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Level two string functions and Rogers Ramanujan type identities

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Abstract

The level two string functions are calculated exactly for all simply laced Lie algebras, using a ladder coset construction. These are the characters of cosets of the type $G/U(1)^r$, where G is the algebra at level two and r is its rank. This coset is a theory of generalized parafermions. A conjectured Rogers Ramanujan type identity is described for these characters. Using the exact string functions, we verify the Rogers Ramanujan type expressions, that are the main focus of this work.

1. Introduction

One of the remarkable properties of two dimensional systems is the expression of many characters of conformal field theories (CFT) as generalized Rogers Ramanujan type sums, which we denote as GRR. Many such examples have been worked out in the literature [1–6]. Yet, the origin of these identities remains somewhat mysterious.

Some of this mystery is explained by the deep connection to two dimensional solvable lattice models, in particular, the so-called rigid solid on solid models (RSOS). This connection was first noted by Baxter, in the context of the Hard Hexagon model [7], and was considerably further developed in Refs. [8,9]. The crux of this connection is that the local state probabilities in the RSOS models are identical to the characters of the fixed point CFT in the appropriate regime. (More precisely, the one dimensional configuration sum, which is the center piece of the local state probability, is equal to the characters.) This connection leads to the conjecture that GRR

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identities exist for every CFT that appears as a fixed point for some RSOS model in some regime, and vice versa.

The connection with RSOS lattice models also implies a way to prove these identities [8,9], where we consider the finite one dimensional configuration sum, which obeys some simple recursion relations. This sum is equal to a version of the characters, which are the finite GRR identities.

In this paper, we describe new GRR identities for the cosets of the type

$$H = \frac{G_2}{U(1)^r}, \tag{1.1}$$

where G is any simply laced algebra, at level two, and r is its rank. The case of A_r was already described in Ref. [6], and we extend this work to all the other algebras. The GRR expression for the identity characters for the models H was conjectured in Ref. [5], and we agree with this conjecture. We find GRR identities for all the characters, where the weight in G_2 has Dynkin index one.

We achieve these new GRR by calculating exactly the string function of G_2, which are the characters of H (up to factors of Dedekind's eta functions). The calculation is done by utilizing a ladder coset construction, which is a product of intermediate cosets, where each of the latter can be solved exactly by identifying it as a simpler CFT.

The connection with RSOS of our GRR identities, and, in particular, their proof in this way, is an interesting question, which we are currently investigating.

2. The general conjecture

Our objective is to describe GRR type equalities (Generalized Rogers Ramanujan identities), for the characters of the cosets of the type

$$H = \frac{G_2}{U(1)^r}, \tag{2.1}$$

where r is the rank of G and G is any of the simply laced algebras which are of the type A_r, D_r, E_6, E_7 and E_8, at level two. The case of A_r was already discussed in Ref. [6], where GRR identities were found for all the characters of H. Our aim here is to generalize these results to all of the simply laced algebras.

The characters of the theory H, for any G at any level, were discussed in Ref. [10], and are related to generalized parafermions. Briefly, they are labeled by two weights, Φ_λ^Λ where Λ is a highest weight of G at level k (here $k = 2$) and λ is an element of the weight lattice of G. The weights obey the condition $\Lambda - \lambda \in M$, where we denote by M the root lattice of G. To each of the fields Φ_λ^Λ, we associate a character χ_λ^Λ, which is given by the corresponding string function of G_k. The string functions are denoted by $c_\lambda^\Lambda(\tau)$ and are defined as

$$c_\lambda^\Lambda(\tau) = \mathrm{Tr}_{\mathcal{H}_\lambda^\Lambda} e^{2\pi i\tau(L_0 - c/24)}, \tag{2.2}$$

where L_0 is the dimension and $\mathcal{H}_\lambda^\Lambda$ is the representation of the affine G_k with highest weight Λ and $U(1)^r$ charge λ. The characters of the coset H are then given by

$$\chi_\lambda^\Lambda(\tau) = \eta(\tau)^r c_\lambda^\Lambda(\tau), \tag{2.3}$$

where $\eta(\tau)$ denotes the Dedekind eta function, λ is defined modulo kM and $\Phi_\lambda^\Lambda = \Phi_{\lambda+\mu}^\Lambda$, where $\mu \in kM$.

Let us present our main result which is a GRR expression for the characters of the theory H. We denote by C_G the Cartan matrix of the algebra G which can be any of the algebras A_r, D_r, E_6, E_7 and E_8. Let Λ stand for any of the fundamental weights which have Dynkin index 1, $\Lambda=\Lambda_i$ or $\Lambda=0$, where $i=1,2,\dots,r$, and $\theta\Lambda_i=1$, where Λ_i is one of the fundamental weights and θ is the highest root. We conjecture, that the characters of H have the following expression,

$$\chi_\lambda^{\Lambda_i}(\tau)=q^{\Delta_\lambda^{\Lambda_i}}\sum_{\substack{\vec{b}=0\\ \vec{b}=\vec{Q}\bmod 2}}^{\infty}\frac{q^{\vec{b}C_G\vec{b}^t/4-b_i/2}}{(q)_{b_1}(q)_{b_2}\dots(q)_{b_r}},\tag{2.4}$$

where $(q)_n$ is

$$(q)_n=\prod_{m=1}^{n}\left(1-q^m\right),\tag{2.5}$$

and for $\Lambda=0$ we take $b_i=0$. Δ_λ^Λ is some dimension which we do not specify. Here $q=e^{2\pi i\tau}$. We use the vector notation $\vec{b}=(b_1,b_2,\dots,b_r)$. Q is a vector of integers modulo 2 defined by

$$\Lambda-\lambda=\sum_{k=1}^{r}Q_k\alpha_k\quad\bmod 2M,\tag{2.6}$$

where α_k denotes the kth simple root.[1] For the algebra $G=A_r$ this reproduces the result of Ref. [6]. For the case of $\Lambda=0$ this agrees with a conjecture which was put forwards in Ref. [5].

