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NMGRS: Neighborhood-based multigranulation rough sets ☆

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ABSTRACT

Recently, a multigranulation rough set (MGRS) has become a new direction in rough set theory, which is based on multiple binary relations on the universe. However, it is worth noticing that the original MGRS can not be used to discover knowledge from information systems with various domains of attributes. In order to extend the theory of MGRS, the objective of this study is to develop a so-called neighborhood-based multigranulation rough set (NMGRS) in the framework of multigranulation rough sets. Furthermore, by using two different approximating strategies, i.e., seeking common reserving difference and seeking common rejecting difference, we first present optimistic and pessimistic 1-type neighborhood-based multigranulation rough sets and optimistic and pessimistic 2-type neighborhood-based multigranulation rough sets, respectively. Through analyzing several important properties of neighborhood-based multigranulation rough sets, we find that the new rough sets degenerate to the original MGRS when the size of neighborhood equals zero. To obtain covering reducts under neighborhood-based multigranulation rough sets, we then propose a new definition of covering reduct to describe the smallest attribute subset that preserves the consistency of the neighborhood decision system, which can be calculated by Chen's discernibility matrix approach. These results show that the proposed NMGRS largely extends the theory and application of classical MGRS in the context of multiple granulations.

1. Introduction

Rough set theory was originally introduced by Pawlak as a tool to deal with vague, uncertain and incomplete data. It has been found applicable in knowledge discovery, decision analysis, conflict analysis and pattern recognition. One of the applications of rough set theory is to obtain a concept approximation of a universe by two definable subsets called lower and upper approximations. It has been known that lower and upper approximation operators in Pawlak's rough set are defined by an equivalence (indiscernibility) relation [24,25]. With respect to different requirements, in the past ten years, various extensions of Pawlak's rough set have been developed. There are two main methods to generalize it. One method is to extend an equivalence relation to other binary relations, such as a similarity relation, a tolerance relation, and dominance relation [2–5,12,21–23,26,27,31–35,37–40,42,43,51,54–56]. The other is to replace a partition of the universe with a covering and obtained the covering rough sets [1,19,52,57–59]. Particularly, in order to deal with an information system with numerical attribute, Lin [13–17] presented the neighborhood-based rough set in the neighborhood information system which was originated by Sierpinski and Krieger [36]. Yao studied the neighborhood information system and proposed an approximation retrieval model based on it [49]. Furthermore, Hu et al. [6–9] introduced a different neighborhood-based rough set for heterogeneous feature selection, which can be used to deal with an information system with heterogeneous attributes including categorical attributes and numerical attributes.

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From above, however, we can find that all extensional rough sets including neighborhood rough sets are constructed on the basis of a single binary relation, which limit the applications of rough set theory. In the view of granular computing, they are constructed on a single granulation. Accordingly, Qian et al. [28,29] proposed multigranulation rough set in complete information system according to a user's different requirements or targets of problem solving. One of important contributions in MGRS is to describe the lower and upper approximations of the rough set by multiple equivalence relations (multiple granulations) instead of a single equivalence relation (a single granulation). In their papers, Qian et al. said that the MGRS are useful in the following cases [28]:

1. We cannot perform the intersection operations between their quotient sets and the target concept cannot be approximated by using $U/(P \cup Q)$ which is called a single granulation in those papers.
2. In the process of some decision making, the decision or the view of each of decision makers may be independent for the same project (or a sample, object and element) in the universe. In this situation, the intersection operations between any two quotient sets will be redundant for decision making.
3. Extract decision rules from distributive information systems and groups of intelligent agents through using rough set approaches.

Since then, many researchers have extended the classical MGRS by using various generalized binary relations. For instance, Qian et al. [29] presented a multigranulation rough set based on multiple tolerance relations in incomplete information systems. Lin et al. [18] proposed a covering-based pessimistic multigranulation rough set, Xu et al. [45] proposed another generalized version, called variable precision multigranulation rough set, and Yang et al. [47] proposed a multigranulation rough set based on a fuzzy binary relation. In fact, the basic idea of multi-granulation has been also discussed by Khan et al. in Ref. [11]. However, the existing multigranulation rough set theory can not be used to describe the inconsistency coming from a neighborhood information system which consists of numerical and categorical attributes. In order to deal with multi-granulation information with heterogeneous attributes, it is necessary to introduce multiple neighborhood relations into a neighborhood information system, and further develop a so-called neighborhood-based multigranulation rough sets (NMGRS). In particular, we will present two types of neighborhood multigranulation rough sets, 1-type NMGRS and 2-type NMGRS. For each NMGRS, we investigate its optimistic version and pessimistic version, respectively, and discuss their properties. In addition, we also given a new definition of covering reducts and propose its calculating method, which is based on a discernibility matrix approach proposed in the literature [1].

The paper is organized as follows. In Section 2, we briefly reviewed some basic concepts of MGRS. In Section 3, a rough set based on multi neighborhood relations is presented, called the neighborhood-based multigranulation rough sets (NMGRS), and some of its important properties are investigated. In Section 4, we first introduce a concept of covering reduct of the neighborhood-based multigranulation rough sets and then employ Chen's discernibility matrix to reduce attributes in the neighborhood-based multigranulation rough sets. Finally, Section 5 concludes this study.

2. Preliminary knowledge on rough sets

In this section, we review some basic concepts, which includes Pawlak's rough set, multigranulation rough sets, and neighborhood-based rough sets (see [8,13,24,28]).

2.1. Pawlak's rough set

In many data analysis applications, knowledge and information presentation in rough set theory are realized by an information system. An information system is a tuple: $S = (U, AT, \{V_a | a \in AT\}, \{f_a | a \in AT\})$, where U is a finite nonempty set of objects, AT is a finite nonempty set of attributes, V_a is a nonempty set of values of $a \in AT$, and $f_a : U \to V_a$ is an information function that maps an object in U to exactly one value in V_a.

In particular, a target information system is given by $S = (U, AT \cup D, \{V_a | a \in AT\}, \{f_a | a \in AT\})$, where AT is a set of condition attributes describing the objects, and D is a set of decision attributes that indicate the classes of objects. In general, we often consider the decision information system with only one decision attribute, because an information system with multi decision attributes can be easily transformed into a system with a single decision attribute by considering the Cartesian product of the original decision attributes [35,50].

Each nonempty subset $B \subseteq AT$ determines an indiscernibility relation, defined as $R_B = \{(x, y) \in U \times U \mid f_a(x) = f_a(y), \forall a \in B\}$.

The relation R_B partitions U into some equivalence classes given by $U/R_B = \{[x]_B | x \in U\}$, where $[x]_B = \{y \in U | (x, y) \in R_B\}$.

For $X \subseteq U$, sets $\underline{R_B}X = \cup\{Y \in U/IND(B) \mid Y \subseteq X\}$ and $\overline{R_B}X = \cup\{Y \in U/IND(B) \mid Y \cap X \neq \emptyset\}$ are called the lower and the upper approximations of X with respect to B, respectively.

