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Abstract

In this study, Betti’s reciprocal theorem and the principle of superposition are used to obtain weight functions in a two-dimensional bi-material interface crack system for any loading, in general and thermal loading, in particular. It is shown that the general expression of weight functions for bi-materials interface crack problems is of the same type as that found in a homogeneous mixed mode loading case. Furthermore, a computational approach has been developed for calculation of thermal stress intensity factors for bi-material interface cracks subjected to thermal loading under quasi-static uncoupled thermo-elasticity assumption. The thermal weight function (TWF) expression and computational scheme have been validated using three examples given in the available literature.

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1. Introduction

Recent increase in demand for durable materials that can cope with extreme thermal environmental conditions are leading to the development of composites that combine two or more materials. Industrial applications such as turbines, combustion chambers, multi-layered electronic packaging structures and nuclear reactors are few such examples which are subjected to transient thermal loads during their service life and need special protection from the extreme thermal environments. Since the strength of thermally stressed structures composed of dissimilar media is significantly influenced by the existence of interfacial cracks, the accurate evaluation of fracture parameters is important.

In a homogeneous elastic body, the local character of thermal stresses at the crack tip has a square root singularity \( r^{-1/2} \), which is the same as for mechanical stresses (Sih, 1962), where \( r \) is the distance from the crack tip. For a thermally insulated crack, it has been proved that the temperature behaves as \( r^{1/2} \) while heat flux as \( r^{-1/2} \) near the crack tip (Chen and Ting, 1985). In dissimilar cracked bodies with an interface crack, the geometric and material discontinuities produce complex stress intensity in the form of an oscillatory singular function, \( r^{-1/2+i\epsilon} \), near the crack tip, where \( \epsilon \) is bi-material constant. As a result, stresses in general behave in an oscillatory manner as the crack tip is approached and being bounded by \( r^{-1/2} \) (Munz and Yang, 1992).

Analytical solutions for thermo-elastic crack problems are available only for a few cases (Sih, 1962; Florence and Goodier, 1960; Olesiak and Sneddon, 1959; Brown and Erdogan, 1968; Kassir and Sih, 1968; Kassir and Sih, 1971; Lee and Shul, 1991), wherein the crack is assumed to be contained in an infinite media under special thermal load conditions. It is difficult to obtain analytical solutions for thermal load cases for most of the finite dimension problems. Finite dimension crack problem of homogeneous bodies (Wilson and Yu, 1979; Hellen and Cesari, 1979; Emmel and Stamm, 1985) and bi-material interface
(Wilson and Meguid, 1995; Yuuki and Cho, 1989; Sun and Jih, 1987; Sun and Quin, 1997; Sun and Ikeda, 2001; Banks-Sills and Dolev, 2004) have been treated numerically using the finite element method.

In case of homogeneous bodies, the approach used in most of the available literature may be categorized as direct methods. These direct methods, in general suffer from a common disadvantage, that it requires repeated finite element analysis in order to obtain the mechanical stress field for varied thermal loading. Alternatively, thermal weight function (TWF) approach (Tsai and Ma, 1992) is best suited to deal with the problems of varied thermal loadings on homogeneous bodies.

Thermal weight function (WF) approach is an extension of the Bueckner’s WF (Bueckner, 1970) for thermal loading, which in turn is based on the principle of superposition. Bueckner’s WF serves as a Green’s function for determining the linear elastic crack tip opening mode (Mode I) stress intensity factor ($K_1$) for any arbitrary distribution of crack face traction and proves to be an easy and computationally efficient method.

Further, WFs are universal functions for a given crack configuration and body geometry. They are interpreted as the displacements which satisfy all the equations of linear elasticity except the concept of uniqueness which distinguishes them from some other displacement fields. The WFs associated with a particular crack tip give rise to unbounded energy in any finite area surrounding that crack tip. The stress fields generated by the WFs are self-equilibrating and have no body forces, produce zero tractions on the crack faces as well as the external boundary of the cracked body. The field of displacements, strains and stresses are also referred to as a fundamental field (Bueckner, 1970).

Rice (1972) defined the WF as a vector field $h$ and applicable throughout the cracked homogeneous body, for an arbitrary and symmetrically prescribed surface tractions $t$ and body forces $f$, such that

$$K_1 = \int_{S} t \cdot h dS + \int_{V} f \cdot h dV.$$

(1)

Here, $V$ is the volume of the body and $S$ is the bounding surface of $V$. Furthermore, Rice (1972) has shown that the WF could easily be determined if the displacement field $u^1$ and stress intensity factor $K_1^1$ are known as a function of crack length $a$, for any particular symmetric reference load system, say $t = t^1$ on $S$. Then

$$h = h(x, y, a) = \frac{H}{2K_1} \frac{\partial u^1}{\partial a},$$

(2)

where $H$ is the appropriate elastic modulus defined as

$$H = \begin{cases} E & \text{for plane stress,} \\ \frac{E}{(1-\nu^2)} & \text{for plane strain}, \end{cases}$$

(3)

where $E$ is Young’s modulus and $\nu$ is Poisson’s ratio.

The concept of WF was extended to three-dimension configurations by Rice (1972) and Bueckner (1973) independently. Since then other works by Paris and McMeeking (1975), Vanderglas (1978), Parks and Kamenetzky (1979), Bortman and Banks-Sills (1983), Rice (1985, 1988), Sham (1987), Bueckner (1987), Sham and Zhou (1989) and Wu and Carlsson (1991) have been done on the generalization and extension of WF for two-dimension and three-dimension problems including mixed mode cases and finite bodies.

Sham (1987) modified the expression given by Rice (1972) to incorporate the prescribed displacements and defined the stress intensity factors as

$$K_b = \int_{S_1} t \cdot h_b dS_1 - \int_{S_u} u \cdot t_b dS_u + \int_{V} f \cdot h_b dV,$$

(4)

where $S_1$ is the bounding surface with prescribed tractions $t$ and $S_u$ is the bounding surface with prescribed displacement $u$, and $t_b$ is the traction generated by the WF $h_b$ on $S_u$ and subscript $b = I, II$ and III denotes Modes I, II and III, respectively.

Using Duhamel-Neumann, initial stress analogies and mechanical WF for a given crack configuration, Heaton (1976) has shown that the stress intensity factors could be computed by applying to the crack surfaces, the stress field that exists in an un-cracked geometry with zero body forces, prescribed tractions and prescribed displacements, provided the thermal and residual field remain elastic upon introduction of crack in the body.

It is to be noted here that the approach proposed by Heaton, using mechanical WF cannot be treated as Green function or WF approach in context to SIF calculations due to thermal loadings, since it requires repeated stress analysis to obtain the tractions along the crack line for the uncracked geometry, thereby making this procedure computationally expensive. Also Heaton’s approach works on the assumption that the temperature field is un-affected by the presence of a crack, which in many cases is not true.