The sum in Eq. (2.4) consists of generalized Rogers Ramanujan identities. To establish this GRR we need to calculate the string functions of $G=D_r,E_6,E_7,E_8$ at level two. We could, in principle, do this numerically using group theoretic recursion relations for the algebras in question. This is, in fact, too difficult to do by a computer program. So, instead, we will calculate the string functions exactly, finding closed analytical expressions for all the string functions. We accomplish this by generalizing the ladder coset construction of Ref. [6]. Once such expressions are found we can verify the GRR, Eq. (2.4).

3. The algebra D_n

We start with $D_n=SO(2n)$ at level two. We can write the coset $H=SO(2n)_2/U(1)^n$ as a product of cosets,

$$H=\prod_{r=1}^{n}\frac{SO(2r)_2}{SO(2r-2)_2\times U(1)},\tag{3.1}$$

where all the intermediate cosets cancel in the product. Of course, care must be taken, to make this correct in conformal field theory. It means that the same representation must be imposed when the same group appears in the numerator and the denominator, and this representation has to be summed over. In other words, for this product coset we take the special modular invariant,

[1] Because of the field identifications in the coset H, $\chi_{\sigma(\lambda)}^{\sigma(\Lambda)}(\tau)=\chi_\lambda^\Lambda(\tau)$, where σ is any automorphism of the Dynkin diagram of the affine $\hat{G}$, at level two, it is enough to consider only Λ which are fundamental weights or zero.

where all the intermediate representations are identical and are summed over. If we denote by G_2^{Λ} the highest weight field Λ for the group G at level two, and by $G_2^{\Lambda}/H_2^{\lambda}$ the appropriate branching function for this coset, we can write the characters as,

$$\chi_{\lambda}^{\Lambda}(\tau) = \sum_{\mu_r,\, r=1,2,\ldots,n-1} \prod_{r=1}^{n} \frac{SO(2r)_2^{\mu_r}}{SO(2r-2)_2^{\mu_{r-1}} \times U(1)^{\lambda\alpha_r}}, \tag{3.2}$$

where $\mu_n = \Lambda$ and α_r is the rth simple root of $SO(2n)$.

So, we proceed to calculate the characters of cosets of the type

$$H_r = \frac{SO(2r)_2}{SO(2r-2)_2 \times U(1)}. \tag{3.3}$$

It is easy to see that for any r this coset has the central charge $c = 1$. Thus, it has to be one of the $c = 1$ models which were completely classified in Ref. [11]. By calculating the dimensions of the fields in this coset we deduce that the coset H_r is given by a Z_2 orbifold of one free boson at the radius $\sqrt{2r(r-1)}$. This implies that all the dimensions in this coset are of the form

$$\Delta_m = \frac{m^2}{4r(r-1)}, \tag{3.4}$$

where m is any integer, or $\Delta = 1/16, 9/16$, which are the dimensions of the twist fields. Eq. (3.4) can be easily checked directly, by calculating the dimensions of the fields in the coset H_r.

Now, using Eq. (3.2), we wish to calculate the characters of the coset $H = SO(2n)_2/U(1)^n$. The calculation is simplified since H is equivalent to the theory which is a product of $n-1$ single boson orbifolds at various radii, implying that H is a Z_2 orbifold of $n-1$ free bosons propagating at some lattice M. If we denote the bosons as ϕ_i, $i = 1, 2, \ldots, n-1$, the orbifold is given by $\phi_i \to -\phi_i$. So it remains to find the lattice M. This lattice can be constructed inductively using the ladder coset expression Eq. (3.2). We use the beta method, Ref. [12], to account for the summation of the intermediate weights, to find that the lattice M is the root lattice of the group $SU(n)$ scaled by $\sqrt{2}$, i.e., $M \approx \sqrt{2}M_{SU(n)}$. This is termed the $SU(n)$ lattice at level two, and the characters are the classical theta functions of $SU(n)$ level two, Ref. [13]. Thus, we established:

Theorem 1. *The coset $SO(2n)_2/U(1)^n$ is equivalent to the theory of $n-1$ free bosons, ϕ_i, $i = 1, 2, \ldots, n-1$, moving on the lattice $\sqrt{2}M_{SU(n)}$, Z_2 orbifolded by the twist $\phi_i \to -\phi_i$.*

Let us demonstrate how the $SU(n)_2$ lattice is built. We do this inductively. First, for $SO(4)$, we consider the theory

$$\frac{SO(4)_2}{U(1)^2}, \tag{3.5}$$

which has the central charge 1 and is equivalent to a Z_2 orbifold of one free boson at the radius 2. This is the $SU(2)$ torus scaled by $\sqrt{2}$, in accordance with the theorem.

To identify the lattice for $SO(2n)_2/U(1)^n$ we consider the character of the identity,

$$\chi_0^0(\tau) = \frac{SO(2n)_2^0}{(U(1)^n)^0}, \tag{3.6}$$

where we denoted by 0 the unit representation. Now, this character is equal to a theta function on some lattice M, which we wish to identify. So, the character is equal to

$$\chi_0^0(\tau) = \sum_{p\in M} e^{\pi i\tau p^2}, \tag{3.7}$$

up to factors of Dedekind's eta function.