The *area of uncertainty or boundary region* is

$$Bn(X) = \overline{R_B}X \setminus \underline{R_B}X.$$

In order to measure the imprecision of a rough set, Pawlak [25] recommended for $X \neq \emptyset$, the ratio $\alpha_{R_B}(X) = \frac{|\underline{R_B}X|}{|\overline{R_B}X|}$, which is called the accuracy measure of X by R_B. Roughness is calculated by subtracting the accuracy from α_{R_B}: $\rho_{R_B}(X) = 1 - \alpha_{R_B}(X)$.

2.2. Multigranulation rough sets (MGRS)

In recent years, Qian et al. [28] have proposed a new extension of Pawlak rough set, i.e., multigranulation rough sets (MGRS). In the multigranulation rough set theory, a target concept is approximated by multiple binary relations. Furthermore, two kinds of important multigranulation rough sets were presented with optimistic and pessimistic strategies, which are called optimistic multigranulation rough sets and pessimistic multigranulation rough sets, respectively [28,30].

Definition 1. Let $S = (U, AT, f)$ be an information system, $A_1, A_2, \ldots, A_m \subseteq AT$, and $X \subseteq U$. The optimistic lower approximation and the upper approximation of X with respect to $A_1, A_2, \ldots, A_m$ are denoted by $\underline{\sum_{i=1}^{m} A_i}^O X$ and $\overline{\sum_{i=1}^{m} A_i}^O X$, respectively, where

$$\underline{\sum_{i=1}^{m} A_i}^O X = \bigcup\{x \in U \mid [x]_{A_i} \subseteq X, \textit{for some } i \leq m\}, \tag{1}$$

$$\overline{\sum_{i=1}^{m} A_i}^O X = \sim \underline{\sum_{i=1}^{m} A_i}^O (\sim X). \tag{2}$$

Then $(\underline{\sum_{i=1}^{m} A_i}^O X, \overline{\sum_{i=1}^{m} A_i}^O X)$ is the optimistic MGRS [24]. The word "optimistic" is used to express the idea that in multiple independent granular structures, one needs only at least one granular structure to satisfy with the inclusion condition between an equivalence class and a target concept. The upper approximation of the optimistic multigranulation rough set is defined by the complement of the lower approximation.
And the *area of uncertainty* or *boundary region* in MGRS is

$$Bn^O_{\sum_{i=1}^{m}}(X) = \overline{\sum_{i=1}^{m} A_i}^O X \Big\backslash \underline{\sum_{i=1}^{m} A_i}^O X .$$

The definition of pessimistic MGRS [30] is defined as follows:

$$\underline{\sum_{i=1}^{m} A_i}^P (X) = \{x \in U \mid [x]_{A_1} \subseteq X \wedge [x]_{A_2} \subseteq X \wedge \cdots \wedge [x]_{A_m} \subseteq X\}, \tag{3}$$

$$\overline{\sum_{i=1}^{m} A_i}^P (X) = \sim \underline{\sum_{i=1}^{m} A_i}^P (\sim X). \tag{4}$$

Then $(\underline{\sum_{i=1}^{m} A_i}^P X, \overline{\sum_{i=1}^{m} A_i}^P X)$ is the pessimistic MGRS [30]. The word "pessimistic" is used to express the idea that in multiple independent granular structures, one needs all granular structures to satisfy with the inclusion condition between an equivalence class and a target concept. The upper approximation of the optimistic multigranulation rough set is also defined by the complement of the lower approximation. And the *area of uncertainty* or *boundary region* in MGRS is

$$BN^P_{\sum_{i=1}^{m} A_i}(X) = \overline{\sum_{i=1}^{m} A_i}^P (X) \Big\backslash \underline{\sum_{i=1}^{m} A_i}^P (X).$$

2.3. Neighborhood-based rough sets

In order to make Pawlak's rough set deal with the information system with heterogeneous attributes, T. Y. Lin et al. [14] gave the concept of neighborhood and proposed neighborhood-based rough sets. Since then, many researchers further studied the theory of the neighborhood-based rough set [6–10,15,41,48]. In this section, we especially introduce some concepts of neighborhood-based rough sets proposed by Hu [8].

Definition 2. Let $S = (U, AT, f)$ be an information system with heterogeneous attributes, $X \subseteq U$ and $A, B \subseteq AT$ are categorical and numerical attributes, respectively. The neighborhood granules of objects x induced by $A, B, A \cup B$ are defined as

Table 1
A target information system with heterogeneous attributes.

	Outlook	Ultra-ray	Temperature	Humidity	Windy	Intensity	Play
x_1	Sunny	Weak	85	85	False	85	No
x_2	Sunny	Strong	80	90	True	95	No
x_3	Overcast	Strong	86	85	False	82	Yes
x_4	Rainy	Middle	70	96	False	91	Yes
x_5	Rainy	Middle	68	80	False	80	Yes
x_6	Rainy	Weak	65	70	True	75	No
x_7	Overcast	Middle	64	65	True	63	Yes
x_8	Sunny	Strong	72	95	False	90	No

(1) $n_A(x) = \{x_i \in U \mid d_A(x, x_i) = 0\}$;
(2) $n_B(x) = \{x_i \in U \mid d_B(x, x_i) \leq \delta\}$;
(3) $n_{(A \cup B)}(x) = \{x_i \in U \mid d_A(x, x_i) = 0 \wedge d_B(x, x_i) \leq \delta\}$,

where d is a distance [40,44] between x and y, δ is a nonnegative number, and "$\wedge$" means "and" operator. (1) is designed for numerical attributes; (2) is designed for categorical attributes, and (3) is designed for heterogeneous attributes, namely, categorical and numerical attributes.

A neighborhood relation N on the universe can be written as a relation matrix $M(N) = (r_{ij})_{n \times n}$, where

$$r_{ij} = \begin{cases} 1, & d(x_i, x_j) \leq \delta, \\ 0, & \text{otherwise.} \end{cases}$$

Accordingly, we say (U, N) a neighborhood approximation space. If there is an attribute subset in the system generating a neighborhood relation on the universe, we can regard this system as a neighborhood information system, denoted by $NIS = (U, AT, N)$, where U is a nonempty finite set and AT is an attribute set. In particular, a neighborhood information system is also called a neighborhood decision information system if we distinguish condition attributes and decision attributes, denoted by $NIS = (U, AT \cup D, N)$.

Example 1. Here, we use an example to illustrate some notions of an information system which consists of categorical and numerical attributes. Table 1 shows data set *play tennis* with heterogeneous attributes, namely, categorical and numerical attributes, where $U = \{x_1, x_2, \ldots, x_8\}$, AT= {*outlook, ultra-ray, temperature, humidity, intensity, windy*}, D= {*play*}. Especially, *Outlook, ultra-ray* , and *windy* are categorical condition attributes, *temperature*, *humidity* and *intensity* are numerical condition attributes, and *play* is a decision attribute. In the sequel, *O*, *U*, *T*, *H*, *W*, *I* will displace *outlook*, *ultra-ray*, *temperature*, *humidity*, *windy*, and *intensity*, respectively. In Table 1, in order to reduce sample classification error rate caused by inconsistent dimension, numerical attribute values are standardized into [0, 1] for computing, see [7].

