{G}$, we have $\tau(G) > 1$ if $\alpha(G) \geq \delta(G) + 1$.

In [4], Chen et al. showed that the conjecture holds when $\alpha(G) = \delta(G) + 2$.

Theorem 2 (Chen et al. [4]). Let $G$ be a 3-connected 3-critical graph with $\alpha(G) = \delta(G) + 2$. Then $G$ is Hamilton-connected.

By a result of Favaron et al. [6] which states that $\alpha(G) \leq \delta(G) + 2$ for any connected 3-critical graph $G$, we see that the conjecture has only one case $\alpha(G) = \delta(G) + 1$ unsolved.

Recently, Chen et al. [5] showed that the conjecture is true for $\alpha(G) = \delta(G) + 1 \geq 5$.

Theorem 3 (Chen et al. [5]). Let $G$ be a 3-connected 3-critical graph with $\alpha(G) = \delta(G) + 1 \geq 5$. Then $G$ is Hamilton-connected.

Since $\tau(G) > 1$ implies $\delta(G) \geq 3$, the case $\alpha(G) = \delta(G) + 1 = 4$ remains open. In this paper, we will show that the conjecture is true when $\alpha(G) = \delta(G) + 1 = 4$. The main result of this paper is the following.

Theorem 4. Let $G$ be a 3-connected 3-critical graph with $\alpha(G) = \delta(G) + 1 = 4$. Then $G$ is Hamilton-connected.

Combining Theorems 1–4, we have the following.

Theorem 5. A connected 3-critical graph $G$ is Hamilton-connected if and only if $\tau(G) > 1$.

By the main result of [3], we have the following.

Theorem 6. Let $G$ be a 3-connected 3-critical graph. Then $G$ is Hamilton-connected if and only if $G$ does not belong to a special infinite family $\mathcal{G}$ described in [3].

Now, we restate a result due to Chen et al. for later use.

Theorem 7 (Chen et al. [1]). Let $G$ be a 3-connected 3-critical graph of order $n$. Then $d^*(G) \geq n - 2$.

2. Some lemmas

Let $G$ be a graph of order $n$, and $x$, $y$ vertices of $G$ such that a longest $(x, y)$-path is of length $n - 2$. Let $P = P_{xy}$ be an $(x, y)$-path of length $n - 2$. We denote by $x_P$ the only vertex not in $P$ and let $d(x_P) = k$ with

\[ N(x_P) = X = \{x_1, x_2, \ldots, x_k\}, \text{ indices following the orientation of } P; \]

\[ A = X^+ = \{a_1, a_2, \ldots, a_s\}, \text{ where } a_i = x_i^+, x_i^+ \in P \text{ and } s \geq k - 1; \]

\[ B = X^- = \{b_1, b_t+1, \ldots, b_k\}, \text{ where } b_t = x_t^-, x_t^- \in P \text{ and } t \leq 2; \text{ and } \]

\[ P_i = a_i \overline{P}_i b_{i+1}, \text{ where } 1 \leq i \leq k - 1. \]

Furthermore, we let $P_0 = x \overline{P}_1 b_1$ if $x \not\in X$ and $P_k = a_k \overline{P}_k y$ if $y \not\in X$. The length of the path $x_i \overline{P}_i x_k$ is denoted by $s(P)$.

**Definition.** A vertex $v \in P_i$ $(1 \leq i \leq k)$ is called an $A$-vertex if $G[P_i \cup \{x_{i+1}\}]$ contains a hamiltonian $(v, x_{i+1})$-path and $v \in P_i$ $(0 \leq i \leq k - 1)$ a $B$-vertex if $G[P_i \cup \{x_i\}]$ contains a hamiltonian $(x_i, v)$-path, where $x_{k+1} = y$ and $x_0 = x$. 


From the definition, we can see that each $a_i$ is an $A$-vertex and each $b_i$ is a $B$-vertex. Furthermore, if $v \in P_i$ ($i \neq 0$) and $v \neq a_i \in E(G)$, then $v$ is an $A$-vertex and if $v \in P_i$ ($i \neq k$) and $v \neq b_{i+1} \in E(G)$, then $v$ is a $B$-vertex.

**Lemma 1** (Chen et al. [5]). If $u_i \in P_i$ and $u_j \in P_j$ are two $A$-vertices ($B$-vertices, respectively) with $i \neq j$, then $x_p u_i \notin E(G)$ and $u_i u_j \notin E(G)$. In particular, both $A \cup \{x_p\}$ and $B \cup \{x_p\}$ are independent sets.

**Lemma 2** (Chen et al. [5]). Let $u_i \in P_i$, $u_j \in P_j$ be $A$-vertices with $i < j$, $Q_i$ and $Q_j$ are hamiltonian $(u_i, x_{i+1})$-path and $(u_j, x_{j+1})$-path in $G[P_i \cup \{x_{i+1}\}]$ and $G[P_j \cup \{x_{j+1}\}]$, respectively, $Q = u_i \overrightarrow{Q} x_{i+1} \overrightarrow{P} x_j$ and $R = u_j \overrightarrow{Q} x_{j+1} \overrightarrow{P} y$. If $v \in N_Q (u_i)$, then $v \neq N(u_j)$ and if $v \in N(u_i) \cap (x \overrightarrow{P} x_j \cup R)$, then $v^+ \notin N(u_i)$. In particular, let $a_i, a_j \in A$ with $i < j$ and $v \in N(a_i)$, then $v^- \notin N(a_j)$ if $v \neq a_i \overrightarrow{P} x_j$ and $v^+ \notin N(a_j)$ if $v \neq x \overrightarrow{P} x_i \cup a_j \overrightarrow{P} y$.

By the symmetry of $A$ and $B$, **Lemma 2** still holds if we exchange $A$ and $B$.

**Lemma 3** (Chen et al. [5]). Let $u, v \in a_i \overrightarrow{P} b_j$ with $j \geq i + 1$ and $G[a_i \overrightarrow{P} b_j]$ contain a hamiltonian $(u, v)$-path. Suppose that $w \in x \overrightarrow{P} x_i \cup j \overrightarrow{P} y$ and $uw \in E(G)$. Then $w^- v \notin E(G)$ if $w^- \in x \overrightarrow{P} x_i \cup x_j \overrightarrow{P} y$ and $w^+ v \notin E(G)$ if $w^+ \in x \overrightarrow{P} x_i \cup x_j \overrightarrow{P} y$. In particular, let $a_i \in A$ and $b_j \in B$ with $j \geq i + 1$. Suppose that $v \in x \overrightarrow{P} x_i \cup x_j \overrightarrow{P} y$ and $a_i v \in E(G)$. Then, $v^- b_j \notin E(G)$ if $v^- \in x \overrightarrow{P} x_i \cup x_j \overrightarrow{P} y$, and $v^+ b_j \notin E(G)$ if $v^+ \in x \overrightarrow{P} x_i \cup x_j \overrightarrow{P} y$.

**Lemma 4** (Chen et al. [5]). Let $u, u^+ \in P_i$. If $u^+ a_i \in E(G)$ for some $l \geq i + 1$, then $b_j u \notin E(G)$ for all $j \leq l$.

**Lemma 5** (Chen et al. [2]). Let $|P_i| \geq 2$, $u, v \notin P_i$ and $\{u, v\} \supset P_i$. If $u a_i, v b_{i+1} \in E(G)$, then there exists some vertex $w$ in $P_i$ such that $u w, v w^+ \in E(G)$.

**Lemma 6** (Chen et al. [5]). Let $i \geq 2$, $z \in P_j$ and $[a_i, z] \rightarrow x_p$. If $|A| \geq 3$ and $j \neq i - 1$, then $A \cup \{z^+, x_p\}$ is an independent set if $z^+ \in P$ and $B \cup \{z^-, x_p\}$ is an independent set if $z^- \in P$.

**Lemma 7.** Let $|A| = |B| = 3$, $z \in P_j$ and $[x_p, z] \rightarrow a_i$. If $z^- \in P$, then $B \cup \{x_p, z^-\}$ is an independent set.

