Comp. & Maths. with Appls. Vol. 12A, No. 3, pp. 363-373, 1986
Printed in Great Britain.

0886-9553/86 $3.00 + .00

INTERPOLATION FUNCTION OF A GENERAL TRIANGULAR MID-EDGE FINITE ELEMENT

CLAUDE LACOMBE† and CLAUDE BÉDARD††
Centre for Building Studies, Concordia University, Montreal, Quebec, H3G 1M8, Canada

(*Received* 15 *November* 1985)

Communicated by E. L. Wachspress

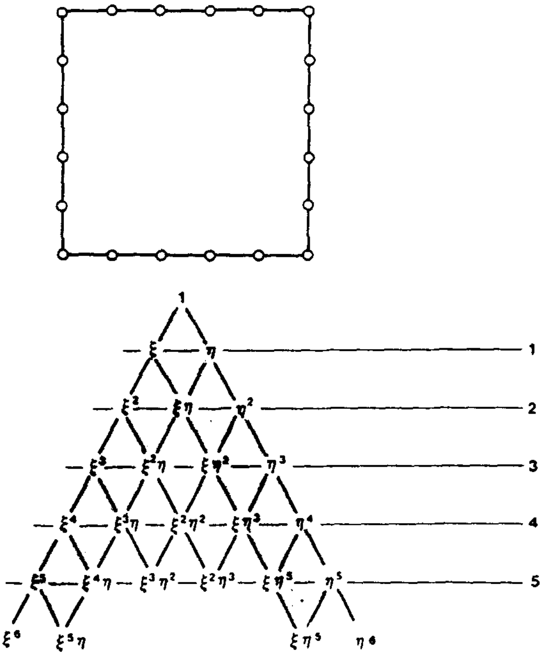
Abstract—An interpolation function for triangular mid-edge finite elements is developed using an algebraic interpolation approach. A convenient method for deriving shape functions of serendipity type directly and explicitly arises also from the interpolation function proposed above.

INTRODUCTION

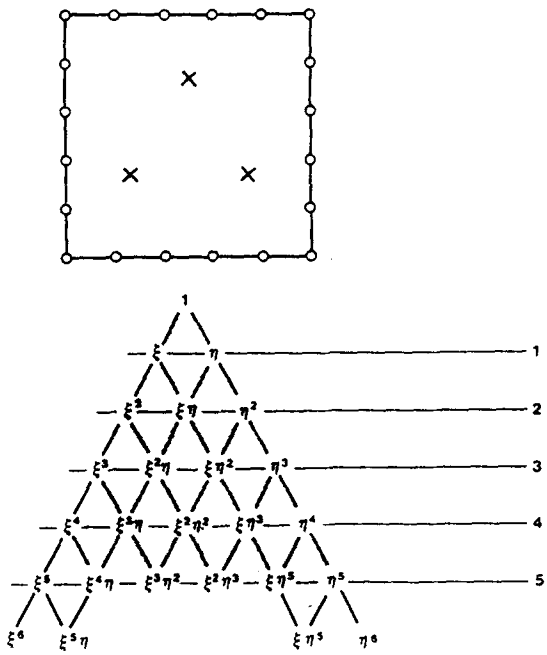
Serendipity shape functions as introduced by Zienkiewicz[1] led to the development of two important families of finite elements, namely the mid-edge family and the serendipity family. For the sake of clarity, recall that, as first pointed out by Taylor[2] and adopting the wording of Okabe[3], only some monomials of the zeroth Pascal hook are attached to elements of the mid-edge family, whereas elements of the serendipity family are m-unisolvent. This distinctive feature is illustrated in Fig. 1 for the fifth member of the families in their rectangular version. The name "serendipity" to characterize shape functions was originally selected to emphasize the fortuity of their construction. Since the chance factor has been removed through successive contributions (Zienkiewicz[1], Taylor[2], Gordon and Hall[4], Zlamal[5] and Ball[6]), "serendipity" for shape functions now has a connotation of directness and explicitness only. Although useful, serendipity shape functions have only been applied so far to the development of rectangular families. Mid-edge and serendipity families thus contrast with, for example, the Lagrange family for which both triangular and rectangular versions were developed in parallel. In fact, a serendipity shape function for the triangular version of both mid-edge and serendipity families does not seem to exist.

Therefore, the purpose of the present work is to develop a serendipity shape function for the triangular version of the mid-edge family. The triangular version of the serendipity family could be generated by producing m-unisolvency with mid-edge members and a Ritz function form. Polynomial completeness is obviously already provided by members of the triangular Lagrange family, but completion in a nodeless fashion is sometimes judged more attractive. Incomplete mid-edge members will also often prove to be useful. We shall first propose a straightforward procedure for the derivation of a general mid-edge rectangular element which is simply a didactic representation of Ball's method[6]. A similar procedure will finally be proposed for the triangle, based on an interpolation function derived by means of an algebraic interpolation approach.

