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The Existence of Cnoidal Water Waves with Surface Tension

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1. INTRODUCTION

In this paper we present a proof of the existence of cnoidal water waves, taking into account the effect of surface tension. Previous proofs have either neglected surface tension or assumed that the water was very shallow. The waves considered are periodic, progressive, two-dimensional flows in a channel with horizontal bottom. They are called cnoidal because their first approximation involves the Jacobian elliptic function cn. These waves are of particular interest because they do not arise in a linear model. We also show that, if surface tension is neglected, the cnoidal waves tend to solitary waves, i.e., waves with a single crest, as the period tends to infinity. It is not known whether solitary waves exist under the influence of surface tension. The method used here is similar to that of [2], where a simplified proof was given for the existence of solitary waves without surface tension.

It will be assumed that the fluid is incompressible and inviscid, and that the fluid motion is determined by gravity, surface tension, and pressure forces. We choose coordinates so that the flow is steady. The flows of interest are then symmetric with respect to a vertical line; we take this line as the y_1-axis and the bottom as the x_1-axis in the plane of the flow. The free surface is described by a curve $y_1 = Y_1(x_1)$, and since the flow will be irrotational, the velocity field is the gradient of a harmonic potential ϕ_1 in the region $0 \leqslant y_1 \leqslant Y_1(x_1)$. If ψ_1 is the harmonic conjugate of ϕ_1, i.e., the stream function, then $\chi_1 = \phi_1 + i\psi_1$ is an analytic function of $z_1 = x_1 + iy_1$. On the free surface one boundary condition is a form of Bernoulli's law:

$$\frac{1}{2}\left|\frac{d\chi_1}{dz_1}\right|^2 + gY_1 - T\frac{d^2Y}{dx_1^2}\left\{1 + \left(\frac{dY_1}{dx_1}\right)^2\right\}^{-3/2} = \text{constant}, \qquad (1.1)$$

where g is the acceleration of gravity, T is the coefficient of surface tension,

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and the density is assumed to be 1. The fact that the velocity is tangent to the streamlines imposes two additional boundary conditions:

$$\frac{\partial \phi_1}{\partial y_1} = \frac{\partial \phi_1}{\partial x_1} \frac{\partial Y_1}{\partial x_1} \qquad \text{on} \quad y_1 = Y_1(x_1), \tag{1.2}$$

$$\frac{\partial \phi_1}{\partial y_1} = 0 \qquad \text{on} \quad y_1 = 0. \tag{1.3}$$

In addition Y_1 and $\nabla\phi_1$ are to be periodic functins of x_1, say with period $2L_1$; Y_1 and ψ_1 will be even, and ϕ_1 odd, with respect to x_1. Finally we assume that the mean depth is a fixed number h, so that

$$\int_0^{L_1} Y_1(x_1)\, dx_1 = hL_1 . \tag{1.4}$$

For a reason to be explained, we assume that $h > .5$ cm.

Our result is that exact solutions of the above problem exist for various values of two parameters, K and a; for each K sufficiently large, there is a set of positive values of a with 0 as a limit point for which solutions exist satisfying

$$Y_1(x_1) = \frac{4}{3}\, a^2 h(1 - 3\beta_0)\, k^2 cn^2 \left(\frac{ax_1}{h}\, ; k\right) + Y_0 + o(a^2), \tag{1.5}$$

$$L_1 = hK/a + o(a), \tag{1.6}$$

as $a \to 0$. Here $\beta_0 = T/gh^2$, $0 \leqslant k \leqslant 1$, $K = K(k)$ is the complete elliptic integral of the first kind, and Y_0 is determined by condition (1.4). The precise result is stated in Section 2 as Theorem 1.

If surface tension is neglected, the last term in (1.1) does not appear. In that case we obtain solutions for all (K, a) with $K \geqslant K_1$, $0 < a \leqslant a_1$, where K_1, a_1 are certain constants. These solutions satisfy (1.5), (1.6) with $\beta_0 = 0$. Under a certain transformation they are C^N with respect to a for any N; thus in principle an expansion about $a = 0$ can be calculated to any desired degree of accuracy. Moreover, for fixed a, the solution converges as $K \to \infty$ to a solitary wave with the profile

$$Y_1(x_1) = h + \frac{4}{3}\, a^2 h \operatorname{sech}^2 \frac{ax_1}{h} + o(a^4). \tag{1.7}$$

These results are formulated in Theorem 2.

The general method of proof is to estimate solutions of a linearized form of the problem and then perform an iteration to obtain exact solutions. Because of the stretching of the horizontal variable which is apparent in (1.5), it is not possible to obtain sharp estimates on the linear problem which are bounded as $a \to 0$. For this reason we use an iteration of the type introduced by Nash and further developed by Moser [9], in which an approximation by Newton's method is alternated with a smoothing operator.

While it seems that surface tension should have little effect on the nature of these waves, the linearized problem mentioned above fails to be solvable for certain values of the parameters if $\beta_0 < \frac{1}{3}$, in contrast to the case where surface tension is neglected. The condition $\beta_0 < \frac{1}{3}$ corresponds to $h > .5$ cm., approximately. It is for this reason that exclusions are made in Theorem 1. The fact that cnoidal waves with surface tension exist for arbitrarily large K leads us to expect that solitary waves also exist in that case, but this does not follow from the arguments given here.

The approximate description (1.5) of cnoidal waves was given by Korteweg and De Vries [6], extending earlier treatments of solitary waves. Keller [5] provided a more systematic derivation based on the shallow water theory. The first proof of the existence of cnoidal and solitary waves, neglecting surface tension, was given by Lavrent'ev [7]. The solitary waves were obtained as limits of cnoidal waves, but it is difficult to compare his convergence result with the one given here. More direct proofs were found by Friedrichs and Hyers [3] for solitary waves and Littman [8] for cnoidal waves. Ter-Krikorov [12] improved Littman's method to obtain solutions for all K sufficiently large and a in a fixed interval. Zeidler [14] gave a proof of the existence of cnoidal waves under the influence of surface tension assuming that $h < .5$ cm. An extensive survey of the existence theory for this class of problems, as well as a more complete bibliography, may be found in [15]. A general account of the theory of surface waves is given in [13].

2. Statement of Results

We begin by nondimensionalizing the problem as in [12]. The conditions (1.2), (1.3) imply that ψ is constant on the top and bottom; we assume that $\psi = 0$ on $y_1 = 0$ and $\psi = Q$ on $y_1 = Y_1(x_1)$. Since the vertical velocity component is zero on $x_1 = L_1$, $\phi(L_1, y_1)$ is a constant, which we denote by cL_1. Then c can be interpreted as the average velocity on any submerged horizontal line. We introduce new variables

$$z_0 = x_0 + iy_0 = cz_1/Q, \qquad \chi = \phi + i\psi = \chi_1/Q$$

and set

$$Y = cY_1/Q, \qquad L = cL_1/Q.$$

The condition (1.1) now can be written as

$$\tfrac{1}{2} \mid D_{z_0}\chi \mid^2 + \nu Y - T_0 Y_{x_0 x_0}(1 + Y_{x_0}^2)^{-3/2} = \text{constant}, \tag{2.1}$$

where

$$\nu = gQ/c^3, \qquad T_0 = T/Qc.$$

In addition we have

$$\psi = 0 \quad \text{on} \quad y_0 = 0, \qquad \psi = 1 \quad \text{on} \quad y_0 = Y(x_0),$$
$$\phi = \pm L \quad \text{on} \quad x_0 = \pm L.$$

We will now transform the problem in a way introduced by Levi-Civita. We set

$$D_{z_0}\chi = \nu^{1/3} \exp\{-i(\theta' + i\lambda')\}$$

and attempt to find $\theta' + i\lambda'$ as an analytic function of χ in the strip $\{\chi : 0 \leqslant \psi \leqslant 1\}$ satisfying appropriate conditions so that a solution of the original problem can be constructed. This transformation is discussed in detail in the case of no surface tension in [11], Secs. 10.9 and 12.2a.

To convert the last term of (2.1) to the new variables, we note that $Y_{x_0} = \phi_{y_0}/\phi_{x_0} = \tan\theta'$, and therefore

$$\begin{aligned} Y_{x_0x_0}(1 + Y_{x_0}^2)^{-3/2} &= D_{x_0}\{Y_{x_0}(1 + Y_{x_0}^2)^{-1/2}\} \\ &= D_{x_0}(\tan\theta'/\sec\theta') = D_{x_0}\sin\theta' \\ &= (D_\phi \sin\theta')\, D_{x_0}\phi(x_0\,, Y(x_0)) \\ &= (\cos\theta')\, \theta'_\phi\{\phi_{x_0} + \phi_{y_0}Y_{x_0}\} \\ &= \nu^{1/3}\cos\theta'\{\cos\theta' + \sin\theta'\tan\theta'\}\, e^{\lambda'}\theta'_\phi \\ &= \nu^{1/3}e^{\lambda'}\theta'_\phi\,. \end{aligned}$$

Differentiating (2.1) with respect to ϕ and using this fact, we obtain as in [11] the condition

$$\theta'_\psi - e^{-3\lambda'}\sin\theta' + \beta e^{-\lambda'}(\theta'_{\phi\phi} - \theta'_\phi\theta'_\psi) = 0 \qquad \text{on} \quad \psi = 1, \tag{2.2}$$

where

$$\beta = T/g^{1/3}Q^{4/3}. \tag{2.3}$$

Also

$$\theta' = 0 \quad \text{on} \quad \psi = 0, \tag{2.4}$$

and $\theta' + i\lambda'$ is periodic in ϕ with period $2L$. Since we are interested in symmetric flows, θ' is odd and λ' even in ϕ.

Three additional conditions are satisfied by $\theta' + i\lambda'$. Since $x_0 = L$ when $\chi = L$, we have

$$L = \nu^{-1/3} \int_0^L e^{-\lambda'(\phi)}\, d\phi. \tag{2.5}$$

The restriction that the mean depth be h implies that

$$hcL/Q = \int_0^L \int_0^{Y(x_0)} 1\, dy_0\, dx_0$$

and thus

$$hcL/Q = \nu^{-2/3} \int_0^L \int_0^1 e^{-2\lambda'}\, d\psi\, d\phi. \tag{2.6}$$

If we are given a $\theta' + i\lambda'$ satisfying (2.2), (2.4), and the periodicity and parity conditions, ν is then determined by (2.5) and hence Q and c by (2.6). On the other hand, Q is determined by β according to (2.3). Thus β cannot be arbitrary in (2.2) but must satisfy a certain relation, which is found to be

$$\beta = \beta_0 \left\{ \int_0^L \int_0^1 e^{-2\lambda'}\, d\psi\, d\phi \Big/ \int_0^L e^{-\lambda'}\, d\phi \right\}^2 \tag{2.7}$$

where

$$\beta_0 = T/gh^2. \tag{2.8}$$

Finally, if $\theta' + i\lambda'$ is determined satisfying the prescribed conditions, including (2.7), and is sufficiently small, then a solution of the original problem can be constructed by finding z_0 as a function of χ using the formula

$$z_0(\chi) = \nu^{-1/3} \int_0^\chi \exp\{i\theta'(\zeta) - \lambda'(\zeta)\}\, d\zeta. \tag{2.9}$$

One checks that this function is one-to-one and the inverse $\chi(z_0)$ satisfies the required conditions; see [11], Sec. 12.2a.

