The Galois algebra with Galois group which is the automorphism group

George Szeto *, Lianyong Xue

Department of Mathematics, Bradley University, Peoria, IL 61625, USA

Received 22 September 2004
Available online 3 March 2005
Communicated by Susan Montgomery

Abstract
Let $B$ be a Galois algebra over a commutative ring $R$ with Galois group $G$. Then it is shown that $G = \text{Aut}_R(B)$ if and only if $B$ is commutative with no idempotents but 0 and 1, or $B \cong R \oplus R$ where $R$ contains no idempotents but 0 and 1.

© 2005 Elsevier Inc. All rights reserved.

1. Introduction

It is well known that the Galois group of a Galois extension of a field is the automorphism group of the field, and S.U. Chase, D.K. Harrison, and A. Rosenberg proved this fact for a commutative Galois extension with no idempotents but 0 and 1 [3, Theorem 3.5]. We are interested in the converse problem: let $B$ be a Galois algebra over a commutative ring $R$ with Galois group $G$. If $G = \text{Aut}_R(B)$, is $B$ a commutative ring with no idempotents but 0 and 1? The present paper will show that $G = \text{Aut}_R(B)$ if and only if either $B$ is commutative with no idempotents but 0 and 1, or $B \cong R \oplus R$ where $R$ contains no idempotents but 0 and 1. We shall employ the general Wedderburn Theorem for an Azumaya algebra over a local ring as given by F.R. DeMeyer [6, Corollary 1] to calculate the inner automorphism

* Corresponding author.
E-mail addresses: szeto@bradley.edu (G. Szeto), lxue@bradley.edu (L. Xue).

0021-8693/$ – see front matter © 2005 Elsevier Inc. All rights reserved.
doi:10.1016/j.jalgebra.2005.01.021
group of a central Galois algebra. Then the problem is reduced to the problem for a Galois algebra $B$ with at most four central idempotents. But then $B$ is either a composition of a central Galois algebra and a commutative Galois algebra with no idempotents but 0 and 1 [5, Theorem 1], or commutative with exactly two minimal central idempotents. This will lead to the conclusion.

2. Basic definitions and notations

Throughout this paper, $B$ will represent a ring with 1, $G$ a finite automorphism group of $B$, $C$ the center of $B$, and $B^G$ the set of elements in $B$ fixed under each element in $G$.

Let $A$ be a subring of a ring $B$ with the same identity 1. We call $B$ a separable extension of $A$ if there exist $\{a_i, b_i \in B, i = 1, 2, \ldots, m\}$ for some integer $m$ such that $\sum a_i b_i = 1$, and $\sum b a_i \otimes b_i = \sum a_i \otimes b_i b$ for all $b$ in $B$ where $\otimes$ is over $A$. An Azumaya algebra is a separable extension of its center. A ring $B$ is called a Galois extension of $R$ if $B$ is a Galois extension of $R$ and $R$ is contained in $C$, and $B$ is called a central Galois algebra if $B$ is a Galois extension of $C$. The characteristic of a ring $C$ is denoted by $\text{Char}(C)$.

3. The inner automorphism group

Let $B$ be a central Galois algebra over its center $C$ with Galois group $G$. We shall show that the rank of $B$ over $C$ is defined and equal to $k^2$ where $k^2 = |G|$, the order of $G$, for some integer $k$ (for the rank of a projective module, see [7, p. 27]). Then by using the general Wedderburn Theorem for Azumaya algebras over a local ring as given by F.R. DeMeyer [6, Corollary 1], we show that the order of the inner automorphism group of $B$ is greater than $|G|$ when either $|G| > 2$, or $|G| = 2$ and $\text{Char}(C) \neq 2$. We begin with the general Wedderburn Theorem.

Proposition A (DeMeyer [6, Corollary 1]). Let $A$ be an Azumaya algebra over a semi-local ring $C$ with no idempotents but 0 and 1. Then $A \cong M_n(D)$, a matrix ring of order $n$ for some integer $n$ over an Azumaya algebra $D$ with no idempotents but 0 and 1 in the same class of $A$ in the Brauer group of $C$.

Lemma 3.1. Let $B$ be a Galois algebra over a commutative ring $R$ with Galois group $G$. Then the rank of $B$ over $R$ is defined and $\text{rank}_R(B) = |G|$.

