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On iterated inverse limits

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Abstract

Let a compact Hausdorff space X be the limit of a cofinite inverse system of compact Hausdorff spaces X_λ, $X = \lim_\lambda X_\lambda$. Then it is possible to express every X_λ as the limit of an inverse system of compact polyhedra Y_λ^μ, $X_\lambda = \lim_\mu Y_\lambda^\mu$, in such a way that the spaces $Y_\nu = Y_\lambda^\mu$ can be organized in an inverse system with $\lim_\nu Y_\nu = \lim_\lambda \lim_\mu Y_\lambda^\mu$. Using ANR-resolutions, the result is generalized to non-compact spaces.

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1. Introduction

Let $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be an inverse system of topological spaces, indexed by a directed set Λ, and let $\boldsymbol{p} : X \to \boldsymbol{X}$ be a mapping of a topological space X to $\boldsymbol{X}$, i.e., a collection of mappings $p_\lambda : X \to X_\lambda$, $\lambda \in \Lambda$, such that

$$p_{\lambda\lambda'} p_{\lambda'} = p_\lambda, \quad \lambda \leqslant \lambda'. \tag{1}$$

Moreover, for every $\lambda \in \Lambda$, let $\boldsymbol{Y}_\lambda = (Y_\lambda^\mu, q_\lambda^{\mu\mu'}, M_\lambda)$ be an inverse system and let $\boldsymbol{q}_\lambda : X_\lambda \to \boldsymbol{Y}_\lambda$ be a mapping consisting of mappings $q_\lambda^\mu : X_\lambda \to Y_\lambda^\mu$. There is no loss of generality in assuming that the index sets M_λ and $M_{\lambda'}$ are disjoint, for $\lambda \neq \lambda'$, and thus, every element ν of the set

$$N = \bigcup_{\lambda \in \Lambda} M_\lambda \tag{2}$$

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admits a unique $\lambda \in \Lambda$ such that $\nu \in M_\lambda$. Consequently, elements $\nu \in N$ can be identified with pairs (λ, μ), where $\lambda \in \Lambda$ and $\mu \in M_\lambda$. For a directed ordering $\leqslant$ of N, we will say that it is *compatible* with the orderings $\leqslant$ of Λ and M_λ, $\lambda \in \Lambda$, provided the following conditions hold:

(i) If $\nu = (\lambda, \mu)$ and $\nu' = (\lambda', \mu')$, then $\nu \leqslant \nu'$ implies $\lambda \leqslant \lambda'$.

(ii) For every $\lambda \in \Lambda$, the ordering $\leqslant$ of N restricted to M_λ coincides with the original ordering $\leqslant$ of M_λ.

Let $\boldsymbol{q} = (q_\nu) : X \to \boldsymbol{Y}$ be a mapping, where $\boldsymbol{Y} = (Y_\nu, q_{\nu\nu'}, N)$ is an inverse system and N is given by (2). We will say that $\boldsymbol{q}$ is *compatible* with $\boldsymbol{p}$ and $\boldsymbol{q}_\lambda$, $\lambda \in \Lambda$, provided the ordering of N is compatible with the orderings of Λ and M_λ, $\lambda \in \Lambda$, and the following conditions hold:

(iii) $Y_\nu = Y_\lambda^\mu$, $\nu = (\lambda, \mu)$.

(iv) $q_\lambda^\mu p_{\lambda\lambda'} = q_{\nu\nu'} q_{\lambda'}^{\mu'}$, for $\nu = (\lambda, \mu) \leqslant (\lambda', \mu') = \nu'$.

(v) $q_{\nu\nu'} = q_\lambda^{\mu\mu'}$, for $\nu = (\lambda, \mu) \leqslant (\lambda, \mu') = \nu'$.

(vi) $q_\nu = q_\lambda^\mu p_\lambda$, for $\nu = (\lambda, \mu)$.

In this paper we will prove the following result.

Theorem 1. *Let $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be a cofinite inverse system of compact Hausdorff spaces and let $\boldsymbol{p} : X \to \boldsymbol{X}$ be its limit. Then there exist cofinite inverse systems of compact polyhedra $\boldsymbol{Y}_\lambda$ with limits $\boldsymbol{q}_\lambda : X_\lambda \to \boldsymbol{Y}_\lambda$, $\lambda \in \Lambda$, which admit a cofinite inverse system $\boldsymbol{Y}$, whose limit $\boldsymbol{q} : X \to \boldsymbol{Y}$ is compatible with $\boldsymbol{p}$ and $\boldsymbol{q}_\lambda$, $\lambda \in \Lambda$.*

The question whether such an assertion holds was raised during a talk, given by Yu.T. Lisitsa, at the 1998 Dubrovnik Conference on Geometric Topology.

We call a directed set Λ *cofinite* if it is ordered (anti-symmetry holds) and every element has a finite number of predecessors. Cofinite systems play an important role in shape theory (see [6,5]).

To realize that Theorem 1 states a non-trivial assertion, we will first prove the following result.

Theorem 2. *There exists a cofinite inverse system of compact metric spaces $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ with limit $\boldsymbol{p} : X \to \boldsymbol{X}$ and there exist inverse sequences of compact polyhedra $\boldsymbol{Y}_\lambda = (Y_\lambda^\mu, q_\lambda^{\mu\mu'}, M_\lambda)$ with limits $\boldsymbol{q}_\lambda : X_\lambda \to \boldsymbol{Y}_\lambda$, $\lambda \in \Lambda$, such that there is no inverse system $\boldsymbol{Y}$, which is formed by polyhedra Y_λ^μ and X is its limit. Consequently, there is no inverse system $\boldsymbol{Y}$ with limit $\boldsymbol{q} : X \to \boldsymbol{Y}$, which is compatible with $\boldsymbol{p}$ and $\boldsymbol{q}_\lambda$, $\lambda \in \Lambda$.*

Our main result is Theorem 3, which is a version of Theorem 1, valid for arbitrary spaces. The role of limits is taken up by resolutions (see I.6 of [6]) and the role of compact polyhedra is taken up by ANR's (for metric spaces).

