Similarity Solutions of the Lubrication Equation

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Abstract—We present a method for constructing closed-form similarity solutions of the fourth-order nonlinear lubrication equation. By extending a technique used to study second-order degenerate diffusion problems, corresponding interface profiles and diffusion coefficient functions can be derived in exact form. Different classes of spreading and shrinking solutions are obtained using this approach.

Keywords—Lubrication equation, Similarity solutions, Degenerate diffusion, Nonlinear parabolic equations.

1. INTRODUCTION

We present a formal method for constructing closed-form similarity solutions of the lubrication equation

\[ \frac{\partial h}{\partial t} + \frac{\partial}{\partial x} \left( D(h) \frac{\partial^3 h}{\partial x^3} \right) = 0, \]  

(1.1)

where the diffusion coefficient function has the asymptotic form \( D(h) \sim h^n \) as \( h \to 0 \) with \( n > 0 \). Several recent studies have focused on examining the types of behaviors possible in interfacial hydrodynamic flows described by this mathematical model. Equation (1.1) can be derived from a lubrication approximation of the Navier-Stokes equations for flow of thin viscous films with surface tension. Use of this model to study flow in Hele-Shaw cells has been extensively considered by Constantin, Kadanoff, Bertozzi et al. [1–3]. The lubrication equation with various \( 1 \leq n \leq 3 \) has been used to describe flows in many applications [1]; Hele-Shaw cells, polymeric liquids, thin viscous films, and contact line motion with slip conditions. Analysis of the solutions of (1.1) has been studied by Bernis et al. [4–6], Bertozzi [8–11], and Beretta et al. [12].

The lubrication equation is a fourth-order nonlinear degenerate parabolic partial differential equation that admits nonnegative weak solutions with compact support. In this letter, we extend a method used for the second-order degenerate diffusion equation to derive exact, closed-form similarity solutions of equation (1.1). Our approach shows that solutions of the lubrication equation divide into distinct classes with different behavior and structure, but all share the same functional relationship with the diffusion coefficient \( D(h) \).

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2. SIMILARITY SOLUTIONS

We seek closed-form nonnegative similarity solutions of (1.1) that satisfy a constant Dirichlet boundary condition, say \( h(0, t) = h_0 \), and zero initial data on the half-line \( 0 < x < \infty \). For \( D(h) \) such that \( D(0) = 0 \), (1.1) is a degenerate diffusion equation that has non-negative weak solutions with compact support \([4,6]\). Such similarity solutions take the form

\[
h(x, t) = [H(\eta)]^+, \quad \eta = xt^{-1/4}, \tag{2.1}
\]

where \([w]^+ = \max(w, 0)\) and \( H(\eta) \) satisfies the nonlinear ordinary differential equation

\[
\frac{\eta}{4} \frac{dH}{d\eta} = \frac{d}{d\eta} \left( D(H) \frac{d^3 H}{d\eta^3} \right), \tag{2.2}
\]

with boundary condition \( H(0) = h_0 \). Solutions of (2.2) can be used to represent the height of thin liquid films spreading onto dry surfaces. The position \( \eta_* \), where \( H(\eta_*) = 0 \) is the edge of the region of support of \([H(\eta)]^+\), \( 0 \leq \eta \leq \eta_* \), and is commonly called the contact line or interface.

A powerful technique for obtaining solutions of the corresponding similarity equation for the second-order porous media equation,

\[
\frac{\partial h}{\partial t} = \frac{\partial}{\partial x} \left( D(h) \frac{\partial h}{\partial x} \right) \tag{2.3}
\]

was used by Philip \([13]\) and others \([14]\) in connection with the study of groundwater diffusion in soils; we illustrate this approach applied to equation (2.2). Instead of attempting to solve the equation directly for \( H = H(\eta) \), it is beneficial to consider the solution in implicit or inverse variable form \( \eta = \eta(H) \). In terms of \( \eta(H) \), (2.2) can be rewritten as

\[
\frac{\eta}{4} \frac{dH}{d\eta} = \frac{\partial}{\partial \eta} \left( D(h) \frac{\partial h}{\partial x} \right) \tag{2.4}
\]