In order to obtain a family of solutions with limiting form as the amplitude tends to zero, it will be necessary to stretch the horizontal variable as was done in [3]. Thus our independent variables will be

$$x = a\phi, \qquad y = \psi,$$

where a is a small positive number. For convenience we also rescale the dependent variables,

$$\theta = a^{-3}\theta', \qquad \lambda = a^{-2}\lambda'.$$

The problem can now be stated as follows: Functions θ and λ are to be found on

$$\mathscr{R} = \{(x, y) : 0 \leqslant y \leqslant 1\}$$

such that

$$\theta_y - e^{-3\epsilon\lambda} S(\theta, \epsilon) + \beta e^{-\epsilon\lambda}\{\epsilon\theta_{xx} - \epsilon^2\theta_x\theta_y\} = 0 \qquad \text{on} \quad y = 1, \tag{2.10}$$

$$\theta_y = -\lambda_x\,, \ \lambda_y = \epsilon\theta_x \qquad \text{on} \quad \mathscr{R} \tag{2.11}$$

$$\theta = 0 \qquad \text{on} \quad y = 0, \tag{2.12}$$

θ is odd and λ even in x, (2.13)

θ and λ have period $2K$ in x. (2.14)

In addition (2.7) must be satisfied. Here $\epsilon = a^2$, $K = aL$, and $S(\theta, \epsilon) = a^{-3} \sin a^3\theta$.

With each $K \geqslant \pi/2$ we associate the number k, $0 \leqslant k < 1$, such that $K = K(k)$ is the complete elliptic integral of the first kind,

$$K(k) = \int_0^{\pi/2} (1 - k^2 \sin \alpha)^{-1/2}\, d\alpha.$$

We denote by k' the number such that $k^2 + k'^2 = 1$. The solutions θ and λ obtained will tend as $\epsilon \to 0$ to the values

$$\lambda^* = (1 - 3\beta_0)\tau, \qquad \theta^* = -(1 - 3\beta_0) y\, D_x\tau, \tag{2.15}$$

where

$$\tau(x; K) = \frac{4}{9}\{2k^2 - 1 - 3k^2 cn^2(x; k)\}, \tag{2.16}$$

cn being the cnoidal function.

We next describe the set of (K, ϵ) for which a solution is obtained. Let Σ be a closed subset of $\{(K, \epsilon)\colon K \geqslant K_1\,, 0 \leqslant \epsilon \leqslant 1\}$ containing $[K_1\,, \infty) \times \{0\}$, where $K_1 > K(2^{-1/2})$. We assume $0 < \beta_0 < \frac{1}{3}$; this holds if $h > .5$ cm. Then the equation

$$\mu \coth \mu = 1 + \beta_0\mu^2$$

has a unique positive solution μ_0 (see [14, Lemma 3]). We require that for all $(K, \epsilon) \in \Sigma$ with $\epsilon > 0$,

$$\left| \frac{\mu_0 K}{a} - n\pi \right| \geqslant c_0 > 0 \tag{2.17}$$

for each integer $n > 0$ and some $c_0 < \pi/2$, and also

$$a \leqslant Ck' \tag{2.18}$$

for some C. As before $a^2 = \epsilon$. Note that Σ can be chosen so that for every $K \geqslant K_1$, there is a $(K, \epsilon) \in \Sigma$ with ϵ arbitrarily small. Given $\epsilon_1 > 0$, we set $\Sigma_1 = \{(K, \epsilon) \in \Sigma : 0 \leqslant \epsilon \leqslant \epsilon_1\}$. We now state the main result.

THEOREM 1. *Assume $0 < \beta_0 < \frac{1}{3}$. Given an even integer m and a set Σ with the properties specified above, there is an ϵ_1 sufficiently small so that functions θ, λ on $\mathscr{R} \times \Sigma_1$ exist with the following properties:*

(i) *For each $(K, \epsilon) \in \Sigma_1$, $D_x^{\alpha_1} D_y^{\alpha_2} \theta(\cdot; K, \epsilon)$ is continuous on $\mathscr{R}$ for $\alpha_1 + \alpha_2 \leqslant m + 2$, and the same for λ.*

(ii) *For each $(K, \epsilon) \in \Sigma_1$, $\theta(\cdot; K, \epsilon)$ and $\lambda(\cdot; K, \epsilon)$ satisfy* (2.10)–(2.14) *and* (2.7).

(iii) *As $\epsilon \to 0$ we have*

$$D_x^{\alpha_1} D_y^{\alpha_2} \theta(x, y; K, \epsilon) = D_x^{\alpha_1} D_y^{\alpha_2} \theta^*(x, y; K) + o(1)$$

and the same for λ, λ^*, with the error uniform in $(x, y; K)$ for K in a bounded interval. Here $\alpha_1 + \alpha_2 \leqslant m + 2$, and θ^* and λ^* are given by (2.15), (2.16).

If surface tension is neglected, this result can be improved, as described earlier.

THEOREM 2. *Let $\Sigma_1 = [K_1, \infty) \times [0, \epsilon_1)$, with K_1 as before. Given integers m and N with m even, there is an ϵ_1 sufficiently small so that functions θ, λ on $\mathscr{R} \times \Sigma_1$ exist with the following properties:*

(i) *$D_\epsilon^{\,j} D_x^{\alpha_1} D_y^{\alpha_2} \theta$ is continuous on $\mathscr{R} \times \Sigma_1$ for $\alpha_1 + \alpha_2 \leqslant m + 2$, $j \leqslant N$, and the same for λ.*

(ii) *For each $(K, \epsilon) \in \Sigma_1$, $\theta(\cdot; K, \epsilon)$ and $\lambda(\cdot; K, \epsilon)$ satisfy* (2.10)–(2.14) *with $\beta = 0$.*

(iii) *For $\epsilon = 0$, θ and λ are given by* (2.15), (2.16) *with $\beta_0 = 0$.*

(iv) *As $K \to \infty$,*

$D_\epsilon^{\,j} D_x^{\alpha_1} D_y^{\alpha_2} \theta(x, y; K, \epsilon) \to D_\epsilon^{\,j} D_x^{\alpha_1} D_y^{\alpha_2} \theta(x, y; \infty, \epsilon)$ for a certain function $\theta(\cdot; \infty, \epsilon)$, uniformly in $(x, y; \epsilon)$ for x in a bounded interval, provided $\alpha_1 + \alpha_2 \leqslant m + 1$, $j \leqslant N - 1$. The same is true for λ.

(v) *The limit functions $\theta(\cdot; \infty, \epsilon)$, $\lambda(\cdot; \infty, \epsilon)$ satisfy* (2.10)–(2.13) *with $\beta = 0$, and $\theta \to 0$, $\lambda \to 4/9$ exponentially as $|x| \to \infty$.*

(vi) *At* $\epsilon = 0$, θ *and* λ *are given by* (2.15) *with* $\beta_0 = 0$ *and*

$$\tau(x; \infty) = \frac{4}{9}(1 - 3 \operatorname{sech}^2 x).$$

Theorem 1 is proved in Section 6 and Theorem 2 in Section 7. The problem is set in appropriate function spaces in Section 3 with β treated as a free parameter to be chosen at the end. Section 4 is concerned with the solution of an associated linear problem. This problem is not solvable for all values of the parameters, and the restrictions (2.17) and (2.18) are needed to obtain uniform bounds on the solution. Finally in Section 6 an iteration of the Nash–Moser type is used to obtain solutions of the nonlinear problem. The value of β is then determined so that (2.7) holds. The proof of convergence as $K \to \infty$ relies on compactness and a uniqueness property for the linearized solitary wave problem derived in [2].

Approximate expressions for various quantities associated with the waves can be obtained from the limiting form at $\epsilon = 0$, with error estimates provided by the theorems. Using (2.5) and (2.6) we find that

$$\nu = 1 - 3(1 - 3\beta_0)\langle\tau\rangle\epsilon + o(\epsilon)$$

$$c^2/gh = 1 + 3(1 - 3\beta_0)\langle\tau\rangle\epsilon + o(\epsilon) \tag{2.19}$$

$$Q = ch + o(\epsilon), \tag{2.20}$$

where

$$\langle\tau\rangle = K^{-1} \int_0^K \tau(x; K)\, dK.$$

Since $\langle\tau\rangle \to 1$ as $K \to \infty$, the flow is supercritical (i.e., $c^2 > gh$) for large K. The errors are uniform for K in a bounded interval, and in the case of no surface tension they are $O(\epsilon^2)$ uniformly for all K. Q/h can be interpreted as the average horizontal velocity over the fluid domain. Both notions of average velocity were introduced by Stokes; see e.g. [13, Sec. 7].

The approximate profile of the wave can be found using (2.9). Substituting for θ' and λ' we have

$$x_0(\phi, 1) = \phi + O(a)$$
$$y_0(\phi, 1) = 1 - \epsilon(1 - 3\beta_0)\{\tau(a\phi) - \langle\tau\rangle\} + o(\epsilon)$$

so that

$$Y(x_0) = 1 - \epsilon(1 - 3\beta_0)\{\tau(ax_0) - \langle\tau\rangle\} + o(\epsilon).$$

Returning to the physical coordinates and using (2.20), we find

$$Y_1(x_1) = h - \epsilon h(1 - 3\beta_0)\{\tau(ax_1/h) - \langle\tau\rangle\} + o(\epsilon),$$

which is equivalent to (1.5). The error is uniform in x_1 and K, for ax_1 and K in bounded intervals. When no surface tension is present, it is $O(\epsilon^2)$, uniformly for all K and ax_1 in a bounded interval. Similar expressions can be obtained for the velocity field; see [8, Sec. 9]. Equation (1.6) follows from (2.20).

As $K \to \infty$ in the case of no surface tension, the parameters have limiting values

$$\nu = \exp(-4\epsilon/3)$$
$$Q = ch,$$
$$c^2/gh = \exp(4\epsilon/3);$$

this can be seen from the nature of convergence of λ described in Section 7. Using these values the solitary wave of Theorem 2(vi) can be transformed back to the physical plane. The profile can be found as above, leading to (1.7).

Because we wish to allow K arbitrarily large, we shall use function spaces with the weight $dn(x; k)$, as was done by Ter-Krikorov in [12]. Here dn is a Jacobian elliptic function, even and of period $2K(k)$; dn decreases on $0 \leqslant x \leqslant K$ from 1 to k'. As $k \to 1$ (and $K \to \infty$), $dn(x; k) \to \operatorname{sech} x$, uniformly for x in bounded interval. The inequality

$$\int_x^K dn\, x'\, dx' \leqslant 2\, dn\, x, \qquad 0 \leqslant x \leqslant K, \tag{2.21}$$

will be used repeatedly; see [12], Theorem 6.3, for proof. The properties of elliptic functions used here can be found in [1], Chapter 16 (but note the parameter m there is k^2 here).

3. The Set-Up

We wish to find solutions of the problem (2.10)–(2.14) for various values of K, ϵ, and β. We suppose that $p = (K, \epsilon, \beta)$ lies in a fixed closed subset Π of $\Pi_0 = [K_1, \infty) \times [0, 1] \times (0, \frac{1}{3})$ such that β and $1 - 3\beta$ have positive lower bounds and $K_1 > K(2^{-1/2})$. Further restrictions on Π will be imposed later. We will treat θ as a single unknown with λ defined as

$$\lambda^0(x, y) = \epsilon \int_1^y \theta_x(K, y')\, dy' + \int_x^K \theta_y(x', y)\, dx' \tag{3.1}$$

plus a constant to be determined. With this definition, $\lambda_x = -\theta_y$; moreover, $\lambda_y = \epsilon\theta_x$ on $x = K$. If θ satisfies $\epsilon\theta_{xx} + \theta_{yy} = 0$ we then have $(\epsilon\theta_x - \lambda_y)_x = 0$ so that $\lambda_y = \epsilon\theta_x$ everywhere.