Theorem 3. *Let $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be a cofinite inverse system of topological spaces and let $\boldsymbol{p} : X \to \boldsymbol{X}$ be a resolution. Then there exist cofinite ANR-resolutions $\boldsymbol{q}_\lambda : X_\lambda \to \boldsymbol{Y}_\lambda$, $\lambda \in \Lambda$, which admit a cofinite resolution $\boldsymbol{q} : X \to \boldsymbol{Y}$ compatible with $\boldsymbol{p}$ and $\boldsymbol{q}_\lambda$, $\lambda \in \Lambda$.*

2. An example which proves Theorem 2

Let X be a compact Hausdorff space, whose covering dimension $\dim X = 1$ and whose inductive dimension $\operatorname{ind} X = 2$. Such spaces were constructed in 1949 by Lunc [3] and Lokucievskiĭ [2]. By a result from [4], $\dim X = 1$ implies the existence of a cofinite system $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ of metric compacta such that $\dim X_\lambda = 1$ and $X = \lim \boldsymbol{X}$. On the other hand, by a classical result of Freudenthal [1] (also see [4]), $\dim X_\lambda = 1$ implies that X_λ is the limit of an inverse sequence $\boldsymbol{Y}_\lambda$ of compact polyhedra Y_λ^μ of dimension $\dim Y_\lambda^\mu = 1$. However, the polyhedra Y_λ^μ cannot be organized in an inverse system $\boldsymbol{Y}$ with limit $\lim \boldsymbol{Y} = X$, because no compact Hausdorff space X with $\dim X = 1$ and $\operatorname{ind} X = 2$ is obtainable as the limit of an inverse system formed by compact 1-dimensional polyhedra (see [7,4]).

3. Resolutions of spaces

Resolutions of a space X are mappings $\boldsymbol{p} = (p_\lambda): X \to \boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$, which satisfy two additional conditions:

(B1) For every normal (numerable) covering $\mathcal{U}$ of X, there is a $\lambda \in \Lambda$ and there is a normal covering $\mathcal{U}_\lambda$ of X_λ such that the covering $p_\lambda^{-1}(\mathcal{U}_\lambda)$ refines $\mathcal{U}$; this is denoted by

$$p_\lambda^{-1}(\mathcal{U}_\lambda) \leqslant \mathcal{U}. \tag{3}$$

(B2) For every $\lambda \in \Lambda$ and every normal covering $\mathcal{U}_\lambda$ of X_λ, there is a $\lambda' \geqslant \lambda$ such that

$$p_{\lambda\lambda'}(X_{\lambda'}) \subseteq \operatorname{St}\big(p_\lambda(X), \mathcal{U}_\lambda\big). \tag{4}$$

If all X_λ are normal spaces, condition (B2) can be replaced by the equivalent condition:

(B2)′ For every $\lambda \in \Lambda$ and every open neighborhood U of the closure $\overline{p_\lambda(X)}$ in X_λ, there is a $\lambda' \geqslant \lambda$ such that

$$p_{\lambda\lambda'}(X_{\lambda'}) \subseteq U. \tag{5}$$

It is well known that, for X_λ Tychonoff and X topologically complete (e.g., for X paracompact), every resolution $\boldsymbol{p}: X \to \boldsymbol{X}$ is an inverse limit. Furthermore, if X and X_λ are compact Hausdorff spaces, also the converse holds, i.e., if $\boldsymbol{p}: X \to \boldsymbol{X}$ is an inverse limit, then $\boldsymbol{p}$ is a resolution. It is also known that every topological space X admits an ANR-resolution $\boldsymbol{p}: X \to \boldsymbol{X}$, i.e., a resolution where all X_λ are ANRs (for metric spaces). Similarly, every compact Hausdorff space X admits a resolution $\boldsymbol{p}: X \to \boldsymbol{X}$, where all X_λ are compact polyhedra, i.e., X is the limit of an inverse system of compact polyhedra. For the proofs of these results see, for instance, the books [6,5].

There is a construction which associates with every inverse system $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$, indexed by a directed ordered set Λ, a cofinite system $\boldsymbol{X}^* = (X_\alpha^*, p_{\alpha\alpha'}^*, \Lambda^*)$ and it associates with every mapping $\boldsymbol{p} = (p_\lambda): X \to \boldsymbol{X}$ a mapping $\boldsymbol{p}^* = (p_\alpha^*): X \to \boldsymbol{X}^*$. By definition, Λ^* consists of all finite subsets α of Λ, with the ordering inherited from Λ, and such that they have a terminal element, denoted by $\overline{\alpha}$. Because of anti-symmetry, $\overline{\alpha}$ is uniquely determined by α. The ordering $\leqslant$ of Λ^* is just the inclusion $\subseteq$. Note that $\alpha_1 \leqslant \alpha_2$ implies $\overline{\alpha}_1 \leqslant \overline{\alpha}_2$. By definition, $X^*_\alpha = X_{\overline{\alpha}}$ and, for $\alpha_1 \leqslant \alpha_2$, $p^*_{\alpha_1\alpha_2} = p_{\overline{\alpha}_1\overline{\alpha}_2}$. Moreover, $p^*_\alpha = p_{\overline{\alpha}}$. Note that every term from $\boldsymbol{X}^*$ is a term from $\boldsymbol{X}$.

Lemma 1. *If $\boldsymbol{p}: X \to \boldsymbol{X}$ is a resolution, then also $\boldsymbol{p}^*: X \to \boldsymbol{X}^*$ is a resolution. Moreover, if $\boldsymbol{X}$ consists of ANR's* (*of compact polyhedra*), *then so does $\boldsymbol{X}^*$.*

The proof is easy (see Lemma 6.31 of [5]) and we omit it.

4. Some technical lemmas on resolutions

To state the first of these lemmas, we describe a construction which applies to any mapping $\boldsymbol{p} = (p_\lambda): X \to \boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$. Let Γ be the set which consists of all pairs $\gamma = (\lambda, G)$, where $\lambda \in \Lambda$ and G is an open neighborhood of the closure $\overline{p_\lambda(X)}$ in X_λ. We order Γ by putting $\gamma \leqslant \gamma' = (\lambda', G')$ provided $\lambda \leqslant \lambda'$ and

$$p_{\lambda\lambda'}(G') \subseteq G. \tag{6}$$

Note that Γ is directed and ordered provided Λ has these properties. For $\gamma = (\lambda, G)$, let $Y_\gamma = G$ and, for $\gamma \leqslant \gamma'$, let $q_{\gamma\gamma'}: Y_{\gamma'} \to Y_\gamma$ be the restriction $p_{\lambda\lambda'}|G': G' \to G$, which is well defined, because of (6). Clearly, $\boldsymbol{Y} = (Y_\gamma, q_{\gamma\gamma'}, \Gamma)$ is an inverse system. We also define a mapping $\boldsymbol{q} = (q_\gamma): X \to \boldsymbol{Y}$. If $\gamma = (\lambda, G)$, $q_\gamma: X \to Y_\gamma = G$ is the restriction $X \to G \subseteq X_\lambda$ of $p_\lambda: X \to X_\lambda$.

