§ 5. Boundary conditions and logarithmic singularity

We now combine the results of § 3 and § 4. In order that the boundary conditions (1.3) be satisfied, \( U_1 \) and \( U_2 \) must be expressible in the form (3.18) by means of sectionally holomorphic functions \( P_1(\xi) \) and \( P_2(\xi) \) which are periodic with period \( 2i\theta \). In order that \( f(r, \varphi) \) shall have the required singularity, \( U_1 \) and \( U_2 \) must make prescribed jumps at \( \text{Im} \, \xi = \varphi_0 \) according to (4.5).

Because of (3.17) this means for \( P_1(\xi) \) and \( P_2(\xi) \) for \( \varepsilon \leq 0 \)

\[
P_1(\eta + i(\varphi_0 + \varepsilon)) - P_1(\eta + i(\varphi_0 - \varepsilon)) = \frac{e^{i\varepsilon \sinh \eta}}{4\pi \phi(\eta + i\varphi_0)}
\]

and

\[
P_2(\eta + i(\varphi_0 + \varepsilon)) - P_2(\eta + i(\varphi_0 - \varepsilon)) = \frac{e^{-i\varepsilon \sinh \eta}}{4\pi \phi(\eta + i\varphi_0)}.
\]

According to Plemelj \(^{7}\) the sectionally holomorphic function

\[
\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t - z} \, dt
\]

makes a jump \( f(z) \) at the real axis. Consequently the function

\[
\frac{1}{4\pi} \int_{-\infty}^{\infty} f(t) \coth \frac{\pi}{2\theta} (t - z) \, dt
\]

makes jumps \( f(\eta) \) at \( \text{Im} \, z = 2m\theta i \), \( \text{Re} \, z = \eta \), where \( m \) is an integer. Moreover this function has the periodicity \( 2\theta i \). It is profitable to take \( \varphi_1 = 0 \) and \( \varphi_2 = \theta \). Then \( P_1(\xi) \) may assume the following form

\[
P_1(\xi) \overset{\text{def}}{=} \frac{1}{16\pi i} \int_{-\infty}^{\infty} \frac{dt}{\phi(t + i\varphi_0)} \left( \frac{e^{i\varepsilon \sinh t}}{\tanh \frac{\pi}{2\theta}^{-1}(t - \xi + i\varphi_0)} - \frac{e^{-i\varepsilon \sinh t}}{\tanh \frac{\pi}{2\theta}^{-1}(t + \xi + i\varphi_0)} \right),
\]

valid for \( \gamma_1 \leq \gamma_2 \).

\(^{7}\) Also cf. Muskhelishvili. Singular Integral Equations, p. 42.
By simple trigonometry (5.3) may be written in the form

\[
P_1(\zeta) = (16\theta\pi i)^{-1} \int_{-\infty}^{+\infty} \frac{dt}{\Phi(t + i\varphi_0)} \cos (r_0 \sinh t) \sinh \pi \theta^{-1} \zeta + i \sin (r_0 \sinh t) \sinh \pi \theta^{-1} (t + i\varphi_0) \left( \cosh \pi \theta^{-1} (t + i\varphi_0) - \cosh \pi \theta^{-1} \zeta \right).
\]

(5.4)

In a similar way, or by applying (3.17), we have

\[
P_2(\zeta) = (16\theta\pi i)^{-1} \int_{-\infty}^{+\infty} \frac{dt}{\Phi(t + i\varphi_0)} \left\{ \frac{e^{\text{ir}_0 \sinh t}}{\tanh \frac{1}{2}\pi \theta^{-1} (t + \zeta + i\varphi_0)} - \frac{e^{-\text{ir}_0 \sinh t}}{\tanh \frac{1}{2}\pi \theta^{-1} (t - \zeta + i\varphi_0)} \right\}.
\]

Substitution of (5.3), (5.4) into (3.8), (2.10) gives ultimately

\[
f(r, \varphi) = (16\theta\pi i)^{-1} \int_{-\infty}^{+\infty} e^{-\text{ir}_0 \sinh \eta} d\eta \int_{-\infty}^{+\infty} e^{\text{ir}_0 \sinh t} \psi(\eta, t) dt
\]

where

\[
\psi(\zeta, s) \overset{\text{def}}{=} \frac{\phi(\zeta)}{\phi(s)} \coth \frac{1}{2}\pi \theta^{-1} (s - \zeta) + \frac{\phi(-\zeta)}{\phi(-s)} \coth \frac{1}{2}\pi \theta^{-1} (s + \zeta) + \frac{\phi(-s)}{\phi(\zeta)} \coth \frac{1}{2}\pi \theta^{-1} (s + \zeta),
\]

(5.6)

with \( \zeta = \eta + i\varphi \) and \( s = t + i\varphi_0 \).

