Generalized Bebiano–Lemos–Providência inequalities and their reverses

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Abstract

We improve Bebiano–Lemos–Providência inequality: For $A, B \geq 0$

$$\left\| A^{\frac{1}{t-s}} B^{\frac{1}{t-s}} A^{\frac{1}{t}} \right\| \leq \left\| A^{\frac{1}{2}} \left( A^{\frac{1}{2}} B^{s} A^{\frac{1}{2}} \right)^{\frac{s}{t}} A^{\frac{1}{2}} \right\|$$

for all $s \geq t \geq 0$. In our discussion, Furuta inequality plays an essential role. Actually we have

$$\left\| A^{\frac{1}{t-s}} B^{1+s} A^{\frac{1}{2}} \right\|^{\frac{p+s}{s(1+s)}} \leq \left\| A^{\frac{1}{2}} \left( A^{\frac{1}{2}} B^{p+s} A^{\frac{1}{2}} \right)^{\frac{1}{p}} A^{\frac{1}{2}} \right\|$$

for $p \geq 1$ and $s \geq 0$.

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1. Introduction

Throughout this note, an operator means a bounded linear operator acting on a Hilbert space $H$. A positive operator $A$ is denoted by $A \geq 0$. Löwner–Heinz inequality (cf. [11]) asserts

$$ A \geq B \geq 0 \quad \text{implies} \quad A^p \geq B^p \quad \text{for all} \quad 0 \leq p \leq 1. \quad (1.1) $$

Recently, Bebiano et al. [2] showed the following norm inequality, say BLP norm inequality: For $A, B \geq 0$

$$ \left\| \left( \frac{1}{A} + \frac{1}{B} \right) \right\| \leq \left\| A^{\frac{1}{2}} \left( \frac{A^{\frac{1}{2}} B A^{\frac{1}{2}}}{t} \right)^{\frac{1}{2}} A^{\frac{1}{2}} \right\| \quad (1.2) $$

for all $s \geq t \geq 0$. The following operator inequality, BLP operator inequality, corresponds to (1.2): For $A, B \geq 0$

$$ A^{\frac{p+s}{p}} B^s \leq A^{1+s} \quad \text{for some} \quad s \geq t \geq 0 \implies B^t \leq A^{1+t}, \quad (1.3) $$

where $A^{\alpha} := A^{\frac{1}{2}} \left( A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^{\alpha} A^{\frac{1}{2}}$ for $A, B > 0$. \quad (1.4)

Here replacing $B$ by $B^{\frac{1+t}{1+t}}$, and putting $p := \frac{s}{t}$ ($\geq 1$) in (1.3), it is rewritten as follows: For $A, B \geq 0$

$$ A^{\frac{p+s}{p}} B^p \leq A^{1+s} \quad \text{for some} \quad p \geq 1 \quad \text{and} \quad s \geq 0 \implies B^{1+s} \leq A^{1+s}. \quad (1.5) $$

In Section 2, we discuss generalizations of BLP inequalities. Our background is Furuta inequality (2.1) cited in below. Precisely we improve (1.5) as follows: Let $A, B \geq 0$. Then

$$ A^{\frac{p+s}{p}} B^p \leq A^{1+s} \quad \text{for some} \quad p \geq 1 \quad \text{and} \quad s \geq 0 \implies B^{1+s} \leq A^{1+s}. \quad (2.2) $$

Moreover we give a norm inequality equivalent to (2.2): Let $A, B \geq 0$. Then

$$ \left\| \left( \frac{1}{A} + \frac{1}{B} \right) \right\|^{\frac{p+s}{p(1+s)}} \leq \left\| A^{\frac{1}{2}} \left( \frac{A^{\frac{1}{2}} B A^{\frac{1}{2}}}{\frac{p+s}{p}} \right)^{\frac{1}{p}} A^{\frac{1}{2}} \right\| \quad (2.3) $$

for all $p \geq 1$ and $s \geq 0$.

In Section 3, we give reverse inequalities of generalized BLP inequalities.

2. Generalized BLP inequalities

In this section we generalize BLP inequalities (1.2) and (1.5). For it we use Furuta inequality [6] (see also [4,7,10,12]): For each $r \geq 0$

$$ A \geq B \geq 0 \implies A^{1+r} \geq \left( A^{\frac{1}{r}} B A^{\frac{1}{r}} \right)^{\frac{1+r}{r}} \quad (2.1) $$

holds for $p \geq 1$. We propose an improvement of BLP operator inequality (1.5).