Lemma 2. *If $\boldsymbol{p}: X \to \boldsymbol{X}$ has property* (B1) *and all X_λ are ANRs, then $\boldsymbol{q}: Y \to \boldsymbol{Y}$ is an ANR-resolution.*

Proof. First note that the spaces $Y_\gamma = G$ are ANRs, because they are open subsets of ANRs X_λ. If $\mathcal{U}$ is a normal covering of X, then (B1) for $\boldsymbol{p}$ yields a $\lambda \in \Lambda$ and a normal covering $\mathcal{V}$ of X_λ such that $p_\lambda^{-1}(\mathcal{V})$ refines $\mathcal{U}$. However, the pair $\gamma = (\lambda, X_\lambda)$ belongs to Γ, $Y_\gamma = X_\lambda$ and $q_\gamma = p_\lambda$. Therefore, $\mathcal{V}$ is an open covering of Y_γ and $q_\gamma^{-1}(\mathcal{V})$ refines $\mathcal{U}$, which proves (B1) for $\boldsymbol{q}$. Now assume that $\gamma = (\lambda, G) \in \Gamma$ and U is an open neighborhood of the closure of $q_\gamma(X)$ in $Y_\gamma = G$. Clearly, this closure coincides with the closure of $p_\lambda(X)$ in X_λ. Therefore, $\gamma' = (\lambda, U)$ belongs to Γ and $\gamma \leqslant \gamma'$, because $p_{\lambda\lambda}(U) = U \subseteq G$. However, $Y_{\gamma'} = U$ and $q_{\gamma\gamma'}(Y_{\gamma'}) = U$, which shows that $\boldsymbol{q}$ also has property (B2)$'$. □

If X is a compact Hausdorff space and $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ is an inverse system of compact polyhedra, we modify the above construction by taking for Γ all pairs (λ, P), where P is a compact polyhedron which is also a neighborhood of $p_\lambda(X)$ in X_λ. Denoting the resulting objects by $\boldsymbol{Y}'$ and $\boldsymbol{q}'$, we have the following lemma.

Lemma 3. *If X and X_λ are compact Hausdorff spaces and $\boldsymbol{p}: X \to \boldsymbol{X}$ has property* (B1), *then $\boldsymbol{Y}'$ is an inverse system of compact polyhedra and $\boldsymbol{q}': X \to \boldsymbol{Y}'$ is its limit.*

In the next section we will also need the following lemma (only property (B1) will be used).

Lemma 4. *Let $(X_\lambda, \lambda \in \Lambda)$ be a family of topological spaces. Then the spaces X_λ admit ANR-resolutions $\boldsymbol{q}_\lambda : X_\lambda \to \boldsymbol{Y}_\lambda = (Y_\lambda^\mu, q_\lambda^{\mu\mu'}, M)$, $\lambda \in \Lambda$, all indexed by the same ordered set M. If all X_λ are compact Hausdorff spaces, one can achieve that all Y_λ^μ are compact polyhedra and thus, $\boldsymbol{q}_\lambda : X_\lambda \to \boldsymbol{Y}_\lambda$, $\lambda \in \Lambda$, are inverse limits.*

Proof. For $\lambda \in \Lambda$, there exist a cofinite inverse system $\boldsymbol{X}_\lambda = (X_\lambda^\mu, p_\lambda^{\mu\mu'}, M_\lambda)$, consisting of ANRs, and a resolution $\boldsymbol{p}_\lambda = (p_\lambda^\mu) : X_\lambda \to \boldsymbol{X}_\lambda$. Let the set

$$M = \prod_{\lambda \in \Lambda} M_\lambda \tag{7}$$

be endowed with the product ordering. Recall that $m, m' \in M$ are functions $m, m' : \Lambda \to \bigcup_{\lambda \in \Lambda} M_\lambda$ such that $m(\lambda), m'(\lambda) \in M_\lambda$, $\lambda \in \Lambda$, and one has $m \leqslant m'$ if and only if $m(\lambda) \leqslant m'(\lambda)$ in M_λ, for all $\lambda \in \Lambda$. Since the sets M_λ, $\lambda \in \Lambda$, are directed and ordered, so is M. Consider an arbitrary $\lambda \in \Lambda$. We define an inverse system $\boldsymbol{Y}_\lambda = (Y_\lambda^m, q_\lambda^{mm'}, M)$ by putting

$$Y_\lambda^m = X_\lambda^{m(\lambda)}, \tag{8}$$

$$q_\lambda^{mm'} = p_\lambda^{m(\lambda)m'(\lambda)}. \tag{9}$$

Moreover, we define a mapping $\boldsymbol{q}_\lambda = (q_\lambda^m) : X_\lambda \to \boldsymbol{Y}_\lambda$ by putting

$$q_\lambda^m = p_\lambda^{m(\lambda)} : X_\lambda \to Y_\lambda^m. \tag{10}$$

Let us verify that $\boldsymbol{q}_\lambda$ has properties (B1) and (B2)$'$ and thus, it is an ANR-resolution.