Definition 3. Let (U, N) be a neighborhood approximation space. For any $X \subseteq U$, the lower approximation and upper approximation of X in U are defined as:

$$\underline{N}X = \{x \in U \mid n(x) \subseteq X\}, \tag{5}$$

$$\overline{N}X = \{x \in U \mid n(x) \cap X \neq \phi\}. \tag{6}$$

One calls $(\underline{N}X, \overline{N}X)$ a neighborhood rough set. Obviously, $\underline{N}X \subseteq X \subseteq \overline{N}X$. The *boundary region* of X in the approximation space is defined as $Bn(X) = \overline{N}X \setminus \underline{N}X$.

The size of boundary region reflects the degree of roughness of set X in the neighborhood approximation space (U, N). In the neighborhood rough set, δ can be considered as a parameter to control the granularity level at which we analyze the classification task.

3. Neighborhood multigranulation rough sets

In this section, we extend the classical MGRS to neighborhood-based multigranulation rough sets (NMGRS). We propose two types of neighborhood multigranulation rough sets according to different representations of neighborhood information granules by Definition 3. In the first case, a granular space induced by a neighborhood relation on the universe can be regarded as a set of mixed information granules induced by both a similarity relation and an indiscernibility relation in the view of granular computing [53]. If the approximations of a target concept are described by these mixed information granules, we call this rough set a 1-type neighborhood multigranulation rough set in this paper, denoted by 1-type NMGRS. In the second case, if the approximations of a target concept are described by multiple neighborhood relations, we call this rough set a 2-type neighborhood multigranulation rough set, denoted by 2-type NMGRS.

In the following, we will give the definitions of optimistic 1-type NMGRS and optimistic 2-type NMGRS and the definitions of pessimistic versions, respectively. Conveniently, we mainly discuss the properties of the optimistic versions. The pessimistic versions can be done similarly. We hence omit them in this paper.

3.1. 1-type neighborhood multigranulation rough sets (1-type NMGRS)

As we know, the incomplete MGRS is based on multiple tolerance relations, which sometimes can be also regarded as a neighborhood relation [7]. However, these existing multigranulation versions still can not deal with data sets with heterogeneous attributes. Therefore, it is necessary to develop a new rough set based on multiple neighborhood relations to deal with hybrid data. Simply, we first investigate the approximation of a target set induced by mixed granules on the universe, which can be regarded as a simple neighborhood multigranulation rough set, just 1-type NMGRS.

Definition 4 (*1-type NMGRS*). Let $NIS = (U, AT, N)$ be a neighborhood information system, $A \subseteq AT$ a categorical attribute set, $B \subseteq AT$ a numerical attribute set, $A \cup B \subseteq AT$ a mixed attribute set; U/A, U/B, and $U/(A \cup B)$ represent a partition and two coverings of the universe U, respectively. For any $X \subseteq U$, the optimistic multigranulation lower and upper approximations of X with respect to A, B in U are defined in the following:

$$\underline{(A+B)}^O X = \{x \in U \mid n_A(x) \subseteq X \vee n_B(x) \subseteq X\}, \tag{7}$$

$$\overline{(A+B)}^O X = \sim \underline{(A+B)}^O (\sim X). \tag{8}$$

By Definition 4, we can see that the lower and upper approximations of X of optimistic 1-type NMGRS satisfy duality property, i.e., the upper approximation can be defined by the complement of the lower approximation. The *area of uncertainty or boundary region* is defined as

$$Bn^O_{(A+B)}(X) = \overline{(A+B)}^O X \setminus \underline{(A+B)}^O X.$$

We call $(\underline{(A+B)}^O X, \overline{(A+B)}^O X)$ an optimistic 1-type NMGRS. Obviously, the optimistic 1-type NMGRS can degenerate into the original optimistic multigranulation while $\delta = 0$. The original MGRS is a special instance of 1-type NMGRS.

Theorem 1. *Let $NIS = (U, AT, N)$ be a neighborhood information system, $A, B \subseteq AT$ categorical and numerical attribute subsets, respectively. For any $X \subseteq U$, then*
$$\overline{(A+B)}^O X = \{x \in U \mid (n_A(x) \cap X \neq \emptyset) \wedge (n_B(x) \cap X \neq \emptyset)\}.$$

Proof. By Definition 4, we have that
$$\begin{aligned} x \in \overline{(A+B)}^O X &\Leftrightarrow x \in \sim \underline{(A+B)}^O (\sim X) \\ &\Leftrightarrow x \notin \underline{(A+B)}^O (\sim X) \\ &\Leftrightarrow n_A(x) \nsubseteq (\sim X) \wedge n_B(x) \nsubseteq (\sim X) \\ &\Leftrightarrow n_A(x) \cap X \neq \emptyset \wedge n_B(x) \cap X \neq \emptyset. \end{aligned}$$
This completes the proof. □

By Theorem 1, we can see that though the optimistic multigranulation upper approximation is defined by the complement of the optimistic multigranulation lower approximation, it also can be constructed by objects with nonempty intersection with the target concept in terms of each granular structure.

Proposition 1. *Let $NIS = (U, AT, N)$ be a neighborhood information system, $\forall A, B \subseteq AT$, and $\forall X \subseteq U$, then*

(1) $\underline{(A+B)}^O X = \underline{A}X \cup \underline{B}X$,
(2) $\overline{(A+B)}^O X = \overline{A}X \cap \overline{B}X$.

Proof. (1) Let $x \in \underline{A}X$ $(x \in U)$, note that $\underline{A}X = \{x \in U \mid n_A(x) \subseteq X\}$, but $x \in \underline{(A+B)}^O X$, hence $\underline{A}X \subseteq \underline{(A+B)}^O X$. Similarly, $\underline{B}X \subseteq \underline{(A+B)}^O X$. So $\underline{(A+B)}^O X \supseteq \underline{A}X \cup \underline{B}X$. And, for $x \in \underline{(A+B)}^O X$, from (7), we have either $n_A(x) \subseteq X$, then $x \in \underline{A}X$ or $n_B(x) \subseteq X$, then $x \in \underline{B}X$, therefore $x \in \underline{A}X \cup \underline{B}X$, namely, $\underline{(A+B)}^O X \subseteq \underline{A}X \cup \underline{B}X$. Therefore, $\underline{(A+B)}^O X = \underline{A}X \cup \underline{B}X$.

(2) From above and (8), we have $\overline{(A+B)}^O X = \sim \underline{(A+B)}^O (\sim X) = \sim (\underline{A}(\sim X) \cup \underline{B}(\sim X)) = \overline{A}(X) \cap \overline{B}(X)$.
This completes the proof. □

Corollary 1. $Bn^O_{(A+B)}(X) \subseteq Bn_A(X) \cup Bn_B(X)$.

In what follows, we will illuminate the difference between the 1-type NMGRS and classical Pawlak's rough sets through employing Example 2.

Example 2 (*Continued from Example 1*). Let $X = \{x_1, x_2, x_3, x_7\}$. Here we compute the neighborhood granules of samples with $\delta = 0.1$. A partition and two coverings are induced from Table 1 as follows:

Let $A = \{O, W\} \subseteq AT$ be a categorical attribute subset. According to Definition 2, the information granules induced by A are listed. $n_A(x_1) = \{x_1, x_8\} = n_A\{x_8\}$, $n_A(x_2) = \{x_2\}$, $n_A(x_3) = \{x_3\}$, $n_A(x_4) = \{x_4, x_5\} = n_A\{x_5\}$,$n_A(x_6) = \{x_6\}$, $n_A(x_7) = \{x_7\}$. Obviously, they form a covering of the universe, i.e., $U/A = \{\{x_1, x_8\}, \{x_2\}, \{x_3\}, \{x_4, x_5\}, \{x_5, x_4\}, \{x_6\}, \{x_7\}, \{x_8, x_1\}\}$ which is a granular structure on U, then $\underline{A}X = \{x_2, x_3, x_7\}$ and $\overline{A}X = \{x_1, x_2, x_3, x_7, x_8\}$.