Proof. Since $B$ is a Galois algebra over $R$ with Galois group $G$, the skew group ring $B \ast G \cong \text{Hom}_R(B, B)$. Hence for each prime ideal $p$ of $R$, $R_p \otimes_R (B \ast G) \cong R_p \otimes_R \text{Hom}_R(B, B)$, that is, $B_p \ast G \cong \text{Hom}_{R_p}(B_p, B_p)$. Thus $\text{rank}_{R_p}(B_p) \cdot |G| = (\text{rank}_{R_p}(B_p))^2$; and so $|G| = \text{rank}_{R_p}(B_p)$ for each $p$. Therefore $\text{rank}_R(B) = |G|$. □
Lemma 3.2. If $B$ is a central Galois algebra over its center $C$ with Galois group $G$, then $|G| = k^2$ for some integer $k$.

Proof. By Lemma 3.1, $\text{rank}_C(B) = |G|$, so $B$ is an Azumaya algebra of rank $|G|$ over $C$. Hence for any prime ideal $p$ of $C$, $C_p \otimes_C B \cong M_n(D)$, a matrix ring of order $n$ for some integer $n$ over an indecomposable Azumaya $C_p$-algebra $D$ by Proposition A. Noting that $\text{rank}_{C_p}(D) = \text{rank}_{C_p}(D/pD) = d^2$ for some integer $d$, we have $\text{rank}_{C_p}(C_p \otimes_C B) = (nd)^2$; and so $|G| = \text{rank}_{C_p}(B) = (nd)^2 = k^2$ where $k = nd$. \qed

For a central Galois algebra over $C$ with Galois group $G$ of order greater than 2, we want to show that the order of the inner automorphism group of $B$, $|\text{Inn}(B)| > |G|$. We first work on $\text{Inn}(B)$ for a matrix ring $B$.

Lemma 3.3. Let $B = M_n(R)$, a matrix ring of order $n$ over a commutative ring $R$. Let $\alpha_A$ and $\alpha_{A'}$ be inner automorphisms of $B$ induced by invertible matrices $A$ and $A'$, respectively. If $\alpha_A = \alpha_{A'}$, then $A = r A'$ for some $r \in R$.

Proof. Since $\alpha_A = \alpha_{A'}$, $AEA^{-1} = A'E(A')^{-1}$ for each $E \in B$. Hence $(A')^{-1}AE = E(A')^{-1}A$ for each $E \in B$. Thus $(A')^{-1}A$ is in the center of $B$. Therefore $(A')^{-1}A = r I$ for some $r \in R$ where $I$ is the identity matrix of $B$; and so $A = r A'$. \qed

Lemma 3.4. Let $B = M_n(R)$ as given in Lemma 3.3 and let $\text{Inn}(B)$ denote the inner automorphism group of $B$. If either $n > 2$, or $n = 2$ and $\Char(R) \neq 2$, then $|\text{Inn}(B)| > n^2$.

Proof. Let $S = \{I + D \mid I$ is the identity matrix of $B$ and $D$ is a strictly upper triangular matrix with 1 as nonzero entries$\}$. Then for any $r \in R$ and any two distinct upper triangular matrices $D$ and $D'$, $I + D$ is an invertible matrix and $I + D \neq r(I + D')$. Hence $\alpha_{(I+D)} \neq \alpha_{(I+D')}$ by Lemma 3.3. Similarly, let $T = \{I + E \mid E$ is a strictly lower triangular matrix with 1 as nonzero entries$\}$. Then $I + E \neq r(I + E')$ for any $E \neq E'$ as given in $T$ and $r \in R$. Hence $\alpha_{(I+E)} \neq \alpha_{(I+E')}$ by Lemma 3.3. Thus $\text{Inn}(B) \geq |S| + |T| = 2 \cdot |\text{Inn}(B)| = 2 \cdot (2^n - 1) > n^2$ for all $n > 2$. For $n = 2$ and $I \neq -I$, $I + D$, $I - D$, $I + E$, and $I - E$ are 5 invertible elements in $B$ which induce 5 distinct inner automorphisms of $B$. Therefore $|\text{Inn}(B)| \geq 5 > 4 = n^2$. \qed

Theorem 3.5. Let $A$ be an Azumaya algebra over its center $C$ of rank $k^2$ for some integer $k$. If either $k > 2$, or $k = 2$ and $\Char(C) \neq 2$, then $|\text{Inn}(A)| > k^2$.