†Research Student
††Assistant Professor



(a)



(b)

Fig. 1. a) Fifth member of the rectangular mid-edge family; b) Fifth member of the rectangular serendipity family.

1. AUTOMATION OF BALL'S METHOD FOR ELEMENTS OF THE RECTANGULAR MID-EDGE FAMILY

The interpolation function for rectangular mid-edge elements derived by Ball[6] is here generally given as:

$$\begin{aligned}\phi(\xi, \eta) = {} & \{1\xi\}[\mathbf{P}_2]\begin{bmatrix}\phi_1^4 & \phi_2^4 & \cdots & \phi_p^4\\ 0 & 0 & \cdots & 0\end{bmatrix}[\mathbf{P}_p]^T\{1\eta \ldots \eta^{p-1}\}^T\\ & + \{1\xi\}[\mathbf{P}_2]\begin{bmatrix}0 & 0 & \cdots & 0\\ \phi_1^2 & \phi_2^2 & \cdots & \phi_n^2\end{bmatrix}[\mathbf{P}_n]^T\{1\eta \ldots \eta^{n-1}\}^T\\ & + \{1\eta\}[\mathbf{P}_2]\begin{bmatrix}\phi_1^1 & \phi_2^1 & \cdots & \phi_m^1\\ 0 & 0 & \cdots & 0\end{bmatrix}[\mathbf{P}_m]^T\{1\xi \ldots \xi^{m-1}\}^T\\ & + \{1\eta\}[\mathbf{P}_2]\begin{bmatrix}0 & 0 & \cdots & 0\\ \phi_1^3 & \phi_2^3 & \cdots & \phi_o^3\end{bmatrix}[\mathbf{P}_o]^T\{1\xi \ldots \xi^{o-1}\}^T\\ & - \{1\xi\}[\mathbf{P}_2]\begin{bmatrix}\phi^{1\cap 4} & \phi^{3\cap 4}\\ \phi^{1\cap 2} & \phi^{2\cap 3}\end{bmatrix}[P_2]^T\{1\eta\}^T \qquad (1)\end{aligned}$$

The related expression for symmetric node distribution was obtained by analytic condensation of internal nodes of a general Lagrange element. Note that (1) is equivalent to the general bilinear interpolation function of Gordon and Hall[4], which is itself a special form of their more general transfinite interpolant. The $[\mathbf{P}_m]$ matrices in (1) are the inverse of Vandermonde matrices constructed from zeros of a polynomial expansion on the $[-1, +1]$ interval. As shown by Ball[6], the use of such matrices results in a direct method for the generation of serendipity shape functions. The method is here presented in such a way that it can serve as the basis for developing a general algorithm for the construction of serendipity shape functions for rectangular mid-edge elements.

We shall demonstrate the method by re-investigating the nodal configuration attempted by Ball[6] and shown in Fig. 2. The inverse of Vandermonde matrices are first assigned to every edge round the element as in Fig. 3. For a given edge, the matrix must be of the same order as the number of nodes on the edge (vertex nodes included). Elements in a given matrix column become the coefficients of a polynomial expansion which, when multiplied by a suitable blending function, constitutes the shape function for the node facing the matrix column. This polynomial expansion is always expressed in the variable of the coordinate axis parallel to the edge on

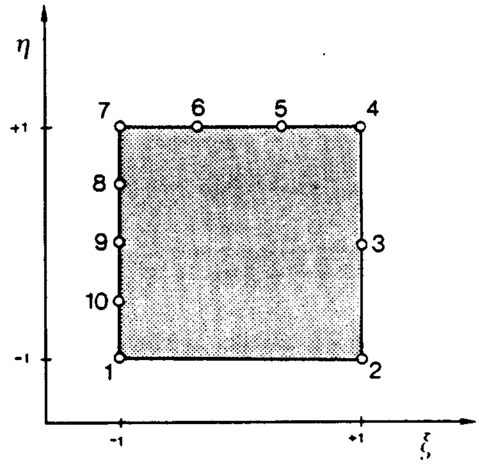


Fig. 2. General rectangular mid-edge element.