**Proof.** Suppose to the contrary that there is some $b_j$ such that $b_j z^- \notin E(G)$. If $l = j + 1$, then $z$ is a $B$-vertex, which contradicts **Lemma 1** since $|B| = 3$ and $B - \{a_i\} \subseteq N(z)$ and $l \neq j + 1$, then $l = 2$ or $3$ for otherwise we have $a_2, a_3 \notin N(z)$ by **Lemma 4**. If $j = 2$ and $l = 1$, then by **Lemma 2** and 4, we have $b_2, a_3 \notin N(z)$, and if $j = 2$ and $l = 2$, then by **Lemma 3** and 4, $a_1, a_3 \notin N(z)$, a contradiction. Thus, we may assume $j = 3$. If $l = 3$, then by **Lemma 3**, $a_1, a_2 \notin N(z)$; if $l = 2$, then by **Lemma 2** and 3, $b_3, a_1 \notin N(z)$; and if $l = 1$, then by **Lemma 2**, $b_2, b_3 \notin N(z)$, a contradiction. If $l > j + 1$, then $b_j z \in E(G)$, by **Lemma 2** we have $j = 0$. If $l = 2$, then by **Lemma 2** and 3, $b_3, a_1 \notin N(z)$ and if $l = 3$, then by **Lemma 3**, $a_1, a_2 \notin N(z)$, a contradiction. Since $|A| = 3$ and $A - \{a_i\} \subseteq N(z)$, by **Lemma 1** we have $z \notin A$, which implies that $z^- x_p \notin E(G)$. Thus, $B \cup \{x_p, z^-\}$ is an independent set.

Now, let $G$ be a 3-critical graph, $\alpha(G) = \delta(G) + 1$ and $v_0 \in V(G)$ with $d(v_0) = \delta(G) = 3$. Suppose that $N(v_0) = \{v_1, v_2, v_3\}$ and $I = \{v_0, \{v_1, v_2, v_3\}\}$ is an independent set. The following lemma restates a lemma due to Sumner and Blitch [7], which has become of considerable utility in dealing with 3-critical graphs. In [7] they considered the case $l \geq 4$, which guarantees $P(W) \cap W = \emptyset$. For the cases $l = 2$ and $l = 3$, **Lemma 8** can be easily verified since $G$ is a 3-critical graph.

**Lemma 8.** Let $G$ be a connected 3-critical graph and $U$ an independent set of $l \geq 2$ vertices. Then there exists an ordering $u_1, u_2, \cdots, u_i$ of the vertices of $U$ and a sequence $P(U) = (y_1, y_2, \cdots, y_{l-1})$ of $l - 1$ distinct vertices such that $[u_i, y_i] \rightarrow u_{i+1}$, $1 \leq i \leq l - 1$.

The next lemma is a useful consequence of **Lemma 8**.

**Lemma 9** (Favaron et al. [6]). Let $U$ be an independent set of $l \geq 3$ vertices of a 3-critical graph $G$ such that $U \cup \{v\}$ is independent for some $v \notin U$. Then the sequence $P(U)$ defined in **Lemma 8** is contained in $N(v)$.

Since $I$ is an independent set of order 4, by **Lemma 8** and 9, we may assume without loss of generality that $[w_i, v_i] \rightarrow w_{i+1}$ for $i = 1, 2$. 
Lemma 10 (Chen et al. [5]). If $[v_0, z] \rightarrow w_i$ for $i \neq 3$, then we have $z \notin N(v_0)$ and if $[v_0, v_i] \rightarrow w_3$, then $l = 2$.

Lemma 11 (Chen et al. [5]). If $[v_0, v_2] \rightarrow w_3$, then we have $v_1, v_2, w_3 \notin N(v_3)$ and $w_1, w_2 \in N(v_3)$.

Lemma 12. Let $G$ be 3-critical, $X = \{x_1, x_2, x_3\} = \{x_i, x_j, x_l\}$ and $\{x_P, a_i, u, v\}$ a maximum independent set. If $[x_P, x_l] \rightarrow a_i$, then we have $x_l x_j \in E(G)$, $x_i x_j \notin N(x_j)$ and $\{x_l, x_j\} \subseteq N(u) \cap N(v)$.

Proof. Let $U = \{a_i, u, v\} = \{u_1, u_2, u_3\}$. By Lemmas 8 and 9, we may assume that $[u_m, x_{q_m}] \rightarrow u_{m+1}$ for $m = 1, 2$. Let $X - \{x_1, x_2\} = \{x_3\}$. If $[x_P, x_l] \rightarrow a_i$, then by Lemma 10, we have $a_i = u_3$ and $x_l = x_3$. Since $[u_1, x_3] \rightarrow u_2$, we have $x_3 a_i \notin E(G)$. By Lemma 11, $x_3 a_i \notin E(G)$. Thus, since $x_i \in X$ and $x_3 a_i \in E(G)$, we have $x_3 a_i = x_i$ and $x_3 = x_j$, that is, $[u_1, x_3] \rightarrow u_2$ and $[u_2, x_1] \rightarrow a_i$. In this case, we have $x_l x_j \in E(G)$ and by Lemma 11, we have $x_l x_j \notin N(x_j)$ and $\{x_l, x_j\} \subseteq N(u) \cap N(v)$. 

The following two lemmas can be extracted from [2].

Lemma 13 (Chen et al. [2]). Suppose that $P$ is a longest $(x, y)$-path such that $|X \cap \{x, y\}|$ is as small as possible and for this path, $d(x_P) = k \geq 4$. If $G$ is 3-critical, then there exists an independent set $I$ such that either $\{x_P\} \cup A \subseteq I$ or $\{x_P\} \cup B \subseteq I$ and $|I| \geq k + 1$.

Lemma 14 (Chen et al. [2]). Let $G$ be a 3-connected 3-critical graph of order $n$, $x, y \in V(G)$ and $p(x, y) = n - 2$. Suppose that $P$ is a longest $(x, y)$-path such that $d(x_P)$ is as large as possible and subject to this, $|X \cap \{x, y\}|$ is as small as possible. If $d(x_P) = 3$, $\{x, y\} \subseteq X$ and $P_i$ is a clique for $i = 1, 2$, then $a_1 b_3 \notin E(G)$, and if $a_2 b_2 \in E(G)$, then $n = 8$ and $\alpha(G) = 3$.

3. Proof of Theorem 4

Let $G$ be a 3-connected 3-critical graph with $\alpha(G) = \delta(G) + 1 = 4$. We still use the notations given in Section 3. Suppose to the contrary that $G$ is not Hamilton-connected. By Theorem 7, there are two vertices $x, y$ such that $p(x, y) = n - 2$. Among all the longest $(x, y)$-paths, we choose $P$ such that

(a) $d(x_P)$ is as large as possible;
(b) subject to (a), $|\{x, y\} \cap N(x_P)|$ is as small as possible;
(c) subject to (a) and (b), $s(P)$ is as small as possible.

Choose an orientation such that $|A| \geq |B|$. Assume without loss of generality that the orientation is from $x$ to $y$. Since $\alpha(G) = \delta(G) + 1 = 4$, by the choice of $P$ and Lemma 13, we have $d(x_P) = 3$.

We consider the following two cases separately.

Case 1. $|A| = 3$

Let $U = N[x_P] \cup A$. If $|A| = 3$, then by Lemmas 8 and 9, we may assume that $[a_{l_i}, x_{l_j}] \rightarrow a_{l_{i+1}}$ for $l = 1, 2$. Thus, noting that $|A| = 3$, we have

$$d_U(x_i) \geq 3 \quad \text{for any} \quad x_i \in X.$$  

If $[a_3, b_3] \rightarrow x_P$, then $b_2 a_3, a_1 b_3 \in E(G)$ by Lemma 1. In this case, we have $|P_2| \geq 2$ and hence $d(x_3) \geq 4$ by (1). Thus, $Q = x_P x_1 x_P x_2 x_P b_3 a_1 b_2 a_3 y$ is an $(x, y)$-path of length $n - 2$ with $x_Q = x_3$, which contradicts the choice of $P$ and hence

$$[a_3, b_3] \rightarrow x_P$$ is impossible.

Claim 1. Let $z \in P_2$ and $[x_P, z] \rightarrow a_i$. If $z^+ \in P$, then $A \cup \{x_P, z^+\}$ is an independent set.