We now introduce function spaces in which to pose the problem. Let m be a large even integer and $j = 0, 1,$ or 2. We denote by $Y'_{m,j}$ the Banach space of functions $\theta: \mathscr{R} \to \mathbf{R}$ such that $D_x^{\alpha_1} D_y^{\alpha_2}\theta$ is bounded and continuous for $\alpha_2 \leqslant j$, $\alpha_1 + \alpha_2 \leqslant m + j$, with the additional requirement for $j = 2$ that $\theta(x\ 0) = 0$. We write Y'_m for $Y'_{m,0}$. Similarly X'_m will be the space of functions $\theta: \mathbf{R} \to \mathbf{R}$ with $D_x{}^\alpha\theta$ bounded and continuous; $\alpha \leqslant m$. Finally we let $Z'_m = Y'_m \times X'_m$ and define the map

$$F = (F^1, F^2) : Y'_{m,2} \times Y'_{m,1} \times \Pi_0 \to Z'_m$$

by the formula

$$\begin{aligned} F^1(\theta, \lambda; p) &= \epsilon\theta_{xx} + \theta_{yy} \\ F^2(\theta, \lambda; p) &= \beta\{\epsilon\theta_{xx} - \epsilon^2\theta_x\theta_y\} + e^{\epsilon\lambda}\theta_y - e^{-2\epsilon\lambda}S(\theta, \epsilon), \end{aligned} \tag{3.2}$$

where $p = (K, \epsilon, \beta)$ and $y = 1$ in the last equation. Clearly F is C^∞. Our problem is to find, for $p \in \Pi$ with $\epsilon > 0$, a $\theta \in Y'_{m,2}$, odd and of period $2K$ in x, for which $F(\theta, \lambda; p) = 0$; here λ is defined as indicated above.

When $\epsilon = 0$, F reduces to the linear map

$$\theta \mapsto (\theta_{yy}, (\theta_y - \theta)|_{y=1}).$$

The null space consists of $\theta \in Y'_{m,2}$ of the form $\theta(x, y) = \theta(x, 1)y$. We introduce a projection P on this subspace defined by $(P\theta)(x, y) = \theta(x, 1)y$. Moreover, if Q is the projection on Z'_m given by

$$Q(g, h) = \left(0, h - \int_0^1 g(\cdot, y)y\, dy\right),$$

an integration by parts show that $QF(\theta, p) = 0$ when $\epsilon = 0$. This leads us to modify F by defining

$$\tilde{F}(\theta, \lambda; p) = \begin{cases} \epsilon^{-1}QF(\theta, \lambda; p) + (I - Q)F(\theta, \lambda; p), & \epsilon > 0, \\ QF_\epsilon(\theta, \lambda; p) + (I - Q)F(\theta, \lambda; p), & \epsilon = 0. \end{cases} \tag{3.3}$$

$\tilde{F}$ is again smooth, and for $\epsilon > 0$, the zeros of F coincide with those of $\tilde{F}$.

Next we wish to incorporate the dependence on parameters into the map. With m and j as before, let $Y^K_{m,j}$ (or $Y^{e,K}_{m,j}$) be the subspace of $Y'_{m,j}$ consisting of functions which are odd (or even) and of period $2K$ with respect to x. Let $Y_{m,j}(\Pi)$ be the space of functions $\theta: \mathscr{R} \times \Pi \to \mathbf{R}$ such that $D_x^{\alpha_1}D_y^{\alpha_2}\theta$ is continuous for $\alpha_1 + \alpha_2 \leqslant m + j$, $\alpha_2 \leqslant j$; $\theta(\cdot; p) \in Y'_{m,j}$; and the norm

$$\|\theta\|_{m,j} = \sup |D_x^{\alpha_1}D_y^{\alpha_2}\theta(x, y; p)|/dn(x; k) \tag{3.4}$$

is finite. If $j = 2$, we again require $\theta = 0$ for $y = 0$. $Y^e_{m,j}(\Pi)$ is defined similarly, and $X_m(\Pi)$, $X_m{}^e(\Pi)$ are the corresponding space of functions $\mathbf{R} \times \Pi \to \mathbf{R}$.

Because of (2.1), equation (3.1) now defines a bounded linear operator

$\theta \mapsto \lambda^0$ from $Y_{m,2}(\Pi)$ to $Y^e_{m,1}(\Pi)$. We will choose the constant $\lambda - \lambda^0$ in such a way that $\lambda = \Lambda\theta + \gamma$, where Λ is a linear operator of the same type and γ is smooth bounded function of p. We now introduce the map

$$H: Y_{m,2}(\Pi) \to Z_m(\Pi) \equiv Y_m(\Pi) \times X_m(\Pi)$$

given by

$$(H\theta)(\cdot; p) = F(\theta(\cdot; p), (\Lambda\theta)(\cdot; p) + \gamma(p), p). \tag{3.5}$$

H is smooth, and its derivatives can be expressed in terms of those of $\tilde{F}$; e.g.,

$$(D_\theta H(\theta)w)(\cdot; p) = D_\theta \tilde{F} w(\cdot; p) + D_\lambda \tilde{F}(\Lambda w)(\cdot; p), \tag{3.6}$$

where $D_\theta \tilde{F}$ and $D_\lambda \tilde{F}$ are evaluated as in (3.5).

We will solve the problem (2.10)–(2.14) for $p \in \Pi$, $\epsilon > 0$, by finding a solution of $H(\theta) = 0$. Our first step is to choose a θ^* for which $H(\theta^*)(\cdot; p) = 0$ when $\epsilon = 0$. Then $H(\theta^*)$ will be arbitrarily small provided $\sup\{\epsilon : p \in \Pi\}$ is sufficiently small. It is easily seen that θ^* satisfies our requirements if and only if $\theta^*(p) \in PY'_{m,2}$ and $QF_\epsilon(\theta^*(p), \lambda^*(p); p) = 0$ when $\epsilon = 0$, $\lambda^* = \Lambda\theta^* + \gamma$. Now for $\theta \in PY'_{m,2}$ and $\epsilon = 0$,

$$QF_\epsilon(\theta, \lambda; p) = (\beta - \tfrac{1}{3})\theta_{xx} + 3\lambda\theta, \tag{3.7}$$

where $y = 1$ on the right. Furthermore, from (3.1) λ^* is independent of y, and $\lambda^*_x = -\theta^*_y = -\theta^*$ for $y = 1$. The choice of θ^* is thus reduced to finding λ^* so that for $\epsilon = 0$,

$$(1/3 - \beta)\lambda^*_{xxx} - 3\lambda^*\lambda^*_x = 0.$$

One solution of this equation is $(1 - 3\beta)\tau(x; K)$, where τ is given by (2.16). We define

$$\theta^*(x, y; p) = -(1 - 3\beta)\, \tau_x(x; K) y; \tag{3.8}$$

then

$$\lambda^* \equiv \Lambda\theta^* + \gamma = (1 - 3\beta)\tau \tag{3.9}$$

for $\epsilon = 0$ provided we choose γ appropriately. From formulas for the derivatives of the elliptic functions it is evident that $\tau \in Y^e_{m,1}(\Pi)$ and $\theta^* \in Y_{m,2}(\Pi)$.

Finally we specify the choice of constants in λ. We define $\Lambda\theta = \lambda^0 + \alpha$, where α is given by

$$\begin{aligned}\alpha(p) &= \lambda^0_{xx}(K, 1; p)/9\tau(K; K)\\ &= -\theta_{xy}(K, 1; p)/4(2k^2 - 1),\end{aligned}$$

so that

$$(\Lambda\theta)_{xx} - 9\tau(\Lambda\theta) = 0 \quad \text{at} \quad (x, y) = (K, 1). \tag{3.10}$$

The reason for this choice will be made clear in the next section. The requirement $K \geqslant K_1$ ensures that α is bounded. Since

$$| \alpha(p)| \leqslant C \mid \theta_{xy}(K, 1; p)| \leqslant Ck' \| \theta \|_{1,1},$$

we have

$$\| \alpha \|_0 \leqslant C \| \theta \|_{1,1}$$

and therefore

$$\| \Lambda\theta \|_{m,1} \leqslant C' \| \theta \|_{m,2},$$

as claimed before.

We now define

$$\gamma(K, \epsilon, \beta) = (1 - 3\beta)\, \tau(K; K) - (\Lambda\theta^*)(K, 1; K, 0, \beta)$$

to be consistent with the earlier requirement that (3.9) holds when $\epsilon = 0$. The operator Λ is of the form

$$(\Lambda\theta)(p) = \Lambda(p)\, \theta(p), \tag{3.11}$$

where $\Lambda(p)$: $Y^K_{m,2} \to Y^{e,K}_{m,1}$ is defined in the obvious way.

4. The Linearized Problem

In order to obtain a solution of $H(\theta) = 0$, we must verify that $D_\theta H(\theta)$ has a right inverse for θ near θ^*. In view of (3.6), this amounts to showing that the operator $\tilde{L}(\theta; p)$: $Y^K_{m,2} \to Z_m{}^K$ given by

$$\tilde{L}(\theta; p)\colon w \mapsto D_\theta \tilde{F} w + D_\lambda \tilde{F} \Lambda(p) w \tag{4.1}$$

has a right inverse for $\theta \in Y^K_{m,2}$, $p \in \Pi$, such that θ is near $\theta^*(p)$. Here $D_\theta \tilde{F}$ and $D_\lambda \tilde{F}$ are evaluated at $(\theta, \Lambda(p)\theta + \gamma(p); p)$. The inverse we obtain will allow for a decrease in m.

As a first step we show that

$$A(\theta; p) = Q\tilde{L}(\theta; p)\colon PY^K_{m,2} \to QZ_m{}^K$$

is invertible for $\epsilon = 0$ and $\theta = \theta^*(p)$. Identifying $PY^K_{m,2}$ with X^K_{m+2} and $QZ_m{}^K$ with $X_m{}^K$, we can treat $A(\theta; p)$ as an operator on functions of x alone.

LEMMA 1. *$A(\theta^*(p); p)$: $X^K_{m,2} \to X_m{}^K$ has a bounded inverse for $\epsilon = 0$.*

Proof. Using (3.7), we have for $\epsilon = 0$,

$$A(\theta^*(p); p)w = (\beta - \tfrac{1}{3})w_{xx} + 3\lambda^* w + 3\theta^*(\Lambda w) \equiv g.$$

Substituting $-(\Lambda w)_x$ for w and θ^*, λ^* from (3.8), (3.9), we can rewrite this as

$$(\Lambda w)_{xxx} - 9(\tau\Lambda w)_x = g/(\tfrac{1}{3} - \beta)$$

or

$$v_{xx} - 9\tau v = h + c,$$

where $v = \Lambda w$, c is a constant, and

$$h(x) = (1/3 - \beta)^{-1} \int_K^x g(x')\, dx'.$$

However, our definition of λ has been made so that the left side is zero when $x = K$, and therefore $c = 0$.

It was shown by Ter-Krikorov [12], Theorems 5.2, 6.1, and 6.2, that the equation

$$v_{xx} - 9\tau v = h$$

has a unique solution $v \in X_2^{e,K}$ for each $h \in X_0^{e,K}$ with the property that

$$\| v \|_2 \leqslant C \| h \|_0 \tag{4.2}$$

where C is independent of K for $k^2 \geqslant \frac{1}{2}$ and the norms are those defined by (3.4). It follows easily by induction that

$$\| v \|_{m+3} \leqslant C_m \| h \|_{m+1}\,.$$

These facts imply that the equation $Aw = g$ has a unique solution for given $g \in X_m{}^K$, and

$$\| w \|_{m+2} \leqslant \| v \|_{m+3} \leqslant C \| h \|_{m+1} \leqslant C' \| g \|_m\,. \quad \blacksquare$$

Since the constants above are independent of K and β, and the derivatives of F are bounded for (θ, λ) in a bounded set, this lemma has the following consesequence.