For monotone decreasing \( H(\eta) \), equation (2.2) is equivalent to (2.4), but in the latter form it can be integrated once. To satisfy conservation of mass for fluid flows, we require that the flux at the interface must vanish \( J(\eta_*) \equiv D(0)H'''(\eta_*) = 0 \). Integrating (2.2), subject to this condition yields

\[
D(H) = \frac{\eta'(H)^5}{4(3\eta''(H)^2 - \eta'''(H)\eta'(H))} \int_0^H \eta(h) \, dh. \tag{2.5}
\]

Equation (2.5) shows that the form of the diffusion coefficient is uniquely determined by the similarity solution. Specifically, the form of the similarity solution \( \eta(H) \) near the interface uniquely selects the asymptotic behavior of \( D(H) \) as \( H \to 0 \). We show that the converse is not true; for \( D(H) = O(H^n) \), with \( n \) in certain ranges, there are multiple qualitatively different coexisting \( \eta(H) \) solutions. It is shown that there are singular solutions for which (2.5) is undefined; these cases lead to different families of similarity solutions.

We begin by applying (2.5) to a simple ansatz. Focusing on the behavior near the interface, we consider the family of solutions for \( \alpha > 0 \),

\[
H(\eta) = (c - \eta)^{1/\alpha}, \quad 0 \leq \eta \leq c, \tag{2.6}
\]

where \( \eta_* = c > 0 \) is the interface position. The corresponding inverse-variable form is

\[
\eta(H) = c - H^\alpha. \tag{2.7}
\]
Substituting (2.7) into (2.5) yields the corresponding diffusion coefficients

\[ D(H) = \frac{\alpha^3}{4(2\alpha - 1)(1 - \alpha)} H^{3\alpha} \left( c - \frac{H^\alpha}{1 + \alpha} \right). \]  

(2.8)

It is important to note that no approximations have been introduced up to this point; (2.6) is an exact solution of equation (2.2) with \(D(H)\) given by (2.8). By deriving corresponding \(H(\eta)\) and \(D(H)\) pairs from a particular inverse variable form \(\eta(H)\), it is thus possible to construct problems for the lubrication equation with exact closed-form solutions. Such problems can serve as the basis for further studies of (1.1), including questions of stability, that might not be analytically tractable using other approaches.

In contrast to most other types of similarity solutions of (1.1), the ansatz (2.1) does not crucially rely on the diffusion coefficient having a special functional form, like a power law \(D(h) = h^n\) (see [6,7]). This allows us the freedom to obtain exact solutions at the price of modifying the diffusion coefficient by adding higher order terms in \(H\). In fact, such polynomial diffusion coefficients are representative of slip models used for thin viscous films [4,15]. Modifying the diffusion coefficient function at finite \(H\) is not expected to qualitatively change the asymptotic structure of the interface as \(H \to 0\), consequently, with respect to the interface behavior, these solutions are representative of the solutions of the power law diffusion problems.

Observe from (2.7) that \(D(H) = O(H^{3\alpha})\) as \(H \to 0\); hence, (2.6) and (2.8) yield the leading order behavior of solutions to (2.4) near the contact line for power law diffusion coefficients with \(n = 3\alpha\). To normalize the diffusion coefficient (2.8) to \(D(H) \sim H^{3\alpha}\) as \(H \to 0\), we take \(c = c_+ \equiv 4(2\alpha - 1)(1 - \alpha)/\alpha^3\). The value of the interface position is an important quantity since it scales the velocity of the contact line \(v(t) = (1/4)ct^{-3/4}\). In order to have a positive region of support, \(c\) must be positive and we must restrict \(1/2 < \alpha < 1\). Then, the corresponding interface velocity is positive and (2.6) represents spreading similarity solutions for \(3/2 < n < 3\) [7,10].

The similarity solution (2.6) satisfies the Dirichlet or time-dependent “pressure” [2,3,11] boundary conditions

\[ h(0, t) = c^{1/\alpha}, \quad h_{xx}(0, t) = \frac{1 - \alpha}{\alpha^2 \sqrt{t}} c^{1/\alpha - 2}. \]  

(2.9)

Different boundary conditions, imposed at finite \(H\) (away from the interface), can be satisfied by assuming more general forms for the ansatz (2.7) with additional higher order terms in \(H\). For \(1/2 < \alpha < 1\), changing the ansatz will not change the qualitative structure of the solution near the interface, which is the central topic of interest for studies of the lubrication equation.