Proof. If $|B| = 3$, then since $B - \{a_i\} \subseteq N(z)$, by Lemma 1 we have $z \notin B$. If $|B| = 2$ and $z = b_2$, then we must have $a_2 = b_3 = a_1$. Since $P_3 \subseteq N(z)$, by Lemmas 1 and 2 we have $N(a_1) \cap P_3 = \emptyset$. Thus, by the choice of $P$, we have $N(a_1) = X$, which contradicts $\tau(G) > 1$ since $\omega(G - X) \geq 3$. If $|B| = 2$ and $z = b_3$, then $a_1 = b_2 = a_i$. Since $P_3 \subseteq N(z)$, by Lemmas 1 and 3 we have $N(a_1) \cap P_3 = \emptyset$. If $a_1 x_3 \in E(G)$, then by the choice of $P$, we have $N(a_1) = X$, which contradicts $\tau(G) > 1$. If $x_3 a_i \notin E(G)$, then $P' = x_P x_2 x_3 y$ is an $(x, y)$-path of length $n - 2$.
such that \( s(P') < s(P) \), a contradiction. Therefore, we have \( z \notin B \) and hence \( z^+x_p \notin E(G) \). Thus, by Lemma 1, we need only to show that \( A \cup \{z^+\} \) is an independent set.

Suppose to the contrary there is some \( a_1 \) such that \( a_1z^+ \in E(G) \). If \( l = j \), then \( z \) is an \( A \)-vertex, which contradicts Lemma 1 since \( |A| = 3 \) and \( A \setminus \{a_i\} \subseteq N(z) \). If \( l < j \), then by Lemmas 2 and 3, we have \( a_{j+1}b_j \notin N(z) \), which implies that \( j = 3 \). If \( l = 1 \), then by Lemma 3, we have \( b_2, b_3 \notin N(z) \) and if \( l = 2 \), then by Lemmas 2 and 3 we have \( a_1, b_3 \notin N(z) \), a contradiction. Thus, we have \( l > j \).

If \( |B| = 3 \), then since \( b_1z \in E(G) \), by Lemma 4 we have \( j = 0 \). Thus, if \( l = 1 \), then by Lemma 3 we have \( b_2, b_3 \notin N(z) \); if \( l = 2 \), then by Lemmas 2 and 3, we have \( a_1, b_3 \notin N(z) \); and if \( l = 3 \), then by Lemma 2, we have \( a_1, a_2 \notin N(z) \), a contradiction. Thus, \( |B| = 2 \).

If \( j = 2 \), then \( l = 3 \). By Lemma 4 we have \( b_2z \notin E(G) \), which implies that \( a_1 = b_2 = a_i \). Let \( Q = xx_pPz \). Obviously, \( |Q| = n - 1 \) and \( xQ = a_1 \). By the choice of \( P \), we have \( d(a_1) = 3 \). If \( N(a_1) \cap P_3 \neq \emptyset \), say \( v \in N(a_1) \cap P_3 \), then the \((x, y)\)-path \( xx_pPz \notin E(G) \), which implies that \( z^+ \) is a \( B \)-vertex. If \( j = 1 \) and \( l = 3 \), then by Lemma 2 we have \( z_3 \notin E(G) \), which implies \( a_2 = b_3 = a_i \). This contradicts Lemma 1 since \( b_2z \notin E(G) \), which implies that \( z^+ \) is a \( B \)-vertex. If \( j = 2 \) and \( l = 3 \), then by Lemma 2 we have \( z_3 \notin E(G) \), which implies \( a_2 = b_3 = a_i \). If \( N(a_2) \cap P_3 \neq \emptyset \), say \( v \in N(a_2) \cap P_3 \), then the \((x, y)\)-path \( x \notin E(G) \), which implies \( a_2 = b_3 = a_i \). This contradicts \( a_2 \) and \( b_3 \) are \( P \)-vertices for any \( a_i \), and hence \( A \cup \{x_p, z^+\} \) is an independent set.

\begin{claim}
Let \( v \in P_1 \), where \( 1 \leq i \leq 3 \). If \( a_i = a_1 \), then \( v \notin E(G) \).
\end{claim}

\textbf{Proof.} If \( v = a_i \), then \( v \in \{a_1, a_2, a_3\} \) and \( \{x_p, z^+\} \) is an independent set of order 5, a contradiction.

\begin{claim}
If \( x^+ = a_1 \in E(G) \), then \( v \notin E(G) \).
\end{claim}

\textbf{Proof.} If \( x^+ = a_1 \in E(G) \), then \( v \notin E(G) \).

\begin{claim}
If \( \{a_2, a_3\} \rightarrow x \), then \( B \cup \{x, z^+\} \) is an independent set.
\end{claim}

\textbf{Proof.} If \( z^+ = b_3 \notin E(G) \), then \( z^+ \) is a \( B \)-vertex. By Lemma 2, \( b_2 \notin E(G) \), which implies that \( b_2 = a_2 \). By Lemma 3, \( b_2 \notin E(G) \), which implies that \( z^+ \) is a \( B \)-vertex. If \( a_2 \notin E(G) \), then \( a_2 \neq a_1 \). Therefore, there is some vertex \( w \) such that \( \{a_1, w\} \rightarrow B \). Thus, we have \( z^+ \) is a \( B \)-vertex. If \( a_2 \notin E(G) \), then \( a_2 \neq a_1 \). Therefore, there is some vertex \( w \) such that \( \{a_1, w\} \rightarrow B \). Thus, we have \( z^+ \) is a \( B \)-vertex.

If \( z^+ = b_3 \notin E(G) \), then by Lemma 1 we have \( a_2 \notin E(G) \) since \( a_1 \notin E(G) \), which implies that \( z^+ \) is an \( A \)-vertex. If \( a_2 \notin E(G) \), then \( x \notin E(G) \), which implies that \( a_2 \neq a_1 \). Therefore, there is some vertex \( w \) such that \( \{a_1, w\} \rightarrow B \). Thus, we have \( z^+ \) is a \( B \)-vertex.
Claim 4. If $z \in P_2$ and $[a_3, z] \rightarrow x_p$, then $B \cup \{x_p, z^-\}$ is an independent set.

Proof. Since $za_1 \in E(G)$, we have $b_2z^- \not\in E(G)$ by Lemma 3. Since $z \in P_2$ and $za_1 \in E(G)$, by Lemma 1, $|P_2| \geq 2$. By (1), $d(x_3) \geq 4$ and $d(x_1) \geq 4$ if $|B| = 3$. If $z^-b_1 \in E(G)$ or $z^-b_3 \in E(G)$, then by Lemma 2, we have $zb_2 \not\in E(G)$, and hence $b_2a_3 \in E(G)$. Thus, $Q = x_p\overrightarrow{P}b_1z^-\overrightarrow{P}x_2x_p\overrightarrow{P}x_3\overrightarrow{P}za_1\overrightarrow{P}b_2a_3\overrightarrow{P}y$ is an $(x, y)$-path of length $n - 2$ with $x_p = x_1$ if $z^-b_1 \in E(G)$ and $R = x_p\overrightarrow{P}x_1x_p\overrightarrow{P}x_2\overrightarrow{P}z^-b_3\overrightarrow{P}za_1\overrightarrow{P}b_2a_3\overrightarrow{P}y$ is an $(x, y)$-path of length $n - 2$ with $x_p = x_3$ if $z^-b_3 \in E(G)$, which contradicts the choice of $P$. Hence, we have $z^-b_1, z^-b_3 \not\in E(G)$. Since $za_1 \in E(G)$, by Lemma 1 we have $z \not\in A$, and hence $z^-x_p \not\in E(G)$. Thus, $B \cup \{x_p, z^-\}$ is an independent set. $\blacksquare$

Since $|A| = 3$, by Lemma 10, there are some vertices $a_i$ with $i \geq 2$ and $z \not\in X$ such that $[x_p, z] \rightarrow a_i$ or $[a_i, z] \rightarrow x_p$. If $|B| = 3$, then by Lemma 7 and Claim 1, we have $[a_i, z] \rightarrow x_p$. By Lemma 6, we have $z \in P_{l-1}$. Thus, by Claims 3 and 4, we see $B \cup \{x_p, z^-\}$ is an independent set of order 5, a contradiction. Hence we have $|B| = 2$.