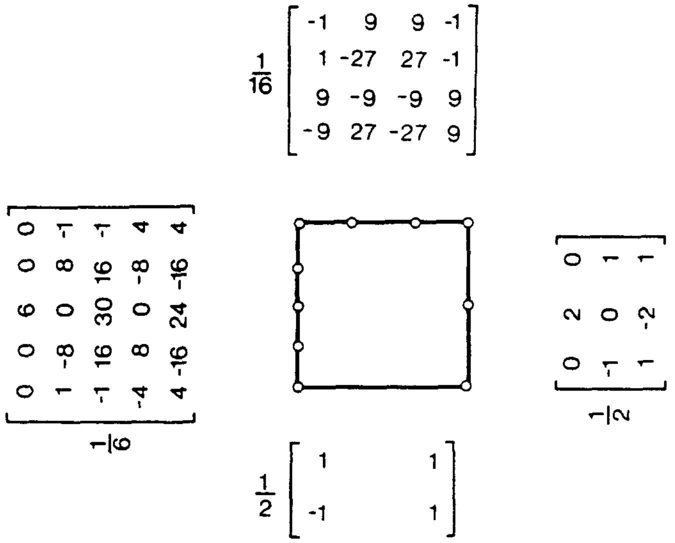


Fig. 3. Assignment of matrices around the rectangular element.

which the node is located. The blending functions are given as:

$$\begin{aligned} &1/2(1 + \xi) \\ &1/2(1 + \eta) \end{aligned} \tag{2}$$

for nodes on the $\xi = +1$ and $\eta = +1$ lines respectively, whereas for nodes on the $\xi = -1$ and $\eta = -1$ lines respectively, they are given as:

$$\begin{aligned} &1/2(1 - \xi) \\ &1/2(1 - \eta) \end{aligned} \tag{3}$$

For vertex nodes, two polynomial expansions with proper blending functions are involved in the expression for the shape function since two matrix columns are facing such nodes. Also, because of the Boolean nature[4] of the interpolation function, a term must be subtracted from the sum of the two polynomial expansions. This term is generally given as:

$$1/4(1 + \xi_i\xi)(1 + \eta_i\eta) \tag{4}$$

where ξ_i and η_i are parameters defined as:

$$\begin{aligned} \xi_i &= \begin{cases} 1, & \text{on } \xi = 1 \\ -1, & \text{on } \xi = -1 \end{cases} \\ \eta_i &= \begin{cases} 1, & \text{on } \eta = 1 \\ -1, & \text{on } \eta = -1 \end{cases} \end{aligned} \tag{5}$$

For example, to obtain the shape function corresponding to node 5, select elements in the corresponding matrix column as:

$$\frac{9}{16}\ \frac{27}{16} - \frac{9}{16} - \frac{27}{16} \tag{6}$$

to form the appropriate polynomial expansion:

$$\frac{1}{16}(9 + 27\xi - 9\xi^2 - 27\xi^3) \tag{7}$$

which finally yields the desired shape function when multiplied by the proper blending function.

$$N_5 = \frac{1}{2}(1 + \eta)\frac{1}{16}(9 + 27\xi - 9\xi^2 - 27\xi^3) \tag{8}$$

A similar procedure can be used for any other node. For vertex node 7, the shape function is derived as:

$$\begin{aligned} N_7 &= \frac{1}{2}(1 - \xi)\frac{1}{6}(-\eta - \eta^2 + 4\eta^3 + 4\eta^4) \\ &+ \frac{1}{2}(1 + \eta)\frac{1}{16}(-1 + \xi + 9\xi^2 - 9\xi^3) \\ &- \frac{1}{2}(1 - \xi)\frac{1}{2}(1 + \eta) \end{aligned} \tag{9}$$

2. INTERPOLATION FUNCTION FOR THE TRIANGLE

The derivation of the interpolation function for the triangle based on an algebraic interpolation approach is now presented. The resulting interpolation function will in fact be similar in form to the one for the rectangle given in (1).

Consider an $\mathbf{R}^2$ plane with three pairwise mutually intersecting straight lines ξ_1, ξ_2 and ξ_3 as shown in Fig. 4. The intersections of pairs of ξ_i-lines have to be distinct singletons for an open bounded region $\mathcal{D}$ of triangular shape to be formed. These singletons are the vertices V_i of the triangle and are related to the ξ_i-lines by:

$$\begin{aligned} \{V_1\} &= \xi_1 \cap \xi_3 \\ \{V_2\} &= \xi_2 \cap \xi_1 \\ \{V_3\} &= \xi_3 \cap \xi_2 \end{aligned} \tag{10}$$