Using the invariance of (1.1) under certain transformations, it is possible to extend the applicability of solution (2.6) to solve other initial-boundary value problems for the lubrication equation. Equation (1.1) is invariant with respect to translation \((x \rightarrow x - x_0)\), reflection \((x \rightarrow -x)\), and time shifts \((t \rightarrow t - t_0)\). Hence, if \(H(\eta)\) is a solution of (2.2), then \(h(x, t) = H((x_0 - x)/(t - t_0)^{1/4})\) is another solution of (1.1) subject to appropriate boundary and initial conditions. The lubrication equation is not invariant under time reversal \(t \rightarrow -t\); however, it is invariant under the transformation \(t \rightarrow -t\) and \(D(H) \rightarrow -D(H)\). Consequently, by using time reversal and time shifting, solution (2.6) can be used for \(0 < \alpha < 1/2\) and \(\alpha > 1\) to yield

\[ h(x, t) = \left( c - \frac{x}{(t_c - t)^{1/4}} \right)^{1/\alpha} \]  

(2.10)

for \(t < t_c\), where \(c = c_- \equiv -4(2\alpha - 1)(1 - \alpha)/\alpha^3 > 0\). These solutions are shrinking similarity solutions for \(n > 3\) and \(0 < n < 3/2\) whose region of support vanishes at a finite critical time \(t_c\). Such shrinking diffusive solutions are a feature of the fourth-order lubrication equation (1.1) that are not present in the second-order porous media equation (2.3). Shrinking solutions could be responsible for rupture of thin films [16] and formation of finite time singularities [1]. We will now go on to consider cases where the simple ansatz (2.6) fails.
3. SINGULARITIES OF $D(H)$

It is clear that the diffusion coefficient (2.8) is undefined for $\alpha = 1/2$ and $\alpha = 1$; these are special singularities of the diffusion coefficient for the lubrication equation. We now demonstrate that these are the only singularities of (2.5) for integrable $\eta(H)$ profiles. $D(H)$ is singular if $\eta(H)$ satisfies the differential equation

$$3 \left( \frac{d^2 \eta}{dH^2} \right)^2 - \frac{d^3 \eta}{dH^3} \frac{dn}{dH} = 0.$$  

(3.1)

This equation has the two-parameter singular solution $\eta(H) = aH + b$, corresponding to $\alpha = 1$, and its general solution is $\eta(H) = c \pm \sqrt{aH + b}$, corresponding to $\alpha = 1/2$ in form (2.7). These inverse solutions clearly correspond to linear and parabolic solutions for $H(\eta)$ which trivially satisfy (2.2) since $H''(\eta) = 0$. In these two cases, the asymptotic behavior of the diffusion coefficient $D(H) = O(H^n)$ as $H \to 0$ is not $n = 3\alpha$, but depends on higher-order terms, as we go on to explore in the following sections.

4. SOLUTIONS FOR $\alpha = 1$ AND $\alpha = 1/2$

Having identified the singular cases $\alpha = 1/2$ and $\alpha = 1$, we now resolve the singular behavior by adding higher order terms to the inverse variable ansatz (2.7). For $\alpha = 1$, consider

$$\eta(H) = c - H - aH^\beta,$$

(4.1)

with $\beta > 1$. Equation (4.1) is a generalization of (2.7) with $\alpha = 1$ and a higher-order term $O(H^\beta)$. We show that this higher order term solves the problem of the singularity at $\alpha = 1$ and introduces no further singularities at higher order. Substituting this form into (2.5) yields

$$D(H) = \frac{-H (1 + \beta H^{\beta-1})^5 (c - (1/2)H - (a/\beta + 1) H^\beta)}{4a\beta(\beta - 1)(2\beta - 1)H^{3\beta-4} - (\beta - 1)(\beta - 2)H^{\beta-3}}.$$  

(4.2)