COROLLARY 2. *There exist δ and ϵ_0 so that if $p \in \Pi$, $\epsilon < \epsilon_0$, $\theta \in Y^K_{m,2}$, and $\| \theta - \theta^*(p) \| < \delta$, then $A(\theta; p)$ has a bounded inverse.*

We next consider a linear problem whose solution can be used in conjunction with the above to construct an inverse for $\tilde{L}$. It is convenient to work with the operator $L(\theta; p)$: $w \mapsto F_\theta w + F_\lambda \Lambda(p) w$ rather than $\tilde{L}$. L and $\tilde{L}$ are related by the equation

$$\tilde{L}w = \epsilon^{-1} Q L w + (I - Q) L w.$$

For $w \in Y^K_{m,2}$, $L(\theta; p) w = (g, h)$, where

$$\begin{gathered} \epsilon w_{xx} + w_{yy} = g \\ \beta\epsilon w_{xx} - \beta\epsilon^2 \theta_x w_y - \beta\epsilon^2 \theta_y w_x + e^{\epsilon\lambda} w_y + \epsilon e^{\epsilon\lambda} \theta_y \Lambda w \\ - e^{-2\epsilon\lambda} S_\theta w + 2\epsilon e^{-2\epsilon\lambda} S \Lambda w = h, \end{gathered} \tag{4.3}$$

the last on $y = 1$. Since $S(\theta, 0) = \theta$ and $S_\theta(\theta, 0) = 1$, it is reasonable to approximate the inverse of L by the solution of the following problem:

$$\begin{aligned} \beta\epsilon w_{xx} + w_y - w &= h, \qquad y = 1 \\ \epsilon w_{xx} + w_{yy} &= g \\ w &= 0, \qquad y = 0, x = 0, x = K. \end{aligned} \tag{4.4}$$

The top equation could of course be written as

$$w_y - w - \beta w_{yy} = h - \beta g.$$

This suggests we make use of the family of eigenfunctions of y satisfying

$$\begin{aligned} \ell_{yy} + \nu^2 \ell &= 0, \qquad 0 \leqslant y \leqslant 1 \\ \ell_y - \ell - \beta \ell_{yy} &= 0, \qquad y = 1 \\ \ell &= 0, \qquad y = 0. \end{aligned} \tag{4.5}$$

LEMMA 3. *The system* (4.5) *has solutions* $\ell = \ell_n$, $\nu = \nu_n$, $n \geqslant 0$, *where* $\nu_0 = 0$ *and* $n\pi < \nu_n < (n + 1)\pi$ *for* $n \geqslant 1$. *In addition there is a solution* $\ell = \ell_-$, $\nu = i\mu$, *where* μ *is the unique positive solution of*

$$\mu \coth \mu = 1 + \beta\mu^2.$$

The vectors $(\ell_-, \ell_-(1))$, $(\ell_n, \ell_n(1))$, $n \geqslant 0$, *form a basis of* $L_2(0, 1) \times \mathbf{R}$ *and are orthogonal with respect to the inner product*

$$\langle (f, q), (g, r) \rangle = \int_0^1 fg \, dy - \beta qr.$$

The proofs of this and subsequent lemmas are given in Section 5. For $f \in L^2(0, 1)$ we will write $f^\sim$ for $(f, f(1)) \in L^2(0, 1) \times \mathbf{R}$. Assume for the moment that we have a solution w of (4.4); let

$$w_n(x) = \langle w(x, \cdot)^\sim, \ell_n^\sim \rangle.$$

An integration by parts shows that w_n satisfies

$$\begin{aligned} \epsilon w_{n,xx} - \nu_n^2 w_n &= g_n \\ w_n(0) = w_n(K) &= 0, \end{aligned} \tag{4.6}$$

where

$$g_n = \langle (g, \beta^{-1}h), \ell_n^\sim \rangle.$$

Similarly for $w_- = \langle w^\sim, \ell_-^\sim \rangle$ we have

$$\begin{aligned} \epsilon w_{-,xx} + \mu^2 w_- &= g_- \\ w_-(0) = w_-(K) &= 0 \end{aligned} \tag{4.7}$$

with

$$g_- = \langle (g, \beta^{-1}h), \ell_-^\sim \rangle.$$

While (4.6) is always solvable, (4.7) will have a solution for arbitrary g_- only if

$$\mu K/a \neq n\pi \tag{4.8}$$

for every positive integer n. Here $a^2 = \epsilon$ as before. We will in fact assume from now on that Π has the property that

$$\left| \frac{\mu K}{a} - n\pi \right| \geqslant c_0 \tag{4.9}$$

for some positive c_0 and all $p \in \Pi$ with $\epsilon > 0$ and all $n > 0$. It can be seen that better estimates for (4.4) are possible in the case where $g_0 = 0$. Since $\ell_0(y) = cy$, this is precisely the condition that $Q(g, h) = 0$. The estimates which we will need for this problem are summarized in the following lemma.

LEMMA 4. *For each $p = (K, \epsilon, \beta)$ satisfying $K \geqslant 1$, $0 < \epsilon \leqslant 1$, $\eta < \beta < \frac{1}{3} - \eta$ for some $\eta > 0$, and (4.9), there is a bounded operator $G(p)$: $(I - Q)Z_m{}^K \to Y_m{}^K$ such that if $w = G(p)(g, h)$ and $\|(g, h)\|_m \leqslant 1$, then w satisfies*

$$| w |_m \leqslant C/a, \qquad a \mid w_x(\cdot, 1)|_m \leqslant C/a,$$
$$| w_y(\cdot, 1)|_m \leqslant CK^{1/2}/a, \qquad |\Lambda w(\cdot, 1)|_m \leqslant CK/a,$$

where single bars denote sup norm without weights. Moreover,

$$\| w \|_{m-4,2} \leqslant C(1 + k'^{-1}a).$$

Here $a^2 = \epsilon$, and the constants depend only on η and C_0.

Because of the second part of this lemma, we can treat $L(\theta; p)\, G(p)$ as an operator taking $(I - Q)Z_m{}^K$ to Z^K_{m-4}. The error $LG - I$ can be estimated in $Z_m{}^K$, however, using (4.3) and the first part of Lemma 4:

LEMMA 5. *For p as in Lemma 4, $\theta \in Y^K_{m,2}$ such that $\| \theta \|_{m,2} \leqslant C_0$, and $z \in (I - Q)Z_m{}^K$ we have*

$$\| L(\theta; p)\, G(p)z - z \|_m \leqslant C_1 K a \| z \|_m .$$

Note that $(I - Q)(LG - I)z = 0$, so that $(\tilde{L}G - I)z = \epsilon^{-1}(LG - I)$, and therefore

$$\| \tilde{L}(\theta; p)\, G(p)z - z \|_m \leqslant C_1 K a^{-1} \| z \|_m .$$

We can now combine Corollary 2 with the above to obtain an inverse for $\tilde{L}$. For arbitrary $z \in Z_m{}^K$ we define $R^1(\theta; p)z \in Y^K_{m-4}$ as

$$\begin{aligned} R^1(\theta; p)z &= A^{-1}(\theta; p)z_1 + G(p)z_2 \\ &\equiv w_1 + w_2 , \end{aligned}$$

where $z_1 = Qz$, $z_2 = (I - Q)z$. To estimate the error $\tilde{L}R^1 - I$, we first note that from the definition of A we have $\tilde{L}w_1 = z_1 + r_1$, where $r_1 \in (I - Q)Z_m{}^K$ and

$$r_1 = (\epsilon w_{1,xx} , (\epsilon/3) w_{1,xx} |_{y=1}).$$

Hence

$$\| r_1 \|_m \leqslant C\epsilon \| w_1 \|_{m,2} \leqslant C\epsilon \| z_1 \|_m .$$

On the other hand we have seen above that $\tilde{L}w_2 = r_2 + z_2$, with $r_2 \in QZ_m{}^K$ and

$$\| r_2 \|_m \leqslant CKa^{-1} \| z_2 \|_m .$$

Thus $M(\theta; p) \equiv \tilde{L}R^1$ is an operator on $Z_m{}^K = QZ_m{}^K \oplus (I - Q)Z_m{}^K$ of the form

$$M = \begin{pmatrix} I & M_2 \\ \epsilon M_1 & I \end{pmatrix} \tag{4.10}$$

with $\| M_1 \| \leqslant C_1$, $\| M_2 \| \leqslant C_2 K a^{-1}$ for $(\theta; p)$ satisfying the conditions that have been imposed. It follows that M has a bounded inverse on $Z_m{}^K$ provided $Ka < 1/C_1C_2$. Finally if we set

$$\tilde{R}(\theta; p) = R^1(\theta; p)\, M(\theta; p)^{-1}\colon Z_m{}^K \to Y^K_{m-4,2},$$

we have $\tilde{L}\tilde{R} = I$ as an operator from $Z_m{}^K$ to Z^K_{m-4}.

In view of Lemma 4 we have the estimate

$$\| \tilde{R}(\theta; p) \| \leqslant C(1 + k'^{-1}a) \tag{4.11}$$

provided Ka is sufficiently small. In order that $\tilde{R}$ be uniformly bounded we impose the additional condition on Π that

$$a/k' < C, \qquad p \in \Pi. \tag{4.12}$$

Now $k'K \to 0$ as $K \to \infty$; therefore under this assumption, there exists K_0 so that

$$Ka < 1/(2C_1C_2) \tag{4.13}$$

for all $p \in \Pi$ with $K > K_0$. But then (4.13) holds for all $p \in \Pi$ for which

$$a < 1/(2K_0C_1C_2).$$

We summarize these results in the following statement:

PROPOSITION 6. *Assume that Π satisfies* (4.9) *and* (4.12). *Then there exist δ and ϵ_1 such that, if $p \in \Pi$, $0 < \epsilon < \epsilon_1$, and $\| \theta - \theta^*(p) \|_{m,2} < \delta$, then $\tilde{L}(\theta; p)$: $Y^K_{m,2} \to Z_m{}^K$ has a right inverse $\tilde{R}(\theta; p)$: $Z_m{}^K \to Y^K_{m-4,2}$, and*

$$\| \tilde{R}(\theta; p) \| \leqslant C.$$

The inverse operator can be extended to $\epsilon = 0$; moreover, it defines an inverse for $D_\theta H$:

PROPOSITION 7. *Let Π, δ, ϵ_1 be as above, and let $\Pi_1 = \{p \in \Pi : \epsilon \leqslant \epsilon_1\}$. Then for $\theta \in Y_{m,2}(\Pi_1)$ such that $\| \theta - \theta^* \|_{m,2} < \delta$, $D_\theta H$: $Y_{m,2}(\Pi_1) \to Z_m(\Pi_1)$ has a right inverse $R(\theta)$: $Z_m(\Pi_1) \to Y_{m-8,2}(\Pi_1)$ of the form*

$$(R(\theta)z)(p) = \tilde{R}(\theta(p); p)\, z(p).$$

Here $\tilde{R}$ is the extension to Π_1 of the operator above.