This solution may also be written in the form

\[
f(r, \varphi) = (8\theta\pi)^{-1} \int_{-\infty}^{+\infty} d\eta \int_{-\infty}^{+\infty} dt \frac{\Phi(\zeta)}{\Phi(s)} \left\{ \frac{\sin (r \sinh \eta + r_0 \sinh t)}{\tanh \frac{1}{2}\pi \theta^{-1} (s + \zeta)} - \frac{\sin (r \sinh \eta - r_0 \sinh t)}{\tanh \frac{1}{2}\pi \theta^{-1} (s - \zeta)} \right\}.
\]

(5.7)

This again may be brought in the form

\[
f(r, \varphi) = (4\theta\pi)^{-1} \int_{-\infty}^{+\infty} d\eta \int_{-\infty}^{+\infty} dt \frac{\phi(\zeta)}{\phi(s)} \left[ \frac{\sinh \pi \theta^{-1} s \cos (r \sinh \eta) \sin (r_0 \sinh t)}{\cosh \pi \theta^{-1} s - \cosh \pi \theta^{-1} \zeta} - \frac{\sinh \pi \theta^{-1} \zeta \sin (r \sinh \eta) \cos (r_0 \sinh t)}{\cosh \pi \theta^{-1} s - \cosh \pi \theta^{-1} \zeta} \right].
\]

(5.8)

According to the remark made in the previous section this expression may be simplified in the following way. If \( r_0 \) is replaced by \( -r_0 \) the right-hand side of (5.9) represents a solution of \((\Delta - 1)f = 0\) satisfying the boundary conditions which is regular in \( A \). By addition and subtraction we obtain resp.

\[
f_1(r, \varphi) = \frac{1}{2\theta\pi} \int_{-\infty}^{+\infty} d\eta \int_{-\infty}^{+\infty} dt \frac{\Phi(\zeta)}{\Phi(s)} \cos (r \sinh \eta) \sin (r_0 \sinh t) \frac{\sinh \pi \theta^{-1} s}{\cosh \pi \theta^{-1} s - \cosh \pi \theta^{-1} \zeta}
\]

(5.9)

\[
f_2(r, \varphi) = \frac{1}{2\theta\pi} \int_{-\infty}^{+\infty} d\eta \int_{-\infty}^{+\infty} dt \frac{\Phi(\zeta)}{\Phi(s)} \cos (r \sinh \eta) \sin (r_0 \sinh t) \frac{\sinh \pi \theta^{-1} \zeta}{\cosh \pi \theta^{-1} s - \cosh \pi \theta^{-1} \zeta}.
\]

(5.10)
and
\[
\left\{ \begin{array}{l}
f_2(r, \varphi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{\infty} dt \frac{\phi(\zeta)}{\phi(s)} \\
\sin (r \sinh \eta) \cos (r_0 \sinh t) \frac{\sinh \pi \theta^{-1} \zeta}{\cosh \pi \theta^{-1} \zeta - \cosh \pi \theta^{-1} s}.
\end{array} \right.
\] (5.11)

The former expression gives Green's function in the case \( \gamma_1 \leq \gamma_2 \). The latter expression gives a function of Green in the case \( \gamma_1 > \gamma_2 \). This function is not unique but (5.11) represents that Green's function that vanishes at \( r=0 \). Both expressions (5.10) and (5.11) are clearly convergent with respect to \( \gamma_1 \) and \( t \).

Using the fact that \( \{\phi(\zeta)\}^{-1} \) is obtained from \( \phi(\zeta) \) by changing the signs of both \( \gamma_1 \) and \( \gamma_2 \) we notice the symmetry between (5.10) and (5.11) which are transformed into each other by

\[
\begin{align*}
& r, \varphi, \gamma_1, \gamma_2 \leftrightarrow r_0, \varphi_0, -\gamma_1, -\gamma_2. \\
& \text{Hence we have proved:}
\end{align*}
\]

**Theorem 5.** Let \( f(r, \varphi) \) satisfy the conditions of Theorem 1, the boundary conditions (1.3) and let \( f(r, \varphi) - (2\pi)^{-1} K_0(|z-z_0|) \) be bounded in \( \bar{A} \). Then, if \( \gamma_1 \leq \gamma_2 \), \( f(r, \varphi) \) is uniquely determined and may be represented by (5.10). If \( \gamma_1 > \gamma_2 \), \( f(r, \varphi) \) is not uniquely determined. However, under the condition of vanishing at the vertex \( r=0 \), it is unique and may be represented by (5.11).