A local axis S_i is assigned to each ξ_i-line. The origin of the S_i-axis is at vertex V_i and the normalization is such that the unit corresponds to the remaining vertex on the ξ_i-line. Region $\mathcal{D}$ is limited by the nondenumerable set of points $\partial\mathcal{D}$ defined as:

$$\partial\mathcal{D} = \{Q(s_i) \in \xi_i | 0 \leq s_i \leq 1, \quad i = 1, 2, 3\} \tag{11}$$

where Q is an arbitrary point. Region $\bar{\mathcal{D}}$ defined as:

$$\bar{\mathcal{D}} = \mathcal{D} \cup \partial\mathcal{D} \tag{12}$$

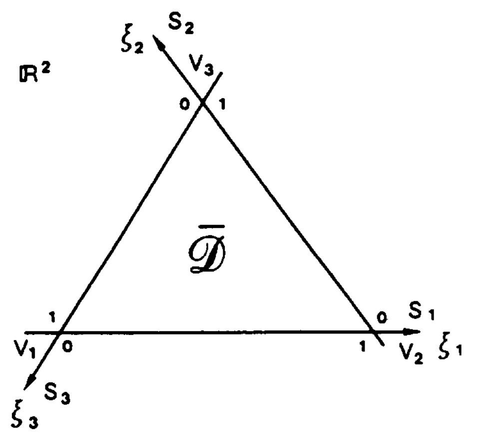


Fig. 4. Triangular domain.

can be assigned an area coordinate system with L_1, L_2 and L_3 variables such that:

$$L_i(V_k) = \delta_{ik}, \quad i, k = 1, 2, 3 \tag{13}$$

Consider now a bivariate function F defined over $\bar{\mathcal{T}}$ and part of the linear space $\mathcal{C}$ of continuous functions defined over $\bar{\mathcal{T}}$:

$$F \in \mathcal{C}(\bar{\mathcal{T}}) \tag{14}$$

Consider also three side-vertex projectors expressed in area coordinates as:

$$\begin{aligned} \mathbf{P}_1[F] &= (1 - L_1)^\alpha F(0, L_2', L_3') + L_1^\alpha F(1, 0, 0) \\ \mathbf{P}_2[F] &= (1 - L_2)^\alpha F(L_1', 0, L_3') + L_2^\alpha F(0, 1, 0) \\ \mathbf{P}_3[F] &= (1 - L_3)^\alpha F(L_1', L_2', 0) + L_3^\alpha F(0, 0, 1) \end{aligned} \tag{15}$$

where

$$L_i' = \frac{L_i}{L_i + L_j}, \; L_j' = \frac{L_j}{L_i + L_j}.$$

Side-vertex projectors were introduced by Marshall[7] and applied to high-order interpolation schemes by Nielson[8]. The $(1 - L_i)$ and L_i terms in (15) are "blending functions" according to the definition of Gordon and Hall[4]. Blending functions are simply partial operators in a given x_i variable that are applied on a multivariable function $F(x_1, x_2, \ldots, x_n)$ parametrized in x_j, $j \neq i$. The $^\alpha$ exponent in (15) was used by Nielson[8] with the following values:

$$\alpha \in \{1, 2\} \tag{16}$$

Blending functions are said to be linear for $\alpha = 1$ and quadratic for $\alpha = 2$.

To every projector $\mathbf{P}_i$ there is an associated precision set $\mathcal{P}_i$ defined as:

$$\begin{aligned} \mathcal{P}_1 &= (\xi_2 \cap \partial\mathcal{T}) \cup \{V_1\} \\ \mathcal{P}_2 &= (\xi_3 \cap \partial\mathcal{T}) \cup \{V_2\} \\ \mathcal{P}_3 &= (\xi_1 \cap \partial\mathcal{T}) \cup \{V_3\} \end{aligned} \tag{17}$$

Side-vertex projectors $\mathbf{P}_i$ are linear interpolating operators that are characterized by their strong commutativity. Because of this strong commutativity, projectors can be taken as generators of a distributive lattice[9] under Boolean sum and tensor product operations. The distributive lattice is partially ordered and all elements from it are also projectors. Since in the present case the set of generators is finite, there are minimal, maximal and intermediate elements. The maximal element of the distributive lattice is given as:

$$\mathbf{P}_1 \oplus \mathbf{P}_2 \oplus \mathbf{P}_3 \tag{18}$$

It can also be shown that in the case of strongly commutative projectors, an isomorphism[9] exists between, on the one hand, tensor product of projectors and intersection of associated precision sets, and on the other hand, Boolean sum of projectors and union of associated precision sets. For example, the isomorphic form for (18) is:

$$\mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 \tag{19}$$

which is also simply:

$$\begin{aligned} \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 &= \{Q(s_i) \in \xi_i \mid 0 \leq s_i \leq 1, \quad i = 1, 2, 3\} \\ &= \partial\mathcal{T} \end{aligned} \tag{20}$$