5. Proofs of the Lemmas

Proof of Lemma 3. For $\nu > 0$ the solutions of (4.5) are of the form $\ell(y) = c \sin \nu y$, where

$$\nu \cot \nu = 1 - \beta\nu^2.$$

On each interval $(n\pi, (n+1)\pi)$, $\nu^{-1} \cot \nu - \nu^{-2}$ decreases from ∞ to $-\infty$, and therefore equals $-\beta$ at exactly one point ν_n in the interval. On $(0, \pi)$ the function decreases from $-\frac{1}{3}$ to $-\infty$, as can be seen from the Laurent series about $\nu = 0$, and there is no root if $\beta < \frac{1}{3}$. For $\nu = i\mu$, $\mu > 0$, $\ell(y) = c \sinh \mu y$, where

$$\mu^{-1} \coth \mu - \mu^{-2} = \beta.$$

The function on the left decreases from $\frac{1}{3}$ to 0 for $0 < \mu < \infty$ (see Zeidler [14, Lemma 3]) so that the equation has a unique root provided $0 < \beta < \frac{1}{3}$.

Next we introduce an operator whose eigenfunctions are the ℓ's. On $V = L_2(0, 1) \times \mathbf{R}$ we define the operator $\mathscr{A}$ by

$$\mathscr{A}(u, q) = (u_{yy}, \beta^{-1}(u_y - u))$$

with

$$D(\mathscr{A}) = \{(u, q) : u \in H^2(0, 1), u(1) = q, u(0) = 0\},$$

where H^2 is the Sobolev space. Then $\mathscr{A}$ is closed and densely defined, and $\mathscr{A}\ell_n^\sim = -\nu_n^2\ell_n^\sim$, $\mathscr{A}\ell_-^\sim = \mu^2\ell_-^\sim$. The identity

$$\langle \mathscr{A}u^\sim, v^\sim \rangle = -\int_0^1 u_y v_y \, dy + u(1)\, v(1), \tag{5.1}$$

$u^\sim, v^\sim \in D(\mathscr{A})$, implies that $\mathscr{A}$ is symmetric with respect to $\langle\ ,\ \rangle$. It also implies by the Schwarz inequality that

$$\langle \mathscr{A}u^\sim, u^\sim \rangle < 0 \text{ unless } u^\sim = cy. \tag{5.2}$$

Now we set $V' = \{v \in V : \langle v, \ell_- \rangle = 0\}$ and show that $\langle\ ,\ \rangle$ is positive definite on V'. First, from (5.2), we have $\langle \ell_-, \ell_- \rangle < 0$, so that

$$\int_0^1 \ell_-^2 \, dy < \beta\ell_-(1)^2.$$

The condition $(u, q) \in V'$ implies

$$q = \int u\ell_- \, dy / \beta\ell_-(1)$$

and therefore

$$q^2 \leqslant \left(\int u^2\right)\left(\int \ell_-^2\right)/\beta^2\ell_-(1)^2$$

so that

$$\beta q^2 \leqslant \rho \int u^2$$

where

$$\rho = \int \ell_-^2/\beta\ell_-(1)^2 < 1.$$

Thus

$$\langle (u, q), (u, q)\rangle > (1 - \rho) \int_0^1 u^2 \, dy,$$

from which it follows that

$$\langle (u, q), (u, q)\rangle > c \left[\int_0^1 u^2 \, dy + q^2\right]$$

for some $c > 0$.

The subspace V' is invariant under $\mathscr{A}$, and $\mathscr{A}\,|_{V'}$ is densely defined. In order to show that $\{\ell_n^{\sim}\}$ is a basis of V', it suffices to show that $(\mathscr{A} - \sigma)^{-1}$ exists and is compact on V' for some $\sigma > 0$. The equation $(\mathscr{A} - \sigma)u^{\sim} = (f, q)$ is equivalent to

$$\begin{aligned} &u_{yy} - \sigma u = f, && 0 < y < 1 \\ &u_y - (1 + \beta\sigma)u = \beta q, && y = 1 \\ &u = 0, && y = 0, \end{aligned}$$

which is a Sturm-Liouville problem with no homogeneous solution provided $\sigma \neq \mu^2$. Thus solutions exist for arbitrary (f, q), and the compactness of the resolvent is a standard fact. ∎

Proof of Lemma 4. The problem (4.4) has a unique solution $w \in Y^K_{m-2,2}$ for arbitrary $(g, h) \in Z_m{}^K$ provided (4.8) is satisfied. This can be seen by expanding g and h in Fourier series with respect to x, noting that the nth term of either series is $O(n^{-m})$. However, to obtain the desired estimates we expand with respect to the eigenfunctions of Lemma 3. With w_n and w_- as before, we have

$$\begin{aligned} w(x, y) &= \sum_{n=0}^{\infty} w_n(x)\, \ell_n(y) - w_-(x)\, \ell_-(y) \\ &= w_+(x, y) - w_-(x)\, \ell_-(y), \end{aligned} \tag{5.3}$$

at least in $L^2(0, 1)$ for fixed x, provided the ℓ's are normalized so that

$$\langle \ell_n^{\sim}, \ell_n^{\sim}\rangle = 1, \qquad \langle \ell_-^{\sim}, \ell_-^{\sim}\rangle = -1.$$

Then

$$| \ell_n(y)| \leqslant C, \qquad | \ell_n(1)| \leqslant Cn^{-1} \tag{5.4}$$

and it suffices to estimate w_n and w_-. We assume that $Q(g, h) = 0$, and therefore $w_0 = 0$.

Estimates for the problem (4.6) with weight function dn were obtained by Ter-Krikorov in [12, Theorem 6.4], by writing the solution in terms of the Green's function; they imply

$$\| w_n \|_{X_0^K} \leqslant Cn^{-2} \| g_n \|_{X_0^K} . \tag{5.5}$$

In a similar manner one may obtain

$$a \| w_{n,x} \|_0 \leqslant Cn^{-1} \| g_n \|_0 \tag{5.6}$$

and, integrating by parts in the usual way,

$$\| w_{n,x} \|_0 \leqslant Cn^{-2} \| g_n \|_1 . \tag{5.7}$$

If we note that $D^{2j}w_n$ satisfies a problem of the type (4.6), it follows inductively from (5.5) and (5.7) that

$$\| w_n \|_m \leqslant Cn^{-2} \| g_n \|_m . \tag{5.8}$$

(Recall we assume m is even.) In the same way we have from (5.6)

$$a \| w_{n,x} \|_m \leqslant Cn^{-1} \| g_n \|_m . \tag{5.9}$$

Clearly

$$\| g_n \|_m \leqslant C \|(g, h)\|_m , \tag{5.10}$$

and we can now estimate w_+ and $w_{+,x}(\cdot, 1)$ using (5.4) and (5.8)–(5.10). The result is that

$$\| w_+ \|_m , \; a \| w_{+,x}(\cdot, 1)\|_m \leqslant C \|(g, h)\|_m . \tag{5.11}$$

In order to estimate $w_{+,y}$, we shall need an L^2-estimate on $w_{+,xx}$. From (4.6) it is easily found that

$$| \epsilon w_{n,xx} |_{L^2}^2 \leqslant C \, | g_n |_{L^2}^2 .$$

Differentiating the sum for w_+ and applying the Schwarz inequality, we have

$$| \epsilon w_{+,xx}(\cdot, 1)|_{L^2}^2 \leqslant C \sum | \epsilon w_{n,xx} |_{L^2}^2 .$$

Now let $v(x) = (g(x, \cdot), \beta^{-1}h(x)) \in V$; then

$$\sum g_n(x)^2 = \langle v(x), v(x)\rangle + g_-(x)^2 \leqslant C \|(g, h)\|_0^2 \, dn^2x,$$

and combining the last three inequalities we obtain

$$\epsilon \mid w_{+,xx}(\cdot, 1)|_{L^2} \leqslant C \|(g, h)\|_0 . \tag{5.12}$$

From (5.11) we also have

$$a \mid w_{+,x}(\cdot, 1)|_{L^2} \leqslant C \|(g, h)\|_0 . \tag{5.13}$$

In order to estimate $w_{+,y}$, we treat w_+ as the solution of a Dirichlet problem

$$\begin{gathered} w_+ = f, \qquad y = 1 \\ \epsilon w_{+,xx} + w_{+,yy} = g_+ \\ w_+ = 0, \qquad y = 0, \qquad x = 0, \qquad x = K, \end{gathered}$$

with $f(x) = w_+(x, 1)$ and $g_+(x, y) = g(x, y) + g_-(x)\,\ell_-(y)$. Expanding in the x-direction, we obtain the expression

$$w_{+,y}(x, 1) = \sum_{j=1}^{\infty} aj\rho(\coth aj\rho) f_j \sin j\rho x + \int_0^1 \int_0^K G(x, \xi, y)\, g_+(\xi, y)\, d\xi\, dy \tag{5.14}$$

where

$$f = \sum f_j \sin j\rho x, \qquad \rho = \pi/K,$$

$$G(x, \xi, y) = \frac{2}{K} \sum_{j=1}^{\infty} \frac{\sinh aj\rho y}{\sinh aj\rho} \sin j\rho x \sin j\rho\xi.$$

We estimate the first term in (5.14) by

$$\begin{aligned} C \sum (1 + aj\rho) \mid f_j \mid &\leqslant C_1 \left(\sum (jf_j)^2\right)^{1/2} + C_2 a K^{-1} \left(\sum (j^2 f_j)^2\right)^{1/2} \\ &\leqslant CK^{1/2}(\mid f_x \mid_{L^2} + a \mid f_{xx} \mid_{L^2}) \\ &\leqslant CK^{1/2} a^{-1} \|(g, h)\|_0 , \end{aligned}$$

using (5.12) and (5.13).

In the second term of (5.14) we can rewrite G as $G_1 + G_2$ with

$$G_1(x, \xi, y) = \frac{2}{K} \sum_{j=1}^{\infty} e^{aj\rho(y-1)} \sin j\rho x \sin j\rho\xi$$

and

$$| G_2 | \leqslant CK^{-1} \sum e^{-aj\rho} \leqslant Ca^{-1}.$$

Thus the contribution of G_2 to (5.14) is bounded by $Ca^{-1} \|(g, h)\|_0$. The remaining part of G is

$$G_1(x, \xi, y) = (2K)^{-1}[P(\xi - x; r) - P(\xi + x; r)],$$

where $r = \exp a\rho(y - 1)$ and P is the Poisson kernel

$$P(x; r) = 1 + 2 \Sigma r^j \cos j\rho x.$$

Consequently

$$| G_1(x, \cdot, \cdot)|_{L^1} \leqslant C,$$

and the term in (5.14) corresponding to G_1 is bounded by $\|(g, h)\|_0$. In summary, we have shown that

$$\sup | w_{+,y}(\cdot, 1)| \leqslant CK^{1/2}a^{-1} \|(g, h)\|_0 .$$

Since $D_x{}^j w_+$ satisfies a similar boundary value problem, it follows that

$$| w_{+,y}(\cdot, 1)|_m \leqslant CK^{1/2}a^{-1} \|(g, h)\|_m .$$

Next we estimate w_- using the formula

$$w_-(x) = \int_0^K G_-(x, \xi)\, g_-(\xi)\, d\xi,$$

$$G_-(x, \xi) = (a^2\sigma \sin \sigma K)^{-1} \sin \sigma\xi \sin \sigma(x - L), \qquad \xi < x,$$

where $\sigma = \mu/a$. By assumption (4.9),

$$| w_-(x)| \leqslant Ca^{-1} \|(g, h)\|_0 .$$

(The constant here depends on η, as well as C_0, since we need a lower bound for μ.) As before,

$$a \,| Dw_-(x)| \leqslant Ca^{-1} \|(g, h)\|_0$$
$$| Dw_-(x)| \leqslant Ca^{-1} \|(g, h)\|_1$$

and by induction

$$\begin{aligned} |\, w_- \,|_m &\leqslant C a^{-1} \|(g, h)\|_m \\ a \,|\, D_x w_- \,|_m &\leqslant C a^{-1} \|(g, h)\|_m \,. \end{aligned} \tag{5.15}$$

The first three estimates of the lemma follow from the above. The fourth may be obtained from the first two by solving the top line of (4.4) for w_y and integrating in x.