Let us show how this lattice is obtained for $SO(6)$. We know that the character $\chi_0^0(\tau)$ may be written as

$$\chi_0^0(\tau) = \sum_\mu \frac{SO(6)_2^0}{SO(4)_2^\mu \times U(1)^0} \times \frac{SO(4)_2^\mu}{SO(2)^0 \times U(1)^0}, \tag{3.8}$$

where we denoted the representation by the superscript. For $\mu = 0$, this is a product of two bosonic $c = 1$ theories, moving on the two-dimensional lattice N, generated by the vectors $\{0, 2\}$ and $\{2\sqrt{3}, 0\}$, as discussed above. Now, the lattice N will be enhanced, i.e., it will be a sub-lattice of M, due to the representations with $\mu \neq 0$. To find M it is enough to consider $\mu = (1, 1)$, which is the second fundamental weight of $SO(4)$. Thus we have to identify the momenta of the two fields twice the fundamental weight

$$P_1 = SO(6)_2^0/SO(4)_2^{(1,1)} \times U(1)^0, \tag{3.9}$$

and

$$P_2 = SO(4)_2^{(1,1)}/SO(2)^0 \times U(1)^0. \tag{3.10}$$

This we do by calculating their dimensions. The dimension of P_1 comes out to be $3/2$, and that of P_2 is $1/2$. Thus, they correspond to the beta vector $\beta = \{\sqrt{3}, -1\}$. So, the lattice M is the lattice generated by β and N. This is the lattice generated by $\{0, 2\}$ and $\beta = \{\sqrt{3}, -1\}$. Multiplying the scalar products of these two generators, we find that the scalar products matrix is

$$H = \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix}, \tag{3.11}$$

which is twice the Cartan matrix of $SU(3)$. Thus M is given by $\sqrt{2}M_{SU(3)}$, in agreement with Theorem 1.

We can continue, in this fashion, for any $SO(2n)$ and we find that M is indeed $\sqrt{2}M_{SU(n)}$, establishing Theorem 1.

Theorem 1 allows to calculate easily the string functions of $SO(2n)_2$, since they are given by the characters of a simple Z_2 orbifold. The fields are labeled by λ, where λ is a weight of $SU(n)$ and their dimensions are

$$\Delta_\lambda = \frac{\lambda^2}{4}. \tag{3.12}$$

In addition, there are the twist fields whose dimensions are $\Delta = \frac{n-1}{16}, \frac{n+7}{16}$. The characters are expressed by a level two $SU(n)$ classical theta functions. These are defined by

$$\Theta_{\lambda,m}(\tau) = \sum_{\mu\in R+\lambda/m} q^{m\mu^2/2}, \tag{3.13}$$

where λ is any element of the weight lattice of $SU(n)$ and R denotes the root lattice of $SU(n)$. We take the level to be $m = 2$ which means that this is a level two theta function.

The characters of the Z_2 orbifold are given by standard argumentation and are divided into the twisted and the untwisted sectors. In the untwisted sector we have three types of characters. The characters of the zero momenta fields are

$$q^{(n-1)/24}\chi_\pm(\tau)=\frac{1}{2}\Theta_{0,2}(\tau)\Big/\prod_{k=1}^{\infty}(1-q^k)^{n-1}\pm\frac{1}{2}\prod_{k=1}^{\infty}(1+q^k)^{-n+1},\tag{3.14}$$

where the second term arises due to the sector twisted in the time direction. The nonzero momenta do not have such a contribution and are given simply by

$$\chi_\lambda(\tau)=\frac{1}{n_\lambda}\frac{\Theta_{\lambda,2}(\tau)}{\eta(\tau)^{n-1}},\tag{3.15}$$

where λ is a weight of $SU(n)$, $n_\lambda=2$ when λ is on the root lattice and $n_\lambda=1$, otherwise. In the twisted sector, we have two fields which have the characters, $\chi_{t,\pm}(\tau)$, where

$$q^{(n-1)/24}q^{(1-n)/16}\chi_{t,\pm}(\tau)=\frac{1}{2}\prod_{k=1}^{\infty}(1-q^{k-1/2})^{-n+1}\pm\frac{1}{2}\prod_{k=1}^{\infty}(1+q^{k-1/2})^{-n+1}.\tag{3.16}$$

We can determine which weight of $SU(n)$ corresponds to each character by simply comparing the dimensions (some care is required when there is more than one field of the same dimension). Thus, we can confirm directly the GRR Eq. (2.4). We will give some examples below.

We first consider the unit character in the coset $SO(2n)_2/U(1)^n$. It is given by the $\lambda=0$ character in the corresponding $SU(n)_2$ orbifold. Thus we have the following identity for any n,

$$\chi_+(\tau)=q^{(1-n)/24}\sum_{\substack{b_i=0\\ b_i=0 \bmod 2}}^{\infty}\frac{q^{\vec{b}C_n\vec{b}^t/4}}{(q)_{b_1}(q)_{b_2}\dots(q)_{b_n}},\tag{3.17}$$

where i goes from 1 to n and $\chi_+(\tau)$ was given by Eq. (3.14). C_n is the Cartan matrix of $D_n\approx SO(2n)$ and $\vec{b}=(b_1,b_2,\dots,b_n)$. This is a beautifully simple identity of the Rogers Ramanujan type that we verified directly using a Mathematica program for $n=2$ to $n=8$ and up to order q^{20}.