**Theorem 2.1.** Let $A$ and $B$ be positive operators. Then

$$ A^{\frac{p+s}{p}} B^p \leq A^{1+s} \quad \text{for some} \quad p \geq 1 \quad \text{and} \quad s \geq 0 \implies B^{1+s} \leq A^{1+s}. \quad (2.2) $$
Proof. We put
\[ C := \left( A^{-\frac{1}{2}} B^{p+s} A^{-\frac{1}{2}} \right)^{\frac{1}{p}}, \quad \text{or} \quad B^{p+s} = A^{-\frac{s}{2}} C^p A^{-\frac{s}{2}}. \]
Then the assumption says that \( A \geq C \geq 0 \), and so Furuta inequality (2.1) ensures that
\[ B^{1+s} = \left( A^{-\frac{s}{2}} C^p A^{-\frac{s}{2}} \right)^{\frac{1+s}{p+1}} \leq A^{1+s}. \]
That is, the desired inequality (2.2) is proved. \( \square \)

Now we have a norm inequality equivalent to the generalized BLP operator inequality (2.2).

**Corollary 2.2.** Let \( A \) and \( B \) be positive operators. Then
\[
\left\| A^{\frac{1+s}{2}} B^{1+s} A^{\frac{1+s}{2}} \right\|^{\frac{p+1}{p+1+t}} \leq \left\| A^{\frac{1}{2}} \left( A^{-\frac{s}{2}} B^{p+s} A^{-\frac{s}{2}} \right)^{\frac{1}{p}} A^{\frac{1}{2}} \right\| \quad (2.3)
\]
for all \( p \geq 1 \) and \( s \geq 0 \).

In addition, Theorem 2.1 and Corollary 2.2 have the following expression by Löwner–Heinz inequality (1.1):

**Corollary 2.3.** Let \( A \) and \( B \) be positive operators. Then
\[
A^{s \frac{1}{p}} B^{p+s} \leq A^{1+s} \quad \text{for some} \quad p \geq 1 \quad \text{and} \quad s \geq 0 \implies B^{1+t} \leq A^{1+t} \quad (2.4)
\]
for \( t \in [0, s] \), or equivalently
\[
\left\| A^{\frac{1+t}{2}} B^{1+t} A^{\frac{1+t}{2}} \right\|^{\frac{p+1}{p(1+t)}} \leq \left\| A^{\frac{1}{2}} \left( A^{-\frac{s}{2}} B^{p+s} A^{-\frac{s}{2}} \right)^{\frac{1}{p}} A^{\frac{1}{2}} \right\| \quad (2.5)
\]
for \( p \geq 1 \) and \( s \geq t \geq 0 \).

**Remark 2.4.** Replacing \( B \) by \( B^{\frac{t}{t+1}} \), (2.5) is expressed as follows: For \( A, B \geq 0 \)
\[
\left\| A^{\frac{1+t}{2}} B^{\frac{t}{t+1}} A^{\frac{1+t}{2}} \right\|^{\frac{p+1}{p(1+t)}} \leq \left\| A^{\frac{1}{2}} \left( A^{-\frac{s}{2}} B^{p+s} A^{-\frac{s}{2}} \right)^{\frac{1}{p}} A^{\frac{1}{2}} \right\| \quad (2.6)
\]
for \( p \geq 1 \) and \( s \geq t \geq 0 \). Thus if we take \( p = \frac{s}{t} \) for \( s \geq t \geq 0 \), then we have the original BLP inequality (1.2) because \( \frac{p+s}{p(1+t)} = 1 \) and \( \frac{t(p+s)}{p(1+t)} = s \).

