Numerical Computation of the Ahlfors Map of a Multiply Connected Planar Domain

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1. INTRODUCTION

N. Kerzman and E. M. Stein discovered in [6] a method for computing the Szegö Kernel of a bounded domain $D$ in the complex plane with $C^\infty$ smooth boundary. In case $D$ is also simply connected, the Kerzman–Stein method yields a powerful technique for computing the Riemann Mapping Function associated to a point $a$ in $D$ (see [6, 7]). In this note, we show how the Kerzman–Stein method can be generalized to yield a method for computing the Ahlfors map associated to a point in a finitely connected, bounded domain in the plane with $C^2$ smooth boundary. The Ahlfors map is a proper holomorphic mapping of $D$ onto the unit disc which maps each boundary component of $D$ one-to-one onto the boundary of the unit disc.

The Ahlfors map might prove to be useful in certain problems arising in fluid mechanics. For example, in the problem of computing the transonic flow past an obstacle in the plane, a conformal map of the outside of the obstacle onto the unit disc is used to set up a grid which is most convenient for making numerical computations (see [5]). The Ahlfors map could be used in a similar way in problems of this sort in which more than one obstacle is involved. These and other applications will be explored in subsequent papers. For the present, we content ourselves with describing our numerical method and proving its validity.

It should be mentioned that in case $D$ is simply connected, the Ahlfors map is equal to the Riemann Mapping Function. Thus, our numerical method yields a new technique for computing the Riemann map. In this setting, however, the method of [7] is to be preferred.

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2. Definition of the Ahlfors Map

If \( D \) is an \( n \)-connected, bounded, planar domain with \( C^2 \) boundary components and \( a \) is a point in \( D \), then the Ahlfors map for \( D \) at \( a \) is the holomorphic function \( f_a(z) \) which is the (unique) solution to the extremal problem: \( f \) maps \( D \) into the unit disc \( \mathbb{D} \) and is holomorphic on \( D \), \( f'(a) \) is real, and \( f'(a) \) is maximum. The mapping \( f_a \) is a branched \( n \) to one covering map of \( D \) onto \( \mathbb{D} \), \( f_a(a) = 0 \), and \( f_a \) and \( f_a' \) extend continuously to \( \partial D \). Each boundary curve of \( D \) is mapped by \( f_a \) one-to-one onto the boundary of \( \mathbb{D} \). Since \( f_a \) agrees with the Riemann Mapping Function associated to \( D \) at \( a \) in case \( D \) is simply connected, we think of \( f_a \) as the natural generalization of the Riemann map to multiply connected domains.

In his Ph.D. thesis, P. Garabedian [4] discovered the connection between the Ahlfors map and the Szegő Kernel Function. We shall give an alternate derivation of Garabedian’s result which will help us to adapt the Kerzman–Stein method to the multiply connected setting.

3. Description of the Method

In this section, we shall write down two Fredholm integral equations of the second kind whose solutions yield the Ahlfors map upon division. In subsequent sections, we justify the steps leading up to this result.

Suppose \( D \) is an \( n \)-connected, bounded plane domain with \( C^2 \) boundary. Let the \( n \) boundary curves \( \gamma_1, \gamma_2, \ldots, \gamma_n \) of \( D \) be parametrized via \( z_1(t), z_2(t), \ldots, z_n(t) \), respectively, where the parameter \( t \) ranges over \( 0 < t < 1 \). We suppose that these boundary curves are parametrized in the standard positive sense. For \( z \) in \( \partial D \), let \( T(z) \) denote the complex number representing the unit tangent vector to \( \partial D \) at \( z \) in the positive sense. Note that \( T(\gamma(t)) = \frac{z_\\prime(t)}{|z_\\prime(t)|} \). For \( w \in \partial D \) and \( z \in \partial D \) with \( z \neq w \), set \( H(w, z) = (2\pi i)^{-1}(z - w)^{-1}T(z) \) and define

\[
A(w, z) = \begin{cases} 
H(z, w) - H(w, z), & w, z \in \partial D, \ z \neq w \\
0, & w = z \in \partial D.
\end{cases}
\]