If $\mathcal{U}$ is a normal covering of X_λ, then, by (B1) for $\boldsymbol{p}_\lambda$, there is an index $\mu \in M_\lambda$ and there is an open covering $\mathcal{V}$ of X_λ^μ such that $(p_\lambda^\mu)^{-1}(\mathcal{V})$ refines $\mathcal{U}$. Let $m \in M$ be a function with $m(\lambda) = \mu$, having arbitrary values $m(\lambda') \in M_{\lambda'}$, for $\lambda' \neq \lambda$. Then $Y_\lambda^m = X_\lambda^{m(\lambda)} = X_\lambda^\mu$ and $\mathcal{V}$ is an open covering of Y_λ^m. Moreover, $q_\lambda^m = p_\lambda^{m(\lambda)} = p_\lambda^\mu$ and thus, $(q_\lambda^m)^{-1}(\mathcal{V}) = (p_\lambda^\mu)^{-1}(\mathcal{V})$ refines $\mathcal{U}$. This establishes property (B1). To verify (B2)$'$, assume that $m \in M$ and U is an open neighborhood of the closure

$$\overline{q_\lambda^m(X_\lambda)} = \overline{p_\lambda^{m(\lambda)}(X_\lambda)}$$

in $Y_\lambda^m = X_\lambda^{m(\lambda)}$. By property (B2)$'$, for $\boldsymbol{p}_\lambda$, there is an index $\mu' \geqslant m(\lambda)$ from M_λ such that $p_\lambda^{m(\lambda)\mu'}(X_\lambda^{\mu'}) \subseteq U$. Choose a function $m' \in M$ such that $m'(\lambda) = \mu'$ and $m'(\lambda') \geqslant m(\lambda')$, for $\lambda' \neq \lambda$. Then $m' \geqslant m$ and

$$q_\lambda^{mm'}\big(Y_\lambda^{m'}\big) = p_\lambda^{m(\lambda)m'(\lambda)}\big(X_\lambda^{m'(\lambda)}\big) = p_\lambda^{m(\lambda)\mu'}\big(X_\lambda^{\mu'}\big) \subseteq U. \tag{11}$$

In the compact case one chooses for $\boldsymbol{Y}_\lambda$ inverse systems of compact polyhedra and one proceeds as in the general case. □

5. Proof of Theorem 3 (Step 1)

The proof of Theorem 3 proceeds in several steps. Let $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be a cofinite inverse system of topological spaces and let $\boldsymbol{p} = (p_\lambda) : X \to \boldsymbol{X}$ be a mapping. By Lemma 4, there exists a directed ordered set M such that every X_λ, $\lambda \in \Lambda$, admits an ANR-resolution $\boldsymbol{p}_\lambda = (p_\lambda^\mu) : X_\lambda \to \boldsymbol{X}_\lambda = (X_\lambda^\mu, p_\lambda^{\mu\mu'}, M)$, indexed by the set M. For every $\lambda \in \Lambda$, we define a new inverse system $\boldsymbol{Z}_\lambda = (Z_\lambda^\mu, r_\lambda^{\mu\mu'}, M)$ as follows. Put

$$Z_\lambda^\mu = \prod_{\zeta \leqslant \lambda} X_\zeta^\mu, \quad \lambda \in \Lambda,\ \mu \in M. \tag{12}$$

Since Λ is cofinite, Z_λ^μ is the product of a finite collection of ANRs, hence, it is an ANR. For $\mu \leqslant \mu'$, define $r_\lambda^{\mu\mu'} : Z_\lambda^{\mu'} \to Z_\lambda^\mu$ as the mapping

$$r_\lambda^{\mu\mu'} = \prod_{\zeta \leqslant \lambda} p_\zeta^{\mu\mu'} : \prod_{\zeta \leqslant \lambda} X_\zeta^{\mu'} \to \prod_{\zeta \leqslant \lambda} X_\zeta^\mu. \tag{13}$$

We also define a mapping $\boldsymbol{r}_\lambda = (r_\lambda^\mu) : X_\lambda \to \boldsymbol{Z}_\lambda$, where $r_\lambda^\mu : X_\lambda \to Z_\lambda^\mu$, $\lambda \in \Lambda$, is determined by the coordinate mappings $p_\zeta^\mu p_{\zeta\lambda} : X_\lambda \to X_\zeta^\mu$, $\zeta \leqslant \lambda$.

Lemma 5. *For every $\lambda \in \Lambda$, the mapping $\boldsymbol{r}_\lambda : X_\lambda \to \boldsymbol{Z}_\lambda$ has property* (B1).

Proof. Let $\mathcal{U}$ be a normal covering of X_λ. By property (B1) for $\boldsymbol{p}_\lambda$, there exist an index $\mu \in M$ and an open covering $\mathcal{V}$ of X_λ^μ such that $(p_\lambda^\mu)^{-1}(\mathcal{V})$ refines $\mathcal{U}$. Consider the open covering $\mathcal{W}$ of Z_λ^μ, consisting of the sets

$$W = \left(\prod_{\zeta < \lambda} X_\zeta^\mu\right) \times V, \tag{14}$$

where $V \in \mathcal{V}$. Clearly,

$$\big(r_\lambda^\mu\big)^{-1}(\mathcal{W}) = \big(p_\lambda^\mu\big)^{-1}(\mathcal{V}), \tag{15}$$

and thus, $(r_\lambda^\mu)^{-1}(\mathcal{W})$ refines $\mathcal{U}$. □

The advantage of the mappings $\boldsymbol{r}_\lambda$ over the mappings $\boldsymbol{p}_\lambda$ lies in the fact that $\boldsymbol{p}$ and $\boldsymbol{r}_\lambda$, $\lambda \in \Lambda$, admit a system $\boldsymbol{Z}$ and a compatible mapping $\boldsymbol{r} : X \to \boldsymbol{Z}$. Indeed, let $\boldsymbol{Z} = (Z_\nu, r_{\nu\nu'}, N)$, where $N = \Lambda \times M$ is ordered coordinatewise, i.e., if $\nu = (\lambda, \mu)$ and $\nu' = (\lambda', \mu')$, then $\nu \leqslant \nu'$ if and only if $\lambda \leqslant \lambda'$ and $\mu \leqslant \mu'$. Note that $N = \bigcup_{\lambda \in \Lambda} N_\lambda$, where $N_\lambda = \{\lambda\} \times M$ is a copy of M. Clearly, the ordering $\leqslant$ of N is compatible with the orderings $\leqslant$ on Λ and N_λ. For $\nu = (\lambda, \mu)$ we put

$$Z_\nu = Z_\lambda^\mu, \tag{16}$$

and for $\nu \leqslant \nu' = (\lambda', \mu')$, we define $r_{\nu\nu'} : Z_{\nu'} \to Z_\nu$ as the composition of the natural projection

$$Z_{\nu'} = Z_{\lambda'}^{\mu'} = \prod_{\zeta \leqslant \lambda'} X_\zeta^{\mu'} \to \prod_{\zeta \leqslant \lambda} X_\zeta^{\mu'} = Z_\lambda^{\mu'} \tag{17}$$

with the mapping $r_\lambda^{\mu\mu'}: Z_\lambda^{\mu'} \to Z_\lambda^\mu$. It is readily verified that $\boldsymbol{Z}$ is indeed an inverse system. Note that, for $\mu = \mu'$, $r_{\nu\nu'}$ is a surjection. We define $\boldsymbol{r} = (r_\nu): X \to \boldsymbol{Z}$ by putting

$$r_\nu = r_\lambda^\mu p_\lambda, \quad \nu = (\lambda, \mu). \tag{18}$$

Again it is easy to verify that $\boldsymbol{r}$ is a mapping.