Other particularly simple identities arise from the twisted sector. They correspond in the coset to the characters $\chi_s^s(\tau)$ and $\chi_{s+\alpha_1}^s(\tau)$, in the notation of Eqs. (2.2) and (2.3), where s stands for the spinor representation and α_1 is the first simple root. The simple roots of D_n are $\alpha_i=\epsilon_i-\epsilon_{i+1}$, for $i=1,2,\dots,n-1$ and $\alpha_n=\epsilon_{n-1}+\epsilon_n$, where ϵ_i are orthogonal unit vectors. Then, we have the following GRR identities,

$$\chi_{t,+}(\tau)=q^{\Delta}\sum_{\substack{b_i=0\\ b_i=0 \bmod 2}}^{\infty}\frac{q^{\vec{b}C_n\vec{b}^t/4-b_n/2}}{(q)_{b_1}(q)_{b_2}\dots(q)_{b_n}},\tag{3.18}$$

where $\chi_{t,+}(\tau)$ was given by Eq. (3.16). For the other twist field we have,

$$\chi_{t,-}(\tau)=q^{\Delta_1}\sum_{\substack{b_i=0\\ b_i=0 \bmod 2\ b_1=1 \bmod 2}}^{\infty}\frac{q^{\vec{b}C_n\vec{b}^t/4-b_n/2}}{(q)_{b_1}(q)_{b_2}\dots(q)_{b_n}},\tag{3.19}$$

where Δ and Δ_1 are some dimensions. The above identities agree with our general GRR conjecture, Eq. (2.4).

Let us give another example. Consider the character $\chi_v^v(\tau)$, for the algebra $SO(8)$, where v is the vector representation. The field has the dimension $3/16$. The dimension determines that it corresponds to the field in the $SU(4)_2$ bosonic theory, which has the weight $\lambda=\nu_1-\nu_2$, where ν_i is the ith fundamental weight of $SU(4)$. Thus, the character is given by

$$\chi_v^v(\tau) = \Theta_{\nu_1-\nu_2,2}(\tau)/\eta(\tau)^3, \tag{3.20}$$

where we denoted with $\Theta_{\lambda,2}(\tau)$ the theta function on the $SU(4)$ lattice, at level two, defined by Eq. (3.13). According to our general GRR conjecture, Eq. (2.4), we have the following identity,

$$\chi_v^v(\tau) = q^{\Delta} \sum_{\substack{b_i=0 \\ b_i=0 \bmod 2}}^{\infty} \frac{q^{\vec{b}C_4\vec{b}^t/4-b_1/2}}{(q)_{b_1}(q)_{b_2}\dots(q)_{b_4}}, \tag{3.21}$$

where C_4 is the Cartan matrix of $D_4 = SO(8)$, Δ is some dimension, and b_1 corresponds to the vector representation. We verified directly this identity up to order q^{20}, using a Mathematica program, and indeed it is correct.

Other examples of this kind can be demonstrated. We checked all the identities for $\chi_\lambda^\Lambda(\tau)$, where $\Lambda\theta = 1$ and any λ using a Mathematica program for $r = 2$ to $r = 8$ up to order q^{20}, and indeed they all hold. However, the condition $\Lambda\theta = 1$, where θ is the highest root (which means that Λ has a Dynkin index one), is not obeyed for some of the weights Λ and, in this case, our conjecture breaks down. Further work is needed to find the GRR for the weights Λ such that $\Lambda\theta \neq 1$.

4. The algebra E_6

Let us consider now the algebra E_6. The relevant coset here is,

$$H = (E_6)_2/U(1)^6. \tag{4.1}$$

We can present this coset as

$$H = P \times \frac{SO(10)_2}{U(1)^5}, \tag{4.2}$$

where the coset P is

$$P_{\mu,q}^{\Lambda} = \frac{((E_6)_2)^{\Lambda}}{(SO(10)_2)^{\mu} \times U(1)^q}, \tag{4.3}$$

where $SO(10) \times U(1)$ is a maximal subgroup of E_6. The characters are given by

$$\chi_{\lambda,q}^{\Lambda}(\tau) = \sum_{\mu} P_{\mu,q}^{\Lambda} \times \frac{SO(10)^{\mu}}{(U(1)^5)^{\lambda}}, \tag{4.4}$$

where we denoted, again, by $\chi_{\lambda,q}^{\Lambda}(\tau)$ the characters with, Λ, μ and q, the weights of the corresponding algebras.

Since we know the characters of $SO(10)_2$, it is enough to determine the characters of P. We note that the central charge of P is

$$c = 8/7. \tag{4.5}$$

It is the same central charge as that of the parafermion theory $W = SU(2)_5/U(1)$ [14]. Calculating the dimensions of the fields in the coset P, reveals that some of them are, indeed, as in the coset W. In fact, the theory P is equivalent to the coset W orbifolded by the symmetry $A \to A^{\dagger}$, where A is any field in the theory. The dimensions of the fields in this orbifold were calculated, along with the characters, in Ref. [15], and there is a complete agreement of all the dimensions, leading us to Theorem 2.