With due observance of this supplementary condition the following skew-symmetry relation holds

\[
f(r, \varphi, r_0, \varphi_0, \gamma_1, \gamma_2) = f(r_0, \varphi_0, r, \varphi, -\gamma_1, -\gamma_2).
\]

§ 6. **Appendix**

In this section a general class of functions \( E_k \) will be considered from which the function \( e(\zeta, \gamma) \) which is used in section 3, and which for a given constant \( \theta \) satisfies the functional relation

\[
e(\zeta + i\theta, \gamma) = \frac{\cosh(\zeta + i\gamma)}{\cosh(\zeta - i\gamma)},
\] (6.1)

may be derived by specialization.

We define for \( k \geq 0 \)

\[
\wedge_k \left( \begin{array}{c}
p_0, \ldots, p_k \\
q_0, \ldots, q_k
\end{array} \right) \overset{\text{def}}{=} \frac{1}{2} \int_{-\infty}^{\infty} dt \prod_{0}^{k} \frac{\sinh p_j t}{\sinh q_j t}
\] (6.2)

for any set \( (p_i, q_i) \) for which the integral converges. With small loss of generality we may assume that all \( \text{Re} q_i \) are positive. We define

\[
\mu \overset{\text{def}}{=} \min \text{Re} q_i > 0.
\] (6.3)

Moreover the \( \wedge_k \) are invariant under arbitrary permutations of the \( p_i \) as well as of the \( q_i \). If one of the \( p_i \) equals one of the \( q_i \), say \( p_k = q_k \), we have obviously a reduction of the following kind

\[
\wedge_k \left( \begin{array}{c}
p_0, \ldots, p_{k-1}, z \\
q_0, \ldots, q_{k-1}, z
\end{array} \right) = \wedge_{k-1} \left( \begin{array}{c}
p_0, \ldots, p_{k-1} \\
q_0, \ldots, q_{k-1}
\end{array} \right).
\] (6.4)
As yet the $\wedge_k$ are defined under the restriction

$$\sum_{j=0}^{k} |\text{Re} \ p_j| < \sum_{j=0}^{k} \text{Re} \ q_j.$$  

(6.5)

However, the region of definition can be extended by expansion of the integrand of 6.2 into exponentials, putting $\frac{1}{2} \int_{-\infty}^{\infty} = \int_{0}^{\infty}$.

Introducing the abbreviations

$$P_\varepsilon = P_{\varepsilon_1, \ldots, \varepsilon_k} \overset{\text{def}}{=} \sum_{j=0}^{k} \varepsilon_j \ p_j$$

(6.6)

$$Q_n = Q_{n_0, \ldots, n_k} \overset{\text{def}}{=} \sum_{j=0}^{k} (2n_j + 1) \ q_j$$

where $n_0, \ldots, n_k$ run independently through the non-negative integers, and $\varepsilon_0, \ldots, \varepsilon_k$ through the pair $-1, +1$, we have

$$2^{k+1} \prod_{j=0}^{k} \sinh \ p_j t = \sum_{\varepsilon} \prod_{j=0}^{k} \varepsilon_j \ \exp \ P_\varepsilon t$$

(6.7)

$$2^{-k-1} \prod_{j=0}^{k} (\sinh \ q_j t)^{-1} = \sum_{n} \exp \ Q_n t,$$

whence

(6.8)

$$\wedge_k \left( P_{\varepsilon_0, \ldots, \varepsilon_k}; q_0, \ldots, q_k \right) = \sum_{n} \sum_{\varepsilon} (Q_n - P_\varepsilon)^{-1} \prod_{j=0}^{k} \varepsilon_j.$$

Since

$$\sum_{\varepsilon} (\prod_{j=0}^{k} \varepsilon_j) \cdot P_\varepsilon = 0 \text{ for } l = 0, 1, \ldots, k; \sum_{\varepsilon} (Q_n - P_\varepsilon)^{-1} \prod_{j=0}^{k} \varepsilon_j \text{ is } 0((\sum n_j)^{-k-2})$$

at infinity, so that the multiple sum on the right-hand side of 6.8 is absolutely convergent as soon as no denominator vanishes. Thus 6.8 gives the continuation of 6.2 where the latter does not exist.