Lemma 6. *The mapping $\boldsymbol{r}: X \to \boldsymbol{Z}$ is compatible with the mappings $\boldsymbol{p}: X \to \boldsymbol{X}$ and $\boldsymbol{r}_\lambda: X_\lambda \to \boldsymbol{Z}_\lambda$, $\lambda \in \Lambda$. Moreover, if $\boldsymbol{p}$ has property* (B1)*, then so does $\boldsymbol{r}$.*

Proof. Condition (iii) coincides with (16). Condition (iv) assumes the form

$$r_\lambda^\mu p_{\lambda\lambda'} = r_{\nu\nu'} r_{\lambda'}^{\mu'}. \tag{19}$$

It holds because, for $\zeta \leqslant \lambda$, the coordinate mappings of the left side of (19) equal $p_\zeta^\mu p_{\zeta\lambda} p_{\lambda\lambda'} = p_\zeta^\mu p_{\zeta\lambda'}$, while the corresponding coordinate mappings of the right side equal $p_\zeta^{\mu\mu'} p_\zeta^{\mu'} p_{\zeta\lambda'} = p_\zeta^\mu p_{\zeta\lambda'}$. If $\lambda = \lambda'$, then $r_{\nu\nu'} = r_\lambda^{\mu\mu'}$, which is condition (v). Finally, (vi) assumes the form (18).

Now assume that $\boldsymbol{p}$ has property (B1), and let $\mathcal{U}$ be a normal covering of X. There exist a $\lambda \in \Lambda$ and a normal covering $\mathcal{V}$ of X_λ such that $p_\lambda^{-1}(\mathcal{V})$ refines $\mathcal{U}$. By Lemma 5, there is a $\mu \in M$ and there is an open covering $\mathcal{W}$ of Z_λ^μ such that $(r_\lambda^\mu)^{-1}(\mathcal{W})$ refines $\mathcal{V}$ and thus,

$$\left(r_\lambda^\mu p_\lambda\right)^{-1}(\mathcal{W}) \leqslant \mathcal{U}. \tag{20}$$

However, for $\nu = (\lambda, \mu)$, $Z_\nu = Z_\lambda^\mu$ and $r_\nu = r_\lambda^\mu p_\lambda$ and thus, $r_\nu^{-1}(\mathcal{W}) \leqslant \mathcal{U}$. □

6. Proof of Theorem 3 (Step 2)

We will now improve the construction described in Section 5 and obtain ANR-resolutions $\boldsymbol{s}_\lambda: X_\lambda \to \boldsymbol{S}_\lambda$, $\lambda \in \Lambda$, and an ANR-resolution $\boldsymbol{s}: X \to \boldsymbol{S}$, compatible with $\boldsymbol{p}$ and $\boldsymbol{s}_\lambda$, $\lambda \in \Lambda$. For $\lambda \in \Lambda$, let Γ_λ consist of all pairs $\gamma = (\mu, G)$, where $\mu \in M$ and G is an open neighborhood of the closure of $r_\lambda^\mu(X_\lambda)$ in Z_λ^μ. Put $\gamma \leqslant \gamma' = (\mu', G')$ provided $\mu \leqslant \mu'$ and

$$r_\lambda^{\mu\mu'}(G') \subseteq G. \tag{21}$$

Moreover, put $S_\lambda^\gamma = G$ and let $s_\lambda^{\gamma\gamma'}: S_\lambda^{\gamma'} \to S_\lambda^\gamma$ be the restriction $r_\lambda^{\mu\mu'}|G': G' \to G$. It is well defined because of (21). Then $\boldsymbol{S}_\lambda = (S_\lambda^\gamma, s_\lambda^{\gamma\gamma'}, \Gamma_\lambda)$ is an inverse system of ANR's. We also define a mapping $\boldsymbol{s}_\lambda = (s_\lambda^\gamma): X_\lambda \to \boldsymbol{S}_\lambda$, where the mappings $s_\lambda^\gamma: X_\lambda \to S_\lambda^\gamma$, $\lambda \in \Lambda$, are obtained by restricting the codomain of $r_\lambda^\mu: X_\lambda \to Z_\lambda^\mu$ to G. An immediate consequence of Lemma 2 is the following lemma.

Lemma 7. *For every $\lambda \in \Lambda$, $\boldsymbol{s}_\lambda: X_\lambda \to \boldsymbol{S}_\lambda$ is an ANR-resolution, whose index set is directed and ordered.*

We will now embed the systems $\boldsymbol{S}_\lambda$ in a system $\boldsymbol{S} = (S_\delta, s_{\delta\delta'}, \Delta)$ as follows. Let

$$\Delta = \bigcup_{\lambda\in\Lambda} \Gamma_\lambda. \tag{22}$$

Then every element δ of Δ can be identified with a pair (ν, G), where $\nu = (\lambda, \mu) \in \Lambda \times M$ and $(\mu, G) = \gamma \in \Gamma_\lambda$. Put $\delta \leqslant \delta'$ provided $\nu \leqslant \nu'$ and

$$r_{\nu\nu'}(G') \subseteq G. \tag{23}$$

Clearly, the ordering of Δ is compatible with the orderings of Λ and Γ_λ, $\lambda \in \Lambda$. For $\delta = (\nu, G)$, put $S_\delta = G$ and, for $\delta \leqslant \delta'$, let $s_{\delta\delta'} : S_{\delta'} \to S_\delta$ be the mapping obtained from $r_{\nu\nu'}$ by restricting its domain to G' and its codomain to G. We also consider a mapping $\boldsymbol{s} = (s_\delta) : X \to \boldsymbol{S}$, where $s_\delta : X \to S_\delta$ is obtained from $r_\nu : X \to Z_\nu$ by restricting its codomain to G. Notice that, by (18),

$$r_\nu(X) = r_\lambda^\mu p_\lambda(X) \subseteq r_\lambda^\mu(X_\lambda) \subseteq G.$$