Claim 5. If $[x_p, y] \rightarrow a_1$, then $B \cup \{x_p, y^-\}$ is an independent set.

Proof. Since $|A| = 3$ and $A - \{a_i\} \subseteq N(y)$, by Lemma 1 we have $y \neq a_3$, which implies that $y^-x_p \not\in E(G)$. If $a_1 \neq a_3$, then by Lemma 3, we have $b_2, b_3 \not\in N(y^-)$. If $a_1 = a_3$, then we have $b_3, a_2 \in N(y)$. By Lemmas 2 and 3, we have $b_2, b_3 \not\in N(y^-)$. Thus, $B \cup \{x_p, y^-\}$ is an independent set. $\blacksquare$

Claim 6. If $[a_2, z] \rightarrow x_p$, then $z = y$.

Proof. By Lemma 6, we have $z \in P_1$ or $z = y$. If $z \neq y$, then $z \in P_1$. Since $x_p a_3 \not\in E(G)$, there is some vertex $w$ such that $[x_p, w] \rightarrow a_3$ or $[a_3, w] \rightarrow x_p$. If $w = y$, then by Lemma 6 or Claim 5, $B \cup \{x_p, y^-\}$ is an independent set. If $z^-y^- \in E(G)$, then the $(x, y)$-path $x_1x_p\overrightarrow{P}x_2\overrightarrow{P}z_a\overrightarrow{P}z^-y^-\overrightarrow{P}a_2y$ is hamiltonian, and hence $z^-y^- \not\in E(G)$. Thus, by Claim 3, we can see that $B \cup \{x_p, y^-, z^-\}$ is an independent set of order 5, and hence $w \neq y$. If $[x_p, w] \rightarrow a_3$, then by Claim 1, we have $w \in \{x_1, x_2\}$. By Lemma 12, we have $a_1x_2 \in E(G)$. By Claim 2, $z$ is an $A$-vertex, which contradicts Lemma 1 since $za_3 \in E(G)$. Thus, we have $[a_3, w] \rightarrow x_p$. By Lemma 6, we have $w \in P_2$. By Claim 4, $B \cup \{x_p, w^-\}$ is an independent set. Noting that $z^-w^- \not\in E(G)$, we have $z^w^- \not\in E(G)$ by Lemma 1. Thus, by Claim 3, $B \cup \{x_p, w^-, z^-\}$ is an independent set of order 5, a contradiction. $\blacksquare$

Claim 7. If $[a_2, y] \rightarrow x_p$ or $[x_p, y] \rightarrow a_2$, then $a_1a_2b_1a_2b_3 \in E(G)$.

Proof. By Lemma 1, $a_3y \in E(G)$. Thus, $y$ is an $A$-vertex. By Lemma 6 or Claim 5, $B \cup \{x_p, y^-\}$ is an independent set. If $a_1b_2 \not\in E(G)$ or $a_2b_3 \not\in E(G)$, then $a_2b_2 \in E(G)$ for otherwise $[x_p, a_1, b_2, a_2, y^-]$, or $[x_2, b_3, b_2, a_2, y^-]$ is an independent set and $a_1b_2 \in E(G)$ for otherwise $[x_p, a_1, b_2, a_2, y^-]$, or $[x_p, b_3, a_1, a_2, y^-]$ is an independent set, which contradicts $a(G) = 4$. Thus, by Lemmas 1 and 3, we have

$$a_1, b_2 \not\in N(x_2) \quad \text{and} \quad a_2, b_2 \not\in N(x_1) \cup N(x_3).$$  \hfill (3)

If $a_1b_2 \not\in E(G)$, then there is some vertex $w$ such that $[a_1, w] \rightarrow b_2$ or $[b_2, w] \rightarrow a_1$. Obviously, $w \neq x_p$. Thus, in order to dominate $x_p$, we have $w \in P_3$. By (3), we have $[b_2, x_3] \rightarrow a_3$. By Lemma 5, there is some vertex $v \in P_2$ such that $b_2v, x_3v^+ \in E(G)$, which implies that the $(x, y)$-path $x_1x_p\overrightarrow{P}b_2\overrightarrow{P}a_1b_3\overrightarrow{P}v^+x_3\overrightarrow{P}y$ is hamiltonian, and hence $a_1b_2 \in E(G)$. If $a_2b_3 \not\in E(G)$, then there is some vertex $u$ such that $[a_2, u] \rightarrow b_3$ or $[b_3, u] \rightarrow a_2$. Clearly, $u \neq x_p$, and hence $u \in X$. By (3), we have $[a_2, x_1] \rightarrow b_3$. By Lemma 5, there is some vertex $v \in P_1$ such that $x_1v, a_2v^+ \in E(G)$, which implies that the $(x, y)$-path $x_1v\overrightarrow{P}a_1b_3\overrightarrow{P}a_2v^+\overrightarrow{P}x_2x_p\overrightarrow{P}y$ is hamiltonian, and hence $a_2b_3 \in E(G)$. $\blacksquare$

Claim 8. If $[x_p, z] \rightarrow a_2$ and $z \in \{x_1, x_3\}$, then $a_1b_2, a_2b_3 \in E(G)$.

Proof. By Lemma 12, we have $a_1x_3 \in E(G)$. By Lemma 3, we have $b_2, b_3 \not\in N(a_3)$. If $a_1b_2 \not\in E(G)$ or $a_2b_3 \not\in E(G)$, then $a_1b_3 \in E(G)$ for otherwise $[x_p, a_1, b_2, b_3, a_3]$, or $A \cup \{x_p, b_3\}$ is an independent set of order 5. Thus by Lemmas 2 and 3, we have $b_2 \not\in N(x_1) \cup N(x_3)$, which contradicts $z \in \{x_1, x_3\}$. $\blacksquare$

Claim 9. If $[x_p, z] \rightarrow a_2$ and $z \in \{x_1, x_3\}$, then $a_3y \in E(G)$.

Proof. Since $x_pa_3 \not\in E(G)$, there is some vertex $w$ such that $[x_p, w] \rightarrow a_3$ or $[a_3, w] \rightarrow x_p$. If $[x_p, w] \rightarrow a_3$, then since $z \in X$, by Lemma 10 we have $w \not\in X$. By Claim 1, $y = w$. If $[a_3, w] \rightarrow x_p$, then by Lemma 6, $w \in P_2$
or \( w = y \). If \( w \in P_2 \), then by Claims 2 and 8, we have \( w = b_3 \), which contradicts (2). Thus, we have \( w = y \) in both cases. By Lemma 6 or Claim 5, \( B \cup \{x_p, y^-\} \) is an independent set. If \( a_3y \notin E(G) \), then since \( z \in X \), by Lemma 10, there is some vertex \( u \in V(G) - N[x_p] \) such that \( [x_p, u] \to a_1 \) or \( [a_1, u] \to x_p \). Since \( a_3y \notin E(G) \), by Claim 1, we can see that \( [x_p, u] \to a_1 \) is impossible. Thus, we have \( [a_1, u] \to x_p \). If \( u \in B \), say \( u = b_1 \), then since \( b_1a_3, a_1y^- \notin E(G) \), by Lemma 5 there is some vertex \( v \in P_3 - \{y\} \) such that \( b_1v, a_1v^+ \in E(G) \), which contradicts Lemma 3. Thus, in order to dominate \( a_3 \), we have \( u \in P_3 - \{y\} \) by Claims 2 and 8. Since \( a_2u \in E(G) \), by Lemma 2, \( a_3u^+ \notin E(G) \). If \( a_1u^+ \in E(G) \) or \( a_2u^+ \in E(G) \), then by Lemma 3, \( b_3u \notin E(G) \), which implies that \( a_1b_3 \in E(G) \). Thus, by Lemmas 2 and 3, we have \( b_3 \notin N(x_1) \cup N(x_3) \), which contradicts \( z \in \{x_1, x_3\} \). Hence, \( a_1, a_2 \notin N(u^+) \), which implies that \( A \cup \{x_p, u^+\} \) is an independent set of order 5, a contradiction. Thus, we have \( a_3y \in E(G) \).