Let $B = \{T, H\} \subseteq AT$ be a numerical attribute subset. Then, we have that $n_B(x_1) = \{x_1, x_2, x_3\} = n_B(x_3)$, $n_B(x_2) = \{x_2, x_1, x_3, x_4, x_8\}$, $n_B(x_4) = \{x_4, x_2, x_8\}$, $n_B(x_5) = \{x_5, x_6\}$, $n_B(x_6) = \{x_6, x_5, x_7\}$, $n_B(x_7) = \{x_7, x_6\}$, $n_B(x_8) = \{x_8, x_2, x_4\}$. Similarly, they form a covering of the universe, i.e., $U/B = \{\{x_1, x_2, x_3\}, \{x_2, x_1, x_3, x_4, x_8\}, \{x_3, x_1, x_2\}, \{x_4, x_2, x_8\}, \{x_5, x_6\}, \{x_6, x_5, x_7\}, \{x_7, x_6\}, \{x_8, x_2, x_4\}\}$. Therefore we have that $\underline{B}X = \{x_1, x_3\}$, $\overline{B}X = \{x_1, x_2, x_3, x_4, x_6, x_7, x_8\}$.

Based on U/A and U/B induced by A and B, we have the optimistic lower and upper approximations of X in U, respectively, $\underline{(A+B)^O}X = \{x_1, x_2, x_3, x_7\} = \underline{A}(X) \cup \underline{B}(X)$, $\overline{(A+B)}^O X = \sim \underline{(A+B)}^O(\sim X) = \{x_1, x_2, x_3, x_7, x_8\} = \overline{A}(X) \cap \overline{B}(X)$.

Furthermore, By the term (3) in Definition 2, we have that $U/(A \cup B) = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_5\}, \{x_6\}, \{x_7\}, \{x_8\}\}$. Obviously, $U/(A \cup B)$ also forms a covering of the universe U. Then, we have $\underline{(A \cup B)}X = \{x_1, x_2, x_3, x_7\}$, $\overline{(A \cup B)}X = \{x_1, x_2, x_3, x_7\}$. Easily, $\underline{(A \cup B)}X \supseteq \underline{(A+B)}^O X$, $\overline{(A \cup B)}X \subseteq \overline{(A+B)}^O X$.

As a result of this example, we have the following results.

Proposition 2. *Let $NIS = (U, AT, N)$ be a neighborhood information system, $A, B \subseteq AT$ categorical and numerical attribute subsets, respectively. For any $X \subseteq U$, then*

(1) $\underline{(A+B)}^O X \subseteq \underline{(A \cup B)}X$,
(2) $\overline{(A+B)}^O X \supseteq \overline{(A \cup B)}X$.

Proof. (1) For any $x \in \underline{(A+B)}^O X$, by Definition 4, it follows that $x \in n_A(x)$ and $x \in n_B(x)$. Hence $x \in n_A(x) \cap n_B(x)$. But $n_A(x) \cap n_B(x) \subseteq n_{(A \cup B)}(x)$ for all $x \in U$. Therefore, $x \in \underline{(A \cup B)}X$, i.e. $\underline{(A+B)}^O X \subseteq \underline{(A \cup B)}X$.

(2) From Pawlak's rough set theory, we know $\overline{(A \cup B)}X = \sim \underline{(A \cup B)}(\sim X)$, applying the result of (1), we have that $\overline{(A \cup B)}(\sim X) \supseteq \underline{(A+B)}^O(\sim X)$. Hence, $\sim \overline{(A \cup B)}(\sim X) \subseteq \sim \underline{(A+B)}^O(\sim X)$, i.e., $\overline{(A+B)}^O X \supseteq \overline{(A \cup B)}X$. This completes the proof. □

Proposition 2 shows that the optimistic lower approximation is not more than the Pawlak's lower approximation, while the optimistic upper approximation is not less than the Pawlak's upper approximation.

Corollary 2. $Bn^O_{(A+B)}(X) \supseteq Bn_{(A \cup B)}(X)$.

Corollary 3. *Let $NIS = (U, AT, N)$ be a neighborhood information system, $A, B \subseteq AT$ categorical and numerical attribute subsets, respectively, and $X \subseteq U$. Then*

$$\alpha_{(A \cup B)}(X) \geq \alpha^O_{(A+B)}(X).$$

Proof. They are straightforward from the definition of accuracy measure of X.

In what follows, we further clarify the difference between multigranulation rough sets and classical rough sets. It can be illustranted from the following four aspects.

(1) Multigranulation rough set theory is a strategy for information fusion through single granulation rough sets. Here, neighborhood-based multigranulation rough sets is a simple information fusion method by operations '$\vee$'(conjunction) or'$\wedge$'(disjunction).
(2) In fact, there are some other fusion strategies [20,45–47]. For instance, in the literature [45], Xu et al. introduced a supporting characteristic function and a variable precision parameter β called information level to investigate a target concept under majority granulations.
(3) It is Proposition 2 that embodies the difference between classic rough sets and multigranulation rough sets.
(4) With regard to some special information systems, such as multi-source information systems, distributive information systems and groups of intelligent agents, the classical rough sets can not deal with these information systems, but multigranulation rough sets can. □

Proposition 3. *Let $NIS = (U, AT, N)$ be a neighborhood information system, $A, B \subseteq AT$ categorical and numerical attribute subsets, respectively, $X \subseteq U$, and δ_1, δ_2 two nonnegative numbers. If $\delta_1 \geq \delta_2$, then*

(1) $\underline{(A+B)_{\delta_1}}^O X \subseteq \underline{(A+B)_{\delta_2}}^O X$,
(2) $\overline{(A+B)_{\delta_1}}^O X \supseteq \overline{(A+B)_{\delta_2}}^O X$.

Proof. (1) Let $X \subseteq U$, assume that $\underline{(A+B)_\delta}^O X = \{x \mid n_A^\delta(x) \subseteq X \vee n_B^\delta(x) \subseteq X\}$. If $\delta_1 \geq \delta_2$, we obviously have $n_A^{\delta_1}(x) \supseteq n_A^{\delta_2}(x)$ and $n_B^{\delta_1}(x) \supseteq n_B^{\delta_2}(x)$. Then, there must exist $x_0 \in X \subseteq U$, such that $n_A^{\delta_1}(x_0) \subseteq X$ but $n_A^{\delta_1}(x_0) \subseteq X$. Similarly, there also exists $y_0 \in X \subseteq U$, such that $n_A^{\delta_2}(y_0) \subseteq X$ but $n_A^{\delta_1}(y_0) \subseteq X$. Based on Definition 4, we have $\underline{(A+B)_{\delta_1}}^O X \subseteq \underline{(A+B)_{\delta_2}}^O X$.