Lemma 8. *The mapping $\boldsymbol{s} : X \to \boldsymbol{S}$ is compatible with $\boldsymbol{p} : X \to \boldsymbol{X}$ and the ANR-resolutions $\boldsymbol{s}_\lambda : X_\lambda \to \boldsymbol{S}_\lambda$, $\lambda \in \Lambda$. Moreover, if $\boldsymbol{p} : X \to \boldsymbol{X}$ is a resolution, then $\boldsymbol{s} : X \to \boldsymbol{S}$ is an ANR-resolution.*

Proof. Compatibility of $\boldsymbol{s}$ is easily verified. In particular, condition (iv) assumes the form $s_\lambda^\gamma p_{\lambda\lambda'} = s_{\delta\delta'} s_{\lambda'}^{\gamma'}$ and it holds because of (19). For $\lambda = \lambda'$, (v) assumes the form $s_{\delta\delta'} = s_\lambda^{\gamma\gamma'}$ and it holds because $r_{\nu\nu'} = r_\lambda^{\mu\mu'}$. Finally, (vi) holds, i.e., $s_\delta = s_\lambda^\gamma p_\lambda$, because of (18).

Now assume that $\boldsymbol{p}$ is a resolution. Let $\mathcal{U}$ be a normal covering of X. By Lemma 6, $\boldsymbol{r}$ has property (B1). Therefore, there exist a $\nu \in N$ and an open covering $\mathcal{V}$ of $Z_\nu = Z_\lambda^\mu$ such that $r_\nu^{-1}(\mathcal{V}) \leqslant \mathcal{U}$. For $G = Z_\lambda^\mu$, we see that $\delta = (\nu, G) \in \Delta$, $S_\delta = Z_\lambda^\mu$ and $s_\delta = r_\nu$. Therefore, $\mathcal{V}$ is an open covering of S_δ such that $s_\delta^{-1}(\mathcal{V}) \leqslant \mathcal{U}$.

To verify condition (B2), consider an index $\delta = (\nu, G) \in \Delta$, an open covering $\mathcal{V}$ of $S_\delta = G \subseteq Z_\nu = Z_\lambda^\mu$ and $\operatorname{St}(s_\delta(X), \mathcal{V})$. Let $\mathcal{V}'$ be an open covering of G, which is a star-refinement of $\mathcal{V}$. Since $\overline{r_\lambda^\mu(X_\lambda)} \subseteq G$, we conclude that

$$\mathcal{U} = \big(r_\lambda^\mu\big)^{-1}(\mathcal{V}') \tag{24}$$

is a normal covering of X_λ. By property (B2) for $\boldsymbol{p}$, there is a $\lambda' \geqslant \lambda$ such that

$$p_{\lambda\lambda'}(X_{\lambda'}) \subseteq \operatorname{St}\big(p_\lambda(X), \mathcal{U}\big). \tag{25}$$

Now consider the pair $\nu' = (\lambda', \mu)$, the mapping $r_{\nu\nu'} : Z_{\nu'} \to Z_\nu$ and the open set

$$G' = r_{\nu\nu'}^{-1}(G) \subseteq Z_{\nu'}. \tag{26}$$

By (19), one has

$$r_{\nu\nu'}\big(\overline{r_{\lambda'}^\mu(X_{\lambda'})}\big) \subseteq \overline{r_{\nu\nu'} r_{\lambda'}^\mu(X_{\lambda'})} = \overline{r_\lambda^\mu p_{\lambda\lambda'}(X_{\lambda'})} \subseteq \overline{r_\lambda^\mu(X_\lambda)} \subseteq G \tag{27}$$

and thus,

$$\overline{r_{\lambda'}^\mu(X_{\lambda'})} \subseteq r_{\nu\nu'}^{-1}(G) = G'. \tag{28}$$

Therefore, $\delta' = (\nu', G') \in \Delta$ and $\delta \leqslant \delta'$. Now consider the open covering $\mathcal{W}$ of G', given by

$$\mathcal{W} = r_{\nu\nu'}^{-1}(\mathcal{V}'). \tag{29}$$

Recall that $s_{\lambda'}^{\gamma}$ is a restriction of $r_{\lambda'}^{\mu}$ and $S_{\lambda'}^{\gamma} = G'$, for $\gamma = (\mu, G') \in \Gamma_{\lambda'}$. Since $\boldsymbol{s}_{\lambda'} : X_{\lambda'} \to \boldsymbol{S}_{\lambda'}$ is a resolution, there exist an index $\mu' \geqslant \mu$ and an open set $G' \subseteq Z_{\lambda'}^{\mu'}$ such that $\delta' = (\nu', G') \in \Delta$, where $\nu' = (\lambda', \mu')$. Moreover,

$$r_{\lambda'}^{\mu\mu'}(G'') \subseteq \mathrm{St}\big(r_{\lambda'}^{\mu}(X_{\lambda'}), \mathcal{W}\big). \tag{30}$$

Consequently, by (29),

$$r_{\nu\nu'} r_{\lambda'}^{\mu\mu'}(G'') \subseteq \mathrm{St}\big(r_{\nu\nu'} r_{\lambda'}^{\mu}(X_{\lambda'}), \mathcal{V}'\big). \tag{31}$$

However, by (19) (for $\mu' = \mu$), by (25) and by (24), one has

$$r_{\nu\nu'} r_{\lambda'}^{\mu}(X_{\lambda'}) = r_{\lambda}^{\mu} p_{\lambda\lambda'}(X_{\lambda'}) \subseteq r_{\lambda}^{\mu}\big(\mathrm{St}\big(p_{\lambda}(X), \mathcal{U}\big)\big) \subseteq \mathrm{St}\big(r_{\lambda}^{\mu} p_{\lambda}(X), \mathcal{V}'\big). \tag{32}$$

Since $s_{\delta\delta'} = r_{\nu\nu'} r_{\lambda'}^{\mu\mu'}$, $S_{\delta'} = G''$ and $s_{\delta} = r_{\lambda}^{\mu} p_{\lambda}$, (30) and (32) yield

$$s_{\delta\delta'}(S_{\delta'}) \subseteq \mathrm{St}\big(\mathrm{St}\big(s_{\delta}(X), \mathcal{V}'\big), \mathcal{V}'\big) \subseteq \mathrm{St}\big(s_{\delta}(X), \mathcal{V}\big), \tag{33}$$

which finally verifies (B2) for $\boldsymbol{s}$. □

Note that Lemma 8 comes very close to proving Theorem 3. Indeed, only cofiniteness of the resolutions $\boldsymbol{s}_{\lambda}$ and $\boldsymbol{s}$ is missing.