The Thermal weight function presented by Tsai and Ma (1992) is an universal function like Bueckner’s WF for mechanical loads and depends only on the crack configuration and body geometry and is independent of thermal loading. Further, since the TWF is independent of time during thermal shock, the whole variation of transient SIFs can be directly computed through integration of the products of TWF and the transient temperature fields (Lu et al., 2004). The repeated determinations of the stress (or displacement) field distributions for individual time instants are thus avoided in the TWF method resulting in savings on computational time. The modified expression for SIFs as given by Tsai and Ma is
\[ K_b = \int_{S_b} \mathbf{t} \cdot \mathbf{h}_b \mathrm{d}S - \int_{S_b} \mathbf{u} \cdot \mathbf{t}_b \mathrm{d}S + \int_Y \mathbf{f} \cdot \mathbf{h}_b \mathrm{d}V + \int_Y \mathbf{h}_b^T \mathrm{d}V. \]  

(5)

where \( T \) is the temperature distribution in the body, \( h_b^T \) is the homogeneous TWF for a given crack configuration and body geometry, defined by

\[ h^T = \frac{\alpha H}{2K_1} \frac{\partial \sigma_{kk}^1}{\partial a}. \]  

(6)

Here, \( \alpha \) is the coefficient of thermal expansion, \( \sigma_{kk}^1 \) is the sum of normal stresses of the reference or known loading system whose stress intensity factor \( K_1^b \) is also known.

Lu et al. (2001), Lu et al. (2004) have proposed a generalized finite element technique of TWF method to calculate the transient stress intensity factors for a three-dimension body subjected to thermal shock in a homogeneous material system.

In the work done on TWFs, certain assumptions have been followed by researchers in their work.

1. The transient thermal stress problem is considered to be linear, decoupled quasi-static process which defines at each time instant, a thermo-elastic equilibrium problem depending only on current temperature and the initial state (Salencon, 2001).
2. Crack face contact does not occur, since this leads to non-linear conditions wherein the superposition principle is not valid.
3. All thermo-elastic coupling effects and the temperature dependence of the thermo-elastic constants are neglected.

Coming to a bi-material cracked body, subjected to mechanical or thermal loads, an oscillatory type of singularity is found to exist at the tip of the crack along the interface. Because of this oscillatory singularity and complexities involved, TWF for thermal stress fields in an elastic solid containing a crack along the interface of two dissimilar bodies is still missing from the literature.

Weight function for a bi-material body subjected to tractions was presented by Banks-Sills (1993). However, this WF approach is not applicable for bi-material bodies with body force or thermal load. Banks-Sills et al. (1997) have extended the (Heaton, 1976) approach involving mechanical WF, to determine the stress intensity factors arising from residual thermal stresses in a bi-material interface crack finite body. But, as in the case of homogeneous bodies, it also requires repeated stress analysis to obtain the tractions along the crack line for the uncoupled geometry and works under the assumption that the crack does not affect the temperature distribution.

In this study, using the Betti’s reciprocal theorem and the principle of superposition, the WFs in a two-dimensional bi-material interface crack system are determined, for any loading in general and thermal loading, in particular. It is shown that the general expression of WFs for bi-materials interface crack problems is of the same type as that found in a homogeneous mixed mode loading case. Furthermore, assuming thermo-elastic problems as decoupled, quasi-static (Nowacki, 1962) and neglecting the temperature dependence of the thermo-elastic constants, the present approach is validated through one steady-state example and two transient thermo-elastic examples given in the literature.

2. Formulation

Consider geometrically equivalent cracked bodies with three different equilibrium configurations as shown in Fig. 1. These configurations are subjected to prescribed traction \( t' \) on boundary \( S_c \) including the crack surface, prescribed displacement \( u' \) on the boundary \( S_b \), body force \( f' \) and temperature \( T' \) in domain \( A \) and designated as case \( (m) \), \( (n) \) and \( (p) \), respectively. The separation of the external boundary \( S_s \) into \( S_c \) and \( S_b \) remains the same in all cases and the value of traction and displacement constraints are specified differently. The cases \( (m) \) and \( (n) \) are known referential loading systems wherein the thermal load, \( T' \), can be taken to be zero and case \( (p) \) is our unknown system for which the SIFs in terms of WFs are sought. Now considering cases \( (m) \) and \( (p) \) as shown in Fig. 1.

Using Betti’s reciprocal theorem for the geometry with crack length \( a \),

\[ \int_{S_b} \mathbf{t}^{(m)} \cdot \mathbf{u}^{(p)}(a) \mathrm{d}S - \int_{S_b} \mathbf{u}^{(m)} \cdot \mathbf{t}^{(p)}(a) \mathrm{d}S + \int_A \mathbf{f}^{(m)} \cdot \mathbf{u}^{(p)}(a) \mathrm{d}A + \sum_{k=1}^{2} \int_{A_k} \alpha_k T^{(m)} \sigma_{kk}^{(p)}(a) \mathrm{d}A \]

\[ = \int_{S_b} \mathbf{t}^{(p)}(a) \cdot \mathbf{u}^{(m)}(a) \mathrm{d}S - \int_{S_b} \mathbf{u}^{(p)}(a) \cdot \mathbf{t}^{(m)}(a) \mathrm{d}S + \int_A \mathbf{f}^{(p)}(a) \cdot \mathbf{u}^{(m)}(a) \mathrm{d}A \]

\[ + \sum_{k=1}^{2} \int_{A_k} \alpha_k T^{(p)}(a) \sigma_{kk}^{(m)}(a) \mathrm{d}A. \]

(7)

where \( \alpha \) is the coefficient of the thermal expansion, \( A \) is the area of the body surrounded by the boundary \( S, S = S_c + S_b \). Also, \( \sigma_{kk} \) is the sum of the normal stress, \( \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \). Now considering same set of cases \( (m) \) and \( (p) \) in Fig. 2, where the crack faces have a virtual crack extension \( \Delta \alpha \). The original stress at position on \( \Delta \alpha \) in case ‘m’ can now be treated as a part of traction \( t^{(m)} \) for the new system. On \( \Delta a \), an equilibrating traction is applied to the upper and lower crack faces, respectively, to nullify the relative displacement along \( \Delta a \), i.e.
Let $S_{\Delta a}$ denote the boundary of the extended crack surface $\Delta a$ and applying the reciprocal theorem to the geometry with crack length $(a + \Delta a)$, we obtain

$$
\delta u = \delta u^m + i\delta u^m = 0.
$$

(8)

Let $S_{\Delta a}$ denote the boundary of the extended crack surface $\Delta a$ and applying the reciprocal theorem to the geometry with crack length $(a + \Delta a)$, we obtain

$$
\begin{align*}
&\int_{S_{\Delta a}} \mathbf{t}^{(m)} \cdot \mathbf{u}^{(p)}(a + \Delta a) dS - \int_{S_{\Delta a}} \mathbf{u}^{(m)} \cdot \mathbf{t}^{(p)}(a + \Delta a) dS \\
&\quad + \int_A \mathbf{t}^{(m)} \cdot \mathbf{u}^{(p)}(a + \Delta a) dA \\
&\quad + \sum_{k=1}^2 \int_{A_k} \alpha_2 T^{(m)} \sigma^{(p)}_{E_k}(a + \Delta a) dA \\
&= \int_{S_{\Delta a}} \mathbf{t}^{(p)}(a + \Delta a) \cdot \mathbf{u}^{(m)}(a) dS - \int_{S_{\Delta a}} \mathbf{u}^{(p)}(a + \Delta a) \cdot \mathbf{t}^{(m)}(a) dS + \int_A \mathbf{t}^{(p)}(a + \Delta a) \cdot \mathbf{u}^{(m)}(a) dA \\
&\quad + \sum_{k=1}^2 \int_{A_k} \alpha_1 T^{(p)}(a + \Delta a) \sigma^{(p)}_{E_k}(a) dA.
\end{align*}
$$