Proof. Let $J$ be the Jacobson radical of $A$, then $A/JA \cong \bigoplus_{i=1}^t A_i$ for some integer $t$ where $A_i$ is a central simple algebra over a field $F_i$ for each $i$. By hypothesis, $\text{rank}_C(A) = k^2$, so $A_i = M_{k_i}(D_i)$, a matrix ring of order $k_i$ over a central division algebra $D_i$ for some integer $k_i$ such that $k_i^2 \dim_{F_i}(D_i) = k^2$ for each $i$. Since $\dim_{F_i}(D_i) = m_i^2$ for some integer $m_i$, $k_i^2 \dim_{F_i}(D_i) = \dim_{F_i}(A_i) = (k_i m_i)^2 = k^2$. We claim that $|\text{Inn}(A_i)| > (k_i m_i)^2 = k^2$ for each $i$. In fact, there are more than $k_i^2$ invertible matrices $\{E_{ij}\}$ over $D_i$ inducing distinct inner automorphisms by Lemma 3.4 and there are $m_i^2$ linearly independent
invertible elements \(\{d_i\}\) in \(D_i\) over \(F_i\), so there are more than \((k_im_i)^2\) invertible elements \(\{E_jd_i\}\) in \(A_i\) which induce more than \((k_im_i)^2\) distinct inner automorphisms of \(A_i\). Thus \(|\text{Inn}(A_i)| > (k_im_i)^2 = k^2\). Therefore \(|\text{Inn}(A)| > \prod_{i=1}^{l}(k_im_i)^2 = k^{2l} \geq k^2. \quad \Box

Next is our first main result for a central Galois algebra derived from Azumaya algebras.

**Theorem 3.6.** Let \(B\) be a central Galois algebra of rank \(k^2\) with Galois group \(G\). If either \(k > 2\), or \(k = 2\) and \(\text{Char}(C) \neq 2\), then \(|\text{Inn}(B)| > |G|\).

**Proof.** Since \(B\) is a central Galois algebra with Galois group \(G\), \(B\) is an Azumaya algebra such that \(\text{rank}_C(B) = |G| = k^2\) for some integer \(k\) by Lemmas 3.1 and 3.2. Thus \(|\text{Inn}(B)| > k^2 = |G|\) by Theorem 3.5. \(\Box\)

4. The Galois group

In this section, we shall show the main theorem for a Galois algebra \(B\) over \(R\) with Galois group \(G\); that is, \(G = \text{Aut}_R(B)\) if and only if either \(B\) is commutative with no idempotents but 0 and 1, or \(B \cong R \oplus R\) where \(R\) contains no idempotents but 0 and 1. We need two results, the first one is the structure of a Galois algebra with no idempotents but 0 and 1 proved by DeMeyer [5, Theorem 1] and the second one is the existence of an automorphism of a Galois algebra which is not in the Galois group.

**Proposition B** (DeMeyer [5, Theorem 1]). Let \(B\) be a Galois algebra with no idempotents but 0 and 1 over \(R\) with Galois group \(G\) and \(K = \{g \in G \mid g(c) = c \text{ for all } c \in C\}\) where \(C\) is the center of \(B\). Then \(B\) is a central Galois algebra with Galois group \(K\) and \(C\) is a commutative Galois algebra over \(R\) with Galois group \(G/K\).

**Lemma 4.1.** Let \(B\) be a Galois algebra over \(R\) with Galois group \(G\) and \(\lambda \in \text{Aut}_R(B)\). If \(e \neq 0\) is a central idempotent in \(B\) such that \(\lambda|_{Be}\) is identity and \(\lambda|_{B(1-e)}\) is not identity, then \(\lambda \notin G\).

**Proof.** Since \(B\) is a Galois algebra over \(R\), \(B\) has a \(G\)-Galois system \(\{a_i, b_i \in B, i = 1, 2, \ldots, m\}\) for some integer \(m\) such that \(\sum_{i=1}^{m} a_i b_i = 1\) and \(\sum_{i=1}^{m} a_i g(b_i) = 0\) for each \(g \neq 1\) in \(G\). Assume \(\lambda \in G\). Then \(\sum_{i=1}^{m} a_i \lambda(b_i) = 0\) because \(\lambda \neq 1\) in \(G\). Hence \(0 = 0 \cdot e = \sum_{i=1}^{m} a_i \lambda(b_i) e = \sum_{i=1}^{m} a_i \lambda(b_i) e = \sum_{i=1}^{m} a_i (b_i e) = e\). This is a contradiction; and so \(\lambda \notin G\). \(\Box\)

**Lemma 4.2.** Let \(B\) be a Galois algebra over \(R\) with Galois group \(G\). If \(G = \text{Aut}_R(B)\), then \(R\) contains no idempotents but 0 and 1.