or in words, and function F in $\mathcal{C}(\bar{\mathcal{T}})$ is interpolated to its exact value by $\mathbf{P}_1 \oplus \mathbf{P}_2 \oplus \mathbf{P}_3$ at the nondenumerable set $\partial \mathcal{T}$. The maximal element (or maximal interpolant) can be expressed in terms of ordinary sum and tensor product operations using a recursive rule similar to a Gram–Schmidt orthogonalization, and can be further reduced using:

$$\mathbf{P}_i \otimes \mathbf{P}_j = \mathbf{P}_1 \otimes \mathbf{P}_2 \otimes \mathbf{P}_3, \quad i, j = 1, 2, 3; i \neq j \tag{21}$$

which is obtained from the isomorphism. The resulting expression for the maximal element then becomes:

$$\mathbf{P}_1 \oplus \mathbf{P}_2 \oplus \mathbf{P}_3 = \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 - 2\mathbf{P}_1 \otimes \mathbf{P}_2 \otimes \mathbf{P}_3 \tag{22}$$

which can be rewritten using (15) and for $\alpha = 1$

$$\begin{aligned} \mathbf{P}_1 \oplus \mathbf{P}_2 \oplus \mathbf{P}_3[F] = {} & (1 - L_1)F(0, L_2, L_3) \\ & + (1 - L_2)F(L_1, 0, L_3) + (1 - L_3)F(L_1, L_2, 0) \\ & - L_1F(1, 0, 0) - L_2F(0, 1, 0) - L_3F(0, 0, 1) \end{aligned} \tag{23}$$

Nielson[8] showed that the span of monomials that are exactly interpolated by (23) is:

$$\text{span}\{\mathbf{P}_1 \oplus \mathbf{P}_2 \oplus \mathbf{P}_3\} = \langle L_1, L_2, L_3 \rangle \tag{24}$$

whereas for $\alpha = 2$

$$\text{span}\{\mathbf{P}_1 \oplus \mathbf{P}_2 \oplus \mathbf{P}_3\} = \langle L_1, L_2, L_3, L_1^2, L_2^2, L_3^2 \rangle \tag{25}$$

The interpolant given in (23) is transfinite according to Gordon and Hall[4]. For use in F.E.M., the above interpolant must also be discretized, i.e., F on $\bar{\mathcal{T}}$ must be approximated by three univariate Lagrange polynomials on the three line segments that make $\partial \mathcal{T}$. On the [0, 1] interval of ξ_i:

$$F|_{s_i} \simeq \{\phi^i\}[\mathbf{P}_{n_i}]^T\{1 s_i \ldots s_i^{n_i - 1}\}^T \tag{26}$$

where:

$$\{\phi^i\} = \left\{ F \Big|_{s_i = \frac{j-1}{n_i - 1}} \right\}_{j=1}^{n_i} \tag{27}$$

and $[\mathbf{P}_{n_i}]$ is the inverse of a Vandermonde matrix of order n_i and of regularly distributed zeros on the $[0, +1]$ interval. Substituting (26) in (23), the interpolant becomes:

$$\begin{aligned} \phi(L_1, L_2, L_3) = {} & (1 - L_1)\{\phi^2\}[\mathbf{P}_{n_2}]^T\{1 s_2 \ldots s_2^{n_2 - 1}\}^T \\ & + (1 - L_2)\{\phi^3\}[\mathbf{P}_{n_3}]^T\{1 s_3 \ldots s_3^{n_3 - 1}\}^T \\ & + (1 - L_3)\{\phi^1\}[\mathbf{P}_{n_1}]^T\{1 s_1 \ldots s_1^{n_1 - 1}\}^T - L_1\phi(1, 0, 0) \\ & - L_2\phi(0, 1, 0) - L_3\phi(0, 0, 1) \end{aligned} \tag{28}$$

or in another form:

$$\begin{aligned} \phi(L_1, L_2, L_3) = {} & \{1 L_1\}[\mathbf{P}_2]\begin{bmatrix} \phi_1^2 & \phi_2^2 & \ldots & \phi_{n_2}^2 \\ 0 & 0 & \ldots & 0 \end{bmatrix}[\mathbf{P}_{n_2}]^T\{1 s_2 \ldots s_2^{n_2 - 1}\}^T \\ & + \{1 L_2\}[\mathbf{P}_2]\begin{bmatrix} \phi_1^3 & \phi_2^3 & \ldots & \phi_{n_3}^3 \\ 0 & 0 & \ldots & 0 \end{bmatrix}[\mathbf{P}_{n_3}]^T\{1 s_3 \ldots s_3^{n_3 - 1}\}^T \\ & + \{1 L_3\}[\mathbf{P}_2]\begin{bmatrix} \phi_1^1 & \phi_2^1 & \ldots & \phi_{n_1}^1 \\ 0 & 0 & \ldots & 0 \end{bmatrix}[\mathbf{P}_{n_1}]^T\{1 s_1 \ldots s_1^{n_1 - 1}\}^T \\ & - L_1\phi(1, 0, 0) - L_2\phi(0, 1, 0) - L_3\phi(0, 0, 1) \end{aligned} \tag{29}$$