(9)
Applying Taylor expansion to the above equation, we obtain

\[
\int_{S_0} \mathbf{t}^{(m)} \cdot \left[ \mathbf{u}^{(p)}(a) + \frac{\partial \mathbf{u}^{(p)}}{\partial a} \Delta a \right] dS + \int_{S_0} \mathbf{t}^{(m)} \cdot \left[ \mathbf{u}^{(p)}(a) + \frac{\partial \mathbf{u}^{(p)}}{\partial a} \Delta a \right] dS - \int_{S_0} \mathbf{t}^{(m)} \cdot \left[ \mathbf{u}^{(p)}(a) + \frac{\partial \mathbf{u}^{(p)}}{\partial a} \Delta a \right] dS
\]

\[
+ \int_A \mathbf{t}^{(m)} \cdot \left[ \mathbf{u}^{(p)}(a) + \frac{\partial \mathbf{u}^{(p)}}{\partial a} \Delta a \right] dA + \sum_{k=1}^2 \int_{\mathcal{A}_k} \alpha_k \mathbf{T}^{(m)} \left[ \sigma^{(p)}_{ik} (a) + \frac{\partial \sigma^{(p)}_{ik}}{\partial a} \Delta a \right] dA
\]

\[
= \int_{S_0} \mathbf{t}^{(p)}(a) + \frac{\partial \mathbf{t}^{(p)}}{\partial a} \Delta a \cdot \mathbf{u}^{(m)}(a) dS - \int_{S_0} \mathbf{t}^{(m)}(a) + \frac{\partial \mathbf{t}^{(m)}}{\partial a} \Delta a \cdot \mathbf{u}^{(p)}(a) dS
\]

\[
+ \int_A \mathbf{t}^{(p)}(a) + \frac{\partial \mathbf{t}^{(p)}}{\partial a} \Delta a \cdot \mathbf{u}^{(m)}(a) dA + \sum_{k=1}^2 \int_{\mathcal{A}_k} \alpha_k \mathbf{T}^{(m)}(a) + \frac{\partial \mathbf{T}^{(m)}}{\partial a} \Delta a \cdot \sigma^{(m)}_{ik} (a) dA.
\]

Differentiating Eq. (7) with respect to crack length \(a\), we get

\[
\int_{S_0} \mathbf{t}^{(m)} \cdot \frac{\partial \mathbf{u}^{(p)}}{\partial a} dS - \int_{S_0} \mathbf{u}^{(p)} \cdot \frac{\partial \mathbf{t}^{(m)}}{\partial a} dS + \int_A \mathbf{t}^{(m)} \cdot \frac{\partial \mathbf{u}^{(p)}}{\partial a} dA + \sum_{k=1}^2 \int_{\mathcal{A}_k} \alpha_k \mathbf{T}^{(m)} \cdot \frac{\partial \sigma^{(p)}_{ik}}{\partial a} dA
\]

\[
= \int_{S_0} \left[ \mathbf{t}^{(p)}(a) + \frac{\partial \mathbf{t}^{(p)}}{\partial a} \right] \cdot \mathbf{u}^{(m)}(a) dS - \int_{S_0} \left[ \mathbf{t}^{(m)}(a) + \frac{\partial \mathbf{t}^{(m)}}{\partial a} \right] \cdot \mathbf{u}^{(p)}(a) dS
\]

\[
+ \int_A \left[ \mathbf{r}^{(p)}(a) + \frac{\partial \mathbf{r}^{(p)}}{\partial a} \right] \cdot \mathbf{u}^{(m)}(a) dA + \sum_{k=1}^2 \int_{\mathcal{A}_k} \alpha_k \left[ \mathbf{T}^{(m)}(a) + \frac{\partial \mathbf{T}^{(m)}}{\partial a} \right] \cdot \sigma^{(m)}_{ik} (a) dA.
\]

Adding Eqs. (7) and (11) and subtracting the result from Eq. (10) yields

\[
- \frac{1}{\Delta a} \int_{S_0} \mathbf{t}^{(m)} \cdot \mathbf{u}^{(p)} dS = \int_{S_0} \mathbf{t}^{(p)} \cdot \frac{\partial \mathbf{u}^{(m)}}{\partial a} dS - \int_{S_0} \mathbf{u}^{(p)} \cdot \frac{\partial \mathbf{t}^{(m)}}{\partial a} dS + \int_A \mathbf{t}^{(m)} \cdot \frac{\partial \mathbf{u}^{(m)}}{\partial a} dA
\]

\[
+ \sum_{k=1}^2 \int_{\mathcal{A}_k} \alpha_k \mathbf{T}^{(m)} \cdot \frac{\partial \sigma^{(m)}_{ik}}{\partial a} dA.
\]

The stresses, \(\sigma_{yy}\) and \(\sigma_{xy}\), at the interface directly ahead of the crack tip, at \(\theta = 0\) are given by

\[
(\sigma)_{y=0} = (\sigma_{yy} + i\sigma_{xy})_{y=0} = \frac{K_1 + iK_2}{\sqrt{2\pi r}} \left( \frac{r}{l_b} \right)^i,
\]

where \(i = \sqrt{-1}\), \(l_b\) is a length parameter and \(\varepsilon\) denotes the bi-material constant defined as

\[
\varepsilon = \frac{1}{2\pi} \log \left( \frac{k_1\mu_2 + k_2\mu_1}{k_2\mu_1 + k_1\mu_2} \right),
\]

where \(\mu_j\) is the shear modulus of material \(j\), \(k_j = (3 - \nu_j)/(1 + \nu_j)\) for plane stress and \(k_j = 3 - 4\nu_j\) for plane strain and \(\nu_j\) is Poisson’s ratio of material \(j\) of the bi-material body.