3. Reverse inequalities of BLP inequalities

In [1], Araki showed a trace inequality which entailed the inequality: For \( A, B \geq 0 \)
\[
\| A^p B^p A^p \| \leq \| ABA \|^p \quad \text{for all} \quad 0 \leq p \leq 1, \quad (3.1)
\]
which is equivalent to Cordes inequality \( \| A^p B^p \| \leq \| AB \|^p \) (cf. [3]). On the other hand, Fujii and Seo [5] gave the following inequality by using a generalized Kantorovich constant
\[
K(h, p) := \frac{1}{h-1} \frac{h^p - h}{p-1} \left( \frac{p-1}{h^p - h} \right)^p \quad (3.2)
\]
for all \( h(\neq 1), p \in \mathbb{R} \) and \( K(1, p) = 1 \), see [8,9].
**Theorem A.** If $A$ and $B$ are positive operators such that $0 < m \leq B \leq M$ for some scalars $0 < m < M$ and $h := \frac{M}{m} (> 1)$, then

$$\|A^p B^p A^p\| \leq K(h, p)\|ABA\|^p \quad \text{for all} \quad p \geq 1.$$  \hfill (3.3)

Theorem A is a reverse inequality of Araki–Cordes inequality (3.1). We next consider a reverse of Corollary 2.2. We use (3.1) and (3.3) as our tool.

**Theorem 3.1.** Let $A$ and $B$ be positive operators such that $0 < m \leq B \leq M$ for some scalars $0 < m < M$ and $h := \frac{M}{m} > 1$. Then

$$\left\| A^{\frac{1}{2}} \left( A^\frac{1}{2} B^{p+s} A^\frac{1}{2} \right)^{\frac{1}{p}} A^{\frac{1}{2}} \right\| \leq K \left( h^{1+t}, \frac{p + s}{1 + t} \right)^{\frac{1}{p}} \left\| A^{\frac{1}{2}} B^{1+t} A^{\frac{1}{2}} \right\|^{\frac{p+s}{p(1+t)}}$$  \hfill (3.4)

for all $p \geq 1$ and $s \geq t \geq 0$.

**Proof.** It follows from (3.1) and (3.3) that for $p > 1$ and $s \geq t \geq 0$

$$\left\| A^{\frac{1}{2}} \left( A^\frac{1}{2} B^{p+s} A^\frac{1}{2} \right)^{\frac{1}{p}} A^{\frac{1}{2}} \right\| \leq \left\| A^{\frac{1}{2}} \left( A^\frac{1}{2} B^{p+s} A^\frac{1}{2} \right)^{\frac{1}{p}} A^{\frac{1}{2}} \right\|^{\frac{1}{p}}$$

$$= \left\| A^{\frac{p+s}{p(1+t)}} B^{(1+t)-\frac{p+s}{p(1+t)}} A^{\frac{1}{2}} \right\|^{\frac{1}{p}}$$

$$\leq K \left( h^{1+t}, \frac{p + s}{1 + t} \right)^{\frac{1}{p}} \left\| A^{\frac{1}{2}} B^{1+t} A^{\frac{1}{2}} \right\|^{\frac{p+s}{p(1+t)}}$$

because $\frac{p+s}{1+t} \geq 1$. So the desired inequality (3.4) holds. \hfill \Box

Now we have an operator inequality equivalent to the norm inequality (3.4) holds.

**Corollary 3.2.** Let $A$ and $B$ be positive operators such that $0 < m \leq B \leq M$ for some scalars $0 < m < M$ and $h := \frac{M}{m} > 1$. Then

$$B^{1+t} \leq A^{1+t} \quad \text{for some} \quad t \geq 0 \iff A^{s\frac{1}{p}} B^{p+s} \leq K \left( h^{1+t}, \frac{p + s}{1 + t} \right)^{\frac{1}{p}} A^{1+s}$$  \hfill (3.5)

for some $p \geq 1$ and $s \geq t$.

**Remark 3.3.** If we put $t = s$ in Theorem 3.1 and Corollary 3.2, then (3.4) and (3.5) are reverse inequalities of (2.3) and (2.2), respectively.