Since \( x_p a_2 \notin E(G) \), there is some vertex \( z \) such that \( [x_p, z] \to a_2 \) or \( [a_2, z] \to x_p \). If \( [a_2, z] \to x_p \), then \( z = y \) by Claim 6. By Claim 7, we have \( a_3y, a_1b_2, a_2b_3 \in E(G) \). If \( [x_p, z] \to a_2 \), then by Claim 1, we have \( z \in \{x_1, x_3, y\} \). Thus, by Claims 7–9, we have \( a_3y, a_1b_2, a_2b_3 \in E(G) \). Hence, by Claim 2, we have

\[
P_i \subseteq N[a_1] \quad \text{for } i = 1, 2, 3.
\]

(4)

If \( z = y \), then by Lemma 1 and (4), we have \( P_3 \subseteq N[y] \). If \( z \neq y \), then by Claims 1 and 6, we have \( [x_p, z] \to a_2 \) and \( z \in \{x_1, x_3\} \). Since \( x_p a_3 \notin E(G) \), there is some vertex \( u \) such that \( [x_p, u] \to a_3 \) or \( [a_3, u] \to x_p \). If \( u \neq y \), then by Lemma 10 and Claim 1, we have \( [a_3, u] \to x_p \). By Lemma 6, we have \( u \in P_2 \). By (4), we have \( u = b_3 \), which contradicts (2). If \( u = y \), then by Lemma 6, \( B \cup \{y^-\} \) is an independent set. Since \( a_1x_p \notin E(G) \), there is some vertex \( w \) such that \( [a_1, w] \to x_p \) or \( [x_p, w] \to a_1 \). Since \( z \in X \), by Lemma 10, \( w \notin X \). If \( w = y \), then by Lemma 1 and (4), \( P_3 \subseteq N[y] \). If \( w \neq y \), then by Claim 1, we have \( [a_1, w] \to x_p \). In order to dominate \( a_2, a_3 \), we have \( w \in B \), which is impossible since \( [a_1, w] \neq y^- \). Therefore, we have

\[
P_3 \subseteq N[y].
\]

(5)

Let \( w \) be a vertex such that \( [x_p, w] \to a_3 \) or \( [a_3, w] \to x_p \). If \( z \in X \), then by Lemma 10, Claim 1 and (4), we have \( [a_3, w] \to x_p \). By Lemma 6, we have \( w \in P_2 \) or \( w = y \). By (2) and (4), we have \( w = y \). If \( z \notin X \), then by Claims 1 and 6, we have \( z = y \). Thus, we have

either \( w = y \) or \( z = y \).

(6)

By (6), we have \( y \neq a_3 \), which implies that \( y^-x_p \notin E(G) \). Let \( v \) be a vertex such that \( [x_p, v] \to y^- \) or \( [y^-, v] \to x_p \). By Lemma 6, Claim 5 and (6), \( B \cup \{x_p, y^-\} \) is an independent set. By (4), \( y^- \) is an \( A \)-vertex. Thus, by Lemma 1 and (4), we have \( N(y^-) \cap P_i = \emptyset \) for \( i = 1, 2 \). If \( [y^-, v] \to x_p \), then we must have \( v = y \), which implies \( \{x_p, y\} \supset V(G) \) by (5), a contradiction. Thus, we have \( [x_p, v] \to y^- \). By (4), we have \( v \in X \). If \( y^- = a_3 \), then by Lemma 12, we have \( N(a_3) \cap \{x_1, x_2\} = \emptyset \), which implies \( d(a_3) = 2 \), a contradiction. Thus, we have \( y^- \neq a_3 \). In this case, \( y^- \notin A \).

By Lemmas 8 and 9, we may assume that \( [a_i, x_{j_i}] \to a_{i+1} \) for \( i = 1, 2 \) and \( X - \{x_{j_1}, x_{j_2}\} = \{x_{j_3}\} \). This implies that \( v = x_{j_3} \). Since \( y^- \) is an \( A \)-vertex, we have \( y^-a_i \notin E(G) \) or \( y^-a_i \notin E(G) \), which implies that either \( y^-x_{j_1} \in E(G) \) or \( y^-x_{j_2} \in E(G) \). Thus, since \( x_{j_1}, x_{j_2} \in E(G) \), we can see that either \( \{x_{j_1}, x_{j_3}\} \supset V(G) \) or \( \{x_{j_2}, x_{j_3}\} \supset V(G) \), a contradiction.

Case 2. \( |A| = 2 \)

In this case, our main idea is to prove that \( P_i \) is a clique for \( i = 1, 2 \). In order to do this, we first show that either \( a_1b_2 \in E(G) \) or \( a_2b_3 \in E(G) \) and then \( a_1b_2, a_2b_3 \in E(G) \).

If \( |P_i| = 1 \) for some \( i \in \{1, 2\} \), then by the choice of \( P \), we have \( N(a_i) = X \), which contradicts \( \tau(G) > 1 \). Thus, we have \( |P_i| \geq 2 \) for \( i = 1, 2 \), which implies that \( b_2, a_2^+ \notin X \). Noting that \( a_2, b_2 \in N(x_2) \), by the choice of \( P \), we see that

there is no \((x, y)\)-path \( Q \) such that \( x_Q = a_2 \) or \( b_2 \).

(7)

Claim 10. If \( a \in P_1 \) is an \( A \)-vertex, then \( a_a^+ \notin E(G) \), and if \( b \in P_2 \) is a \( B \)-vertex, then \( b \notin E(G) \).

Proof. Let \( Q \) be a hamiltonian \((a, x_2)\)-path in \( G[P_1 \cup \{x_2\}] \). If \( a_a^+ \in E(G) \), then \( R = x_1x_px_2 \overrightarrow{Q} a_a^+ \overrightarrow{P} x_3 \) is an \((x, y)\)-path of length \( n - 2 \) with \( x_R = a_2 \), which contradicts (7). As for the latter part, the proof is similar. ■
Claim 11. If \( a \in P_2 \) is an A-vertex and \( aa_1^+ \in E(G) \), then \( N(a_1) = \{x_1, x_3, a_1^+\} \). Similarly, if \( b \in P_1 \) is a B-vertex and \( bb_3^- \in E(G) \), then \( N(b_3) = \{x_1, x_3, b_3^-\} \).

Proof. Let \( Q \) be a hamiltonian \((a, x_3)\)-path in \( G[P_2 \cup \{x_3\}] \). If \( aa_1^+ \in E(G) \), then \( R = x_1 x_P x_2 \tilde{P} a_1^+ a_1^+ Q x_3 \) is an \((x, y)\)-path of length \( n - 2 \) with \( x_R = a_1^+ \). By the choice of \( P \), we have \( d(a_1) = 3 \) and \( x_1, x_3 \in N(a_1) \), which implies that \( N(a_1) = \{x_1, x_3, a_1^+\} \). As for the latter part, the proof is similar. □

Let \( a \in P_2 - \{b_2\} \) and \( b \in P_2 - \{a_2\} \). Suppose that \( P' \) is an \((a, b^-)\)-path with \( V(P') = P_2 - \{b_2\} \) and \( P'' \) an \((a_2^+, b^-)\)-path with \( V(P'') = P_2 - \{a_2\} \). We have the following two claims.

Claim 12. If \( (N(x_1) \cup N(x_3)) \cap \{b_2, a_2^+\} = \emptyset \), then \( ab \notin E(G) \).

Proof. By symmetry, we may assume that \( N(x_1) \cap \{b_2, a_2^+\} = \emptyset \). If \( ab \in E(G) \), then \( Q = x_1 b_2 P\tilde{P} a_1^+ a_2^+ a_2 x_P x_3 \) is an \((x, y)\)-path of length \( n - 2 \) with \( x_Q = b_2 \) if \( x_1 b_2 \in E(G) \), and \( R = x_1 a_2^+ P\tilde{R} a_2^+ b_2 x_2 x_P x_3 \) is an \((x, y)\)-path of length \( n - 2 \) with \( x_R = a_2^+ \). Therefore, we have \( d(b_2) = 3 \) and \( x_1 \in N(b_2) \), which contradicts (7). As for the latter part, the proof is similar. □

Claim 13. If \( v \in P_2 \) and \( av \in E(G) \), then \( v^+, v^- \notin N(b_2^-) \) and if \( u \in P_1 \) and \( bu \in E(G) \), then \( u^+, u^- \notin N(a_2^+) \).