Further we define for $k \geq 0$

$$E_k(z) = E_k \left( z; P_{1}, \ldots, P_{k}; q_0, q_1, \ldots, q_k \right) \overset{\text{def}}{=} \exp \left[ - \int_{0}^{z} \wedge_k \left( P_{\varepsilon_0, \ldots, \varepsilon_k}; q_0, q_1, \ldots, q_k \right) d\varepsilon \right]$$

(6.9)

so that $- \wedge_k$ is the logarithmic derivative of $E_k(p_0)$. Evidently

$$\ln E_k \left( z; P_{1}, \ldots, P_{k}; q_0, q_1, \ldots, q_k \right) = -\frac{1}{2} \int_{-\infty}^{\infty} \frac{dt}{t} \left( \cosh tz - 1 \right) \prod_{j=0}^{k} \frac{\sinh p_j t}{\sinh q_j t}$$

(6.10)

for those $z$ for which the right-hand side exists.

If $\varkappa = \sum_{j=0}^{k} \text{Re} \ q_j - \sum_{j=1}^{k} |\text{Re} \ p_j| > 0$ then 6.10 holds in the strip $|\text{Re} \ z| < \varkappa$. From 6.8 we obtain the continuation

$$E_k \left( z; P_{1}, \ldots, P_{k}; q_0, q_1, \ldots, q_k \right) = \prod_{n=0}^{k} \prod_{\varepsilon} \left[ 1 - \frac{z^2}{(Q_n - P_\varepsilon)^2} \right]^{\Pi \varepsilon_j}$$

(6.11)

where now $\varepsilon_j$ runs through $\varepsilon_1, \ldots, \varepsilon_k$.  

\footnote{In what follows $\Pi$ is an abbreviation for $\prod_{j=0}^{k}$ and $\Pi^*$ for $\prod_{j=1}^{k}$.}
The points \( z = \pm (Q_n - P_n) \) are poles if \( \prod_e \chi_j = -1 \), zeros if \( \prod_e \chi_j = +1 \).

We note the particular cases

\[
\wedge_0 \left( \frac{p}{q} \right) = \frac{\pi}{2q} \tan \frac{\pi p}{2q}
\]

and

\[
E_0 \left( \frac{z}{q} \right) = \cos \frac{\pi z}{2q} = \prod_{n=0}^{\infty} \left( 1 - \frac{z^2}{(2n+1)^2 q^2} \right).
\]

From 6.2 the following difference equation may be obtained

\[
\begin{align*}
\wedge_k \left( \frac{p_0 + q_0; p_1, \ldots, p_k}{q_0, q_1, \ldots, q_k} \right) & - \wedge_k \left( \frac{p_0 - q_0; p_1, \ldots, p_k}{q_0, q_1, \ldots, q_k} \right) \\
& = \wedge_{k-1} \left( \frac{p_0 + q_1; p_2, \ldots, p_k}{q_1, q_2, \ldots, q_k} \right) - \wedge_{k-1} \left( \frac{p_0 - q_1; p_2, \ldots, p_k}{q_1, q_2, \ldots, q_k} \right).
\end{align*}
\]

According to the principle of continuation this holds everywhere with the exception of an enumerable set.

Similarly

\[
\begin{align*}
E_k \left( \frac{z + q_0; p_1, \ldots, p_k}{q_0, q_1, \ldots, q_k} \right) & - E_k \left( \frac{z - q_0; p_1, \ldots, p_k}{q_0, q_1, \ldots, q_k} \right) \\
& = E_{k-1} \left( \frac{z + q_1; p_2, \ldots, p_k}{q_1, q_2, \ldots, q_k} \right) - E_{k-1} \left( \frac{z - q_1; p_2, \ldots, p_k}{q_1, q_2, \ldots, q_k} \right).
\end{align*}
\]

Taking in particular \( k = 1, q_1 = \frac{1}{2} \pi \) we obtain in view of 6.13

\[
E_1 \left( \frac{z + q\ell}{q}, \frac{1}{2} \pi \right) / E_1 \left( \frac{z - q\ell}{q}, \frac{1}{2} \pi \right) = \cos (z + \pi) / \cos (z - \pi)
\]

which is the identity underlying 6.1. We need only take

\[
e(z, \gamma) \overset{\text{def}}{=} E_1 \left( \frac{-iz, \gamma}{0, \frac{1}{2} \pi} \right).
\]

If \( \text{Im}(q_1/q_0) \neq 0 \), \( E_1(z) \) is the quotient of two products of two double gamma functions each, having their poles in opposite angles, viz. \( z = Q + p \) and \( z = -Q - p \) for the numerator, \( z = Q - p \) and \( z = -Q + p \) for the denominator.