For the last statement we need an estimate on w_- which is bounded as $a \to 0$. If we set $\delta w = w_- - \mu^{-2} g_-$, then

$$\epsilon(\delta w)_{xx} + \mu^2(\delta w) = -\epsilon\mu^{-2} g_{-,xx} \,,$$

and by (5.15),

$$|\, \delta w \,|_{m-2} \leqslant C a \, \|(g, h)\|_m \leqslant C a,$$

so that

$$\|\, w_- \,\|_{m-2} \leqslant C(a k'^{-1} + 1).$$

Combining this with the estimate on w_+, we have

$$\|\, w \,\|_{m-2} \leqslant C(a k'^{-1} + 1).$$

It remains to show that

$$\|\, w_y \,\|_{m-3} + \|\, w_{yy} \,\|_{m-4} \leqslant C(a k'^{-1} + 1).$$

In view of (5.3), it will suffice to show that $\|\, w_{+,y} \,\|_{m-3}$ and $\|\, w_{+,yy} \,\|_{m-4}$ are bounded. In fact, since $w_{+,yy} = g_+ - \epsilon w_{+,xx}$, (5.11) implies that $\|\, w_{+,yy} \,\|_{m-2} \leqslant C$. Finally, the identity

$$v_y(y) = \int_0^1 \int_z^y v_{yy}(t)\, dt\, dz + v(1) - v(0)$$

applied to $v = w_+(x, \cdot)$ shows that $\|\, w_{+,y} \,\|_{m-2} \leqslant C$. ∎

Proof of Lemma 5. From (4.3) and (4.4), $(LG - 1)z = (0, r)$, where

$$\begin{aligned} r = &-\beta\epsilon^2\theta_x w_y - \beta\epsilon^2\theta_y w_x + (e^{\epsilon\lambda} - 1) w_y \\ &+ \epsilon e^{\epsilon\lambda}\theta_y(\Lambda w) - (e^{-2\epsilon\lambda} S_\theta - 1) w \\ &+ 2\epsilon e^{-2\epsilon\lambda} S(\Lambda w), \end{aligned}$$

all evaluated at $y = 1$. The stated estimate can be shown for each term using Lemma 4 and the trivial observation that

$$\| fv \|_m \leqslant C \mid f \mid_m \| v \|_m .$$

We verify this only for the fifth term, the others being more straightforward. Assume $\| z \|_m \leqslant 1$. Then

$$\begin{aligned} \|(e^{-2\epsilon\lambda} S_\theta - 1) w \|_m &\leqslant C \| e^{-2\epsilon\lambda} S_\theta - 1 \|_m \mid w \mid_m \leqslant C a^{-1} \| e^{-2\epsilon\lambda} S_\theta - 1 \|_m \\ &\leqslant C a^{-1} [\| e^{-2\epsilon\lambda} (S_\theta - 1) \|_m + \| e^{-2\epsilon\lambda} - 1 \|_m] \\ &\leqslant C a^{-1} [\| S_\theta - 1 \|_m + \| e^{-2\epsilon\lambda} - 1 \|_m] \\ &\leqslant C a^{-1} [C' \epsilon^3 + C'' \epsilon] \\ &\leqslant C a. \quad \blacksquare \end{aligned}$$

Proof of Prop. 7. Let J: $Y^K_{m,2} \to Y^1_{m,2}$ be the operator $[Ju](x, y) = u(Kx, y)$; in the same way we can treat J as an operator on various other spaces. We will show that

$$\tilde{R}_J(J^{-1}u; p) = J\tilde{R}(J^{-1}u; p) J^{-1}\colon Z_m{}^1 \to Y^1_{m-8,2}$$

has an extension to Π_1 such that $\tilde{R}_J$ depends continuously on $(u; p) \in Y_{m,2} \times \Pi_1$, where u is close to $J\theta^*(p)$. It then follows that $\tilde{R} = J^{-1}\tilde{R}_J J$ is a right inverse for $\tilde{L}$ for $\epsilon = 0$ as well as $\epsilon > 0$. It also follows that $R(\theta)z$ and its (x, y)-derivatives are continuous in $(x, y; p)$. Since we have already obtained a uniform bound on $\tilde{R}$, the proposition will be proved.

First we note that, because of the smoothness of $\tilde{F}$, $\tilde{L}_J(J^{-1}u; p)$: $Y^1_{m,2} \to Z_m{}^1$ depends smoothly on $(u, p) \in Y^1_{m,2} \times \Pi_1$; here $\tilde{L}_J = J^{-1}LJ$. As a consequence, $A_J^{-1}(J^{-1}u; p)$: $QZ_m{}^1 \to Y^1_{m,2}$ is also smooth in $(u; p)$ where defined. It is clear that for $p \in \Pi$ with $\epsilon > 0$, $G_J(p)$: $(I - Q)Z_m{}^1 \to Y^1_{m-4,2}$ is a uniform sum of operators depending continuously on p, and is therefore also continuous. We now examine the limit $\epsilon \to 0$. Given $p = (K, 0, \beta) \in \Pi$, let $G(p)$: $Z_m{}^K \to Y^K_{m-2,2}$ be the operator $(g, h) \mapsto w^0$ solving the problem

$$\begin{aligned} w_y{}^0 - w^0 &= h \quad &&\text{on} \quad y = 1, \\ w_{yy} &= g \quad &&\text{in} \quad \mathscr{R}, \\ w^0 &= 0 \quad &&\text{on} \quad y = 0. \end{aligned} \tag{5.12}$$

An argument like the proof of Lemma 4 shows that $\| G(p) \| \leqslant C$. Now let $p' = (K, \epsilon, \beta)$, $\epsilon > 0$, and set $w = G(p)(g, h)$. Then $\delta w = w - w^0$ satisfies

$$\begin{aligned} (\delta w)_y - (\delta w) &= -\beta\epsilon w_{xx} \quad &&\text{on} \quad y = 1, \\ (\delta w)_{yy} &= -\epsilon w_{xx} \quad &&\text{in } \mathscr{R}, \\ \delta w &= 0 \quad &&\text{on} \quad y = 0. \end{aligned}$$

Since $Q(w, \beta w) = 0$, $\delta w = -\epsilon G(p)(w_{xx}, \beta w_{xx})$. Hence

$$\| \delta w \|_{m-4,2} \leqslant C\epsilon \| w_{xx} \|_{m-2} \leqslant C\epsilon \| w \|_{m-4,2} \leqslant C\epsilon \|(g, h)\|_m .$$

Thus $G(p') \to G(p)$ in the space of linear operators $(I - Q)Z_m{}^K \to Y^K_{m-4,2}$, as $\epsilon \to 0$, uniformly in K, β. It follows that $G_J(p)$: $(I - Q)Z^1_m \to Y^1_{m-4,2}$ is continuous on Π.

We have now shown that $\tilde{R}_J{}^1(J^{-1}u; p)$: $Z_m{}^1 \to Y^1_{m-4,2}$, or $Z^1_{m-4} \to Y^1_{m-8,2}$, depends continuously on $(u; p)$, and it remains only to show that $(M(J^{-1}u; p)^{-1})_J$: $Z_m{}^1 \to Z^1_{m-4}$ does so. From the construction of G and the estimates of Lemmas 4 and 5, it is clear that M_J: $Z_m{}^1 \to Z_m{}^1$ is continuous for $\epsilon > 0$, and therefore so is $(M_J)^{-1} = (M^{-1})_J$. On the other hand, writing M as in (4.10), we can express M^{-1} as a series with the form

$$M^{-1} = \begin{pmatrix} I & -M_2 - \epsilon M_2 M_1 M_2 \\ 0 & I \end{pmatrix} + N,$$

where $\| N \| \leqslant Ca$, N: $Z_m{}^K \to Z_m{}^K$ uniformly for p in a bounded subset of Π_1. Using our extension of G, we can extend M_2: $Z_m{}^K \to Z^K_{m-4}$ to $\epsilon = 0$ so that $(M_2)_J$ is continuous. It then follows that $(M^{-1})_J$: $Z_m{}^K \to Z^K_{m-4}$ has an extension depending continuously on $(u; p) \in Y^1_{m,2} \times \Pi_1$, and the proof of the proposition is complete. ∎

6. Proof of Theorem 1

It will be convenient to perform the iteration leading to the solution of $H(\theta) = 0$ in a space without weighted norms. Let $Y_m{}^W(\Pi_1)$ be the space defined precisely as $Y_m(\Pi_1)$ except that for $u \in Y_m{}^W(\Pi_1)$, $D_x{}^\alpha u$ is required to be a bounded continuous function on $\mathscr{R} \times \Pi_1$, $\alpha \leqslant m$, and the norm is

$$\sup | D_x{}^\alpha u(x, y; p)|,$$

for $(x, y) \in \mathscr{R}$, $p \in \Pi_1$, and $\alpha \leqslant m$. $Y^W_{m,2}(\Pi_1)$ and $Z_m{}^W(\Pi_1)$ are defined similarly. Now dn has the property that

$$| D^\alpha \, dn \, x | \leqslant C_\alpha \, dn \, x,$$

with C_α independent of K; therefore the operator

$$W\colon Y^W_{m,2}(\Pi_1) \to Y_{m,2}(\Pi_1)\colon u \mapsto u \, dn \, x$$

is bounded. On the other hand, if $\theta \in Y_{m,2}(\Pi_1)$ and $u = W^{-1}\theta = \theta/dn$, it is easily seen by induction that

$$| u |_{m,2} \leqslant C_m \| \theta \|_{m,2} ,$$

i.e., W^{-1} is bounded. The same applies to $Z_m{}^W(\Pi_1)$ and $Z_m(\Pi_1)$.

We can now work with the smooth function $H^W = W^{-1}HW$: $Y^W_{m,2}(\Pi_1) \to Z_m{}^W(\Pi_1)$ rather than H. Let $u^* = W^{-1}\theta^*$; according to Prop. 7, the derivative $DH^W(u)$ has a right inverse $R^W(u) = W^{-1}R(Wu)W$: $Z_m{}^W(\Pi_1) \to Y^W_{m-8,2}(\Pi_1)$, provided $| u - u^* |_{m,2} < \delta_1$. Moreover u^* was chosen so that $H(u^*) = 0$ on $\epsilon = 0$, and therefore $| H(u^*)|_m$ is arbitrarily small if ϵ_1 is sufficiently small. H has the further property that if $| u |_{m,2} \leqslant C$ and $u \in Y^W_{m+\ell,2}(\Pi_1)$ for some $\ell > 0$, then

$$| H^W(u)|_{m+\ell} < C_\ell(| u |_{m+\ell,2} + 1).$$

(See, e.g., [4, (A.5) and (A.1)].)

We will use smoothing operators on $Y^W_{m,2}(\Pi_1)$ of the type introduced by Nash. Let $\psi \in C^\infty(\mathbf{R})$ be such that the Fourier transform $\hat{\psi}$ is 1 near $x = 0$ and of compact support. For $t \geqslant 1$ and $u \in Y^W_{0,2}(\Pi_1)$ we set

$$(S(t)u)(x, y; p) = t \int_{\mathbf{R}} \psi(t(x' - x))\, u(x', y; p)\, dx'.$$

Then $S(t)u \in Y^W_{r,2}(\Pi_1)$ for any r, and

$$| S(t)u |_{r+s,2} \leqslant Ct^s \, | u |_{r,2}$$
$$| u - S(t)u |_{r,2} \leqslant Ct^{-s} \, | u |_{r+s,2} ,$$

$r, s \geqslant 0$. (See, e.g., [10, pp. 38–39].)