Theorem 2. *The characters of the theory $(E_6)_2/U(1)^6$ are given by the characters of the orbifolded fifth parafermion times the characters of $SO(10)_2/U(1)^5$, known from Theorem 1, where there is summation on the intermediate weight, Eq. (4.4).*

The central charge of the kth parafermion, $k = 1, 2, \ldots$, is given by

$$c = \frac{3k}{k+2} - 1, \tag{4.6}$$

and the dimensions are given by

$$\Delta^l_m = \frac{l(l+2)}{4(k+2)} - \frac{m^2}{4k}, \tag{4.7}$$

where $l = 0, 1, \ldots, k$ and $m = 0, 1, \ldots, k$ and $l = m \bmod 2$. The characters are given by the string functions of $SU(2)$, up to a factor of Dedekind's eta function, and can be expressed as Hecke indefinite modular forms [16]. For the fifth parafermionic theory, we use the expression for the characters appearing in Ref. [16] (page 219). Define

$$\epsilon(m,n) = \exp\left(2\pi i \frac{m-n+2}{8}\right), \quad n = 0 \bmod 2, \tag{4.8}$$

$$\epsilon(m,n) = \exp\left(2\pi i \frac{n+1}{4}\right), \quad n = 1 \bmod 2. \tag{4.9}$$

Then, the characters of the fifth parafermion are given by

$$d^p_r(\tau) = (-1)^p \eta(\tau)^{-2} \sum_{\substack{m,n=-\infty \\ m=r \bmod 5 \\ n=2p+2 \bmod 7 \\ 7m^2+5n^2=4 \bmod 16}}^{\infty} \epsilon(m,n) q^{(7m^2+5n^2)/560}, \tag{4.10}$$

where $p = r \bmod 2$.

Due to the orbifold, we have additional twist (disorder) fields. Their dimensions are, Ref. [15],

$$\Delta_\gamma = \frac{(\gamma + p/2)^2}{4(k+2)} - \frac{1}{48} + \frac{c}{24}, \tag{4.11}$$

where k is the level, here $k = 5$, and $p = k \bmod 2$, $p = 0$ or 1. The label $\gamma = 0, 1, \ldots, k+1$, and the central charge c was given by Eq. (4.6).

It can be verified that all the fields in the coset P have dimensions which correspond either to the untwisted sector, Eq. (4.7), or the twisted sector, Eq. (4.11), in agreement with Theorem 2.

The characters of W, orbifolded by the twist $A \to A^\dagger$, were given in Ref. [15]. In the untwisted sector we have two types of characters.

$$y^l_m(\tau) = d^l_m(\tau), \tag{4.12}$$

for $l = 0, 1, \ldots, k$ and $m = 1, 2, \ldots, k-1$. I.e., the characters with $m \neq 0$ are the same as in the untwisted theory. For $m = 0$ we have

$$y^l_{0,\pm}(\tau) = d^l_0/2 \pm B^l(\tau)/2, \tag{4.13}$$

where

$$B^l(\tau) = \eta(2\tau)^{-1} \sum_{n=-\infty}^{\infty} (-1)^{nk} e^{2(k+2)\pi i\tau(n+\frac{l+1}{2(k+2)})^2}. \tag{4.14}$$

In the twisted sector, the characters are given by

$$d^{\gamma}_{t,\pm}(\tau) = w^{\gamma}_t(\tau)/2 \pm w^{\gamma}_t(\tau+1)/2, \tag{4.15}$$

where

$$w^{\gamma}_t(\tau) = \eta(\tau/2)^{-1}\big[\theta_{\gamma+p/2,k+2}(\tau) - \theta_{\gamma+p/2+k+2,k+2}(\tau)\big], \tag{4.16}$$

and where $\theta_{l,k}(\tau)$ stands for the $SU(2)$ classical theta function,

$$\theta_{l,k}(\tau) = \sum_{j=-\infty}^{\infty} e^{2\pi i k\tau(j+\frac{l}{2k})^2}, \tag{4.17}$$

and l is any integer.

Let us give some examples. We take the basis for the simple roots of E_6 which corresponds to the breaking of E_6 to $SO(10)\times U(1)$,

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}, \tag{4.18}$$

where we denoted by α_i the ith simple root. The simplest fields to consider in the P coset are

$$\frac{(E_6)^0_2}{SO(10)^{\lambda}_2 \times U(1)^{-\sqrt{3}/2}}, \tag{4.19}$$

where $\lambda = s$ or $v+\bar{s}$, where s ($\bar{s}$) denotes the spinor (anti spinor) representations and v is the vector representation of $SO(10)$. The upper index is the representation, which, here, is the singlet of E_6. These fields have the dimensions $1/4$ or $3/4$, and are thus fields in the twisted sector with $\gamma = 2$, $\Delta_\gamma = 1/4, 3/4$, Eq. (4.11). There are two characters for these fields which are $d^2_{t,\pm}$, Eq. (4.15), corresponding respectively to $\lambda = s$ or $\lambda = v+\bar{s}$. These fields are multiplied by the fields in the $SO(10)_2/U(1)^5$ theory, which are the twist fields of the $SU(5)_2$ orbifold, whose characters are given by Eq. (3.16),

$$u_\pm(\tau) = q^{-4/24} q^{4/16}\left(\frac{1}{2}\prod_{j=0}^{\infty}\big(1-q^{j+1/2}\big)^{-4} \pm \frac{1}{2}\prod_{j=0}^{\infty}\big(1+q^{j+1/2}\big)^{-4}\right). \tag{4.20}$$

Thus, the two characters of the coset

$$\frac{(E_6)^0_2}{(U(1)^6)^{\zeta}}, \tag{4.21}$$

where ζ is either the weight $\zeta_+ = \{1/2, 1/2, 1/2, 1/2, 1/2, -\sqrt{3}/2\}$ or the weight $\zeta_- = \{3/2, 1/2, 1/2, 1/2, -1/2, -\sqrt{3}/2\}$ are given by

$$r_+(\tau) = u_+(\tau)d^2_{t,+}(\tau) + u_-(\tau)d^2_{t,-}, \tag{4.22}$$

and

$$r_-(\tau) = u_+(\tau)d^2_{t,-}(\tau) + u_-(\tau)d^2_{t,+}(\tau). \tag{4.23}$$

We observe, now, that these two characters are given by the GRR type identity

$$r_\pm(\tau) = \sum_{\substack{\vec{b}=0 \\ \vec{b}=0 \bmod 2\ b_5=e \bmod 2}}^{\infty} \frac{q^{\vec{b}C_{E6}\vec{b}^t/4}}{(q)_{b_1}(q)_{b_2}\dots(q)_{b_6}}, \tag{4.24}$$

where $e=0$ for r_+, $e=1$ for r_- and C_{E6} stands for the Cartan matrix of E_6. We checked these two identities, by a Mathematica program, up to order q^{20}, and indeed they hold.