The function \( H(w, z) \) is the Cauchy Kernel for \( D \), and \( A(w, z) \) is the Kerzman–Stein Kernel which measures the extent to which the Cauchy projection associated to \( D \) is not self-adjoint. Note that

\[
F(w) = \int_{\partial D} H(w, z) \, F(z) \, ds_z
\]

whenever \( F \) is a holomorphic function in \( C(\partial D) \). The symbol \( ds_z \) denotes arc
length in the $z$ variable. The kernel $A(w, z)$ is continuous on $bD \times bD$ because $bD$ is $C^2$ (see [6, 7]), and $A(w, z) = -A(z, w)$. Furthermore, $A(w, z)$ is easy to compute because, as Kerzman and Stein [6] observed, $A(w, z) = (2\pi i)^{-1}(z-w)^{-1}(T_R(w, z) - T(z))$ where $T_R(w, z)$ denotes the (complex) vector obtained by reflecting the vector $T(w)$ about the chord joining $z$ to $w$.

We shall now state our main results, deferring all proofs to Section 4. Fix a point $a$ in $D$. For $z$ in $bD$, let $g(z) = \overline{H(a, z)}$ and let $h(z) = (2\pi)^{-1}(z-a)^{-1}$. The integral equation

$$S(z) + \int_{w \in bD} A(z, w) S(w) \, ds_w = g(z), \quad z \in bD \quad (3.1)$$

has a unique continuous solution $S(z)$ which extends to be holomorphic on $D$ and in $C(\overline{D})$. The integral equation

$$L(z) + \int_{w \in bD} A(z, w) L(w) \, ds_w = h(z), \quad z \in bD \quad (3.2)$$

has a unique continuous solution $L(z)$. The function $L(z)$ extends to be holomorphic on $D - \{a\}$ and in $C(\overline{D} - \{a\})$. Furthermore, $L(z) \neq 0$ on $\overline{D} - \{a\}$ and $L(z)$ has a simple pole with residue $(2\pi)^{-1}$ at $a$. The Ahlfors map for $D$ at $a$ is given by $f_a(z) = S(z)/L(z)$ for $z$ in $\overline{D} - \{a\}$.

We close this section by rewriting Eq. (3.1) and (3.2) in a form suitable for numerical computation. Following [7], (3.1) can be written

$$\sigma_i(t) + \sum_{j=1}^{n} \int_{0}^{1} A_{ij}(t, u) \sigma_j(u) \, du = \phi_i(u) \quad (3.1)'$$

where $\sigma_i(t) = |z_i'(t)|^{1/2} S(z_i(t))$, $\phi_i(t) = |z_i'(t)|^{1/2} g(z_i(t))$, and $A_{ij}(t, u) = |z_i(t)|^{1/2} A(z_i(t), z_j(u)) |z_j'(u)|^{1/2}$. Similarly, (3.2) becomes

$$\lambda_i(t) + \sum_{j=1}^{n} \int_{0}^{1} A_{ij}(t, u) \lambda_j(u) \, du = \psi_i(t) \quad (3.2)'$$

where $\lambda_i(t) = |z_i'(t)|^{1/2} L(z_i(t))$, and $\psi_i(t) = |z_i'(t)|^{1/2} h(z_i(t))$. In this setting, $f_a(\tau_i(t)) = \sigma_i(t)/\lambda_i(t)$ for $0 \leq t \leq 1$, and $f_a(w)$ can be evaluated at any point $w$ in $D$ via the Cauchy Integral Formula.

Equations (3.1)' and (3.2)' can be viewed as Fredholm integral equations of the second kind and, as such, are amenable to numerical computation. Indeed, if we introduce the vector notation $\Sigma(t) = (\sigma_1, \sigma_2, ..., \sigma_n)$,
\[ A(t) = (\lambda_1, \ldots, \lambda_n), \Phi(t) = (\phi_1, \ldots, \phi_n), \text{ and } \Psi(t) = (\psi_1, \ldots, \psi_n), \text{ and if we denote} \]
\[ \mathbf{K}(t, u) \] by \( \mathbf{K}(t, u) \), then (3.1)' and (3.2)' can be written
\[ \Sigma(t) + \int_0^1 \mathbf{K}(t, u) \Sigma(u) \, du = \Phi(t) \]
\[ \Lambda(t) + \int_0^1 \mathbf{K}(t, u) \Lambda(u) \, du = \Psi(t). \]

These equations are Fredholm integral equations of the second kind with skew hermitian kernels. Therefore, the integral operators have pure imaginary eigenvalues and the Fredholm alternative yields that \( \Sigma(t) \) and \( \Lambda(t) \) exist and are unique. Kerzman and Trummer [7] have made computations using similar integral equations in the simply connected case and have achieved excellent results.