Following Moser [9], we now construct a sequence u_n approximating the solution. Let $u_0 = u^*$, and for $n \geqslant 0$,

$$u_{n+1} = u_n + S(t_{n+1})v_n ,$$

where

$$v_n = -R^W(u_n)\, H^W(u_n)$$

and $\{t_n\}$ is a sequence defined by $t_0 > 1$, $t_{n+1} = t_n^{3/2}$. The proof of Theorem 1 of [9] shows that, if $H^W(u^*)$ is sufficiently small, then u_n converges in $Y^W_{m,2}(\Pi_1)$ to a solution of $H^W = 0$. Moreover, since $H^W(u^*) = 0$ on $\epsilon = 0$, $u_n = u^*$ on $\epsilon = 0$ for each n, and the same is true for the limit function. We summarize this result:

PROPOSITION 8. *If Π satisfies* (4.9) *and* (4.12), $\Pi_1 = \Pi \cap \{\epsilon < \epsilon_1\}$, *and* ϵ_1 *is sufficiently small, then there exists* $\theta \in Y_{m,2}(\Pi_1)$ *such that* $H(\theta) = 0$ *and* $\theta = \theta^*$ *for* $\epsilon = 0$.

To prove Theorem 1 we must choose β in terms of K and ϵ so that condition (2.7) is satisfied. Assuming that a set Σ is given satisfying (2.17) and (2.18), we first define an appropriate Π so that the choice can be made. Let

$$\Pi = \{(K, \epsilon, \beta) : (K, \epsilon) \in \Sigma, \mid \beta - \beta_0 \mid \leqslant \beta_1 \epsilon, \mid \beta - \beta_0 \mid \leqslant \delta, \epsilon \leqslant \epsilon_2\}.$$

Here β_1 is a constant to be specified later and δ is small enough so that $\beta_0 > \delta$ and $\beta_0 + \delta < \frac{1}{3}$. We verify that Π satisfies (4.9) provided ϵ_2 is sufficiently small. By assumption

$$|(\mu_0 K/a) - n\pi \mid > 2c_0 ,$$

for each n and $(K, \epsilon) \in \Sigma$ and some c_0 ; here μ_0 is the eigenvalue μ at $\beta = \beta_0$. The restriction on β implies $\mid \mu - \mu_0 \mid < C\epsilon$, so that

$$K \mid \mu - \mu_0 \mid / a < CKa.$$

Now by an argument like that preceding Prop. 6, $CKa < c_0$ for $(K, \epsilon) \in \Pi$ if ϵ_2 is small enough. Therefore

$$|(\mu K/a) - n\pi \mid > c_0$$

for $p \in \Pi$.

Applying Prop. 8 to Π, we have a solution $\theta(\cdot; p)$ of $H(\theta) = 0$ on Π_1 ; given $(K, \epsilon) \in \Sigma$ we wish to choose β so that $(K, \epsilon, \beta) \in \Pi$ and (2.7) holds. This condition can be stated as

$$\beta = \beta_0(\mathscr{I}_2/\mathscr{I}_1)^2, \tag{6.1}$$

where

$$\mathscr{I}_1 = K^{-1} \int_0^K e^{-\epsilon\lambda}\, dx, \qquad \mathscr{I}_2 = K^{-1} \int_0^K \int_0^1 e^{-2\epsilon\lambda}\, dy\, dx.$$

To estimate $\mid \mathscr{I}_1 - 1 \mid$, we note that

$$\mid e^{-\epsilon\lambda} - 1 \mid < 3\epsilon \mid \lambda \mid \leqslant 3\epsilon[C\, dn\, x + \mid \gamma \mid]$$

if $\epsilon < 1/\sup \mid \lambda \mid$. Thus

$$\mid \mathscr{I}_1 - 1 \mid \leqslant 3\epsilon[C_1/K + \mid \gamma \mid]. \tag{6.2}$$

Here C_1 is independent of p but depends on Π. Now $\lambda(x, y; p)$ is uniformly continuous where $K \leqslant C_1$ and $x \leqslant C_1$; since $\lambda = (1 - 3\beta)\tau$ when $\epsilon = 0$, we have for ϵ sufficiently small and $K \leqslant C_1$,

$$|\lambda(x, y; K, \epsilon, \beta) - (1 - 3\beta)\, \tau(x; K)| < 1.$$

Consequently we can estimate

$$|\mathscr{I}_1 - 1| \leqslant 3\epsilon(|\tau|_0 + 1), \qquad K \leqslant C_1,$$

where $|\tau|_0 = \sup_{(x;K)} |\tau(x; K)|$, while from (6.2),

$$|\mathscr{I}_1 - 1| \leqslant 3\epsilon(|\gamma| + 1), \qquad K \geqslant C_1.$$

In any case, we have

$$|\mathscr{I}_1 - 1| \leqslant C_2\epsilon,$$

where C_2 is a universal constant. Estimating $\mathscr{I}_2$ similarly, we obtain

$$|(\mathscr{I}_1/\mathscr{I}_2)^2 - 1| \leqslant b\epsilon,$$

where b is a universal constant. Thus if β_1 above is chosen so that $\beta_1 > b\beta_0$, there is a solution of (6.1) satisfying $|\beta - \beta_0| < \beta_1\epsilon$ for ϵ sufficiently small.

Finally, with β chosen as a function of (K, ϵ) as above, we can treat θ and λ as functions on $\mathscr{R} \times \Sigma_1$, for some ϵ_1, satisfying (ii) of Theorem 1. Because of Prop. 8, assertions (i) and (iii) hold for $D_x^{\alpha_1} D_y^{\alpha_2}\theta$ if $\alpha_1 \leqslant 2$ and $\alpha_1 + \alpha_2 \leqslant m + 2$. However, the other derivatives of θ and λ of order $\leqslant m + 2$ may be expressed in terms of these using (2.11). Thus the proof of Theorem 1 is complete.

7. Proof of Theorem 2

Our formulation here will be the same as in Section 3 except that the parameter β does not occur and we require that θ be differentiable with respect to ϵ. Let

$$\Sigma = \{p = (K, \epsilon) : K \geqslant K_1,\, 0 \leqslant \epsilon \leqslant 1\},$$

where $K_1 > K(2^{-1/2})$. As in (3.2) we define

$$F = (F^1, F^2) \colon Y'_{m,2} \times \Sigma \to Z'_m$$

by

$$\begin{aligned} F_1(\theta, \lambda; p) &= \epsilon\theta_{xx} + \theta_{yy} \\ F_2(\theta, \lambda; p) &= [\theta_y - e^{-3\epsilon\lambda} S(\theta, \epsilon)]_{y=1}. \end{aligned} \tag{7.1}$$

Then F is C^∞, and $\tilde{F}$, defined by (3.3), is also. Now with m, j, and $Y^K_{m,j}$ as before, let $Y_{m,j,N}(\Sigma)$ be the space of functions $\theta\colon \mathscr{R} \times \Sigma \to \mathbf{R}$ such that $D_\epsilon^{\alpha_3} D_x^{\alpha_1} D_y^{\alpha_2}\theta$ is continuous for $\alpha_1 + \alpha_2 \leqslant m + j$, $\alpha_2 \leqslant j$, and $\alpha_3 \leqslant N$; $\theta(\cdot\,; p) \in Y^K_{m,j}$; and the norm

$$\| \theta \|_{m,j,N} = \sup | D_\epsilon^{\alpha_3} D_x^{\alpha_1} D_y^{\alpha_2} \theta(x, y; p)| / dn(x; k)$$

is finite. If $j = 2$, we require $\theta = 0$ on $y = 0$. Spaces $Y^e_{m,j,N}(\Sigma)$, $X_{m,N}(\Sigma)$, and $Z_{m,N}$ are defined correspondingly. The operator Λ is bounded from $Y_{m,2,N}$ to $Y^e_{m,1,N}(\Sigma)$. Finally, if we define

$$H\colon Y_{m,2,N}(\Sigma) \to Z_{m,N}(\Sigma)$$

by (3.5) and θ^* as before with $\beta = 0$, we have $H(\theta^*) = 0$ on $\epsilon = 0$.

We now proceed to construct an inverse for $D_\theta H$. To approximate the inverse we use solutions of the problem

$$\begin{aligned} w_y - w &= h \qquad \text{on} \quad y = 1 \\ \epsilon w_{xx} + w_{yy} &= g \qquad \text{in} \quad \mathscr{R} \\ w &= 0 \qquad \text{on} \quad y = 0. \end{aligned} \tag{7.2}$$

We can estimate w using the eigenfunctions $\ell_n(y)$ of (4.5) with $\beta = 0$. In this case we again have $\nu_0 = 0$, but all other eigenvalues are negative. The ℓ_n's form an orthogonal basis of $L^2(0, 1)$, and if

$$w_n(x) = \int_0^1 w(x, y)\, \ell_n(y)\, dy, \qquad g_n(x) = \int_0^1 g(x, y)\, \ell_n(y)\, dy,$$

we have

$$\begin{aligned} \epsilon w_{n,xx} - \nu_n^2 w_n &= g_n - \ell_n(1) h \\ w_n(0) = w_n(K) &= 0. \end{aligned}$$

In place of Lemmas 4 and 5 we have the following:

LEMMA 9. *For $0 < \epsilon \leqslant 1$ and $K \geqslant 1$ there is a bounded operator $G(K, \epsilon)$: $(I - Q)Z_m{}^K \to Y_m{}^K$ such that, if $w = G(K, \epsilon)(g, h)$, then w satisfies (7.2). If $\|(g, h)\| \leqslant 1$, then*

$$\| w \|_m \leqslant C, \qquad \|(\Lambda w)(\cdot, 1)\|_m \leqslant C,$$
$$\| w \|_{m-3,2} \leqslant C,$$

where C is independent of (K, ϵ).

LEMMA 10. *For* $p = (K, \epsilon)$ *as above,* $\theta \in Y^K_{m,2}$, $\|\theta\|_{m,2} \leqslant C_0$, *and* $z \in (I - Q)Z_m{}^K$, *we have*

$$\| L(\theta, p)\, G(p) z - z \|_m \leqslant C_1 \epsilon \, \| Z \|_m .$$

The first estimate of Lemma 9 is proved as in Lemma 4, and the second follows from the fact that $w_y = w + h$ on $y = 1$. Next, we have $w_{yy} = g - \epsilon w_{xx} \in Y^K_{m-2}$; from this and the facts that $w_y(\cdot, 1) \in Y_m{}^K$, $w(\cdot, 0) = 0$, we can conclude that $w_y \in Y^K_{m-2}$ and the last estimate of the lemma holds. Lemma 10 follows from the first two estimates as in Lemma 5.

If we now define $R^1(\theta; p)$ and $M(\theta; p)$ just as before, we find that M is of the form (4.10) with M_1 and M_2 uniformly bounded. Therefore

$$\tilde{R}(\theta; p) = R^1(\theta; p)\, M(\theta; p)^{-1} \colon Z_m{}^K \to Y^K_{m-4}$$

is a right inverse for $L(\theta; p)$, uniformly bounded in norm, for $\theta \in Y^K_{m,2}$ near $\theta^*(p)$ and $\epsilon > 0$ sufficiently small. This inverse can be extended to $\epsilon = 0$ and provides us with an inverse for $D_\theta H$:

LEMMA 11. *There exist* $\delta > 0$ *and* $\epsilon_1 > 0$ *so that if* $\Sigma_1 = \Sigma \cap \{p : \epsilon \leqslant \epsilon_1\}$, *then* $D_\theta H\colon Y_{m,2,N}(\Sigma_1) \to Z_{m,N}(\Sigma_1)$ *has a right inverse* $R(\theta)\colon Z_{m,N}(\Sigma_1) \to Y_{m-\sigma,2,N}(\Sigma_1)$ *of the form*

$$(R(\theta)z)(p) = \tilde{R}(\theta(p); p)\, z(p)$$

provided $\| \theta - \theta^* \| < \delta$ *in* $Y_{m,2,0}(\Sigma_1)$. *Here* $\sigma > 0$ *is a number depending only on* N.