(2) Similarly, we can prove that $\overline{(A+B)_{\delta_1}}^O X \supseteq \overline{(A+B)_{\delta_2}}^O X$.
This completes the proof. □

Proposition 3 shows that the size of lower approximation of X under a 1-type optimistic neighborhood-based multigranuation rough set will become much larger with the value of the parameter δ being much bigger. Its upper approximation has the inverse conclusion.

Proposition 4. *Let $NIS = (U, AT, N)$ be a neighborhood information system, $A, B \subseteq AT$ categorical and numerical attribute subsets, respectively, and $X, Y \subseteq U$. If $X \subseteq Y$, then*

(1) $\underline{(A+B)}^O X \subseteq \underline{(A+B)}^O Y$,
(2) $\overline{(A+B)}^O X \subseteq \overline{(A+B)}^O Y$.

Proof. (1) If $X \subseteq Y$, then $X \cap Y = X$. Then we have

$$\begin{aligned}\underline{(A+B)}^O X &= \underline{(A+B)}^O (X \cap Y)\\ &= \underline{A}(X \cap Y) \cup \underline{B}(X \cap Y)\\ &= ((\underline{A}X \cap \underline{A}Y) \cup (\underline{B}X \cap (\underline{B}Y)\\ &= ((\underline{A}X \cap \underline{A}Y) \cup \underline{B}X) \cap ((\underline{A}X \cap \underline{A}Y) \cup \underline{B}Y)\\ &= ((\underline{A}X \cup \underline{B}X) \cap (\underline{A}Y \cup \underline{B}X)) \cap (\underline{A}X \cup \underline{B}Y) \cap (\underline{A}Y \cup \underline{B}Y)\\ &= ((\underline{(A+B)}^O X \cap \underline{(A+B)}^O Y) \cap ((\underline{A}Y \cup \underline{B}X) \cap (\underline{A}X \cup \underline{B}Y))\\ &\subseteq ((\underline{(A+B)}^O X \cap \underline{(A+B)}^O Y) \subseteq \underline{(A+B)}^O Y.\end{aligned}$$

Hence, $\underline{(A+B)}^O X \subseteq \underline{(A+B)}^O Y$.

(2) If $X \subseteq Y$, then $X \cup Y = Y$. Then we have

$$\begin{aligned}\overline{(A+B)}^O Y &= \overline{(A+B)}^O (X \cup Y)\\ &= \overline{A}(X \cup Y) \cap \overline{B}(X \cup Y)\\ &= (\overline{A}X \cup \overline{A}Y) \cap (\overline{B}X \cup \overline{B}Y)\\ &= ((\overline{A}X \cup \overline{A}Y) \cap \overline{B}X) \cup ((\overline{A}X \cup \overline{A}Y) \cap \overline{B}Y)\\ &= (\overline{A}X \cap \overline{B}X) \cup (\overline{A}Y \cap \overline{B}X) \cup (\overline{A}X \cap \overline{B}Y) \cup (\overline{A}X \cap \overline{B}Y)\\ &= \overline{(A+B)}^O X \cup \overline{(A+B)}^O Y \cup (\overline{A}X \cap \overline{B}Y) \cup (\overline{A}X \cup \overline{B}Y)\\ &\supseteq ((\overline{(A+B)}^O X \cup \overline{(A+B)}^O Y) \supseteq \overline{(A+B)}^O X.\end{aligned}$$

Hence, $\overline{(A+B)}^O Y \supseteq \overline{(A+B)}^O X$.
This completes the proof. □

Corollary 4. *Let $NIS = (U, AT, N)$ be a neighborhood information system, $A, B \subseteq AT$ categorical and numerical attribute subsets, respectively, and $X \subseteq U$. If δ_1, δ_2 are two nonnegative numbers and $\delta_1 \geq \delta_2$, then*

$$\alpha^O_{(A+B)_{\delta_1}}(X) \leq \alpha^O_{(A+B)_{\delta_2}}(X).$$

Proof. It is straightforward from Proposition 3.

Similar to the classical pessimistic MGRS's definition [26], let $NIS = (U, AT, N)$ be a neighborhood information system, where $A, B \subseteq AT$ are categorical and numerical attributes, respectively. For any $X \subseteq U$, the lower and upper approximations of X of the pessimistic 1-type NMGRS in U are described as:

$$\underline{(A+B)}^P X = \{x \in U \mid n_A(x) \subseteq X \wedge n_B(x) \subseteq X\}, \tag{9}$$

$$\overline{(A+B)}^P X = \sim \underline{(A+B)}^P (\sim X). \tag{10}$$

Analogously, this *multigranulation boundary region* of X is

$$Bn^P_{(A+B)}(X) = \overline{(A+B)}^P X \setminus \underline{(A+B)}^P X.$$

We call $(\underline{(A+B)}^P X, \overline{(A+B)}^P X)$ a pessimistic 1-type neighborhood multigranulation rough set. □

Theorem 2. *Let $NIS = (U, AT, N)$ be a neighborhood information system, where $A, B \subseteq AT$ are categorical and numerical attributes, respectively. For any $X \subseteq U$, then $\overline{(A+B)}^P X = \{x \in U \mid (n_A(x) \cap X \neq \emptyset) \ \vee \ (n_B(x) \cap X \neq \emptyset)\}$.*

Proof. By the above definitions, we have

$$\begin{aligned} x \in \overline{(A+B)}^P X &\Leftrightarrow x \in \sim \underline{(A+B)}^P(\sim X) \\ &\Leftrightarrow x \notin \underline{(A+B)}^P(\sim X) \\ &\Leftrightarrow n_A(x) \nsubseteq (\sim X) \vee n_B(x) \nsubseteq (\sim X) \\ &\Leftrightarrow n_A(x) \cap X \neq \phi \vee n_B(x) \cap X \neq \emptyset. \end{aligned}$$

This completes the proof. □

Different from the upper approximation of optimistic 1-type neighborhood multigranulation rough set, the upper approximation of pessimistic 1-type neighborhood multigranulation rough set is represented as a set in which objects have non-empty intersection with the target in terms of at least one granular structure.

From the above analysis, we can obtain the following two corollaries and one proposition.

Corollary 5. *Let $NIS = (U, AT, N)$ be a neighborhood information system, $A, B \subseteq AT$ categorical and numerical attributes, respectively. For any $X \subseteq U$, then $\overline{(A+B)}^P X = \overline{A}X \cup \overline{B}X$.*

Proof.

$$\begin{aligned} \overline{(A+B)}^P X &= \sim \underline{(A+B)}^P(\sim X) \\ &= \sim (\underline{A}(\sim X) \cap \underline{B}(\sim X)) \\ &= \sim \underline{A}(\sim X) \cup \sim \underline{B}(\sim X) \\ &= \overline{A}X \cup \overline{B}X. \end{aligned}$$

This completes the proof. □

Similarly, other properties of the pessimistic version can be proved by the same method.