4. Justification of the Method

Let \( L^2(bD) \) denote the space of square integrable complex valued functions on \( bD \) with respect to arc length measure \( ds \), and let \( H^2(bD) \) denote the usual Hardy space consisting of functions in \( L^2(bD) \) which are boundary values of holomorphic functions on \( D \) (see [3]). The inner product on \( L^2(bD) \) is given by \( (u, v) = \int_{bD} u \overline{v} \, ds \). Since \( H^2(bD) \) is a closed subspace of \( L^2(bD) \), the orthogonal projection \( P: L^2(bD) \rightarrow H^2(bD) \) is defined and is known as the Szegő projection. The Szegő Kernel Function \( S(w, z) \) is defined on \( D \times \overline{D} \) and is related to the Szegő projection via
\[ (P\phi)(w) = \int_{bD} S(w, z) \phi(z) \, ds_z, \]
which holds for all functions \( \phi \) in \( L^2(bD) \). The Szegő Kernel is holomorphic in \( w \) and antiholomorphic in \( z \), \( S(w, z) = \overline{S(z, w)} \), and \( F(z) = S(z, w) \) is in \( H^2(bD) \) for each \( w \) in \( D \). Since \( bD \) is of class \( C^2 \), \( S(w, z) \) extends continuously to \( \overline{D} \times D - \{ (z, z) : z \in bD \} \). We shall also need to define the projection \( P^\perp = I - P \) which is the orthogonal projection of \( L^2(bD) \) onto the orthogonal compliment \( H^2(bD) \perp \) of \( H^2(bD) \) in \( L^2(bD) \).

Another important kernel related to the Szegő Kernel is the L-Kernel \( L(w, z) \), discovered by Garabedian in [4]. The two kernels are connected via the differential identity
\[ S(a, z) \, ds_z = (1/i) L(z, a) \, dz \quad \text{for } z \in bD, \, a \in D. \quad (4.1) \]
Garabedian proved in [4] that the Ahlfors map \( f_a(z) \) is given by

\[
f_a(z) = S(z, a)/L(z, a).
\]

The kernel \( L(z, a) \) is holomorphic in \( z \) on \( D - \{a\} \) and holomorphic in \( a \) on \( D - \{z\} \). Furthermore, \( L(z, a) \neq 0 \) if \( z \neq a \), and \( F(z) = L(z, a) \) has a simple pole at \( a \) with residue \((2\pi)^{-1}\). Proofs of these facts can be found in Garabedian [4], Nehari [8, 9], or Bergman [2]. Since \( bD \) is of class \( C^2 \), the Szegő Kernel is in \( C(bD) \) as a function of \( z \) and (4.1) implies that \( L(z, a) \) extends continuously to \( bD \) as a function of \( z \).

It follows from a result of Nehari [8] that \( L(z, a) = (P^+h)(z) \) where \( h(w) = (2\pi)^{-1}(w-a)^{-1} \). We shall give an alternative proof of this formula which is shorter and simpler than Nehari's original proof. Let \( h(w) \) be a parametrization of \( bD \) with respect to arc length \( s \). We shall prove Nehari's formula by showing that

\[
S(z, \zeta(s)) = (1/i) \zeta'(s)(P^+h)(\zeta(s)).
\]  

Then the formula follows from (4.1).

It is known that a function \( F \) in \( C(bD) \) extends to be holomorphic on \( D \) if and only if \( \int_{bD} F(z) A(z) \, dz = 0 \) for all functions \( A(z) \) holomorphic on \( D \) and in \( C(bD) \) (see [1]). Since the space of holomorphic functions in \( C(bD) \) is dense in \( H^2(bD) \), we conclude that the class of functions \( A \) in \( L^2(bD) \) defined via \( A = \{ A(z) \, T(z): A \in H^2(bD) \cap C(bD) \} \) forms a dense subspace of \( H^2(bD)^2 \). Thus, to prove (4.2), it will suffice to show that

\[
\int_{bD} S(a, z) \phi(z) \, dz = -\frac{1}{i} \int_{bD} (P^+h)(z) \phi(z) \, dz
\]  

for all \( \phi \) in \( H^2(bD) \) and all \( \phi \) in \( A \). In case \( \phi \) is in \( H^2(bD) \), the left-hand side of (4.3) is simply \((P\phi)(a) = \phi(a)\), and the right-hand side is \((1/i)(P^+h, \phi T) = (h, T) = \int_{bD} H(a, z) \phi(z) \, dz = \phi(a)\). In case \( \phi \) is in \( A \), i.e., in case \( \phi = AT \) for some function \( A(z) \) in \( H^2(bD) \) and \( C(bD) \), the left-hand side of (4.3) is \((P\phi)(a) = 0\) because \( A \) is a subset of \( H^2(bD)^1 \). The right-hand side is \((1/i)(P^+h, A) = 0\). Thus, we conclude that \( L(z, a) = (P^+h)(z) \).