Proof. As in Prop. 7, it is enough to show that $D_\epsilon{}^\alpha \tilde{R}_J(J^{-1}u; p)\colon Z_m{}^1 \to Y^1_{m-\sigma,2}$ is defined and depends continuously on $(u; p) \in Y^1_{m,2,N} \times \Sigma_1$, u near $J\theta^*(p)$. Here $J\colon Y^K_{m,2} \to Y^1_{m,2}$ as before. First we consider the dependence of G on ϵ. We define $G(K, 0)\colon (I - Q)Z_m{}^1 \to Y_m{}^K \cap Y^K_{m-2,2}$ to be the operator $(g, h) \mapsto w^0$ of (5.12). For fixed K, G is then continuous in $\epsilon \geqslant 0$; in fact, arguing as in Prop. 7, we obtain the estimate

$$\| G(K, \epsilon) z - G(K, \epsilon') z \|_{m-2} \leqslant C \mid \epsilon - \epsilon' \mid \| z \|_m \tag{7.3}$$

for $\epsilon, \epsilon' \geqslant 0$. Moreover, the proof of Lemma 5 in [2] shows that $G(K, \cdot)$ is differentiable as a map

$$G(K, \cdot)\colon [0, 1) \to L((I - Q)Z_m{}^K : Y^K_{m-4} \cap Y^K_{m-8,2})$$

and

$$D_\epsilon G = -G(D_{xx}, 0) G.$$

It follows inductively that higher ϵ-derivatives exist with further decrease in x-differentiability. Since $G_J(K, \epsilon) = G(1, K^2\epsilon)$, $D_\epsilon{}^j G_J$ depends continuously on (K, ϵ) in appropriate norms.

Because A^{-1} was a bounded inverse, $(A^{-1})_J$ depends smoothly on $(u; p)$. It follows from this and the properties of G that $D_\epsilon{}^\alpha R^1(J^{-1}u; p)_J$ and $D_\epsilon{}^\alpha M(J^{-1}u; p)_J$ exist and are continuous in $(u; p)$ with decrease in x-differentiability. In particular, $M(J^{-1}u; p): Z_m{}^K \to Z_{m-4}^K$ is differentiable with respect to ϵ. The continuous dependence of M_J implies the same for $(M^{-1})_J$.

It remains to show that M_J^{-1} can be differentiated with respect to ϵ. With u and K fixed, it follows from (7.3) that

$$\| M(\epsilon)z - M(\epsilon')z \|_{m-2} \leqslant C\epsilon \| z \|_m . \tag{7.4}$$

We now show that $M(\epsilon)^{-1}: Z_m{}^K \to Z_{m-4}^K$ is differentiable at $\epsilon = \epsilon_0$. First we write

$$M^{-1} = (M_0 + (M - M_0))^{-1} = (1 + \Delta)^{-1} M_0^{-1},$$

where $M = M(\epsilon)$, $M_0 = M(\epsilon_0)$, and $\Delta = M_0^{-1}(M - M_0)$. Using the identity

$$(1 + \Delta)^{-1} = 1 - \Delta + \Delta^2(1 + \Delta)^{-1}$$

we have

$$M^{-1} = M_0^{-1} - \Delta M_0^{-1} + \Delta^2(1 + \Delta)^{-1} M_0^{-1}.$$

Because of (7.4), the last term is $O((\epsilon - \epsilon_0)^2)$ in $L(Z_m{}^K: Z_{m-4}^K)$. Therefore,

$$D_\epsilon M^{-1} |_{\epsilon=\epsilon_0} = -M_0(D_\epsilon M |_{\epsilon=\epsilon_0}) M_0 .$$

It follows inductively that higher ϵ-derivatives of M^{-1} exist, and $D_\epsilon{}^\alpha M_J^{-1}$ depends continuously on $(u; p)$ in an appropriate norm. Since $\tilde{R} = R^1 M^{-1}$, this completes the proof. ∎

In the iteration of Section 6 we used θ^* as the first approximation to the exact solution. Here we will need a better approximation, since $D_\epsilon{}^j H(\theta^*)$ may not be small for ϵ small, $j > 0$.

Lemma 12. *There exists $\theta^N \in Y_{m,2,N}(\Sigma_1)$ such that $D_\epsilon{}^j H(\theta^N) = 0$ for $\epsilon = 0$, $0 \leqslant j \leqslant N$.*

Proof. We will choose θ^N to be a polynomial in ϵ of degree N with coefficients in $Y_{m,2}(\Sigma_1)$, independent of ϵ:

$$\theta^N = \theta_0 + \epsilon\theta_1 + \cdots + \epsilon^N \theta_N .$$

First we take $\theta_0 = \theta^*$. If H is written as in (3.5), it is clear that $D_\epsilon{}^j H(\theta)$ is a sum of terms, each of which is a derivative of $\tilde{F}$ applied to an ϵ-derivative of θ, or to an ϵ-derivative of Λ applied to an ϵ-derivative of θ. The only term in which $D_\epsilon^j\theta$ appears is $\tilde{L}(\theta(p); p)\, D_\epsilon^j\theta(p)$. Thus we may solve for $\theta_j = (j!)^{-1} D_\epsilon^j\theta\,|_{\epsilon=0}$ successively, using $\tilde{R}$, so that the conditions are satisfied. Since $\theta^* \in Y_{m+s,2}(\Sigma_1)$ for every s, $\theta_j \in Y_{m,2}(\Sigma_1)$. ▮

It follows from Lemma 12 that $H(\theta^N)$ is arbitrarily small in $Z_{m,N}(\Sigma_1)$ provided ϵ_1 is sufficiently small. Letting $u_0 = W^{-1}\theta^N$, we can now iterate as in Section 6 and obtain a solution $\theta \in Y_{m,2,N}(\Sigma_1)$ of $H(\theta) = 0$ such that $\theta = \theta^*$ for $\epsilon = 0$. As in Section 6, $D_\epsilon^j D_x^{\alpha_1} D_y^{\alpha_2}\theta$ is continuous in $(x, y; K, \epsilon)$ for $j \leqslant N$ and $\alpha_1 + \alpha_2 \leqslant m + 2$, and the same for $\lambda = \Lambda\theta + \gamma$. We have now verified (i)–(iii) of Theorem 2.

Now suppose we choose a sequence $K_n \to \infty$ and set

$$u_n(x, y; \epsilon) = \theta(x, y; K_n, \epsilon)/dn(x; k).$$

According to the remarks at the beginning of Section 6, $D_\epsilon^j D_x^{\alpha_1} D_y^{\alpha_2} u_n$ is uniformly bounded for $\alpha_1 + \alpha_2 \leqslant m + 2$, $j \leqslant N$. Therefore by Ascoli's Theorem, there is a subsequence on which $D_\epsilon^j D_x^{\alpha_1} D_y^{\alpha_2} u_n$ converges to $D_\epsilon^j D_x^{\alpha_1} D_y^{\alpha_2} u$ for some u, uniformly in $(x, y; \epsilon)$ for x in a bounded interval, provided $\alpha_1 + \alpha_2 \leqslant m + 1$, $j \leqslant N - 1$. Furthermore $dn(x; k) \to \operatorname{sech} x$ as $K \to \infty$, uniformly for x in a bounded interval, and similarly for the derivatives. Thus if we let

$$\theta(x, y; \infty, \epsilon) = u(x, y; \epsilon) \operatorname{sech} x$$

we have

$$D_\epsilon^j D_x^{\alpha_1} D_y^{\alpha_2}\theta(x, y; K_n, \epsilon) \to D_\epsilon^j D_x^{\alpha_1} D_y^{\alpha_2}\theta(x, y; \infty, \epsilon)$$

on a subsequence, uniformly for x in a bounded interval, where $\alpha_1 + \alpha_2 \leqslant m + 1$, $j \leqslant N - 1$. Moreover,

$$| D^j D_x^{\alpha_1} D_y^{\alpha_2}\theta(x, y; \infty, \epsilon)| < Ce^{-|x|}. \tag{7.5}$$

For $\epsilon = 0$, we have

$$\theta(x, y; \infty, 0) = \frac{4}{3} y D_x(\operatorname{sech}^2 x). \tag{7.6}$$

Similarly $\Lambda\theta$ converges in the same way on a subsequence of $\{K_n\}$ to a limit $\lambda^0(x, y; \infty, \epsilon)$ with exponential decay as in (7.5). Finally, as $K \to \infty$,

$$\lim \gamma(K) = \lim \tau(K; K) = 4/9.$$

If we set $\lambda(\cdot; \infty, \epsilon) = \lambda^0(\cdot; \infty, \epsilon) + 4/9$, we now have a solution of (2.10)–(2.13) such that $\theta \to 0$ and $\lambda - 4/9 \to 0$ exponentially as $|x| \to \infty$. To complete the proof of Theorem 2, it only remains to show that this limit is independent of the sequence $\{K_n\}$. This follows for ϵ sufficiently small from the lemma below.

LEMMA 13. *Suppose* $\theta(x, y; \infty, \epsilon)$ *and* $\lambda(x, y; \infty, \epsilon)$ *satisfy* (2.10)–(2.13) *and the following properties*: $D_x^{\alpha_1} D_y^{\alpha_2} \theta$ *and* $D_x^{\alpha_1} D_y^{\alpha_2} \lambda$ *are continuous*; $\alpha_1 + \alpha_2 \leqslant m + 1$; $\theta(\cdot; 0)$ *is given by* (7.6); *and*

$$| D_x^{\alpha_1} D_y^{\alpha_2} \theta(x, y; \infty, \epsilon)| \leqslant Ce^{-|x|}$$

$$\left| \lambda(x, y; \infty, \epsilon) - \frac{4}{9} \right| \leqslant Ce^{-|x|}.$$

If θ' *and* λ' *satisfy the same conditions and*

$$\sup e^{|x|} \, | D_x^{\alpha_1} D_y^{\alpha_2} (\theta - \theta')|$$

is sufficiently small, then $\theta' = \theta$ *and* $\lambda' = \lambda$.

Proof. In [2] we obtained a solution of a problem similar to the above in spaces of functions analytic with respect to x. The only differences were that $\theta = 3D_x \operatorname{sech}^2(3x/2)$ at $\epsilon = 0$ and $\lambda \to 1$ as $|x| \to \infty$. It is easy to check that if θ, λ solve the problem above, then

$$\theta^{\sim}(x, y; \epsilon) = (\tfrac{3}{2})^3 \, \theta(3x/2, y; \infty, \epsilon)$$
$$\lambda^{\sim}(x, y; \epsilon) = (\tfrac{3}{2})^2 \, \lambda(3x/2, y; \infty, \epsilon)$$

solve the problem of [2]. Thus it is sufficient to verify uniqueness for the latter.

The problem of [2] can be formulated in spaces like the ones used in this paper, using the weight function $e^{|x|}$. In this case the right inverse $\tilde{R}(\theta; \epsilon)$ has no null vectors; this is shown in Section 2, part (i), of [2]. Consequently, $\tilde{R}$ is also a left inverse of $\tilde{L}$. The uniqueness statement now follows from Theorem 2 of [9]. ∎

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