Also, the associated crack surface displacement, \(u_x\) and \(u_y\), at a distance \(r\) behind the crack tip \((\theta = \pm \pi)\), are given by

\[
\delta u = \delta u_y + i\delta u_x = \frac{(K_1 + iK_2) \left( \frac{1 + \alpha_1}{\mu_1} + \frac{1 + \alpha_2}{\mu_2} \right)}{2(1 + 2\alpha)} \left( \frac{1}{l_b} \right)^i \sqrt{\frac{r}{2\pi}} \left( \frac{1}{l_b} \right)^i \sqrt{\frac{r}{2\pi}}.
\]

Using Eqs. (13) and (15) and taking the limit as \(\Delta a \to 0\), the left-hand side of Eq. (12) reduces to

\[
- \lim_{\Delta a \to 0} \frac{1}{\Delta a} \int_{S_0} \mathbf{t}^{(m)} \cdot \mathbf{u}^{(p)} dS
\]

\[
= \text{Re} \left[ \lim_{\Delta a \to 0} \frac{1}{\Delta a} \int_{S_0} (\sigma_{yy} + i\sigma_{xy}) (\Delta a - r) \cdot (u_y - iu_x)^{(p)} (r) \right]
\]

\[
= \lim_{\Delta a \to 0} \frac{1}{\Delta a} \int_0^\Delta \left[ \sigma_{yy}^{(p)} (\Delta a - r) u_y^{(p)} (r) + \sigma_{xy}^{(p)} (\Delta a - r) u_x^{(p)} (r) \right] dr
\]

\[
= \text{Re} \left[ \int \left[ (K_1 - K_2) + i(K_1 + K_2) \right] \left( \frac{1 + \alpha_1}{\mu_1} + \frac{1 + \alpha_2}{\mu_2} \right) \frac{1}{\text{cosh} \pi \epsilon} \sqrt{\frac{1 - \xi}{1 + 2\xi}} \int_0^1 \left( \frac{1 - t}{1 + t} \right)^{1/2-i} dt \right].
\]

The integral in the end of the above equation is recognized as the complex Beta function \(B(1/2 + i\epsilon, 3/2 - i\epsilon)\). Further, the orthogonality of the stress modes guarantees that they decouple such that there are no terms of the form \(K_1 K_2\) (Shih and Asaro, 1988). Upon evaluation of \(B\), we obtain

\[
- \lim_{\Delta a \to 0} \frac{1}{\Delta a} \int_{S_0} \mathbf{t}^{(m)} \cdot \mathbf{u}^{(p)} dS = \frac{2(K_1 - K_2) \left( \frac{1 + \alpha_1}{\mu_1} + \frac{1 + \alpha_2}{\mu_2} \right)}{16 \text{cosh}^2 \pi \epsilon} = \frac{2(K_1 + K_2) (K_1 - K_2)}{\text{H}}.
\]
where $H$ is bi-material constant and is given by

$$H = \frac{16\cosh^2 \pi e}{\left(\frac{1 - \nu_1}{\mu_1} + \frac{1 - \nu_2}{\mu_2}\right)}.$$  \hspace{1cm} (18)

From Eqs. (12) and (17), we can write

$$\frac{2}{H} (K_1^{(n)} K_1^{(p)} + K_2^{(n)} K_2^{(p)}) = \int_{S_1} t^{(p)} \cdot \frac{\partial u^{(m)}}{\partial a} \, dS - \int_{S_1} u^{(p)} \cdot \frac{\partial t^{(m)}}{\partial a} \, dS + \int_A f^{(p)} \cdot \frac{\partial u^{(m)}}{\partial a} \, dA + \sum_{k=1}^2 \int_{A_k} \alpha_k T^{(p)} \frac{\partial \sigma_{kk}^{(m)}}{\partial a} \, dA.\hspace{1cm} (19)$$

Now, considering the loading configurations (n) and (p) and using a similar procedure as above, we can write

$$\frac{2}{H} (K_1^{(n)} K_1^{(p)} + K_2^{(n)} K_2^{(p)}) = \int_{S_1} t^{(p)} \cdot \frac{\partial u^{(n)}}{\partial a} \, dS - \int_{S_1} u^{(p)} \cdot \frac{\partial t^{(n)}}{\partial a} \, dS + \int_A f^{(p)} \cdot \frac{\partial u^{(n)}}{\partial a} \, dA + \sum_{k=1}^2 \int_{A_k} \alpha_k T^{(p)} \frac{\partial \sigma_{kk}^{(n)}}{\partial a} \, dA.\hspace{1cm} (20)$$

Solving Eqs. (19) and (20) for the unknown values of stress intensity factors, $K_1^{(p)}$ and $K_2^{(p)}$ for loading configuration (p) we obtain

$$K_1 = \frac{H}{2K^2} \left[ \int_{S_1} t^{(p)} \cdot \left( K_2^{(n)} \frac{\partial u^{(m)}}{\partial a} - K_2^{(m)} \frac{\partial u^{(n)}}{\partial a} \right) \, dS - \int_{S_1} u^{(p)} \cdot \left( K_2^{(m)} \frac{\partial t^{(m)}}{\partial a} - K_2^{(m)} \frac{\partial t^{(n)}}{\partial a} \right) \, dS + \int_A f^{(p)} \cdot \left( K_2^{(n)} \frac{\partial u^{(m)}}{\partial a} - K_2^{(m)} \frac{\partial u^{(n)}}{\partial a} \right) \, dA + \sum_{k=1}^2 \int_{A_k} \alpha_k T^{(p)} \frac{\partial \sigma_{kk}^{(m)}}{\partial a} \, dA \right].\hspace{1cm} (21)$$

and

$$K_2 = \frac{H}{2K^2} \left[ \int_{S_1} t^{(p)} \cdot \left( K_1^{(m)} \frac{\partial u^{(n)}}{\partial a} - K_1^{(n)} \frac{\partial u^{(n)}}{\partial a} \right) \, dS - \int_{S_1} u^{(p)} \cdot \left( K_1^{(m)} \frac{\partial t^{(n)}}{\partial a} - K_1^{(n)} \frac{\partial t^{(n)}}{\partial a} \right) \, dS + \int_A f^{(p)} \cdot \left( K_1^{(m)} \frac{\partial u^{(n)}}{\partial a} - K_1^{(n)} \frac{\partial u^{(n)}}{\partial a} \right) \, dA + \sum_{k=1}^2 \int_{A_k} \alpha_k T^{(p)} \frac{\partial \sigma_{kk}^{(n)}}{\partial a} \, dA \right].\hspace{1cm} (22)$$

where

$$K^2 = K_1^{(m)} K_1^{(n)} - K_2^{(m)} K_2^{(n)} \neq 0.\hspace{1cm} (23)$$

Eqs. (21) and (22) can be represented in terms of WFs as

$$K_p = \int_{S_1} t^{(p)} \cdot h_0^{(n)} \, dS - \int_{S_1} u^{(p)} \cdot h_0^{(m)} \, dS + \int_A f^{(p)} \cdot h_0^{(n)} \, dA + \sum_{k=1}^2 \int_{A_k} T^{(p)} (h_0^{(n)}) \, dA.\hspace{1cm} (24)$$