**Proof.** Assume \(e^2 = e \in R\) and \(e \neq 0, 1\). Then \(B = Be \oplus B(1-e)\), \(G(Be) = Be\), and \(G(B(1-e)) = B(1-e)\). Hence for each \(g \neq 1\) in \(G\), \(g|_{Be} \neq 1\) or \(g|_{B(1-e)} \neq 1\). Without loss of generality, assume \(g|_{B(1-e)} \neq 1\). Therefore \(\lambda = 1 \oplus g|_{B(1-e)} \in \text{Aut}_R(B)\) but
Theorem 4.4. \( \lambda \notin G \) by Lemma 4.1. Thus \(|\text{Aut}_R(B)| > |G|\). This is a contradiction; and so \( R \) contains no idempotents but 0 and 1. \( \square \)

Lemma 4.3. Let \( B \) be a Galois algebra over \( R \) with Galois group \( G \). If \( G = \text{Aut}_R(B) \) and \( |G| > 2 \), then \( B \) is a Galois algebra with no central idempotents but 0 and 1.

Proof. By Lemma 4.2, \( R \) contains no idempotents but 0 and 1. Next we claim that \( B \) contains no central idempotents but 0 and 1. Since \( B \) is a Galois algebra over \( R \) with Galois group \( G \), \( B \) is a finitely generated \( R \)-module. Hence \( C \) contains only finitely many minimal idempotents \( \{e_i \mid i = 1, 2, \ldots, q\} \) for some integer \( q \). If \( q = 1 \), then \( B \) is a Galois algebra with no central idempotents but 0 and 1. Next we show that \( q > 1 \) leads to a contradiction. Assume that \( q > 1 \). Since \( \{e_i \mid i = 1, 2, \ldots, q\} \) are the minimal idempotents in \( C \), \( g \) permutes \( \{e_i\} \) for each \( g \in G \). Let \( g \neq 1 \) in \( G \) and \( g(e_1) = e_j \) for some \( j \). Then \( g(Be_1) = Be_j \). There are 4 cases.

Case 1: \( j = 1 \) and \( g|_{Be_1} \neq 1 \). We have that \( g|_{B(1-e_1)} \neq 1 \) since \( g \neq 1 \) in \( G \). But then by Lemma 4.1, \( g \notin G \). This is a contradiction.

Case 2: \( j = 1 \) and \( g|_{Be_1} \neq 1 \). We have an automorphism \( \lambda \in \text{Aut}_R(B) \) such that \( \lambda|_{Be_1} = g|_{Be_1} \) and \( \lambda|_{Be_i} = 1 \) for \( i \neq 1 \). Thus \( \lambda \in \text{Aut}_R(B) \) but \( \lambda \notin G \) by Lemma 4.1. Therefore \( \text{Aut}_R(B) \neq G \), a contradiction again.

Case 3: \( j \neq 1 \) and \( q > 2 \). We have an automorphism \( \lambda \in \text{Aut}_R(B) \) such that \( \lambda|_{Be_1} = g|_{Be_1} \), \( \lambda|_{Be_i} = g^{-1}|_{Be_i} \), and \( \lambda|_{Be_i} = 1 \) for \( i \neq 1 \). Thus \( \lambda \in \text{Aut}_R(B) \) but \( \lambda \notin G \) by Lemma 4.1. Therefore \( \text{Aut}_R(B) \neq G \). This contradiction leads to \( q = 1 \) or 2.

Case 4: \( j \neq 1 \) and \( q = 2 \). Then \( g(e_1) = e_2 \). Since \(|G| > 2 \), there exists a \( g' \neq 1 \), \( g \) in \( G \) such that \( g'(e_1) = e_1 \) or \( g'(e_1) = e_2 \). Noting that \( g'(e_1) = e_1 \) is either Case 1 or Case 2, we can assume that \( g'(e_1) = e_2 \). But then \( (g^{-1}g')(e_1) = e_1 \) and \( g^{-1}g' \neq 1 \) in \( G \). This is either Case 1 or Case 2 which leads to a contradiction. Hence \( q = 1 \), and so \( B \) is a Galois algebra with no central idempotents but 0 and 1. \( \square \)

Next we show the structure of \( B \) when \( G = \text{Aut}_R(B) \) for either \(|G| > 2 \) or \(|G| = 2 \), respectively.