7. Proof of Theorem 3 (Step 3)

In this section we will establish additional properties of the resolutions $\boldsymbol{s}_{\lambda}$ and $\boldsymbol{s}$, needed in the final step of the proof. First observe that, in the above construction, we have associated with every index $\gamma = (\mu, G) \in \Gamma_{\lambda}$ an index $\gamma' = (\mu, G') \in \Gamma_{\lambda'}$, where $G' = r_{\nu\nu'}^{-1}(G)$, $\nu = (\lambda, \mu)$ and $\nu' = (\lambda', \mu)$. This defines a function $\rho_{\lambda\lambda'} : \Gamma_{\lambda} \to \Gamma_{\lambda'}$, $\rho_{\lambda\lambda'}(\gamma) = \gamma'$.

Lemma 9. *The function $\rho_{\lambda\lambda'} : \Gamma_{\lambda} \to \Gamma_{\lambda'}$ is strictly increasing. For every $\gamma \in \Gamma_{\lambda}$, $\rho_{\lambda\lambda'}(\gamma) \geqslant \gamma$ in Δ. Moreover, $\rho_{\lambda\lambda} = id$ and, for $\lambda \leqslant \lambda' \leqslant \lambda''$,*

$$\rho_{\lambda'\lambda''}\rho_{\lambda\lambda'} = \rho_{\lambda\lambda''}. \tag{34}$$

Proof. Let $\gamma_1, \gamma_2 \in \Gamma_{\lambda}$ and let $\gamma_1 = (\mu_1, G_1) \leqslant (\mu_2, G_2) = \gamma_2$. Then $\mu_1 \leqslant \mu_2$ and $r_{\lambda}^{\mu_1\mu_2}(G_2) \subseteq G_1$. Therefore, $\rho_{\lambda\lambda'}(\gamma_i) = \gamma'_i$, $i = 1, 2$, where $\gamma'_i = (\mu_i, G'_i)$, $G'_i = r_{\nu_i\nu_i'}^{-1}(G_i)$, $\nu_i = (\lambda, \mu_i)$, $\nu_i' = (\lambda', \mu_i)$, $i = 1, 2$. Note that $\nu_1 \leqslant \nu_2 \leqslant \nu_2'$ and $\nu_1 \leqslant \nu_1' \leqslant \nu_2'$. Therefore, $r_{\nu_1\nu_2} r_{\nu_2\nu_2'} = r_{\nu_1\nu_2'} = r_{\nu_1\nu'_1} r_{\nu'_1\nu'_2}$. Since $r_{\nu_1\nu_2} = r_{\lambda}^{\mu_1\mu_2}$ and $r_{\nu'_1\nu'_2} = r_{\lambda'}^{\mu_1\mu_2}$, we conclude that

$$r_{\lambda}^{\mu_1\mu_2} r_{\nu_2\nu'_2} = r_{\nu_1\nu'_1} r_{\lambda'}^{\mu_1\mu_2}. \tag{35}$$

Since $r_{\nu_2\nu'_2}(G'_2) \subseteq G_2$ and $r_\lambda^{\mu_1\mu_2}(G_2) \subseteq G_1$, we conclude that

$$r_{\nu_1\nu'_1} r_{\lambda'}^{\mu_1\mu_2}\big(G'_2\big) = r_\lambda^{\mu_1\mu_2} r_{\nu_2\nu'_2}\big(G'_2\big) \subseteq r_\lambda^{\mu_1\mu_2}(G_2) \subseteq G_1, \tag{36}$$

and thus,

$$r_{\lambda'}^{\mu_1\mu_2}\big(G'_2\big) \subseteq r_{\nu_1\nu'_1}^{-1}(G_1) = G'_1, \tag{37}$$

which shows that $\gamma'_1 \leqslant \gamma'_2$, i.e., the function $\rho_{\lambda\lambda'}$ is increasing. Now assume that $\gamma_1, \gamma_2 \in \Gamma_\lambda$ and $\rho_{\lambda\lambda'}(\gamma_1) = \rho_{\lambda\lambda'}(\gamma_2) = \gamma' = (\mu, G')$. Then $\mu_1 = \mu_2 = \mu$ and $r_{\nu_1\nu_2}^{-1}(G_1) = r_{\nu_1\nu_2}^{-1}(G_2) = G'$. However, in this case $r_{\nu_1\nu_2}$ is a surjection and thus, $G_1 = G_2$, i.e., $\gamma_1 = \gamma_2$, which shows that $\rho_{\lambda\lambda'}$ is injective. If $\gamma = (\mu, G) \in \Gamma_\lambda$ and $\rho_{\lambda\lambda'}(\gamma) = \gamma' = (\mu, G') \in \Gamma_{\lambda'}$, then $G' = r_{\nu\nu'}^{-1}(G)$ and thus, (23) holds. Consequently, $\gamma \leqslant \rho_{\lambda\lambda'}(\gamma)$ in Δ. Next note that $\rho_{\lambda\lambda} = \mathrm{id}$ is obviously fulfilled. To prove (34), let $\gamma = (\mu, G) \in \Gamma_\lambda$, let $\rho_{\lambda\lambda'}(\gamma) = \gamma'$ and let $\rho_{\lambda'\lambda''}(\gamma') = \gamma''$. Then $\gamma' = (\mu, G') \in \Gamma_{\lambda'}$, $\gamma'' = (\mu, G'') \in \Gamma_{\lambda''}$, where $G' = r_{\nu\nu'}^{-1}(G)$, $G'' = r_{\nu'\nu''}^{-1}(G')$ and $\nu = (\lambda, \mu)$, $\nu' = (\lambda', \mu)$, $\nu'' = (\lambda'', \mu)$. Note that $r_{\nu\nu'} r_{\nu'\nu''} = r_{\nu\nu''}$ because $\nu \leqslant \nu' \leqslant \nu''$. Therefore, $G'' = r_{\nu\nu''}^{-1}(G)$, which shows that $\rho_{\lambda\lambda''}(\gamma) = \gamma'' = \rho_{\lambda'\lambda''}\rho_{\lambda\lambda'}(\gamma)$. □