where:

$$\Phi_j^i = F\Big|_{s_i = \frac{j-1}{n_i - 1}} \tag{30}$$

with the s_i variables expressed in area coordinates as:

$$s_1 = \frac{L_2}{L_1 + L_2}$$

$$s_2 = \frac{L_3}{L_2 + L_3}$$

$$s_3 = \frac{L_1}{L_3 + L_1} \tag{31}$$

which has the effect of constraining end-segment zeros of Lagrange polynomials to remain on $\partial \mathcal{T}$ as a blending on $\bar{\mathcal{T}}$ is carried out.

Equation (29) is similar in form to the interpolation function of a general rectangular mid-edge element and is valid for a general triangular mid-edge element with variable number of nodes on edges and for $\alpha = 1$ (i.e. linear blending).

3. SERENDIPITY SHAPE FUNCTIONS OF TRIANGULAR MID-EDGE ELEMENTS

The similarity between (29) and (1) suggests that inverse of Vandermonde matrices can be assigned to the edges of triangular mid-edge elements to generate shape functions, as was the case for rectangular mid-edge elements. Once again, the coefficient matrix must be of the same order as the number of nodes on a given edge (including vertex nodes). In the present case, coefficient matrices are obtained by inverting Vandermonde matrices of zeros of polynomial on the [0, +1] interval. Some of the coefficient matrices are listed in the Appendix. For each node, there is thus a corresponding column in the coefficient matrix. Elements in a given column are the coefficients of a polynomial expansion in the variable of the ξ_i-axis coinciding with the

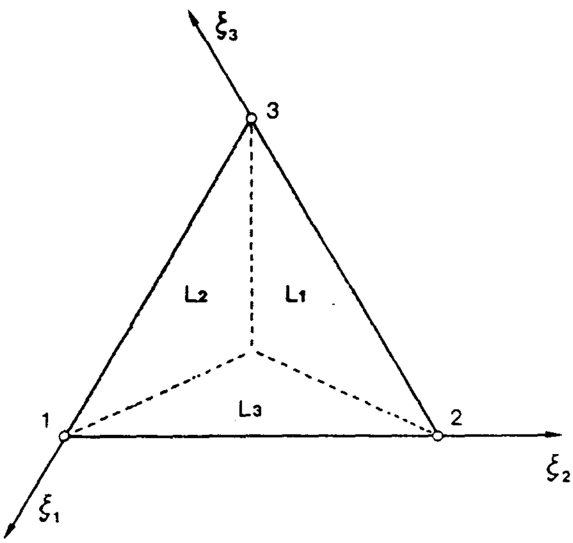


Fig. 5. Constant strain triangular element.

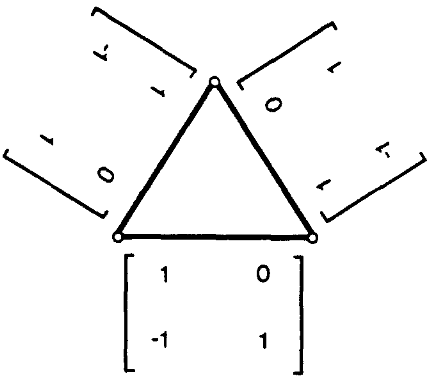


Fig. 6. Assignment of matrices around the CST element.

edge. When multiplied by a suitable blending function, the polynomial expansion becomes the shape function for the node facing the matrix column. Note that each coefficient matrix has a column increase in the direction of the ξ_i-axis (see on Fig. 6). Two matrix columns are assigned to vertex nodes and a term arising from the Boolean nature of the interpolation function must be subtracted as in the case of the rectangle. The blending function for nodes on the $L_i = 0$ edge is:

$$(1 - L_i)^{\alpha} \tag{32}$$

whereas for vertex nodes with coordinate $L_i = 1$, the term to be subtracted is simply given by:

$$L_i^{\alpha} \tag{33}$$

As a first example, we shall derive shape functions of the constant strain triangular element shown in Fig. 5. Coefficient matrices are first assigned to every edge as in Fig. 6. Using linear