Proof. If \( v^+ b_2^- \in E(G) \), then \( Q = x_1 x_P x_2 \tilde{P} v^+ P b_2^- v^+ P x_3 \) is an \((x, y)\)-path of length \( n - 2 \) with \( x_Q = b_2^- \) and if \( v^- b_2^- \in E(G) \), then \( R = x_1 x_P x_2 \tilde{P} v^- P b_2^- v^+ P x_3 \) is an \((x, y)\)-path of length \( n - 2 \) with \( x_R = b_2^- \). Therefore, we have \( d(b_2) = 3 \) and \( x_1 \in N(b_2) \), which implies that \( N(b_2) = \{x_1, x_3, b_3^-\} \). □

Claim 14. If \( a_2 b_2 \in E(G) \) and \( [a_1, x_2] \rightarrow b_3 \), then \( P_2 - \{b_3\} \subseteq N(x_2) \) and \( N(b_3) = \{x_1, x_3, b_3^-\} \).

Proof. If \( v \in P_2 \) and \( a_1 v \in E(G) \), then by Lemma 5, there is some vertex \( u \in a_2 \tilde{P} v \) such that \( u x_2, u^+ a_1 \in E(G) \). Thus, \( x_1 x_P x_2 u \tilde{P} a_2 b_2 \tilde{P} a_1 u^+ \tilde{P} v x_3 \) is a hamiltonian \((x, y)\)-path, and hence \( N(a_1) \cap P_2 = \emptyset \), which implies that \( P_2 - \{b_3\} \subseteq N(x_2) \). On the other hand, since \( Q = x_1 \tilde{P} b_2 a_2 \tilde{P} b_3^- x_2 x_P x_3 \) is an \((x, y)\)-path of length \( n - 2 \) with \( x_Q = b_3 = b_3^- \), by the choice of \( P \), we have \( d(b_3) = 3 \) and \( x_1 \in N(b_3) \), which implies that \( N(b_3) = \{x_1, x_3, b_3^-\} \). □

Claim 15. If \( a_1 b_2, a_2 b_3 \notin E(G) \), then either \( a_1 b_3 \in E(G) \) or \( a_2 b_2 \in E(G) \).

Proof. Otherwise, \( \{x_P, a_1, a_2, b_2, b_3\} \) is an independent set of order 5 by Lemma 1, a contradiction. □
If \([b_2, x_3] \rightarrow a_1\), then by Lemma 5, there is some vertex \(u \in P_2\) such that \(ub_2, u^+ x_3 \in E(G)\). If \(a_1 b_3 \in E(G)\), then \(x_1 x_p x_2 PB \cap a_1 b_3 PB \cap u^+ x_3\) is a hamiltonian \((x, y)\)-path, and hence \(a_1 b_3 \notin E(G)\). By Claim 15, \(a_2 b_2 \in E(G)\). By Lemma 3, \(a_1, b_3 \notin N(x_2)\). Consider \(ab_3 \notin E(G)\). Since \([b_3, x_3] \rightarrow a_2\), \([b_2, x_1] \rightarrow a_2\) and \([a_2, x_2] \rightarrow b_3\) are impossible, we have \([a_2, x_1] \rightarrow b_3\). If \(a_1^+ a_2^+ \in E(G)\), then \(Q = x_1 x_p x_2 PB \cap a_1^+ a_2^+ PB x_3\) is an \((x, y)\)-path of length \(n - 2\) with \(x_q = a_1\), which contradicts the choice of \(P\) since \(a_1 x_3 \notin E(G)\). By Claim 11, \(a_1^+ a_2 \notin E(G)\).

Consider \(a_1^+ a_2 \notin E(G)\), we have \([a_1^+, x_1] \rightarrow a_2\) or \([a_1^+, x_3] \rightarrow a_2\). If \([a_1^+, x_1] \rightarrow a_2\), then \(a_1^+ b_3, x_1 a_2^+ \in E(G)\), which implies that \(R = x_1 a_1^+ PB b_3 a_2^+ PB x_2 x_3 PB x_3\) is an \((x, y)\)-path of length \(n - 2\) with \(x_r = a_1\), a contradiction. Hence, we have \([a_1^+, x_3] \rightarrow a_2\). Since \([b_2, x_3] \rightarrow a_1\) and \([a_2, x_1] \rightarrow b_3\), by symmetry, we have \([b_3^+, x_1] \rightarrow b_2\). Thus, \(x_1 b_2, x_2 \notin E(G)\). Now, consider \(a_1 b_3 \notin E(G)\), we have \([a_1, x_2] \rightarrow b_3\) or \([b_3, x_2] \rightarrow a_1\). By symmetry, we may assume that \([a_1, x_2] \rightarrow b_3\).

By Claim 14, \(x_1 b_3 \notin E(G)\), which contradicts \([a_2, x_1] \rightarrow b_3\). Therefore, \([b_2, x_3] \rightarrow a_1\) is impossible.

It follows from the argument above that either \(a_1 b_2 \in E(G)\) or \(a_2 b_3 \in E(G)\).

Since \(a_1 b_2 \in E(G)\) or \(a_2 b_3 \in E(G)\), by symmetry, we may assume that \(a_1 b_2 \in E(G)\). If \(a_2 b_3 \notin E(G)\), then there is some vertex \(z\) such that \([a_2, z] \rightarrow b_3\) or \([b_3, z] \rightarrow a_2\). Obviously, \(z \notin x_p\) and hence \(z \in X\). It is not difficult to see that there are four cases: \([a_2, x_1] \rightarrow b_3\), \([a_2, x_2] \rightarrow b_3\), \([b_3, x_1] \rightarrow a_2\) or \([b_3, x_3] \rightarrow a_2\), and at least one of the four cases occurs.

In order to prove \(a_2 b_3 \in E(G)\), we need the following four claims.
order 5, and hence $a_2z^− \in E(G)$. If $Q_1 \not\subseteq N[a_2]$, then since $a_2z^− \in E(G)$, there is some vertex $v \in Q_1$ such that $va_2 \not\in E(G)$ and $va_2v^+ \in E(G)$. Clearly, $v$ is an $A$-vertex. If $vz^− \not\in E(G)$, then $z$ is an $A$-vertex, a contradiction. Thus, $[a_1, a_2, v, z^+, xP]$ is an independent set of order 5, and hence $Q_1 \not\subseteq N[a_2]$. In this case, $N(z^+) \cap Q_1 = \emptyset$ for otherwise $z$ is an $A$-vertex. If $u, v \in Q_1$ and $uv \not\in E(G)$, then $\{a_1, u, v, z^+, xP\}$ is an independent set of order 5, and hence $Q_1$ is a clique. If $u_i \in Q_1$ for $i = 1, 2$ and $u_1u_2 \in E(G)$, then $u_1 \neq a_2, z^−$, and hence $x_2a_2Pv_1z^−Pv_1u_2Pb_3v_2Pz$ is a hamiltonian $(x_2, z)$-path in $G[P_2 \cup \{x_2\}]$, which implies that $z$ is a $B$-vertex, a contradiction. Thus, we have $E(Q_1, Q_2) = \emptyset$.

Claim 18. If $a_2b_3 \not\in E(G)$, then for any $z \in P_2$, both $[xP, z] \rightarrow a_2$ and $[a_2, z] \rightarrow xP$ are impossible.