Eqs. (21), (22) and (24) yield different types of WF for bi-material interface crack systems namely, WFs for prescribed traction, $h_0^{(n)}$, WFs for prescribed displacement, $h_0^{(m)}$, WFs for body force, $h_0^{(n)}$, and WFs for thermal loading, $h_0^{(n)}$, which are given by

$$h_0^{(n)} = \frac{H}{2K^2} \left( K_2^{(n)} \frac{\partial u^{(m)}}{\partial a} - K_2^{(m)} \frac{\partial u^{(n)}}{\partial a} \right),$$

$$h_0^{(m)} = \frac{H}{2K^2} \left( K_1^{(m)} \frac{\partial u^{(n)}}{\partial a} - K_1^{(n)} \frac{\partial u^{(n)}}{\partial a} \right),$$

$$h_1^{(n)} = \frac{H}{2K^2} \left( K_2^{(n)} \frac{\partial t^{(m)}}{\partial a} - K_2^{(m)} \frac{\partial t^{(n)}}{\partial a} \right),$$

$$h_1^{(m)} = \frac{H}{2K^2} \left( K_1^{(m)} \frac{\partial t^{(n)}}{\partial a} - K_1^{(n)} \frac{\partial t^{(n)}}{\partial a} \right),$$

$$h_2^{(n)} = \frac{H}{2K^2} \left( K_2^{(n)} \frac{\partial u^{(m)}}{\partial a} - K_2^{(m)} \frac{\partial u^{(n)}}{\partial a} \right),$$

$$h_2^{(m)} = \frac{H}{2K^2} \left( K_1^{(m)} \frac{\partial u^{(n)}}{\partial a} - K_1^{(n)} \frac{\partial u^{(n)}}{\partial a} \right).$$
\( (h_1^*)_k = \frac{H}{2K^2} \left( K^{(n)}_i \frac{\partial \sigma_{il}^{(m)}}{\partial a} - K^{(m)}_2 \frac{\partial \sigma_{il}^{(n)}}{\partial a} \right). \)

\( (h_2^*)_k = \frac{H}{2K^2} \left( K^{(m)}_1 \frac{\partial \sigma_{il}^{(n)}}{\partial a} - K^{(n)}_1 \frac{\partial \sigma_{il}^{(m)}}{\partial a} \right). \)

The expression for prescribed traction WF given in Eq. (25) is same as the one given by Banks-Sills (1993). It should also be noted that the expression for body force WF (Eq. (27)) and traction WF (Eq. (25)) are the same in case of bi-material interface crack configuration system. When the bi-material constant \( \varepsilon = 0 \), then all the WFs get reduced to the expressions given by Tsai and Ma (1992) for homogeneous material.

If only thermal loading is considered, then the stress intensity factors under known temperature distribution can be obtained from Eq. (24) as

\[
K_i^T = \frac{H}{2K^2} \int_{A_h} \left[ T^T \left( K^{(n)}_2 \frac{\partial u^{(m)}}{\partial a} - K^{(m)}_2 \frac{\partial u^{(n)}}{\partial a} \right) - K^{(m)}_1 \frac{\partial u^{(n)}}{\partial a} + K^{(n)}_1 \frac{\partial u^{(m)}}{\partial a} \right] dA.
\]

3. Finite element implementation of weight function method

The WFs in Eq. (28) may be determined analytically for certain bodies. In the case of a finite body, however, a numerical procedure in most of the cases are more appropriate and is developed in this section. The thermal SIFs using the WFs for two-dimensional crack configuration in a bi-material body is formulated as shown by Eq. (29).

Since, \( \sigma_{il} \) is sensitive to the crack size, it is convenient to reduced it in terms of displacement field (Tsai and Ma, 1992) for implementation in a finite element code. For an auxiliary known field without any thermal load, \( \sigma_{il} \) can be simplified as

\[
\sigma_{il} = \left\{ \begin{array}{ll}
\frac{E}{1-\nu^2} \sigma_{il} & \text{for 3D cases,} \\
\frac{E}{2(1+\nu)} \left( \sigma_{il} + \frac{\nu}{1-\nu} \sigma_{ij} \right) & \text{for plane stress,} \\
\frac{E}{2} \left( \sigma_{il} + \frac{\nu}{1-\nu} \sigma_{ij} \right) & \text{for plane strain.}
\end{array} \right.
\]

Using the above expression of \( \sigma_{il} \) in Eq. (29), we obtain

\[
K_i^{(p)} = \frac{H}{2K^2} \sum_{k=1}^{2} \int_{A_h} \beta_k T^{(p)} \left\{ \left( K^{(n)}_2 \frac{\partial u^{(m)}}{\partial a} - K^{(m)}_2 \frac{\partial u^{(n)}}{\partial a} \right) + \left( K^{(m)}_1 \frac{\partial u^{(n)}}{\partial a} - K^{(n)}_1 \frac{\partial u^{(m)}}{\partial a} \right) \right\} dy \\
+ \int_{A_h} \beta_k \left( \frac{\partial T^{(p)}}{\partial x} \left( K^{(m)}_1 \frac{\partial u^{(n)}}{\partial a} - K^{(n)}_1 \frac{\partial u^{(m)}}{\partial a} \right) \right) dx \\
+ \int_{A_h} \beta_k \left( \frac{\partial T^{(p)}}{\partial y} \left( K^{(m)}_2 \frac{\partial u^{(n)}}{\partial a} - K^{(n)}_2 \frac{\partial u^{(m)}}{\partial a} \right) \right) dy,
\]

\[
K_i^{(s)} = \frac{H}{2K^2} \sum_{k=1}^{2} \int_{A_h} \beta_k T^{(s)} \left\{ \left( K^{(n)}_2 \frac{\partial u^{(m)}}{\partial a} - K^{(m)}_2 \frac{\partial u^{(n)}}{\partial a} \right) + \left( K^{(m)}_1 \frac{\partial u^{(n)}}{\partial a} - K^{(n)}_1 \frac{\partial u^{(m)}}{\partial a} \right) \right\} dy \\
+ \int_{A_h} \beta_k \left( \frac{\partial T^{(s)}}{\partial x} \left( K^{(m)}_1 \frac{\partial u^{(n)}}{\partial a} - K^{(n)}_1 \frac{\partial u^{(m)}}{\partial a} \right) \right) dx \\
+ \int_{A_h} \beta_k \left( \frac{\partial T^{(s)}}{\partial y} \left( K^{(m)}_2 \frac{\partial u^{(n)}}{\partial a} - K^{(n)}_2 \frac{\partial u^{(m)}}{\partial a} \right) \right) dy,
\]

where \( \beta \) is defined as

\[
\beta = \left\{ \begin{array}{ll}
\frac{E_k}{(1-2\nu_k)} & \text{for plane strain,} \\
\frac{E_k}{(1-\nu_k)} & \text{for plane stress.}
\end{array} \right.
\]
The first part of the above two equations with line integral sign indicates that the integration is carried out around the whole of outer boundary $S$, incorporating boundaries of both the materials and shows that the stress intensity factor is a function of the temperature. The second and last part of the integral is take over the whole of domain $A$ of the bi-material configuration system and is a function of temperature gradients.

The partial derivatives of displacements and temperature with respect to the coordinates $x$ and $y$ can be obtained using the shape functions of any given finite element. Also, the partial derivatives of the displacement with respect to crack length $a$ may be obtained using the stiffness derivative technique or virtual crack extension technique as shown in Lu et al. (2001).

4. Validation

To demonstrate the accuracy, stability and computational suitability of the proposed method, three illustrative example problems, one for constant temperature and the other two related to transient thermo-elastic state have been treated, and the results compared with solutions available in the literature.