The asymptotic behavior of $D(H)$ as $H \to 0$ reduces to two cases; for $\beta = 2$,

$$D(H) \sim -\frac{cH}{48a^2},$$

(4.3)

and for $\beta \neq 2$

$$D(H) \sim \frac{cH^{4-\beta}}{4a\beta(\beta - 1)(\beta - 2)}.$$  

(4.4)

For any $a \neq 0$, a well-defined diffusion coefficient is given by (4.3) or (4.4). For $\beta = 2$, we obtain shrinking solutions with $c = c_- = 48a^2 > 0$ and $D(H) \sim H$. For $\beta \neq 2$, we get both spreading and shrinking solutions, depending on the sign of $a$, $c = c_+ \equiv \pm 4a\beta(\beta - 1)(\beta - 2) > 0$, with $D(H) \sim H^{4-\beta}$. To obtain the interface profile corresponding to the inverse-variable form $\eta(H)$, we expand the algebraic equation

$$aH^\beta + H^\alpha - (c - \eta) = 0,$$

(4.5)

yielding

$$H(\eta) \sim (c - \eta)^{1/\alpha} - \frac{a}{\alpha} (c - \eta)^{(\beta-\alpha+1)/\alpha}, \quad \text{as } \eta \to c.$$  

(4.6)

Two special cases arising from (4.3) and (4.4) are worth mentioning at this point. $D(H)$ given by (4.4) with $\beta = 4$ is $O(1)$ as $H \to 0$. The resulting linear lubrication equation $h_t + h_{xxxx} = 0$, is nondegenerate and has no compact support solutions. Our formal method for constructing the local expansion of $H(\eta)$ as $H \to 0$, does not guarantee that these solutions have compact support;
it merely gives the forms possible for local behavior of the solutions. In the linear case, smooth solutions can pass through \( H = 0 \) and become negative. Second, if we search for solutions of the lubrication equation with \( D(H) \sim H \), then (4.3) yields the shrinking solution

\[
H(\eta) \sim (c - \eta) - a(c - \eta)^2, \quad \text{as } \eta \to c, \tag{4.7}
\]

while (4.4) with \( \beta = 3 \) yields both spreading and shrinking solutions

\[
H(\eta) \sim (c - \eta) \pm a(c - \eta)^3, \quad \text{as } \eta \to c. \tag{4.8}
\]

The coexistence of these different classes of solutions is representative of many questions of uniqueness currently being studied [12].

The singular behavior at \( \alpha = 1/2 \) can be handled similarly with

\[
\eta(H) = c - H^{1/2} - aH^\beta, \tag{4.9}
\]

where \( \beta > 1/2 \). The resulting diffusion coefficient is

\[
D(H) = \frac{-H \left( (1/2)H^{-1/2} + a\beta H^{\beta-1} \right) \left( c - (2/3)H^{1/2} - (a/\beta + 1)H^\beta \right)}{2a\beta (2a\beta(2\beta - 1)(\beta - 1)H^{2\beta-4} - (\beta^2 - (1/4))H^{\beta-7/2})} \tag{4.10}
\]

with asymptotic behavior as \( H \to 0 \),

\[
D(H) \sim \frac{cH^{2-\beta}}{16a\beta (4\beta^2 - 1)}. \tag{4.11}
\]

As in (4.4), spreading and shrinking solutions can be obtained with \( c \equiv \pm 16a\beta(4\beta^2 - 1) > 0 \), and

\[
H(\eta) \sim (c - \eta)^2 \pm 2a(c - \eta)^{2\beta+1}, \quad \text{as } \eta \to c. \tag{4.12}
\]

Again, note that for \( \beta = 2 \), (4.12) is a local expansion of a solution of the linear lubrication equation, and for \( \beta = 1 \) we get yet another class of solutions for \( D(H) \sim H \).