Proof. Suppose to the contrary that there is some vertex $z \in P_2$ such that $[xP, z] \rightarrow a_2$ or $[a_2, z] \rightarrow xP$. If $[xP, z] \rightarrow a_2$, then $z \neq b_3$. If $[a_2, b_3] \rightarrow xP$, then by Lemmas 1 and 5, there is some vertex $u \in P_1$ such that $ub_3, u \rightarrow \in E(G)$, which contradicts Lemma 3. Thus, we have $z \neq b_3$ in both cases. Let $P' = a_2Pb_3$ and $P'' = z^−Pb_3$. Since $a_1b_2 \in E(G)$, by Lemma 1, we have $b_7a_2 \not\in E(G)$. Thus, $a_1, b_7, b_3 \in N(z)$. Since $z a_1 \in E(G)$, by Lemma 3 and Claim 13, we have $b_2, b_5 \not\in N(z^−)$. By Lemma 1 and Claim 16, we have $a_1Pb_2 \subseteq N(z)$. By Claim 17, $P'' \subseteq N(z)$. Since $[b_2, b_3, xP]$ is a maximum independent set, by Lemma 10, there is some vertex $v \in \{b_2, b_3\}$ and a vertex $w \in V(G) - N[xP]$ such that $[u, w] \rightarrow xP$ or $[xP, w] \rightarrow u$. If $[xP, w] \rightarrow u$, then $w \neq z$. If $u = b_7$, then since $wb_7 \in E(G)$, by Claim 16 and 17, we have $w \rightarrow P''$ which is impossible since $ua_2 \not\in E(G)$. If $u = b_3$, then $w \neq P''$ by Claim 17. Since $b_2z^− \not\in E(G)$, we have $w \neq b_2, z^−$. Thus, by Lemma 1 and Claim 16 and 17, we see that $w \not\in P_1 \cup P_2$, a contradiction. Hence, we have $[u, w] \rightarrow xP$. If $u = b_7$, then in order to dominate $a_2$ and $b_3$, we have $w = z$ by Lemma 1 and Claims 16 and 17. If $u = b_3$, then in order to dominate $P_1$ and $P''$, it is easy to see that $w = z$ by Lemma 1 and Claims 16 and 17. In both cases, we have $P'' \subseteq N(z)$ by Lemma 1 and Claim 17. Thus, we have $[a_2, z] \rightarrow xP$. If $b_2z \in E(G)$, then we have $[xP, z] \rightarrow V(G)$. If $b_2z \not\in E(G)$, then $b_2b_3 \in E(G)$. By Lemma 3, $b_2, b_3 \not\in N(x_2)$. If $z^−x_2 \in E(G)$, then $x_1Pz^−x_2Pb_2Pb_3Pz$ is a hamiltonian $(x_2, y)$-path. Thus, $b_2, z^−, b_3 \not\in N(x_2)$. Noting that $[b_7, z^−, b_3, xP]$ is an independent set, by Lemma 8 and 9, we have $x_1, x_3 \in N(x_2)$, which implies $\{x_2, z\} \rightarrow V(G)$, a contradiction.

Claim 19. If $a_2b_3 \not\in E(G)$ and $x_1, x_3, b_2 \not\in N(a_2)$, then $[xP, a_2^+] \rightarrow a_1$ is impossible.

Proof. If $[xP, a_2^+] \rightarrow a_1$, then $a_2^+b_3 \in E(G)$. If $a_2^+Pb_3 \not\in N[b_3]$, then there is some vertex $v \in a_2^+Pb_3$ such that $v \rightarrow b_3 \in E(G)$ and $vb_3 \not\in E(G)$. Clearly, $v$ is a $B$-vertex. By Claim 10, $v, b_3 \not\in N(b_2^+)$. By Lemma 1 and Claim 16, $a_2 \not\in N(b_2^+)$. If $a_2v \in E(G)$, then it is easy to see that $a_2^+$ is a $B$-vertex, which contradicts Lemma 1 since $b_2a_2^+ \in E(G)$. Thus, $[b_2^+, a_2, v, b_3, xP]$ is an independent set of order 5, a contradiction. Hence, we have $a_2^+Pb_3 \subseteq N[b_3]$, which implies that $N(a_2) \cap a_2^+Pb_3 = \{a_2^+\}$. Thus, noting that $x_1, x_3, b_2 \not\in N(a_2)$, by Lemma 1 and Claim 16, we have $d(a_2) = 2$, a contradiction. Hence, $[xP, a_2^+] \rightarrow a_1$ is impossible.

We now begin to prove $a_2b_3 \not\in E(G)$. Suppose to the contrary that $a_2b_3 \not\in E(G)$.

Since $xPa_2 \not\in E(G)$, there is some vertex $z$ such that $[xP, z] \rightarrow a_2$ or $[a_2, z] \rightarrow xP$. By Claim 16, we have $z \not\in P_1$. By Claim 18, we have $z \not\in X$. In this case, we have $[xP, x_1] \rightarrow a_2$ or $[xP, x_3] \rightarrow a_2$.

If $[a_2, x_1] \rightarrow b_3$, then $[xP, x_1] \rightarrow a_2$ is impossible. If $[xP, x_3] \rightarrow a_2$, then by Lemma 12, we have $x_1x_3 \not\in E(G)$, which is impossible since $a_2x_3 \not\in E(G)$ and $[a_2, x_1] \rightarrow b_3$. Thus, $[a_2, x_1] \rightarrow b_3$ is impossible. If $[a_2, x_1] \rightarrow b_3$, we let $[i, j] = \{1, 3\}$. If $[xP, x_j] \rightarrow a_2$, then by Lemma 12, we have $x_2x_j \in E(G)$. Since $[a_2, x_3] \rightarrow b_3$, we have $x_2x_j \in E(G)$ or $a_2x_j \in E(G)$, which implies $[x_1, x_2] \rightarrow V(G)$ or $[x_1, a_2] \rightarrow V(G)$, a contradiction. Thus, $[a_2, x_1] \rightarrow b_3$ is impossible. Therefore, we have $[x_1, x_3] \rightarrow a_2$ or $[b_3, x_3] \rightarrow a_2$.

By Claim 16, $[xP, a_2, b_3]$ is a maximum independent set. Since $[xP, x_1] \rightarrow a_2$ or $[xP, x_3] \rightarrow a_2$, by Lemma 12, we have $x_1, x_3 \in N(a_2) \cap N(b_3)$ and $x_1, x_3 \not\in N(a_2)$. If $[xP, b_2] \rightarrow b_3$, then since $[b_2, x_3] \rightarrow a_2$ or $[b_3, x_3] \rightarrow a_2$, we have $[b_2, x_3] \rightarrow V(G)$ or $[b_2, x_3] \rightarrow V(G)$ by Lemma 1, a contradiction. Obviously, $[xP, b_3] \rightarrow b_3$ is impossible. Thus, there is some vertex $u \in U$ such that $[b_2, u] \rightarrow b_3$ or $[b_3, u] \rightarrow b_3$. Since $[b_2, x_3] \neq \in a_2$ for $i = 1, 3$ and $x_1, x_3 \in N(b_3)$, we have $[b_2, x_1] \rightarrow b_3$.

Since $[xP, x_1] \rightarrow a_2$ or $[xP, x_3] \rightarrow a_2$, by Lemma 12, we have $x_2x_3 \not\in E(G)$ or $x_1x_2 \not\in E(G)$. Noting that $[b_2, x_3] \rightarrow b_3$, by Lemma 1, we have $b_2x_3 \in E(G)$ or $b_2x_1 \in E(G)$. Thus, if $a_2b_3 \in E(G)$, then we have $[b_2, x_3] \rightarrow V(G)$ or $[b_2, x_3] \rightarrow V(G)$, and hence $a_2b_3 \not\in E(G)$. Thus, we have $x_1, x_3, b_2 \not\in N(a_2)$. 

By Claim 10, $a_1a_2^+ \not\in E(G)$. Since $x_1, x_3, b_2 \not\in N(a_2)$, by Claim 19, $[x_p, a_2^+] \to a_1$ is impossible. Obviously, $[x_p, a_1] \to a_2^+$ is impossible. Thus, there is some vertex $w \in X$ such that $[a_1, w] \to a_2^+$ or $[a_1, w] \to a_1$. Since $[b_2, x_2] \to b_3$, we have $x_2b_3 \not\in E(G)$. Thus, noting that $[a_1, x_1] \not\sim a_2$ for $i = 1, 3$, $[a_1, x_2] \not\sim b_3$ and $x_1, x_3 \in N(a_1)$, we have $[a_1^+, x_3] \to a_1$. If $[x_p, x_1] \to a_2$, then by Lemma 12, we have $x_1x_2 \in E(G)$ and $x_2x_3 \not\in E(G)$. In this case, we have $a_2^+x_3 \in E(G)$, which implies $[x_1, a_2^+] > V(G)$, a contradiction. If $[x_p, x_3] \to a_2$, then by Lemma 12, we have $x_2x_3 \in E(G)$ and $x_1x_3 \not\in E(G)$. In this case, we have $a_2^+x_1 \in E(G)$, which implies $[x_3, a_2^+] > V(G)$, again a contradiction. Thus, we have $a_2b_3 \in E(G)$.