In all these examples, it has been assumed that no contact of the crack faces occurs. Also the transient thermal problems are treated as uncoupled and quasi-static (Nowacki, 1962). Though, the time history of temperature distributions are obtained using a transient thermal analysis, the stress analysis is performed for a series of static cases applying corresponding temperature distribution. In other words, SIFs have been computed for this quasi-static case at different time instances applying corresponding temperature distribution. As a static stress analysis is performed for different time distribution, a constant WF is sufficient for computation of SIFs. For this purpose, both materials are assumed homogeneous, isotropic and linear elastic. It is also assumed that the bi-material structure is initially stress free.

Further, it should be mentioned here that although, both the decoupled stress intensity factors and displacement derivatives depend strongly on the reference loading system, the WF obtained according to Eq. (28) are indeed invariant with respect to loading conditions for a given crack geometry of pre-selected constrained conditions under different loading conditions. In principle, the use of two simple loadings such as pair of a concentrated point load acting on both side of the crack surface and far field tensile force acting on the outer edge of the crack configuration system in case of bi-material interface crack problem is sufficient, provided it satisfies Eq. (23), for explicit finite element determination of WFs, which depends on the geometry, composition, and constraint conditions but are independent of loading conditions.

The numerical implementation of Eqs. (31) and (32), given explicitly in Lu et al. (2001), has been incorporated into a locally developed finite element code. This code performs two-dimensional, linearly elastic analyses using the conventional displacement method of finite elements as well as two-dimensional steady and transient thermal analysis. Preprocessing inputs required for this code is obtained from ANSYS. For all the mechanical analysis in present case, eight-noded quadrilateral elements (PLANE82) are used so that the displacement components have quadratic expansions which make the stresses vary linearly within the elements. Further, eight-noded PLANE77 elements have been used in the thermal analysis. Here, in all the examples, the geometry and loading condition are symmetrical. Therefore, the reference loadings $m$ and $n$ are also taken to be symmetrical. Due to this symmetry, only one half of the geometry is modeled with same displacement constraint throughout the analysis.

4.1. Dissimilar semi-infinite plate with double edge crack subjected to uniform temperature change

A jointed dissimilar semi-infinite plate with two collinear semi-infinite cracks with crack tips at a distance $2a$ apart along the interface (Fig. 3) and subjected to uniform temperature change of 100 °C (Sun and Ikeda, 2001) is analyzed under plane strain conditions. Due to symmetrical nature of the plate and the thermal loading with respect to $y$-axis, only the left half is analyzed. The model along with the displacement boundary conditions are shown in Fig. 4. In order to simulate ‘semi-infinite’ body, the dimensions of the plate are taken to be 200 units by 400 units with jointed interface of 1 unit. The material properties used in the analysis are shown in Table 1.

The finite element model used in this example contained a total of 9821, eight-noded iso-parametric elements with 30,088 nodes. The same finite element model is used for thermal analysis to obtain the temperature distribution, displacement analysis of the fundamental reference loading systems used in order to get the stress intensity factors and for the displacement analysis of the fundamental reference loading systems used for computing the WFs.

The solution in terms of the stress intensity factors for this problem was given by Erdogan (1965) as

$$K_1 = -2i\sigma_0(\zeta_2\eta_2 - \zeta_1\eta_1)T\sqrt{\pi b}(2b)^{2i},$$

$$K_2 = \sigma_0(\zeta_2\eta_2 - \zeta_1\eta_1)T\sqrt{\pi b}(2b)^{2i},$$

where, $\eta_1 = 1$ for plane stress and $\eta_1 = 1 + \nu$ for plane strain case, $\sigma_0$ is defined by

$$\sigma_0 = \frac{4\mu_1\mu_2\cosh(\nu\pi)}{(\mu_1 + \mu_2 K_1 + \mu_1 K_2 + \mu_2)},$$

where $\zeta_1$ and $\zeta_2$ are the coefficients of linear thermal expansion for material 1 and 2, respectively. $T$ is the temperature excursion.
Let $K_0 = \sigma \sqrt{\pi b}$, where $\sigma = \sigma_0 (\alpha_2 \eta_2 - \alpha_1 \eta_1) T$. The normalized complex stress intensity factors is defined as

$$K = \frac{K L}{\sigma \sqrt{\pi a}},$$

where $L$ and $a$ are length parameters and $\sigma$ is stress. For the present case, normalized stress intensity factors with $L = 2a$ and $a = b$ can be written as

![Fig. 3. Semi-infinite bi-material plate subjected to constant uniform temperature load.](image)

![Fig. 4. Displacement boundary conditions (Example 1).](image)

### Table 1

Material properties used in the analysis

<table>
<thead>
<tr>
<th>Examples</th>
<th>1</th>
<th>2 and 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Material</td>
<td>1</td>
<td>Material 2</td>
</tr>
<tr>
<td>Young modulus (Pa)</td>
<td>1000e9</td>
<td>100e9</td>
</tr>
<tr>
<td>Poisson ratio</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>Thermal expansion coefficient (°C⁻¹)</td>
<td>$1.0 \times 10^{-6}$</td>
<td>$1.0 \times 10^{-7}$</td>
</tr>
<tr>
<td>Material density (kg/m⁻³)</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>Coefficient of heat conduction (W/m °C)</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>Specific heat of material (J/kg °C)</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>
As derived in the previous section, the computation of thermal stress intensity factors through Eqs. (31) and (32) requires the displacement fields and stress intensity factors of two reference loading systems (m) and (n) for the same geometry (Fig. 5).

Load Case 'm': Here, in this reference loading case the far field tensile stress of $1e4$ units is applied to the given crack configuration system as shown in Fig. 5(a). The reference stress intensity factors $K_m^1$ and $K_m^2$ calculated through domain $M$ integral concept (Shih et al., 1988) are shown in Table 2.

Load Case 'n': In this reference loading case we choose a loading system wherein an isolated tensile force of magnitude $1e7$ acts on both left and right crack surfaces at $x = -2$ units, respectively, on each side of the crack faces (Fig. 5(b)). The FE mesh and displacement boundary condition used in this case is same as the previous one. The stress intensity factors $K_n^1$ and $K_n^2$ calculated through domain $M$ integral concept are given in Table 2.

Using Eqs. (31) and (32) the thermal stress intensity factor solutions obtained from the present approach is $K_p^1 = -0.1512$, $K_p^2 = 0.9810$ with an error of $0.277\%$ and $1.9\%$, respectively, from the analytical solution. The results seem to be encouraging.

4.2. Thermal barrier coating system with an interfacial central crack subjected to a cooling shock

A transient fracture problem of thermal barrier coating (TBC) as shown in Fig. 6 is considered. A coating layer of thickness $H_c$ and a substrate of thickness $H_s = 10H_c$ demarcate the whole body into two parts with the length of the coating-substrate medium being kept infinite. An interfacial central crack of length $2a = 2(H_s/8)$ is considered to exist between the two layers. At time $t = 0$, the top coating surface is subjected to a cooling environment of temperature $T_a = 25^\circ C$ and cooled by surface convection at a heat transfer coefficient of $h = 50 \text{ W/}^\circ\text{C m}^2$. The initial temperature of the TBC system is at $1000^\circ C$ and is assumed to be stress free. Also, it is assumed that the crack surfaces are insulated. The aforementioned conditions make sure that the singular character of the heat flux field appearing near the crack tip is not eliminated. This problem has been analyzed under plain stress condition.