Theorem 4.4. Let \( B \) be a Galois algebra over \( R \) with Galois group \( G \). If \( G = \text{Aut}_R(B) \) and \(|G| > 2 \), then \( B \) is a commutative Galois algebra with no idempotents but 0 and 1.

Proof. Since \( G = \text{Aut}_R(B) \) and \(|G| > 2 \), by Lemma 4.3, \( B \) is a Galois algebra with no central idempotents but 0 and 1. Hence by Proposition B, \( B \) is a central Galois algebra with Galois group \( K \) where \( K = \{g \in G \mid g(c) = c \text{ for all } c \in C\} \), and \( C \) is a commutative Galois algebra over \( R \) with Galois group \( G/K \). But then \(|K| = k^2 \) for some integer \( k \) by Lemma 3.2. Since \( G = \text{Aut}_R(B) \), \( K = \text{Aut}_C(B) \). We have three cases.

Case 1: either \( k > 2 \) or \( k = 2 \) and \( \text{Char}(C) \neq 2 \). Then \(|\text{Aut}_C(B)| \geq |\text{Inn}(A)| > k^2 = |K| \) by Theorem 3.6. This contradicts to \( K = \text{Aut}_C(B) \).

Case 2: \( k = 2 \) and \( \text{Char}(C) = 2 \). Then \(|K| = 2^2 = 4 \), and so \( B \) is a central Galois algebra with Galois group \( K \) of order 4. Thus \( 4 \) is a unit in \( C \) [8, Corollary 3]. This is impossible because \( \text{Char}(C) = 2 \) by hypothesis in this case.

Consequently, we are left with only
Case 3: \( k = 1 \). That is, \( K = \{1\} \), so \( B = C \) which is a commutative Galois algebra with no idempotents but 0 and 1. \( \square \)

**Theorem 4.5.** Let \( B \) be a Galois algebra over \( R \) with Galois group \( G \). If \( G = \text{Aut}_R(B) \) and \( |G| = 2 \), then either \( B \) is a commutative Galois algebra with no idempotents but 0 and 1, or \( B \cong R \oplus R \) where \( R \) contains no idempotents but 0 and 1.

**Proof.** Since \( |G| = 2 \), \( G \) is a cyclic group. Hence \( B \) is commutative [4, Theorem 11]. By hypothesis, \( B \) is a Galois algebra over \( R \) with Galois group \( G \) and \( G = \text{Aut}_R(B) \). Hence \( R \) contains no idempotents but 0 and 1 by Lemma 4.2. But then by [9, Corollary 12(ii)], either \( B \) contains no idempotents but 0 and 1, or \( B \cong R \oplus R \). Thus either \( B \) is a commutative Galois algebra with no idempotents but 0 and 1, or \( B \cong R \oplus R \) where \( R \) contains no idempotents but 0 and 1. \( \square \)

**Theorem 4.6.** Let \( B \) be a Galois algebra over \( R \) with Galois group \( G \). Then \( G = \text{Aut}_R(B) \) if and only if either \( B \) is commutative with no idempotents but 0 and 1, or \( B \cong R \oplus R \) where \( R \) contains no idempotents but 0 and 1.

**Proof.** (\( \Rightarrow \)) is consequence of Theorems 4.4 and 4.5.

(\( \Leftarrow \)) If algebra \( B \) is commutative with no idempotents but 0 and 1, then \( G = \text{Aut}_R(B) \) [3, Theorem 3.5]. Next, assume \( B \cong R \oplus R \) where \( R \) contains no idempotents but 0 and 1. Then \( B \) contains only two minimal idempotents, \( e_1 = (1, 0) \) and \( e_2 = (0, 1) = 1 - e_1 \). Hence for any \( \alpha \neq 1 \) in \( \text{Aut}_R(B) \), \( \alpha(e_1) = e_2 \) and \( \alpha(e_2) = e_1 \). Thus \( \text{Aut}_R(B) = \{1, \alpha \mid \alpha(e_1 + be_2) = ae_2 + be_1 \text{ for each } ae_1 + be_2 \in B\} \); and so \( G = \text{Aut}_R(B) \). \( \square \)