The justification of the numerical method of Section 3 will be finished if we show that \( S(z) = S(z, a) \) and \( L(z) = L(z, a) \).

Kerzman and Stein [6] proved that \( S(z) = S(z, a) \). It will be instructive to repeat their argument here. The function \( H(w, z) \) is the Cauchy Kernel. The Cauchy Projection is a (non-orthogonal) projection of \( L^2(bD) \) onto \( H^2(bD) \) given by

\[
(C\phi)(w) = \int_{bD} H(w, z) \phi(z) \, ds_z
\]
for $\phi$ in $L^2(bD)$. (It follows from the Plemelj formula and elementary facts about the Hilbert transform that $C\phi$ extends to be in $H^2(bD)$ when $\phi$ is in $L^2(bD)$.) Note that if $F$ is in $H^2(bD)$, then $(CF)(w) = F(w)$ for $w$ in $D$ via the Cauchy integral formula.

Since $P$ and $C$ reproduce holomorphic functions, and because $P = P^*$, the following identities are apparent:

(a) $PC = C$,
(b) $CP = P$,
(c) $C^*P = C^*$,
(d) $PC^* = P$.

Subtracting (a) from (d) yields

$$P(C^* - C) = P - C. \quad (4.4)$$

Applying (4.4) to a continuous function $\phi(z)$ on $bD$, and noting that the kernel of the operator $C^* - C$ is $A(w, z)$, we obtain via Fubini's theorem

$$\int_{z \in bD} \left( \int_{w \in bD} A(w, z) S(a, w) ds_w \right) \phi(z) ds_z = \int_{bD} (S(a, z) - H(a, z)) \phi(z) ds_z.$$

Thus, we are able to deduce that

$$\int_{bD} A(w, z) S(a, w) ds_w = S(a, z) - H(a, z).$$

Taking the complex conjugate of this equation, and noting that $A(z, w) = -\overline{A(z, w)}$ and $S(w, a) = \overline{S(a, w)}$, we obtain that $S(z) = S(z, a)$ is the unique solution to (3.1).

We shall now use similar ideas to show that $L(z) = L(z, a)$. Since $P^\perp = (P^\perp)^*$, and because $P^\perp = I - P$, the operators $P^\perp$ and $C$ are seen to be related via

(A) $P^\perp C = 0$,
(B) $CP^\perp = C - I + P^\perp$,
(C) $C^*P^\perp = 0$,
(D) $P^\perp C^* = C^* - I + P^\perp$.

Subtracting (B) from (C), we obtain

$$(C^* - C) P^\perp = I - C - P^\perp.$$
Applying this operator identity to the function \( h(z) = (2\pi)^{-1}(z - a)^{-1} \), we obtain

\[
\int_{\partial D} A(w, z) \, L(z, a) \, dz = h(w) - (Ch)(w) - L(w, a), \quad w \in \partial D
\]

because \( L(w, a) = (P^1h)(w) \). But

\[
2\pi (Ch)(w) = \frac{1}{2\pi i} \int_{\partial D} \frac{dz}{(z - w)(z - a)} = 0
\]

for \( w \in D \) by the Residue Theorem. Hence, \( (Ch)(w) = 0 \) for \( w \) in \( \partial D \) too. We conclude that \( L(z) = L(z, a) \) is the unique solution to (3.2). This completes the justification.

*Note added in proof.* Because of Eq. (4.1), the Ahlfors map can be written \( f_\nu(z) = S(z, a)/L(z, a) \) = \((T(z)/L(z, a)) = S(z, a)/S(a, z)\) for \( z \) in \( \partial D \). Thus, it is enough to compute either \( S(z, a) \) or \( L(z, a) \). I wish to thank Norberto Kerzman for pointing this out to me.

**REFERENCES**