3.2. 2-Type neighborhood multigranulation rough sets (2-type NMGRS)

When multiple neighborhood relations are used in the neighborhood information system, we call such a multigranulation rough set a 2-type neighborhood multigranulation rough set, denoted by 2-type NMGRS. Simply, we first investigate how to approximate a target concept through two neighborhood relations. For simpleness, we use the denotations $\underline{A+B}X = \underline{N}X$, and $\overline{A+B}X = \overline{N}X$ in the following:

Definition 5 (*2-type NMGRS*)**.** Let $NIS = (U, AT, N)$ be a neighborhood information system, N_1, N_2 two neighborhood relations on the universe U, N_1 induced by A_1 and B_1, N_2 induced by A_2 and B_2, where A_1, A_2 are two categorical attribute subsets and B_1, B_2 are two numerical attribute subsets , and $U/A_1, U/A_2, U/B_1, U/B_2$ are four coverings on the universe U. Then for any $X \subseteq U$, the optimistic lower approximation and upper approximation of X in U are defined as

$$\underline{(N_1+N_2)}^O X = \{x \in U \mid n_{(A_1+B_1)}(x) \subseteq X \vee n_{(A_2+B_2)}(x) \subseteq X\}, \tag{11}$$

$$\overline{(N_1+N_2)}^O X = \sim \underline{(N_1+N_2)}^O(\sim X). \tag{12}$$

The *area of uncertainty or boundary region* is defined as:

$$Bn^O_{(N_1+N_2)}(X) = \overline{(N_1+N_2)}^O X \setminus \underline{(N_1+N_2)}^O X.$$

We call $(\underline{(N_1+N_2)}^O X, \overline{(N_1+N_2)}^O X)$ an optimistic 2-type NMGRS based on two neighborhood relations.

In 2-type NMGRS, $n_{(A+B)}(x)$ represents a neighborhood induced by a heterogeneous attribute subset and $n_{(A+B)}(x) = \{x \in U \mid n_A(x) \leq \delta \vee n_B(x) \leq \delta\}$. However, by the (3) of Definition 2, $n_{(A\cup B)}(x) = \{x_i \in U \mid d_A(x, x_i) = 0 \wedge d_B(x, x_i) \leq \delta\}$.

It is deserved to point out that let $NIS = (U, AT, N)$ be a neighborhood information system, a partition U/A induced by a categorical attribute subset A, and a covering U/B induced by a numerical attribute subset B, then $U/(A \cup B)$ induced by $A \cup B$ is also a covering of the universe.

Example 3 (*Continued from Example 1*)**.** Let $X = \{x_1, x_2, x_3, x_7\}$, four coverings on the universe U are induced from Table 1 as follows:

Let $A_1 = \{O, W\} \subseteq AT$ be a categorical attribute subset, from Example 2, it follows that $U/A_1 = \{\{x_1, x_8\}, \{x_2\}, \{x_3\}, \{x_4, x_5\}, \{x_6\}, \{x_7\}\}$. Then $\underline{A_1}X = \{x_2, x_3, x_7\}$, $\overline{A_1}X = \{x_1, x_2, x_3, x_7, x_8\}$.

Let $A_2 = \{O, U\} \subseteq AT$ be a categorical attribute subset, from Table 1, it follows that $U/A_2 = \{\{x_1\}, \{x_2, x_8\}, \{x_3\}, \{x_4, x_5\}, \{x_6\}, \{x_7\}\}\}$. Then $\underline{A_2}X = \{x_1, x_3, x_7\}$, $\overline{A_2}X = \{x_1, x_2, x_3, x_7, x_8\}$.

Let $B_1 = \{T, H\} \subseteq AT$ be numerical attribute subset, from Example 2, it follows that $U/B_1 = \{\{x_1, x_2, x_3\}, \{x_2, x_1, x_3, x_4, x_8\}, \{x_3, x_1, x_2\}, \{x_4, x_2, x_8\}, \{x_5, x_6\}, \{x_6, x_5, x_7\}, \{x_7, x_6\}, \{x_8, x_4, x_2\}\}$, we have that $\underline{B_1}X = \{x_1, x_3\}$, $\overline{B_1}X = \{x_1, x_2, x_3, x_4, x_6, x_7, x_8\}$.

Let $B_2 = \{T, I\} \subseteq AT$ be a numerical attribute subset, from Table 1, it follows that $U/B_2 = \{\{x_1, x_2, x_3\}, \{x_2, x_1, x_4, x_8\}, \{x_3, x_1\}, \{x_4, x_2, x_8\}, \{x_5, x_6, x_8\}, \{x_6, x_5\}, \{x_7\}, \{x_8, x_2, x_4, x_5\}\}$. We have that $\underline{B_2}X = \{x_1, x_3, x_7\}$, $\overline{B_2}X = \{x_1, x_2, x_3, x_4, x_7, x_8\}$. From the definition of the optimistic 1-type NMGRS, by computing, we have that $\underline{(A_1 + B_1)}^O X = \{x_1, x_2, x_3, x_7\}$, $\overline{(A_1 + B_1)}^O X = \{x_1, x_2, x_3, x_7, x_8\}$. And $\underline{(A_2 + B_2)}^O X = \{x_1, x_3, x_7\}$, $\overline{(A_2 + B_2)}^O X = \{x_1, x_2, x_3, x_4, x_7, x_8\}$. Then $\underline{(N_1 + N_2)}^O X = \{x_1, x_2, x_3, x_7\}$, $\overline{(N_1 + N_2)}^O X = \{x_1, x_2, x_3, x_4, x_7, x_8\}$.

From Example 2, it follows that $\underline{A_1 \cup B_1}X = \{x_1, x_2, x_3, x_7\}$, $\overline{A_1 \cup B_1}X = \{x_1, x_2, x_3, x_7\}$.

For $U/(A_2 \cup B_2) = \{\{x_1\}, \{x_2, x_8\}, \{x_3\}, \{x_4\}, \{x_6\}, \{x_7\}\}$, then $\underline{(A_2 \cup B_2)}X = \{x_1, x_3, x_7\}$, $\overline{(A_2 \cup B_2)}X = \{x_1, x_2, x_3, x_7, x_8\}$.

In addition, $U/((A_1 \cup B_1) \cup (A_2 \cup B_2)) = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_5\}, \{x_6\}, \{x_7\}, \{x_8\}\}$, one has $\underline{(N_1 \cup N_2)}X = \underline{(A_1 + B_1)}^O X \cup \underline{(A_2 + B_2)}^O X = \{x_1, x_2, x_3, x_7\}$ and $\overline{N_1 \cup N_2}X = \sim \underline{(N_1 \cup N_2)}(\sim X) = \{x_1, x_2, x_3, x_7\}$.

Obviously, for the optimistic 2-type neighborhood multigranulation rough set, we have that $\underline{(N_1 + N_2)}^O X = \{x_3, x_7\} \subseteq \{x_1, x_3, x_7\} = \underline{(N_1 \cup N_2)}X$, and $\overline{(N_1 + N_2)}^O X = \{x_1, x_2, x_3, x_7, x_8\} \supseteq \{x_1, x_2, x_3, x_7\} = \overline{(N_1 \cup N_2)}X$.

From the definition of approximation and the discussion above, we can get the following properties of the lower and upper approximations.

Proposition 5. *Let $NIS = (U, AT, N)$ be a neighborhood information system, N_1, N_2 two neighborhood relations on the universe U. Then for any $X \subseteq U$, then*

(1) $\underline{(N_1 + N_2)}^O X \subseteq \underline{(N_1 \cup N_2)}X$,
(2) $\overline{(N_1 + N_2)}^O X \supseteq \overline{(N_1 \cup N_2)}X$.