Up to now, we have shown that $a_1b_2, a_2b_3 \in E(G)$. In the following, we will show that $P_i$ is a clique for $i = 1, 2$.

If $P_i \not\subseteq N[a_1]$, then since $a_1b_{i+1} \in E(G)$, there is some vertex $u \in P_i$ such that $a_1u \not\in E(G)$ and $a_1u^+ \in E(G)$. We let $u_i \in P_i$ be such a vertex if $P_i \not\subseteq N[a_1]$, where $i = 1, 2$.

If $P_1 \not\subseteq N[a_1]$, then $[a_1, u_1, a_2, x_p] \to a_2$, $[a_1, x_3] \to a_2$, $[a_2, x_2] \to a_1$ or $[a_2, x_3] \to a_1$.

If $P_2 \not\subseteq N[a_2]$, then $[a_1, u_1, a_2, u_2, x_p] \to a_2$, $[a_1, x_2] \to a_2$, $[a_2, x_2] \to a_1$ or $[a_2, x_3] \to a_1$.

If $P_2 \not\subseteq N[a_2]$, then $[a_1, u_1, a_2, u_2, x_p] \to a_2$, $[a_1, x_2] \to a_2$, $[a_2, x_2] \to a_1$ or $[a_2, x_3] \to a_1$.

Let $a \in \{a_1, u_1\}$ and $w \in V(G) - N[x_p]$. If $[x_p, w] \to a$ or $[a, w] \to x_p$, then by (8), we have $w \in P_1$ and $P_2 \subseteq N(w)$. Thus, by Lemma 1, $w$ is neither an A-vertex nor a B-vertex. Obviously, $|P_1| \geq 3$. Since $a_1b_2 \in E(G)$, it is easy to see that $G[P_i]$ contains a Hamiltonian $(w, w^+)$-path.

If $[a_1, x_1] \to a_2$ or $[a_1, x_3] \to a_2$, then since $w$ is not an A-vertex, we have $a_1w^+ \not\in E(G)$, and hence $w_1w^+ \in E(G)$ or $w_3w^+ \in E(G)$. If $x_1w^+ \in E(G)$, then since $a_1b_2 \in E(G)$, we see that $w$ is a B-vertex, a contradiction. If $x_3w^+ \in E(G)$, then by (9) and Lemma 3, we have $wb_3 \not\in E(G)$, which contradicts $P_2 \subseteq N(w)$. If $[a_2, x_2] \to a_1$, then since $a_2w, b_2x_2 \in E(G)$, by Lemma 5, there is some vertex $v \in w \bar{P}b_2$ such that $wv_2, v^+x_2 \in E(G)$, which contradicts Lemma 3 since $a_1b_2 \in E(G)$, which implies that $G[P_i]$ contains a Hamiltonian $(v, v^+)$-path. Since $a_2, b_2 \in N(w)$, by (9) and Lemma 3, we have $w^+x_3, w^+a_2 \not\in E(G)$, which implies that $[a_2, x_3] \to a_1$ is impossible. Thus, for any $a \in \{a_1, u_1\}$ and $w \in V(G) - N[x_p]$, both $[x_p, w] \to a$ and $[a, w] \to x_p$ are impossible, which contradicts Lemma 10 since $[a_1, u_1, a_2, x_p]$ is an independent set. Therefore, we have $P_1 \subseteq N[a_1]$. If $P_2 \not\subseteq N[a_2]$, then since $P_1 \subseteq N[a_1]$, by symmetry, we have $P_2 \subseteq N[b_3]$. Thus, $u_2$ is both an A-vertex and a B-vertex. By Lemma 1, $P_1 \cap N(u_2) = \emptyset$. Since $x_pa_2 \not\in E(G)$, there is some vertex $w$ such that $[x_p, w] \to a_2$ or $[a_2, w] \to x_p$. If $[a_2, w] \to x_p$, then $w \not\in P_i$ for otherwise $[a_2, w] \not\sim u_2$. Thus, we have $w \in P_2$. Since $P_2 \subseteq N[b_3]$, by Lemma 1, we have $a_2b_2, wb_2^+ \in E(G)$, which contradicts Lemma 3. Thus, we have $[x_p, w] \to a_2$. If $w \in P_1$, then $wv_2 \not\in E(G)$ and if $w \in P_2$, then $wb_2 \not\in E(G)$. Thus, we have $w \in [x_1, x_3]$. If $[x_p, x_1] \to a_2$, then $x_1x_2 \in E(G)$ by Lemma 12. In this case, we have $[x_1, b_3] > V(G)$. If $[x_p, x_3] \to a_2$, then by Lemma 12, we have $x_2x_3 \in E(G)$ and $x_1x_2 \not\in E(G)$. Thus, we have $x_2, a_2, a_2^+ \not\in N(x_1)$ for otherwise $\gamma(G) = 2$. Since $b_2u_2 \not\in E(G)$, there is some vertex $v$ such that $[b_2, v] \to u_2$ or $[u_2, v] \to b_2$. Obviously, $v \not\sim x_1$, and hence $v \in X$. Since $[x_p, x_3] \to a_2$ implies that $b_2, b_3 \in N(x_3)$, we have $v \not\sim x_3$. Since $[u_2, x_1] \not\sim a_2$ and $[b_2, x_1] \not\sim a_2^+$, we have $v \not\sim x_1$, and hence $v = x_2$, which implies that $[b_2, x_2] \to u_2$. Since $x_1x_2 \not\in E(G)$, we have $x_1b_2 \in E(G)$. If $a_2b_2 \in E(G)$, then $x_1b_2 > V(G)$, and hence $a_2b_2 \not\in E(G)$. Now, consider $x_pu_2 \not\in E(G)$. Since $[a_1, a_2, u_2]$ is an independent set and $[x_p, x_1] \to a_2$, by Lemma 10, there is some vertex $v \in V(G) - N[x_p]$ such that $[x_p, u] \to u_2$ or $[u_2, u] \to x_p$. Since $N(a_2) \cap P_1 = \emptyset$ and $N(u_2) \cap P_1 = \emptyset$, we have $u \in P_2$ in both cases. This is impossible since $[u_2, u] \not\sim b_2$. Thus, we have $P_2 \subseteq N[a_2]$.

By symmetry, we have $P_1 \subseteq N(a_1) \cap N(b_{1+i})$ for $i = 1, 2$. If $P_1$ is not a clique, then there are two vertices $u, v \in P_1 - \{a_1, b_2\}$ such that $uv \not\in E(G)$. Obviously, $u$ and $v$ are both A-vertices and B-vertices. Thus, $(N(u) \cup N(v)) \cap P_2 = \emptyset$. Since $\{u, v, a_2, x_p\}$ is an independent set, by Lemma 10, there is some $w \in V(G) - N[x_p]$. Then, by (8), we have $w \in P_1$ and $P_2 \subseteq N(w)$. Thus, by Lemma 1, $w$ is neither an A-vertex nor a B-vertex. Obviously, $|P_1| \geq 3$. Since $a_1b_2 \in E(G)$, it is easy to see that $G[P_i]$ contains a Hamiltonian $(w, w^+)$-path.
and a vertex in \( \{u, v\} \), say \( u \), such that \([u, w] \rightarrow x_P \) or \([x_P, w] \rightarrow u\). It is easy to see that such a vertex \( w \) does not exist, and hence \( P_1 \) is a clique. By symmetry, \( P_2 \) is a clique.

Since \( P_i \) is a clique for \( i = 1, 2 \), by Lemmas 1 and 14, we have \( E(P_1, P_2) \subseteq \{a_2b_2\} \). If \( a_2b_2 \notin E(G) \), then \( X \) is a 3-cutset such that \( \omega(G - X) = 3 \), which contradicts \( \tau(G) > 1 \). If \( a_2b_2 \in E(G) \), then by Lemma 14, we have \( \alpha(G) = 3 \), again a contradiction.

The proof of Theorem 4 is complete. ■

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References