Fig. 7 shows the displacement and thermal boundary conditions used for right half of the TBC system which has been modeled, taking into account $y$-axis of symmetry (Fig. 6). On the right surface of the model, which are subject to far field boundary conditions, symmetry conditions are imposed, for both displacement and thermal fields. The same boundary

![Fig. 5](image_url)

**Fig. 5.** (a) Reference loading 'm' – crack configuration subjected to far field tensile stress. (b) Reference loading 'n' – crack configuration subjected to pairs of concentrated force $P$.

<table>
<thead>
<tr>
<th>Ref. case</th>
<th>m</th>
<th>n</th>
<th>$K^2_{m}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Example 1</td>
<td>$2.32e6$</td>
<td>$-1.24e5$</td>
<td>$1.34e7$</td>
</tr>
<tr>
<td>Example 2</td>
<td>$2.95e4$</td>
<td>$-5.28e3$</td>
<td>$2.61e7$</td>
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<tr>
<td>Example 3</td>
<td>$3.49e4$</td>
<td>$-712.55$</td>
<td>$6.27e6$</td>
</tr>
</tbody>
</table>

Table 2

Reference stress intensity factors (SIFs) calculated for validation examples

The material properties and dimensions are shown in Fig. 5.
conditions have been imposed on left symmetrical surface. As has been stated in Giannopoulos and Anifantis (2005), a length $L = 10a$ is sufficient for the time-dependent solution to converge to the solution of the infinite problem, that is, the constraint on the right vertical edge would not have any effect on the solution. The material properties used in the analysis are shown in Table 2.

Fig. 8 shows the discretized mesh used in the FE analysis with the region around the crack tip enlarged for clarity. A total of 7743 elements (eight-noded) and 23,952 nodes have been used in the finite element model. 

As both the reference loading systems which are required to obtain the TWFs are considered to be symmetrical, only one half of the system is sufficient for the analysis and the same FE mesh could be used for all the analysis. The two reference loading systems related to this example are shown in Fig. 9.

The value of far field stress used is $1e4$ units for reference case ‘m’ (Fig. 9(a)). The value of $P$ for reference case ‘n’ used is $1e7$ units and is applied at a distance of $0.5$ unit from the origin (Fig. 9(b)). The SIFs computed for reference loading cases using the M-integral method are shown in Table 2.
Fig. 8. (a) FEM mesh used for the TBC model. (b) Zone around the crack tip.

Fig. 9. (a) Reference loading 'm' – crack configuration subjected to far field tensile stress. (b) Reference loading 'n' – crack configuration m subjected to pairs of concentrated force $P$. 
Table 3 shows the comparison of results of SIFs computed through TWF and thermal M-integral (Banks-Sills and Dolev, 2004) at four different time instances. It is seen that the computed results by these two methods are in close agreement. The difference between two methods are less than 2%.

Fig. 10(a) and (b) shows the variation of Mode 1 and Mode 2 stress intensity factors for different ratios of modulus of elasticity, $E_c/E_s$, with respect to time, keeping all other material properties same for the coating and the substrate. It is seen that both Mode 1 and Mode 2 SIFs are increased by a factor of approximately 5 for an increase of $E_c/E_s$ ratio from 10 to 1000, which is relatively small. This implies that the variation in the $E_c/E_s$ ratio has less impact on the thermal stress intensity factors.

Fig. 11(a) and (b) shows the variation of Mode 1 and Mode 2 stress intensity factors for different ratio of coefficient of thermal expansion, $\alpha_c/\alpha_s$, with respect to time, keeping rest of the material properties same for the coating and the substrate. Here, there is a significant increase in both Mode 1 and Mode 2 SIFs for a increase of $\alpha_c/\alpha_s$ ratio from 10 to 1000.

<table>
<thead>
<tr>
<th>Time (s)</th>
<th>$K_1$ (N m$^{-2}$ √mm$^{-1}$) (M integral)</th>
<th>Error %</th>
<th>$K_1$ (N m$^{-2}$ √mm$^{-1}$) (TWF)</th>
<th>Error %</th>
<th>$K_2$ (N m$^{-2}$ √mm$^{-1}$) (M integral)</th>
<th>Error %</th>
<th>$K_2$ (N m$^{-2}$ √mm$^{-1}$) (TWF)</th>
<th>Error %</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>58.77</td>
<td>1.58</td>
<td>57.84</td>
<td>2.06</td>
<td>23.93</td>
<td>1.25</td>
<td>24.23</td>
<td>1.35</td>
</tr>
<tr>
<td>5e4</td>
<td>1.017e6</td>
<td>2.06</td>
<td>1.038e6</td>
<td>2.60</td>
<td>3.23e6</td>
<td>1.85</td>
<td>3.17e6</td>
<td>1.97</td>
</tr>
<tr>
<td>5e5</td>
<td>6.358e7</td>
<td>1.67</td>
<td>6.464e7</td>
<td>2.31</td>
<td>2.013e8</td>
<td>0.49</td>
<td>2.003e8</td>
<td>0.51</td>
</tr>
<tr>
<td>2.5e7</td>
<td>3.269e8</td>
<td>1.87</td>
<td>3.330e8</td>
<td>2.44</td>
<td>1.063e9</td>
<td>1.13</td>
<td>1.051e9</td>
<td>1.15</td>
</tr>
</tbody>
</table>

Fig. 10. Variation of Mode 1 (a) and Mode 2 (b) stress intensity factors for different ratios of modulus of elasticity with respect to time.
Fig. 12(a) and (b) shows the variation of Mode 1 and Mode 2 stress intensity factors for different ratio of thermal conductivities, $C_s/C_c$, with respect to time, keeping all other material properties constant. It is seen that for the same thermal conductivities of both coating and substrate, both Mode 1 and Mode 2 SIFs are negative, but with increase in $C_s/C_c$ ratio, the Mode 1 SIF becomes positive and the Mode 2 SIF remains negative. Also, the overall percentage increase in both Mode 1 and Mode 2 SIFs are not much with increase in $C_s/C_c$ from 22 to 1000.

Fig. 11. Variation of Mode 1 (a) and Mode 2 (b) stress intensity factors for different ratios of coefficient of thermal expansion with respect to time.

Fig. 12(a) and (b) shows the variation of Mode 1 and Mode 2 stress intensity factors for different ratio of thermal conductivities, $C_s/C_c$, with respect to time, keeping all other material properties constant. It is seen that for the same thermal conductivities of both coating and substrate, both Mode 1 and Mode 2 SIFs are negative, but with increase in $C_s/C_c$ ratio, the Mode 1 SIF becomes positive and the Mode 2 SIF remains negative. Also, the overall percentage increase in both Mode 1 and Mode 2 SIFs are not much with increase in $C_s/C_c$ from 22 to 1000.

Thus, from Figs. 10(a)–12(b), it can be said that both Mode 1 and Mode 2 thermal stress intensity factors are most significantly influenced by $\alpha_c/\alpha_s$ ratio when compared to both $E_c/E_s$ and $C_s/C_c$ ratios.