By Theorem 3.1 and Corollary 3.2, we have reverse inequalities of (1.2) and (1.5) as follows:

**Corollary 3.4.** Suppose the hypothesis of Theorem 3.1. Then the following inequalities hold:

$$\left\| A^{\frac{1}{2}} \left( A^\frac{1}{2} B^s A^\frac{1}{2} \right)^{\frac{1}{2}} A^{\frac{1}{2}} \right\| \leq K \left( h', \frac{s}{t} \right)^{\frac{1}{2}} \left\| A^{\frac{1+t}{2}} B^t A^{\frac{1+t}{2}} \right\|$$  \hfill (3.6)

for $s \geq t \geq 0$, or equivalently
\[ B^{1 + \frac{s}{p}} \leq A^{1 + \frac{s}{p}} \quad \text{for some} \quad p \geq 1 \quad \text{and} \quad s \geq 0 \implies A^{\frac{sp}{1 + \frac{s}{p}}} B^{p+s} \leq K \left( h^{1 + \frac{s}{p}}, p \right)^{\frac{1}{p}} A^{1+s}. \] (3.7)

**Proof.** The inequality (3.7) is given by taking \( p := \frac{1}{t} \geq 1 \) in Corollary 3.2. Moreover the inequality (3.6) is given by replacing \( B \) with \( B^{\frac{t}{t+1}} \) in Theorem 3.1. \( \square \)

As a consequence, we have the following corollary which includes a complement inequality of Löwner–Heinz inequality (1.1) below.

**Corollary 3.5.** Let \( A \) and \( B \) be positive operators such that \( 0 < m \leq B \leq M \) for some scalars \( 0 < m < M \) and \( h := \frac{M}{m} > 1 \). Then

\[ A \geq B > 0 \implies A^{\frac{sp}{1 + \frac{s}{p}}} B^{p+s} \leq K(h, p+s) A^{1+s} \quad \text{for} \quad p \geq 1 \quad \text{and} \quad s \geq 0. \] (3.8)

In particular,

\[ A \geq B > 0 \implies B^{1+s} \leq K(h, 1+s) A^{1+s} \quad \text{for} \quad s \geq 0. \] (3.9)

**Proof.** If we put \( t = 0 \) in (3.5), then (3.8) holds. (3.9) follows from putting \( p = 1 \) in (3.8). \( \square \)

### 4. Complementary inequalities

We give an upper bound of \( \| A^{\frac{1}{q}} \left( A^{\frac{1}{q}} B^{p+s} A^{\frac{1}{q}} \right)^{\frac{1}{p}} A^{\frac{1}{q}} \|^p - \lambda \| A^{\frac{1}{q}} B^{1+t} A^{\frac{1}{q}} \|^p \) for \( \lambda > 0 \) under the hypothesis of Theorem 3.1. We use the following notation: For \( q \geq 1 \) and \( h := \frac{M}{m} \) (0 < \( m < M \))

\[ I_q = I_{q,m,M} := \left[ \frac{q(h - 1)}{h^q - 1}, \frac{q(h^q - h^q - 1)}{h^q - 1} \right] \]

and

\[ F(m, M, q; \lambda) := \begin{cases} (1 - \lambda)M^q & \text{if} \quad 0 < \lambda < \frac{q(h - 1)}{h^q - 1}, \\ \lambda m^q \left( \frac{h^q - h^q - 1}{h^q - 1} \right)^{\frac{1}{p-1}} - 1 & \text{if} \quad \lambda \in I_q, \\ (1 - \lambda)m^q & \text{if} \quad \lambda > \frac{q(h^q - h^q - 1)}{h^q - 1}. \end{cases} \] (4.1)

The function \( f(\lambda) := F(m, M, q; \lambda) \) for \( \lambda > 0 \) is monotone decreasing and \( \lambda = K(h, q)^{-1} \) (\( \in I_q \)) is a unique solution of \( f(\lambda) = 0 \).

Now we cite a complementary inequality of Araki–Cordes inequality (3.1) by Fujii and Seo [5]: If \( A \) and \( B \) are positive operators such that \( 0 < m \leq B \leq M \) for some scalars \( m < M \), then for each \( \lambda \geq K(h, p)^{-1} \)

\[ \| ABA \|^q \geq \lambda \| A^q B^q A^q \| + F(m, M, q; \lambda) \| A \|^{2q} \quad \text{for all} \quad q > 1. \] (4.2)

By applying this, we show the following complementary inequality for (2.3).
Theorem 4.1. If $A$ and $B$ are positive operators such that $0 < m \leq B \leq M$ for some scalars $0 < m < M$ and $h := M_m > 1$, then for each $\lambda \in \left(0, K \left(h^{1+t}, \frac{p+s}{1+t} \right) \right)$

$$
\left\| A^{\frac{1}{2}} \left( A^\frac{1}{2} B^{p+s} A^\frac{1}{2} \right)^\frac{1}{p} A^{\frac{1}{2}} \right\|^p \leq \lambda \left\| A^{\frac{1}{2+t}} B^{1+t} A^{\frac{1}{2+t}} \right\|^{\frac{p+s}{1+t}} - \lambda F \left(m^{1+t}, M^{1+t}, \frac{p+s}{1+t} ; \frac{1}{\lambda} \right) \|A\|^{p+s}
$$

(4.3)

for $p \geq 1$ and $s \geq t \geq 0$.