Let us give another example. Consider the character,

$$b = \frac{(E_6)_2^{\mu_1}}{(U(1)^5)^{\lambda_3} \times U(1)^{1/\sqrt{3}}}, \tag{4.25}$$

where the superscript denotes the representation and where μ_1 is the first fundamental weight of E_6 (the 27) and λ_3 is the third fundamental weight of $SO(10)=D_5$. Using Theorem 2, this string function can be written as

$$b = \sum_\lambda \frac{E_6^{\mu_1}}{SO(10)^\lambda \times U(1)^{1/\sqrt{3}}} \times \frac{SO(10)^\lambda}{(U(1)^5)^{\lambda_3}}, \tag{4.26}$$

where the sum is over λ which are equal to $\lambda=\lambda_1,\lambda_3,\lambda_5$, where λ_i denotes the ith fundamental weight of $SO(10)$. We identify each of the sub-characters in Eq. (4.26) according to the dimensions, which are

$$b = \left[\frac{3}{35}\right] \times [7/10] + \left[\frac{17}{35}\right] \times [3/10] + \left[\frac{2}{7}\right] \times [1/2], \tag{4.27}$$

where $[x]$ denotes the character of the corresponding field with the dimension x. We used the following expressions for the dimensions of the fields. The dimension of the field $(E_6)_2^\mu/SO(10)_2^\lambda \times U(1)^q$ is

$$\Delta^\mu_{\lambda,q} = \mu(\mu+2\rho)/28 - \lambda(\lambda+2\rho_{10})/20 - q^2/4, \tag{4.28}$$

where ρ is the sum of E_6 fundamental weights and ρ_{10} that of $SO(10)$. The dimension of $SO(10)_2^\lambda/(U(1)^5)^\omega$ is

$$\Delta^\lambda_\omega = \lambda(\lambda+2\rho_{10})/20 - \omega^2/4. \tag{4.29}$$

This translates to the following character formula,

$$b = d_2^2\Theta(\nu_1+\nu_3) + d_1^3\Theta(\nu_1-\nu_3) + d_0^2\Theta(\nu_1+\nu_4), \tag{4.30}$$

where we denoted by ν_i the ith fundamental weight of $SU(5)$ and where $\Theta(\nu)$ is the classical theta function of $SU(5)$ at level two and the weight ν, Eq. (3.13),

$$\Theta(\nu) = \Theta_{\nu,2}(\tau)/\eta(\tau)^4, \tag{4.31}$$

which expresses the string functions of $SO(10)_2$, in accordance with Theorem 1. Finally, we can verify the following GRR in accordance with the conjecture, Eq. (2.4),

$$b = q^\Delta \sum_{\substack{\vec{b} \\ \vec{b}=0 \bmod 2 \\ b_2=b_3=1 \bmod 2}}^{\infty} \frac{q^{\vec{b}C_{E_6}\vec{b}^t/4 - b_1/2}}{(q)_{b_1}\dots(q)_{b_6}}, \tag{4.32}$$

where Δ is some dimension. We verified, by Mathematica, this GRR up to order q^{20} and indeed it holds.

5. The algebra E_7

Let us turn now to the string functions of E_7 at level two. The relevant coset here is

$$P = \frac{(E_7)_2}{U(1)^7}. \tag{5.1}$$

Here we use the following ladder coset construction,

$$P = G \times \frac{SO(12)_2}{U(1)^6}, \tag{5.2}$$

where G is the coset

$$G = \frac{(E_7)_2}{SO(12)_2 \times U(1)}, \tag{5.3}$$

and where there is a summation over the intermediate representations,

$$\chi_\lambda^\Lambda(\tau) = \sum_\mu \frac{(E_7)_2^\Lambda}{SO(12)_2^\mu \times U(1)^{\lambda'}} \times \frac{SO(12)_2^\mu}{(U(1)^6)^{\bar{\lambda}}}, \tag{5.4}$$

where λ' and $\bar{\lambda}$ are the projections of λ to the $U(1)$ and $U(1)^6$, respectively.

We can calculate the central charge of G, and it is given by $c = 13/10$. We note that it is the same central charge as the product of minimal models $M_3 \times M_5$, where M_3 is the Ising and M_5 is the three state Pott model, whose central charge is $c = 1/2 + 4/5 = 13/10$. Calculating the dimensions in the coset, we note that the dimensions are given by the sum of the dimensions of the two minimal models. Thus, we establish that G is given by $M_3 \times M_5$. Since we know the string functions of $SO(12)_2$ we have a closed formula for the string functions of $(E_7)_2$, which are the characters of the coset $(E_7)_2/U(1)^7$. This is expressed in the following Theorem 3.