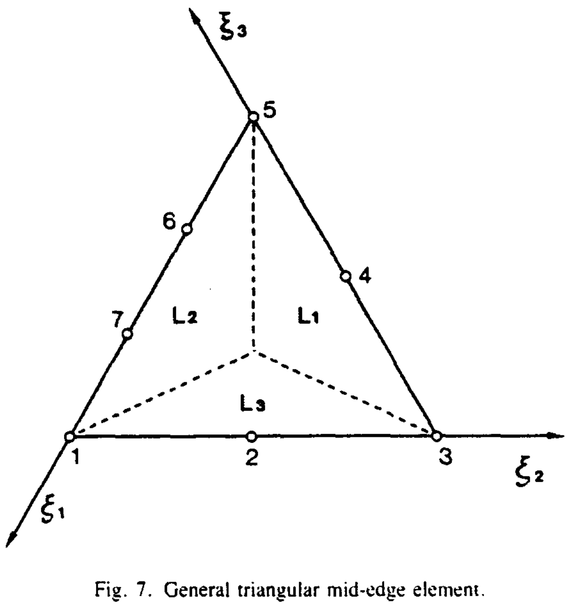


Fig. 7. General triangular mid-edge element.

blending functions, shape functions are simply obtained as:

$$
\begin{aligned}
N_1 &= (1 - L_3)\left[1 - 1\left(\frac{L_2}{1 - L_3}\right)\right] + (1 - L_2)\left[0 + 1\left(\frac{L_1}{1 - L_2}\right)\right] - L_1 \\
&= 1 - L_3 - L_2 + L_1 - L_1 = L_1 \\
N_2 &= (1 - L_3)\left[0 + 1\left(\frac{L_2}{1 - L_3}\right)\right] + (1 - L_1)\left[1 - 1\left(\frac{L_3}{1 - L_1}\right)\right] - L_2 \\
&= L_2 + 1 - L_1 - L_3 - L_2 = L_2 \\
N_3 &= (1 - L_1)\left[0 + \left(\frac{L_3}{1 - L_1}\right)\right] + (1 - L_2)\left[1 - 1\left(\frac{L_1}{1 - L_2}\right)\right] - L_3 \\
&= L_3 + 1 - L_2 - L_1 - L_3 = L_3
\end{aligned}
\tag{34}
$$

As a second example, we shall consider the case of a general triangular mid-edge element as shown in Fig. 7. Coefficient matrices are first assigned to every edge as in Fig. 8. Shape functions are obtained using quadratic blending functions for degree two precision as:

$$
\begin{aligned}
N_1 &= (1 - L_3)^2\left[1 - 3\left(\frac{L_2}{1 - L_3}\right) + 2\left(\frac{L_2}{1 - L_3}\right)^2\right] \\
&\quad + (1 - L_2)^2\frac{1}{2}\left[0 + 2\left(\frac{L_1}{1 - L_2}\right) - 9\left(\frac{L_1}{1 - L_2}\right)^2 + 9\left(\frac{L_1}{1 - L_2}\right)^3\right] - L_1^2 \\
N_2 &= (1 - L_3)^2\left[0 + 4\left(\frac{L_2}{1 - L_3}\right) - 4\left(\frac{L_2}{1 - L_3}\right)^2\right] \\
N_3 &= (1 - L_3)^2\left[0 - 1\left(\frac{L_2}{1 - L_3}\right) + 2\left(\frac{L_2}{1 - L_3}\right)^2\right] \\
&\quad + (1 - L_1)^2\left[1 - 3\left(\frac{L_3}{1 - L_1}\right) + 2\left(\frac{L_3}{1 - L_1}\right)^2\right] - L_3^2 \\
N_4 &= (1 - L_1)^2\left[0 + 4\left(\frac{L_3}{1 - L_1}\right) - 4\left(\frac{L_3}{1 - L_1}\right)^2\right] \\
N_5 &= (1 - L_1)^2\left[0 - 1\left(\frac{L_3}{1 - L_1}\right) + \left(\frac{L_3}{1 - L_1}\right)^2\right] \\
&\quad + (1 - L_2)^2\frac{1}{2}\left[2 - 11\left(\frac{L_1}{1 - L_2}\right) + 18\left(\frac{L_1}{1 - L_2}\right)^2 - 9\left(\frac{L_1}{1 - L_2}\right)^3\right] - L_3^2 \\
N_6 &= (1 - L_2)^2\frac{1}{2}\left[0 + 18\left(\frac{L_1}{1 - L_2}\right) - 45\left(\frac{L_1}{1 - L_2}\right)^2 + 27\left(\frac{L_1}{1 - L_2}\right)^3\right] \\
N_7 &= (1 - L_2)^2\frac{1}{2}\left[0 - 9\left(\frac{L_1}{1 - L_2}\right) + 36\left(\frac{L_1}{1 - L_2}\right)^2 - 27\left(\frac{L_1}{1 - L_2}\right)^3\right].
\end{aligned}
\tag{35}
$$