4.3. Time-dependent problem of an interfacial central crack subjected to an instantaneous heat source

A transient fracture problem of a non-insulated interfacial central crack of length $2a$ located between two dissimilar elastic semi-infinite plate subjected to an instantaneous heat source of strength $q$ located at origin of the plate as shown in Fig. 13 is considered. The thermo–elastic constants for material occupying the upper ($S^+$) and lower half ($S^-$) planes are $E_1, v_1, \alpha_1, C_1, S_{p1}$ and $E_2, v_2, \alpha_2, C_2, S_{p2}$, respectively, where $E_k$ is the modulus of elasticity, $v_k$ is the Poisson ratio, $\alpha_k$ is the coefficient of linear thermal expansion, $C_k$ is the thermal conductivity and $S_k$ is the specific heat of the material. The numerical values of all these constants are given in Table 1. The subscript $k = 1, 2$ represents the material occupying the upper and lower portion, respectively, of the bi-material interface. This problem is analyzed under plane stress condition.

Fig. 14 shows the right half of the plate modeled along with displacement and thermal boundary conditions, taking into account the $y$-axis of symmetry (Fig. 13). In order to simulate an ‘infinite’ body, the dimensions of the plate used are $80 \times 160$ units with the crack of length 4 units. The finite element model for this example consisted of a total of
Fig. 12. Variation of Mode 1 (a) and Mode 2 (b) stress intensity factors for different ratios of coefficient of conductivity with respect to time.

Fig. 13. Finite crack subjected to an impulse heat source at the origin.
8923 elements (eight-noded) with 27,190 nodes. The same finite element model is used for analyzing the reference configurations too.

As in the previous case studies, two isothermal reference field solutions to obtain the TWFs are considered. The value of far field stress used is 1e4 units for reference case 'm' (Fig. 15(a)). The value of $P$ for reference case 'n' used is 1e7 units and is applied at $2.104$ units from the origin (Fig. 15(b)). The computed SIFs of reference loadings used for determining the TWFs are given in Table 2.

Table 4 shows the comparison of results of SIFs computed through present method (TWF) and thermal M-integral (Banks-Sills and Dolev, 2004) for four different time instants. Once again it is seen that the computed results by these two methods are in close agreement within an error of 2%.

Fig. 16(a) and (b) shows the variation of Mode 1 and Mode 2 stress intensity factors for different ratios of modulus of elasticity, $E_1/E_2$, with respect to time, keeping all other material properties constant. Here, for the same value of modulus of elasticity in both the materials, $K_2$ remain positive throughout, whereas $K_1$ increases with a positive value immediately upon the application of heat source and later on, after some time it attain a negative value and increases negatively. As seen in the previous example of TBC system, the effect of $E_1/E_2$ ratio is not much on the both Mode 1 and Mode 2 thermal stress intensity factors.

Fig. 17(a) and (b) show the variation of Mode 1 and Mode 2 stress intensity factors for different ratio of coefficient of thermal expansion, $\alpha_c/\alpha_s$, with respect to time keeping all other material properties constant. As seen in these figures, both Mode

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Fig. 14. Displacement boundary condition (Example 3).

Fig. 15. (a) Reference loading 'm' – crack configuration subjected to far field tensile stress. (b) Reference loading 'n' – crack configuration subjected to pairs of concentrated force $P$. 

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Fig. 16. Variation of Mode 1 and Mode 2 stress intensity factors with respect to time for different ratios of modulus of elasticity, $E_1/E_2$. 

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Fig. 17. Variation of Mode 1 and Mode 2 stress intensity factors with respect to time for different ratio of coefficient of thermal expansion, $\alpha_c/\alpha_s$. 

---
1 and Mode 2 stress intensity factors initially remain negative for $\frac{x_c}{x_s}$ ratio of 1 but with increase in this ratio both Mode 1 and Mode 2 stress intensity factors attain a high positive value.

Fig. 16(a) and (b) shows the variation of Mode 1 and Mode 2 stress intensity factors for different ratios of thermal conductivities with respect to time, keeping all other material properties constant. As seen in these figures, the Mode 1 stress intensity factor, $K_1$, remains positive initially when the heat source is applied and later on it decreases and attain a negative value. With increasing ratio of thermal conductivities, an early attainment of negative value is seen. The Mode 2 stress intensity factor, $K_2$ always attain a positive values for all ratios of thermal conductivities. Also, changing

### Table 4
Stress intensity factors (SIFs) at different time instants (Example 3)

<table>
<thead>
<tr>
<th>Time (s)</th>
<th>$K_1$ (N m$^{-2}$/$\sqrt{m}$)</th>
<th>Error %</th>
<th>$K_2$ (N m$^{-2}$/$\sqrt{m}$)</th>
<th>Error %</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(M integral)</td>
<td></td>
<td>(TWF)</td>
<td></td>
</tr>
<tr>
<td>1e-3</td>
<td>1.496e7</td>
<td>-1.74</td>
<td>1.173e8</td>
<td>1.27</td>
</tr>
<tr>
<td>1e3</td>
<td>7.226e6</td>
<td>-1.55</td>
<td>2.254e8</td>
<td>1.47</td>
</tr>
<tr>
<td>1e6</td>
<td>-3.583e7</td>
<td>2.2</td>
<td>2.758e8</td>
<td>1.67</td>
</tr>
<tr>
<td>5e7</td>
<td>-4.744e7</td>
<td>1.53</td>
<td>2.850e8</td>
<td>1.79</td>
</tr>
</tbody>
</table>

Fig. 16. Variation of Mode 1 (a) and Mode 2 (b) stress intensity factors for different ratios of modulus of elasticity with respect to time.
The ratio of thermal conductivities of both the materials does not have significant effect on SIFs as compared to changing ratios of

c

c

c

5. Conclusions

Being a universal function, TWF is independent of temperature fields and depends only on the crack configuration and body geometry. Once the TWF of a specific crack tip and body geometry is determined, the SIFs for the crack tip of the body subjected to any temperature field can be directly calculated through integration of proper products of the TWF and temperature fields. In the present study, a computational approach has been presented to derive TWFs for any type of thermal loading and two-dimensional geometry of bi-material interface. The general expressions of WFs for bi-materials interface problems has been derived and it has been observed that these expressions are of same type as that of the homogeneous mixed mode loading cases. Further, the FE implementation of the TWF method for bi-material crack configuration system in plane stress, plane strain problems are presented and validated using one constant temperature and two thermal shock problems.

The present method is suitable for determining the variation of transient SIFs of a cracked body subjected to thermal shock. TWF is independent of time during thermal shock, so the whole variation of transient SIFs can be directly calculated through integration of the products of TWF and transient temperature fields. The repeated determination of the distribution of stress fields at individual time instants as being done in the existing method (M-integral) are avoided in the

Fig. 17. Variation of Mode 1 (a) and Mode 2 (b) stress intensity factors for different ratios of coefficient of thermal expansion with respect to time.
present TWF method, thus making it a computationally efficient technique for determination of the thermal stress intensity factors.

References