Theorem 3. *The coset $(E_7)_2/U(1)^7$ is equivalent to the conformal field theory which is the product of the two minimal models $M_3 \times M_5$, times the orbifold of free bosons on $SU(6)_2$, where the product is according to Eq. (5.4).*

The unitary minimal model M_k, Refs. [17,18], has the central charge

$$c = 1 - \frac{6}{k(k+1)}, \quad k = 3, 4, \ldots, \tag{5.5}$$

and the dimensions of the primary fields are,

$$\Delta_{n,m} = \frac{(n(k+1) - mk)^2 - 1}{4k(k+1)}, \tag{5.6}$$

where $n = 1, 2, \ldots, k-1$ and $m = 1, 2, \ldots, k$. The characters of the primary fields can be expressed in terms of $SU(2)$ classical theta functions,

$$\chi_{n,m}^{(k)}(\tau) = \frac{\theta_{n(k+1)-mk,k(k+1)}(\tau) - \theta_{n(k+1)+mk,k(k+1)}(\tau)}{\eta(\tau)}, \tag{5.7}$$

where the classical $SU(2)$ theta functions, $\theta_{r,s}(\tau)$ were defined by Eq. (4.17). The dimension of the field in the G theory

$$\Phi^{\Lambda}_{\mu,q} = \frac{(E_7)^{\Lambda}_2}{(SO(12)_2)^{\mu} \times U(1)^q}, \tag{5.8}$$

where Λ is a highest weight of E_7 at level 2, μ is a weight of $SO(12)_2$, q is the weight of the $U(1)$, is given by

$$\Delta^{\Lambda}_{\mu,q} = \frac{\Lambda(\Lambda + 2\rho)}{40} - \frac{\mu(\mu + 2\rho_{12})}{24} - \frac{q^2}{4}, \tag{5.9}$$

where ρ, or ρ_{12}, are the sum of the fundamental weights of E_7, or $SO(12)$, respectively. We take a basis for the simple weights of E_7, which corresponds to the breaking into $SO(12) \times U(1)$,

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \sqrt{2}/2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}. \tag{5.10}$$

Let us give an example. Consider the fields in the $(E_7)_2/U(1)^7$ theory which are

$$S_{\pm} = \frac{(E_7)^0_2}{(U(1)^7)^{\nu_{\pm}}}, \tag{5.11}$$

where $\nu_+ = -\alpha_6$ for S_+ and $\nu_- = -\alpha_6 + (1, 0, 0, 0, 0, -1, 0)$ for S_-, and where α_6 is the sixth fundamental root of E_7, $-\alpha_6 = \{1, 1, 1, 1, 1, 1, -\sqrt{2}\}/2$. According to Theorem 3 the character of the fields $S_{\pm}$ can be written as

$$S_{\pm}(\tau) = \sum_{\mu} \frac{(E_7)^0_2}{SO(12)^{\mu}_2 \times U(1)^{-\sqrt{2}/2}} \times \frac{SO(12)^{\mu}_2}{(U(1)^6)^{\kappa_{\pm}}}, \tag{5.12}$$

where $\kappa_+ = s$ and $\kappa_- = \bar{s} + v$ and s is the weight of the spinor representation, $\bar{s}$ is the anti-spinor and v is the vector, all of the algebra $SO(12)$. μ is summed over the two values $\mu_+ = s$ and $\mu_- = v + \bar{s}$. We can identify the fields in the G theory,

$$g_{\pm} = \frac{(E_7)^0_2}{SO(12)^{\mu_{\pm}}_2 \times U(1)^{-\sqrt{2}/2}}, \tag{5.13}$$

by determining the dimensions of these fields, using Eqs. (5.9), (5.6), which are $3/16$ and $27/16$, we infer that they correspond to the fields in the minimal models $M_3 \times M_5$, which are $(1, 2, 1, 2)$ and $(1, 2, 1, 4)$, for g_+ and g_-, respectively, where we denoted by (n_1, m_1, n_2, m_2) the field in the product of minimal models $M_3 \times M_5$, with dimensions Δ_{n_1,m_1} and Δ_{n_2,m_2}, respectively. The two characters of these fields are given by Eq. (5.7) and are

$$t_{\pm}(\tau) = \chi^{(3)}_{n_1,m_1} \chi^{(5)}_{n_2,m_2}, \tag{5.14}$$

where (n_1, m_1) and (n_2, m_2) take the two values indicated before. In accordance with Theorem 3, these characters are multiplied by fields in the bosonic orbifold $SU(6)_2$. Since the representations of $SO(12)_2$ are spinorial, in this example, these fields are the two twist fields of the orbifolded $SU(6)_2$ theory, which are given by

$$q^{-5/16+5/24}u_\pm(\tau)=\frac{1}{2}\prod_{m=1}^{\infty}\left[1-q^{m-1/2}\right]^{-5}\pm\frac{1}{2}\prod_{m=1}^{\infty}\left[1+q^{m-1/2}\right]^{-5}. \tag{5.15}$$

Thus, the two characters $S_\pm(\tau)$ are given by

$$\begin{aligned} r_+ &= t_+u_+ + t_-u_-, \\ r_- &= t_-u_+ + t_+u_-. \end{aligned} \tag{5.16}$$

$r_\pm$ are the corresponding exact characters of the coset $(E_7)_2/U(1)^7$. Thus we can check our GRR conjecture Eq. (2.4), for this specific case. We find the following identity,

$$q^{y_\pm}r_\pm(\tau)=\sum_{\substack{\vec{b}=0 \\ \vec{b}=0 \bmod 2,\ b_1=1 \bmod 2,\ b_6=e \bmod 2}}^{\infty}\frac{q^{\vec{b}C_{E7}\vec{b}^t/4}}{(q)_{b_1}(q)_{b_2}\dots(q)_{b_7}}, \tag{5.17}$$

where $e=0$ for r_+ and $e=1$ for r_-, and C_{E7} is the Cartan matrix of E_7. $y_\pm$ are some dimensions. We checked these two identities up to order q^{20}, by Mathematica, and indeed they hold.