Hence the pattern of poles of the numerator is obtained from that of the denominator by translations over \( \pm 2p \). In our application 6.17 \( q_1/q_0 \) is real and positive. Then the pattern of poles of \( e(z, \gamma) \) degenerates into countable sets on lines parallel to the imaginary axis viz.

\[
z = \pm i(m \theta + \frac{1}{2} \pi n + \gamma)
\]

where \( m, n \) run through odd positive integers.

We further need some knowledge of the asymptotic behaviour of \( E_k \) for \( \eta = \text{Im} z \rightarrow \pm \infty \). We first consider \( E_k \) in the strip of convergence...
\[ |\text{Re} \ z| < \kappa \text{ of the integral representation 6.10 only. Then it is easily seen}
\text{that } E_k(0|\eta|) \text{ for } \eta \to \pm \infty, \text{ and, more precisely, that}
\]
\[
(6.18) \quad \lim_{|\eta| \to \infty} |\eta|^{-1} \ln E_k \left( \frac{z + i\eta}{q_0}, \frac{p_1, \ldots, p_k}{q_1, \ldots, q_k} \right) = \frac{1}{2\pi} \prod p_i/\prod q_i.
\]

We restrict \( z \) to the strip \( |\text{Re} \ z| < \kappa \) where \( p_0 \) is some constant with \( \text{Re} \ p_0 > 0 \). Then it follows from (6.18) and the corresponding relation for \( k=0 \)
\[
(6.19) \quad \ln E_k \left( \frac{z + i\eta}{q_0}, \frac{p_1, \ldots, p_k}{q_1, \ldots, q_k} \right) - \prod p_i \ln E_0 \left( \frac{z + i\eta}{p_0} \right) = o(|\eta|).
\]

Actually it has a limit for \( |\eta| \to \infty \) which can easily be determined as follows.

The left-hand side of (6.19) equals \(-\frac{1}{2} \int_{-\infty}^{\infty} \{\cosh (z + i\eta) t - 1\} g(t) dt \) if we define
\[
(6.20) \quad g(t) \overset{\text{def}}{=} \frac{1}{t \sinh p_o t} \left\{ \prod p_i \frac{\sinh p_i t}{\sinh q_i t} - \prod q_i \right\}.
\]

Since \( g(t) \) is regular in the strip \( |\text{Im} \ t| < \lambda \) where
\[ \lambda \overset{\text{def}}{=} \min \left( \text{Re} \frac{\pi}{p_0}, \text{Re} \frac{\pi}{q_j} \right) \quad j = 0, \ldots, k, \]
we have
\[
(6.21) \quad \int_{-\infty}^{\infty} \cosh (z + i\eta) t g(t) dt = 0(e^{-\lambda|\eta|}) \text{ for any } \lambda' < \lambda.
\]

Thus we have proved

**Theorem 6.** If \( k \geq 0, \ min \{ \text{Re} p_0, \text{Re} q_0, \text{Re} q_1, \ldots, \text{Re} q_k \} > 0, \)
\[ \kappa \overset{\text{def}}{=} \sum_0^k \text{Re} q_i - \sum_1^k |\text{Re} p_i| > 0, \]
then for \( |\text{Re} \ z| < \min (\text{Re} p_0, \kappa) \) and \( |\text{Im} \ z| \to \infty, \)
\[ \ln E_k \left( \frac{z}{q_0}, \frac{p_1, \ldots, p_k}{q_1, \ldots, q_k} \right) - \prod p_i \ln E_0 \left( \frac{z}{p_0} \right) = \]
\[ \frac{1}{2} \int_{-\infty}^{\infty} \prod \left\{ \frac{\sinh p_i t}{\sinh q_i t} - \frac{p_i}{q_i} \right\} \frac{dt}{t \sinh p_o t} + o(e^{-\lambda'|\text{Im} \ z|}) \]
where
\[ \lambda' < \lambda \overset{\text{def}}{=} \min \left( \frac{\pi}{p_0}, \frac{\pi}{q_0}, \frac{\pi}{q_1}, \ldots, \frac{\pi}{q_k} \right). \]

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REFERENCES