CONCLUSION

The interpolation function of a general triangular mid-edge element has been derived. As for the case of a general rectangular mid-edge element, a straightforward method for obtaining serendipity shape functions of a general triangular mid-edge element arises from the interpolation function developed. Storage in "potential" of various finite element nodal configurations can be achieved using the method of construction of serendipity shape functions illustrated, i.e. only the inverse of Vandermonde matrices, which are generated once during the initial steps of an analysis, need to be kept in memory for use with any element nodal configuration.

Acknowledgements—This research was supported by the Natural Sciences and Engineering Research Council of Canada.

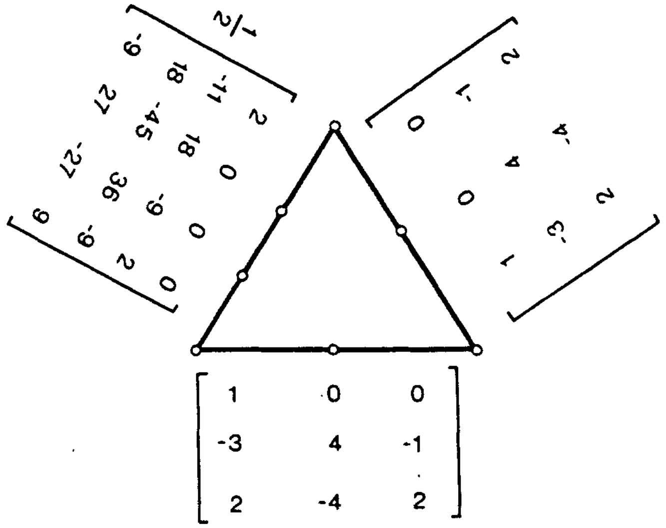


Fig. 8. Assignment of matrices around the general element.

REFERENCES

1. O. C. Zienkiewicz, *The Finite Element Method*, 3rd edn. McGraw-Hill, London (1977).
2. R. L. Taylor, On completeness of shape functions for finite element analysis. *Int. J. Num. Meth. Engng* **4**, 17–22 (1972).
3. M. Okabe, Complete Lagrange family for the cube in finite element interpolations. *Comput. Meths, Appl. Mech. Engng.* **29**, 51–66 (1981).
4. W. J. Gordon and C. A. Hall, Transfinite element methods: blending-function interpolation over arbitrary curved element-domains. *Numer. Mat.* **21**, 109–129 (1973).
5. M. Zlamal, A remark on the "serendipity family." *Int. J. Num. Meth. Engng.* **7**, 98–100 (1973).
6. A. A. Ball, The interpolation function of a general serendipity rectangular element. *Int. J. Num. Meth. Engng.* **15**, 773–778 (1980).
7. J. A. Marshall, *Some Applications of Blending Function Techniques in Finite Element Methods*, Ph.D. thesis, University of Dundee, 1975.
8. G. M. Nielson, The side-vertex method for interpolation in triangles, *J. Approximation Theory* **25**, 318–336 (1979).
9. W. J. Gordon, Distributive lattices and the approximation of multivariate functions, in *Approximation with Special Emphasis on Spline Functions*, I. J. Schoenberg ed., 223–277, Academic Press, New York (1969).

APPENDIX

$$[P_2] = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$[P_3] = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 4 & -1 \\ 2 & -4 & 2 \end{bmatrix}$$

$$[P_4] = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 0 \\ -11 & 18 & -9 & 2 \\ 18 & -45 & 36 & -9 \\ -9 & 27 & -27 & 9 \end{bmatrix}$$

$$[P_5] = \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ -25 & 48 & -36 & 16 & -3 \\ 70 & -208 & 228 & -112 & 22 \\ -80 & 288 & -384 & 224 & -48 \\ 32 & -128 & 192 & -128 & 32 \end{bmatrix}$$

$$[P_6] = \frac{1}{24}\begin{bmatrix} 24 & 0 & 0 & 0 & 0 & 0 \\ -274 & 600 & -600 & 400 & -150 & 24 \\ 1125 & -3850 & 5350 & -3900 & 1525 & -250 \\ -2125 & 8875 & -14750 & 12250 & -5125 & 875 \\ 1875 & -8750 & 16250 & -15000 & 6875 & -1250 \\ -625 & 3125 & -6250 & 6250 & -3125 & 625 \end{bmatrix}$$