Remark 1. For $\lambda \leqslant \lambda'$ we can define a mapping $\boldsymbol{p}_{\lambda\lambda'} : \boldsymbol{S}_{\lambda'} \to \boldsymbol{S}_\lambda$ as follows. For the index function we take $\rho_{\lambda\lambda'} : \Gamma_\lambda \to \Gamma_{\lambda'}$. For $p_{\lambda\lambda'}^\gamma : S_{\lambda'}^{\rho_{\lambda\lambda'}(\gamma)} \to S_\lambda^\gamma$, $\gamma = (\mu, G)$, we take $r_{\nu\nu'} : G' \to G$, where $\nu = (\lambda, \mu)$ and $\nu' = (\lambda', \mu)$. By (35), $r_\lambda^{\mu_1\mu_2} p_{\lambda\lambda'}^{\gamma_2} = p_{\lambda\lambda'}^{\gamma_1} r_{\lambda'}^{\mu_1\mu_2}$, for $\gamma_1 \leqslant \gamma_2$, which implies that $\boldsymbol{p}_{\lambda\lambda'} = (\rho_{\lambda\lambda'}, p_{\lambda\lambda'}^\gamma)$ is indeed a mapping of systems. Note that, for $\lambda \leqslant \lambda'$,

$$\boldsymbol{s}_\lambda p_{\lambda\lambda'} = \boldsymbol{p}_{\lambda\lambda'} \boldsymbol{s}_{\lambda'}. \tag{38}$$

Moreover, for $\lambda \leqslant \lambda' \leqslant \lambda''$,

$$\boldsymbol{p}_{\lambda\lambda'} \boldsymbol{p}_{\lambda'\lambda''} = \boldsymbol{p}_{\lambda\lambda''}. \tag{39}$$

Formula (38) shows that $\boldsymbol{p}_{\lambda\lambda'}$ is an ANR-resolution of $p_{\lambda\lambda'}$.

8. Proof of Theorem 3 (Step 4)

Let $\boldsymbol{p} = (p_\lambda) : X \to \boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be a cofinite resolution of topological spaces. Consider the ANR-resolutions $\boldsymbol{s}_\lambda = (s_\lambda^\gamma) : X_\lambda \to \boldsymbol{S}_\lambda = (S_\lambda^\gamma, s_\lambda^{\gamma\gamma'}, \Gamma_\lambda)$, $\lambda \in \Lambda$, and the ANR-resolution $\boldsymbol{s} = (s_\delta) : X \to \boldsymbol{S} = (S_\delta, s_{\delta\delta'}, \Delta)$ from Lemmas 7 and 8. Moreover, consider the functions $\rho_{\lambda\lambda'} : \Gamma_\lambda \to \Gamma_{\lambda'}$ from Lemma 9. Application of the $*$-construction from Lemma 1 to $\boldsymbol{s}_\lambda$ yields cofinite ANR-resolutions $\boldsymbol{q}_\lambda = (q_\lambda^\alpha) : X_\lambda \to \boldsymbol{Y}_\lambda = (Y_\lambda^\alpha, q_\lambda^{\alpha\alpha'}, A_\lambda)$. Here A_λ are disjoint copies of Γ_λ^* and thus, consist of finite subsets $\alpha \subseteq \Gamma_\lambda$ having a terminal element $\overline{\alpha} \in \Gamma_\lambda$, while $Y_\lambda^\alpha = S_\lambda^{\overline{\alpha}}$, $q_\lambda^{\alpha\alpha'} = s_\lambda^{\overline{\alpha}\overline{\alpha'}}$ and $q_\lambda^\alpha = s_\lambda^{\overline{\alpha}}$. Put $B = \bigcup A_\lambda$ and note that every element $\beta \in B$ can be viewed as a pair $\beta = (\lambda, \alpha)$, where $\lambda \in \Lambda$, $\alpha \in \Gamma_\lambda^*$. Order B by putting $\beta \leqslant \beta' = (\lambda', \alpha')$, whenever $\lambda \leqslant \lambda'$ and

$$\rho_{\lambda\lambda'}(\alpha) \subseteq \alpha'. \tag{40}$$

That $\leqslant$ is indeed an ordering is an immediate consequence of Lemma 9. Antisymmetry and directedness of $\leqslant$ are also easily verified. To prove cofiniteness, consider an element $\beta' = (\lambda', \alpha') \in B$ and assume that $\beta = (\lambda, \alpha) \leqslant \beta'$. Then $\lambda \leqslant \lambda'$ and cofiniteness of Λ implies that there are only finitely many possible indices λ. Now fix such a λ. Since α' is a finite set, and by Lemma 9, $\rho_{\lambda\lambda'}$ is an injection, there are only finitely many subsets α satisfying (40).

For $\beta = (\lambda, \alpha) \in B$ put $Y_\beta = S_{\overline{\alpha}}$ and $q_\beta = s_{\overline{\alpha}}$. Moreover, for $\beta \leqslant \beta' = (\lambda', \alpha')$, put $q_{\beta\beta'} = s_{\overline{\alpha}\overline{\alpha'}}$. Note that (40) implies

$$\rho_{\lambda\lambda'}(\overline{\alpha}) = \overline{\rho_{\lambda\lambda'}(\alpha)} \leqslant \overline{\alpha'}. \tag{41}$$

Moreover, by Lemma 9, $\overline{\alpha} \leqslant \rho_{\lambda\lambda'}(\overline{\alpha})$ and thus, $\overline{\alpha} \leqslant \overline{\alpha'}$. Therefore, $q_{\beta\beta'}$ is well defined. It is now easy to see that $\boldsymbol{Y} = (Y_\beta, q_{\beta\beta'}, B)$ is an inverse system and $\boldsymbol{q} = (q_\beta) : X \to \boldsymbol{Y}$ is a mapping. Moreover, $\boldsymbol{q} : X \to \boldsymbol{Y}$ is an ANR-resolution, which is compatible with $\boldsymbol{p}$ and $\boldsymbol{q}_\lambda$, $\lambda \in \Lambda$.

9. Proof of Theorem 1

This proof is a variation of the proof of Theorem 3. In the first step of the proof one uses the compact version of Lemma 4. Note that a product of finitely many compact polyhedra is a compact polyhedron. Therefore, the spaces Z_λ^μ are compact polyhedra. In the second step, instead of Lemma 2, one uses Lemma 3. All other steps remain unchanged.

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