### 5. Solutions for \( n = 3/2 \) and \( n = 3 \)

At this point, we note that despite having exhausted all possible algebraic forms of the solution near the interface we still cannot obtain diffusion coefficients \( D(H) \sim H^n \) with \( n = 3/2 \) or \( n = 3 \). Motivated by the results of Bernis, Peletier and Williams [7] for \( n = 3/2 \), we consider logarithmic corrections to the solution. Such forms are reminiscent of Frobenius series or generalizations thereof used for nonlinear differential equations [17]. In analogy to (2.7), consider

\[
\eta(H) = c - H^\alpha \ln^\beta \left( \frac{1}{H} \right). \tag{5.1}
\]

Substitution of (5.1) into (2.5) is somewhat algebraically cumbersome. However, we note that the integral of \( \eta \) in (2.5) can be expressed in terms of an incomplete gamma function [18],

\[
\int_0^H \eta(h) \, dh = cH - (1 + \alpha)^{-\beta-1} \Gamma(\beta + 1, -(1 + \alpha) \ln H), \tag{5.2}
\]

which yields the asymptotic behavior as \( H \to 0 \),

\[
\int_0^H \eta(h) \, dh \sim H \left( c - \frac{1}{1 + \alpha} H^\alpha \ln^\beta \left( \frac{1}{H} \right) \right). \tag{5.3}
\]

There are two significant special cases where the leading order behavior of the diffusion coefficient is a power law. These are the singular cases \( \alpha = 1 \) and \( \alpha = 1/2 \), both with \( \beta = -1/3 \);

\[
\eta(H) = c - \frac{\sqrt{H}}{\ln^{1/3} (1/H)}, \quad D(H) \sim \frac{3c}{32} H^{3/2}, \tag{5.4}
\]

\[
\eta(H) = c - \frac{H}{\ln^{1/3} (1/H)}, \quad D(H) \sim -\frac{3c}{4} H^3. \tag{5.5}
\]

While (5.4) is not trivially invertible to yield the result of Bernis et al. [7], numerical plots show that both solutions represent the same asymptotic behavior as \( H \to 0 \). We note that (5.4) yields spreading solutions for \( n = 3/2 \), while (5.5) yields shrinking solutions for \( n = 3 \).
6. CONCLUSIONS

We conclude by summarizing our results in Table 1. While other studies of the lubrication equation have generally focused on finding similarity solutions subject to different constraints, several features appear closely analogous. Despite some differences in properties of solutions due to details of the similarity forms used, there seem to be certain shared characteristics that are due to the fundamental form of equation (1.1). We make no claims about our results other than the fact that they are formal solutions of the similarity differential equation (2.2). Some of these solutions are not the “most regular” forms [10] possible under the imposed conditions, but they may still be relevant to studies of the dynamics of (1.1). In this letter, we have derived exact closed-form solutions that can serve as a convenient starting point for studies of more delicate structures involved in topological transitions and stability analysis. Some of these solutions can be used to study merging of spreading droplets and rupture of thin films [14].

Table 1.

<table>
<thead>
<tr>
<th>$D(H) \sim H^n$</th>
<th>$H(\eta)$</th>
<th>Motion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 &lt; n &lt; \frac{3}{2}$</td>
<td>$H = (c - \eta)^{3/n}$</td>
<td>−</td>
</tr>
<tr>
<td>$\frac{3}{2} &lt; n &lt; 3$</td>
<td>$H = (c - \eta)^{3/n}$</td>
<td>+</td>
</tr>
<tr>
<td>$n &gt; 3$</td>
<td>$H = (c - \eta)^{3/n}$</td>
<td>−</td>
</tr>
<tr>
<td>$n &lt; 3, n \neq 2$</td>
<td>$H \sim (c - \eta) \pm a(c - \eta)^{4-n}$</td>
<td>±</td>
</tr>
<tr>
<td>$n = 1$</td>
<td>$H \sim (c - \eta) - a(c - \eta)^{2}$</td>
<td>−</td>
</tr>
<tr>
<td>$n &lt; \frac{3}{2}$</td>
<td>$H \sim (c - \eta)^{2} \pm 2a(c - \eta)^{5-2n}$</td>
<td>±</td>
</tr>
<tr>
<td>$n = \frac{3}{2}$</td>
<td>$\eta = c - \frac{\sqrt{H}}{\ln^{1/3}(1/H)}$</td>
<td>+</td>
</tr>
<tr>
<td>$n = 3$</td>
<td>$\eta = c - \frac{H}{\ln^{1/3}(1/H)}$</td>
<td>−</td>
</tr>
</tbody>
</table>

Motion: + spreading solutions, − shrinking solutions.

REFERENCES