Proof. (1) For any $x \in \underline{(N_1 + N_2)}^O X$, from Definition 5, it follows that $x \in n_{(A_1+B_1)}$ and $x \in n_{(A_2+B_2)}$. Hence, $x \in n_{(A_1+B_1)}(x) \cap n_{(A_2+B_2)}(x)$, $n_{(A_1+B_1)}(x) \wedge n_{(A_2+B_2)}(x) \subseteq n_{(N_1 \cup N_2)}(x)$, we have $x \in \underline{(N_1 \cup N_2)}X$, i.e., $\underline{(N_1 + N_2)}^O X \subseteq \underline{(N_1 \cup N_2)}X$.

(2) Due to duality property of the lower and upper approximations, $\overline{(N_1 \cup N_2)}X = \sim \underline{(N_1 \cup N_2)}(\sim X)$. Applying the result of (1), we have that $\overline{(N_1 \cup N_2)}X = \sim \underline{(N_1 \cup N_2)}(\sim X) \subseteq \sim \underline{(N_1 + N_2)}^O(\sim X) = \overline{(N_1 + N_2)}^O X$, i.e., $\overline{(N_1 \cup N_2)}X \subseteq \overline{(N_1 + N_2)}^O X$.
This completes the proof. □

Corollary 6. $Bn_{N_1}(X) \cup Bn_{N_2}(X) \subseteq Bn^O_{(N_1+N_2)}(X)$.

Corollary 7. *Let $NIS = (U, AT, N)$ be a neighborhood information system, N_1, N_2 two neighborhood relations on the universe U. Then, for $X \subseteq U$, one has*

$$\alpha^O_{(N_1+N_2)}(X) \leq \alpha_{(N_1 \cup N_2)}(X).$$

Proof. This is straightforward from the definition of the accuracy measure of X. □

Proposition 6. *Let $NIS = (U, AT, N)$ be a neighborhood information system, N_1, N_2 two neighborhood relations on the universe U, and $X \subseteq U$. If δ_1, δ_2 are two nonnegative numbers and $\delta_1 \geq \delta_2$, then*

(1) $\underline{(N_1 + N_2)_{\delta_1}}^O X \subseteq \underline{(N_1 + N_2)_{\delta_2}}^O X$,
(2) $\overline{(N_1 + N_2)}^O_{\delta_1} X \supseteq \overline{(N_1 + N_2)}^O_{\delta_2} X$.

Proof. It can be easily proved similar to Proposition 3.

Proposition 6 states that the size of lower approximation of X under a 2-type optimistic neighborhood-based multigranulation rough set will become much larger with the value of the parameter δ being much bigger. Its upper approximation has the inverse conclusion. □

Corollary 8. *Let $NIS = (U, AT, N)$ be a neighborhood information system, N_1, N_2 two neighborhood relations on the universe U, and $X \subseteq U$. If δ_1, δ_2 are two nonnegative numbers and $\delta_1 \geq \delta_2$, then,*

$$\alpha^{O}_{(N_1+N_2)_{\delta_1}}(X) \leq \alpha^{O}_{(N_1+N_2)_{\delta_2}}(X).$$

Proposition 7. *Let $NIS = (U, AT, N)$ be a neighborhood information system, N_1, N_2 two neighborhood relations on the universe U, and $X, Y \subseteq U$. If $X \subseteq Y$, then*

(1) $\underline{(N_1+N_2)}^O X \subseteq \underline{(N_1+N_2)}^O Y$,
(2) $\overline{(N_1+N_2)}^O X \subseteq \overline{(N_1+N_2)}^O Y$.

Proof. (1) If $X \subseteq Y, X \cap Y = X$. Then

$$\begin{aligned}\underline{(N_1+N_2)}^O X &= \underline{(N_1+N_2)}^O (X \cap Y)\\ &= \underline{N_1}(X \cap Y) \cup \underline{N_2}(X \cap Y)\\ &= (\underline{N_1}X \cap \underline{N_1}Y) \cup (\underline{N_2}X \cap \underline{N_2}Y)\\ &= ((\underline{N_1}X \cap \underline{N_1}Y) \cup \underline{N_2}X) \cap ((\underline{N_1}X \cap \underline{N_1}Y) \cup \underline{N_2}Y)\\ &= (\underline{N_1}X \cup \underline{N_2}X) \cap (\underline{N_1}Y \cup \underline{N_2}X) \cap (\underline{N_1}X \cup \underline{N_2}Y) \cap (\underline{N_1}Y \cup \underline{N_2}Y)\\ &= \underline{(N_1+N_2)}^O X \cap \underline{(N_1+N_2)}^O Y \cap (\underline{N_1}Y \cup \underline{N_2}X) \cap (\underline{N_1}X \cup \underline{N_2}Y)\\ &\subseteq \underline{(N_1+N_2)}^O X \cap \underline{(N_1+N_2)}^O Y\\ &\subseteq \underline{(N_1+N_2)}^O Y.\end{aligned}$$

So $\underline{(N_1+N_2)}^O X \subseteq \underline{(N_1+N_2)}^O Y$.

(2) If $X \subseteq Y$, $\sim X \supseteq \sim Y$, from the result of (1), $\underline{(N_1+N_2)}^O(\sim X) \supseteq \underline{(N_1+N_2)}^O(\sim Y)$. Then, $\sim (\underline{(N_1+N_2)}^O(\sim X)) \subseteq \sim \underline{(N_1+N_2)}^O(\sim Y)$, then $\overline{(N_1+N_2)}^O X \subseteq \overline{(N_1+N_2)}^O Y$.
This completes the proof. □

Similarly, the pessimistic 2-type neighborhood multigranulation rough set with two neighborhood granulations can be also defined as follows:

$$\underline{(N_1+N_2)}^P X = \{x \mid n_{(A_1+B_1)}(x) \subseteq X \wedge n_{(A_2+B_2)}(x) \subseteq X\}, \tag{13}$$

$$\overline{(N_1+N_2)}^P X = \sim \underline{(N_1+N_2)}^P(\sim X). \tag{14}$$

The *area of uncertainty or boundary region* is defined as:

$$Bn^P_{(N_1+N_2)}(X) = \overline{(N_1+N_2)}^P X \setminus \underline{(N_1+N_2)}^P X.$$

Parallelly, we can present the corresponding properties of this pessimistic version.

Based on the above conclusions, we extend 2-type NMGRS based on two neighborhood relations to that based on multiple neighborhood relations.

Definition 6. Let $NIS = (U, AT, N)$ be a neighborhood information system, $A_1, A_2, \ldots, A_m$ categorical attribute subsets of AT; $B_1, B_2, \ldots, B_m$ numerical attributes of AT, N_i induced by A_i and B_i for $i = 1, 2, \ldots, m$, and $X \subseteq U$. We define an optimistic multigranulation lower approximation and an upper approximation of X by the following:

$$\underline{\sum_{i=1}^{m} N_i}^O X = \bigcup\{x \in U \mid n_{(A_i+B_i)}(x) \subseteq X, i \leq m\}, \tag{15}$$

$$\overline{\sum_{i=1}^{m} N_i}^O X = \sim \underline{\sum_{i=1}^{m} N_i}(\sim X). \tag{16}$$

Similarly, the *area of uncertainty or boundary region* is defined as:

$$Bn^O_{\sum_{i=1}^m N_i}(X) = \overline{\sum_{i=1}^{m} N_i}^O X \Big\backslash \underline{\sum_{i=1}^{m} N_i}^O X.$$

We call $(\underline{\sum_{i=1}^m N_i}^O X, \overline{\sum_{i=1}^m N_i}^O X)$ an optimistic 2-type NMGRS based on multiple neighborhood relations.