6. The algebra E_8

The remaining algebra is E_8. Our objective is to calculate the string functions of E_8 at level two. To do this, we use the following ladder coset construction,

$$H_\lambda^\Lambda=\frac{(E_8)_2^\Lambda}{(U(1)^8)^\lambda}=\sum_\mu\frac{(E_8)_2^\Lambda}{SO(16)_2^\mu}\times\frac{SO(16)_2^\mu}{(U(1)^8)^\lambda}, \tag{6.1}$$

where, as usual, we denoted by the superscripts the highest weights of the representations. Since, we know from Theorem 1 the string functions of $SO(16)_2$ it is enough to resolve the coset

$$G=\frac{(E_8)_2}{SO(16)_2}. \tag{6.2}$$

The central charge of this coset is calculated to be $c=1/2$. Since the theory G is unitary, it has to be equivalent to the Ising model, previously denoted as M_3. This yields an analytic expression for the string functions of $(E_8)_2$, and Theorem 4 follows.

Theorem 4. *The conformal field theory, the coset $E_8/U(1)^8$ at level two, is equivalent to a product of the Ising model times the orbifold of free bosons on the $SU(8)$ torus at level two.*

In view of Theorem 4, we can calculate the string functions of $(E_8)_2$ and verify our GRR conjecture, Eq. (2.4). Let us give an example. Consider the field in the H theory, H_μ^0 where $\mu=s$ or $\mu=\bar{s}+v$, and where we denoted by s, $\bar{s}$ and v the spinor, anti-spinor and vector representations of $SO(16)$. In the ladder coset construction, Eq. (6.1), this translates to the product of characters

$$H_\mu^0(\tau)=G\times R_\pm, \tag{6.3}$$

where

$$G = \frac{(E_8)_2^0}{SO(16)^s}, \tag{6.4}$$

and

$$R_\pm = \frac{SO(16)_2^s}{(U(1)^8)^\mu}, \tag{6.5}$$

where we took R_+ for $\mu = s$ and R_- for $\mu = v + \bar{s}$. Since G is a field in the Ising model, it can be identified by its dimension, which is $1/16$, where we used Eq. (5.6). Thus, $G = \chi_{2,2}^{(3)}$, which is the character of the spin field in the Ising model. Now, $R_\pm$ are given by the two characters of the twist fields of the bosonic $SU(8)_2$, which are

$$q^{-7/16+7/24} R_\pm(\tau) = \frac{1}{2}\prod_{j=1}^{\infty}\left(1 - q^{j-1/2}\right)^{-7} \pm \frac{1}{2}\prod_{j=1}^{\infty}\left(1 + q^{j-1/2}\right)^{-7}. \tag{6.6}$$

Thus, Eq. (6.6) yields these two characters, enabling the verification of our GRR conjecture, Eq. (2.4). We have the expression,

$$H_\mu(\tau) = q^{z_\pm} \sum_{\substack{\vec{b}=0 \\ \vec{b}=Q_\pm \bmod 2}}^{\infty} \frac{q^{\vec{b} C_{E8} \vec{b}^t/4}}{(q)_{b_1}(q)_{b_2}\dots(q)_{b_8}}, \tag{6.7}$$

where $Q_+ = (1, 0, 0, 0, 0, 0, 0, 0)$ for R_+ and $Q_- = (1, 0, 0, 0, 0, 1, 0, 0)$ for R_-, where $Q_\pm$ is defined modulo two and where C_{E_8} is the Cartan matrix of E_8 and $z_\pm$ are some dimensions. Eq. (6.7) agrees with our general GRR conjecture, Eq. (2.4). We verified directly these two identities and, indeed, they hold up to order q^{10}.

The reader might wonder about the result we would get if we take in the GRR, Eq. (2.4), the summation on $Q \bmod 2$, other than the ones given by Eq. (2.6). The result is surprising. For any choice of $Q \bmod 2$, we find that the GRR gives one of the characters of the corresponding coset. This amounts to a great deal of interesting and nontrivial identities, and it generalizes the identity by Slater, Ref. [19], for the Ising model. These identities we term generalized Slater identities. Some of these identities were already noted in Ref. [6].

7. Discussion

In this paper, we described generalized Rogers Ramanujan (GRR) identities for the string functions of all the simply laced algebras at level two. These are the characters of the conformal field theory (CFT) of generalized parafermions [10], specialized to the appropriate algebra and level. We find identities for all the characters, such that the weight in the algebra has Dynkin index one. These identities are surprisingly simple and pretty, and they center around the Cartan matrix of the algebra in question, in a type of ADE classification for such identities. This adds considerably to the GRR identities which were already established for characters of certain conformal field theories [1–6].

An interesting question is how to extend these identities to the weights which have Dynkin index greater than one. Another important question is how to extend these GRR identities to the levels which are greater than two and to the non-simply laced algebras.

Perhaps, the most interesting issue, from the physics point of view, is the relation to RSOS lattice models. For the case of A_r these models were investigated by Jimbo et al. [20]. The relation between GRR and RSOS lattice models was established before in a great deal of cases [7–9,4]. Importantly, this relation, once established, will furnish a way to prove the GRR identities, described here, and will also provide an expression for the finite one dimensional configuration sum in the RSOS models, which is the main piece of the local state probabilities (or, the one point functions), a central problem in this field.

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