Proof. Since $p \geq 1$ and $s \geq t \geq 0$ implies $\frac{p+s}{1+t} > 1$, it follows from (3.1) and (4.2) that for each $\lambda \in \left(0, K \left(h^{1+t}, \frac{p+s}{1+t} \right) \right)$

$$
\left\| A^{\frac{1}{2}} \left( A^\frac{1}{2} B^{p+s} A^\frac{1}{2} \right)^\frac{1}{p} A^{\frac{1}{2}} \right\|^p \leq \left\| A^\frac{1}{2} \left( A^\frac{1}{2} B^{p+s} A^\frac{1}{2} \right) A^\frac{1}{2} \right\|
$$

$$
= \left\| A^{\frac{p+s}{1+t}} B^{p+s} A^{\frac{p+s}{1+t}} \right\| \leq \lambda \left\| A^{\frac{1}{1+t}} B^{1+t} A^{\frac{1}{1+t}} \right\|^{\frac{p+s}{1+t}} - \lambda F \left(m^{1+t}, M^{1+t}, \frac{p+s}{1+t} ; \frac{1}{\lambda} \right) \|A\|^{p+s}
$$

So the desired inequality (4.3) holds. □

Remark 4.2. (i) If we take $p = \frac{s}{t}$ in Theorem 4.1, then a complementary inequality of (1.2) holds as follows: For each $\lambda \in \left(0, K \left(h^{1+t}, \frac{s}{t} \right) \right)$

$$
\left\| A^{\frac{1}{2}} \left( A^\frac{1}{2} B^{p+s} A^\frac{1}{2} \right)^\frac{1}{s} A^{\frac{1}{2}} \right\|^s \leq \lambda \left\| A^{\frac{1}{1+t}} B^{1+t} A^{\frac{1}{1+t}} \right\|^{\frac{s}{t}} - \lambda F \left(m^{1+t}, M^{1+t}, \frac{s}{t} ; \frac{1}{\lambda} \right) \|A\|^{\frac{s}{t}+s}
$$

for $s \geq t \geq 0$.

(ii) Under the same condition of Theorem 4.1 we have for each $\lambda \geq K \left(h^{1+t}, \frac{p+s}{1+t} \right)$

$$
\left\| A^{\frac{1}{2}} \left( A^\frac{1}{2} B^{p+s} A^\frac{1}{2} \right)^\frac{1}{p} A^{\frac{1}{2}} \right\|^p \leq \lambda \left\| A^{\frac{1}{1+t}} B^{1+t} A^{\frac{1}{1+t}} \right\|^{\frac{p+s}{1+t}} - \lambda F \left(m^{1+t}, M^{1+t}, \frac{p+s}{1+t} ; \frac{1}{\lambda} \right) \|A\|^{-(p+s)}
$$

for $p \geq 1$ and $s \geq t \geq 0$.

(iii) By Theorem 4.1 we have a difference reverse inequality of (2.3)

$$
\left\| A^{\frac{1}{2}} \left( A^\frac{1}{2} B^{p+s} A^\frac{1}{2} \right)^\frac{1}{p} A^{\frac{1}{2}} \right\|^p \leq \left\| A^{\frac{1}{2+t}} B^{1+t} A^{\frac{1}{2+t}} \right\|^{\frac{p+s}{1+t}} - m^{\frac{p+s}{1+t}} \frac{h - h^{\frac{p+s}{1+t}}}{h - 1}
$$

$$
\times \left\{ \left(K \left(h^{1+t}, \frac{p+s}{1+t} \right) \right)^{\frac{p+s}{1+t} - 1} - 1 \right\} \|A\|^{p+s}
$$

for $p \geq 1$ and $s \geq t \geq 0$.  


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