Theorem 4.6 can be applied to a Galois Azumaya extension as studied in [1,2]. Ring \( B \) is called a Galois Azumaya extension of \( B^G \) with Galois group \( G \) if \( B \) is a Galois extension of \( B^G \) with Galois group \( G \) and \( B^G \) is an Azumaya \( C^G \)-algebra (see [1,2]). Ring \( B \) is called a DeMeyer–Kanzaki Galois extension of \( B^G \) with Galois group \( G \) if \( B \) is an Azumaya algebra over \( C \) and \( C \) is a Galois algebra over \( C^G \) with Galois group induced by and isomorphic with \( G \) (see [4,8]).

**Corollary 4.7.** Let \( B \) be a Galois Azumaya extension of \( B^G \) with Galois group \( G \). Then \( G = \text{Aut}_{BG}(B) \) if and only if either \( B \) is a DeMeyer–Kanzaki Galois extension with no central idempotents but 0 and 1, or \( B \cong B^G \oplus B^G \) which is a Galois extension of \( B^G \) with Galois group \( G \) of order 2 where \( G = \{1, g\} \) such that \( g(be) = b(1 - e) \) and \( g(b(1 - e)) = be \) for each \( b \in B^G \), and \( e = 1 \oplus 0 \) is a minimal central idempotent of \( B \).

**Proof.** (\( \Rightarrow \)) Since \( B \) is a Galois Azumaya extension of \( B^G \) with Galois group \( G \), \( B = B^G \cdot V_B(B^G) \cong B^G \otimes_{C^G} V_B(B^G) \) such that \( V_B(B^G) \) is Galois algebra over \( C^G \) with Galois group \( G \mid G_{V_B(B)} \cong G \) [1, Theorem 2]. By hypothesis, \( G = \text{Aut}_{BG}(B) \), so \( G \mid G_{V_B(B)} = \text{Aut}_{C^G}(V_B(B^G)) \). Thus by Theorem 4.6, either \( V_B(B^G) \) is commutative with no idempotents but 0 and 1, or \( V_B(B^G) \cong C^G \oplus C^G \) where \( C^G \) contains no idempotents but 0 and 1. Noting that \( B = B^G \cdot V_B(B^G) \) implies that the center of \( V_B(B^G) \) is \( C \), we conclude that either \( B \) is a DeMeyer–Kanzaki Galois extension with no central idempotents but 0
and $1$, or $B \cong B^G \oplus B^G$ where $B^G$ contains no central idempotents but 0 and 1; and so $B$ contains only two minimal central idempotents, $e = 1 \oplus 0$ and $1 - e$. Thus for a $g \neq 1$ in $G$, $g(e) = 1 - e$ and $g(1 - e) = e$. Therefore $G = \{1, g\}$ is of order 2 such that $g(be) = b(1 - e)$ and $g(b(1 - e)) = be$ for each $b \in B^G$.

$(\Leftarrow)$ In case $B$ is a DeMeyer–Kanzaki Galois extension with no central idempotents but 0 and 1, $C$ is a commutative Galois algebra with Galois group $G|_C \cong G$ with no idempotents but 0 and 1. Hence $G \cong G|_C = \text{Aut}_{C^G}(C)$ [3, Theorem 3.5]. But $B = B^G \cdot C \cong B^G \otimes_{C^G} C$ [4, Lemma 2], so $G \cong G|_C = \text{Aut}_{C^G}(C) \cong \text{Aut}_{B^G}(B)$. Next, assume $B = B^G \oplus B^G$ which is a Galois extension of $B^G$ with Galois group $G$ of order 2 where $G = \{1, g\}$ such that $g(be) = b(1 - e)$ and $g(b(1 - e)) = be$ for each $b \in B^G$, and $e = 1 \oplus 0$ is a minimal central idempotent of $B$. Since any non identity element in $\text{Aut}_{B^G}(B)$ permutes $\{e, 1 - e\}$ and $B = B^G \oplus B^G$, $\text{Aut}_{B^G}(B) = \{1, g\} = G$. $\square$

**Acknowledgments**

This paper was revised under the suggestions of the referee. The authors thank the referee for the valuable suggestions. Also this work was done under the support of a Caterpillar Fellowship at Bradley University. We thank Caterpillar Inc. for the support.

**References**