Proposition 8. *Let $NIS = (U, AT, N)$ be a neighborhood information system, $N_1, N_2, \ldots, N_m$ m neighborhood relations on the universe U, and $X \subseteq U$. Then,*

(1) $\underline{\sum_{i=1}^{m} N_i}^{O} X \subseteq \underline{(N_1 \cup N_2 \cup \cdots \cup N_m)} X$,
(2) $\overline{\sum_{i=1}^{m} N_i}^{O} X \supseteq \overline{(N_1 \cup N_2 \cup \cdots \cup N_m)} X$.

Proof. If $m = 1$, they are straightforward.
If $m > 1$, we prove them as follows:
(1) It can be easily proved from Definition 6.
(2) $\overline{\sum_{i=1}^{m} N_i}^{O} X = \sim \underline{\sum_{i=1}^{m} N_i}^{O} (\sim X) \supseteq \sim \underline{(N_1 \cup N_2 \cup \cdots \cup N_m)} (\sim X) = \overline{(N_1 \cup N_2 \cup \cdots \cup N_m)} X$.
This completes the proof. □

Corollary 9. *Let $NIS = (U, AT, N)$ be a neighborhood system, $N_1, N_2, \ldots, N_m$ m neighborhood relations on the universe U, and $X \subseteq U$. Then,*

$$\alpha^{O}_{\sum_{i=1}^{m} N_i}(X) \leq \alpha_{(N_1 \cup N_2 \cup \cdots \cup N_m)}(X).$$

Proposition 9. *Let $NIS = (U, AT, N)$ be a neighborhood information system, $N_1, N_2, \ldots, N_m$ m neighborhood relations on the universe U, $X \subseteq U$, and δ_1, δ_2 two nonnegative numbers. If $\delta_1 \geq \delta_2$, then,*

(1) $\underline{(\sum_{i=1}^{m} N_i)_{\delta_1}}^{O} X \subseteq \underline{(\sum_{i=1}^{m} N_i)_{\delta_2}}^{O} X$,
(2) $\overline{(\sum_{i=1}^{m} N_i)}^{O}_{\delta_1} X \supseteq \overline{(\sum_{i=1}^{m} N_i)}^{O}_{\delta_2} X$.

Proof. It can be proved similar to Proposition 3. □

Corollary 10. *Let $NIS = (U, AT, N)$ be a neighborhood information system, $N_1, N_2, \ldots, N_m$ m neighborhood relations on the universe U, and $X \subseteq U$. If δ_1, δ_2 are two nonnegative numbers, and $\delta_1 \geq \delta_2$, then the following properties hold.*

$$\alpha^{O}_{(\sum_{i=1}^{m} N_i)_{\delta_1}}(X) \leq \alpha^{O}_{(\sum_{i=1}^{m} N_i)_{\delta_2}}(X).$$

Proposition 10. *Let $NIS = (U, AT, N)$ be a neighborhood information system, $N_1, N_2, \ldots, N_m$ m neighborhood relations on the universe U, and $X, Y \subseteq U$. If $X \subseteq Y$, then*

(1) $\underline{\sum_{i=1}^{m} N_i}^{O} X \subseteq \underline{\sum_{i=1}^{m} N_i}^{O} Y$,
(2) $\overline{\sum_{i=1}^{m} N_i}^{O} X \subseteq \overline{\sum_{i=1}^{m} N_i}^{O} Y$.

Proof. It is similar to the proof of Proposition 4. □

Similarly, we can also define the pessimistic 2-type neighborhood multigranulation rough set as the following:

$$\underline{\sum_{i=1}^{m} N_i}^{P} X = \{x \in U \mid n_{(A_1+B_1)}(x) \subseteq X \wedge \cdots \wedge n_{(A_m+B_m)}(x) \subseteq X\}, \tag{17}$$

$$\overline{\sum_{i=1}^{m} N_i}^{P} X = \sim \underline{\sum_{i=1}^{m} N_i}^{P} (\sim X). \tag{18}$$

Similarly, the *area of uncertainty or boundary region* is defined as:

$$Bn^{P}_{\sum_{i=1}^{m} N_i}(X) = \overline{\sum_{i=1}^{m} N_i}^{P} X \Big\backslash \underline{\sum_{i=1}^{m} N_i}^{P} X.$$

Analogously, we can gain the same results of the pessimistic version with multiple neighborhood granulations.

4. Attribute reduction of neighborhood multigranulation rough sets

In this section, we investigate the reduction of coverings induced by the multiple neighborhood relations. A discernibility matrix will be used to compute all the reducts of neighborhood multigranulation rough set. The objective of reduction is to select a subset of coverings that can preserve consistence of the neighborhood decision system [1]. Let $\Omega = \{C_1, C_2, \ldots, C_m\}$ be a family of coverings of U. $C_i = \{K_{i1}, K_{i2}, \ldots, K_{it_i}\}$, where K_{ij} is nonempty subset of U for $j = \{1, 2, \ldots, t_i\}$. For any $x \in U$, $(C_i)_x = \bigcap\{K_{ij} \mid K_{ij} \in C, x \in K_{ij}\}$, $Cov(C_i) = \{(C_i)_x | x \in U\}$, $\Omega_x = \bigcap\{K_{i_x} \in Cov(C_i), x \in C_{i_x}\}$, and $Cov(\Omega) = \{\Omega_x \mid x \in U\}$. As a result, $Cov(C_i) = \{(C_i)_x | x \in U\}$ and $Cov(\Omega) = \{\Omega_x \mid x \in U\}$ are two coverings of U.

Definition 7. Let $\Omega = \{C_1, C_2, \ldots, C_m\}$ be a family of coverings of U, $D = \{d\}$ a decision attribute set, and $U/D = \{D_1, D_2, \ldots, D_q\}$ a decision partition on U. If for any$x \in U$, there exists $D_j \in U/D$ such that $\Omega_x \subseteq D_j$, then decision system (U, Ω, D) is called a consistent covering decision system and denoted by $Cov(\Omega) \leq U/D$.

Definition 8. Let $NIS = (U, AT \cup D, N)$ be a neighborhood decision information system, where $D = \{d\}$, C_i induced by a categorical attribute subset A_i or a numerical attribute subset B_i, $i = 1, 2, \ldots, m$, and $\Omega = \{C_1, C_2, \ldots, C_m\}$ m coverings of U. We call (U, Ω, D) a covering neighborhood decision system.

Definition 9. Let $(U, \Omega, D = \{d\})$ be a covering neighborhood decision information system. For $C_i \in \Omega$, if $Cov(\Omega - C_i) \leq U/D$, then C_i is called a superfluous covering relative to D in Ω, otherwise C_i is called indispensable relative to D in Ω. For every $P \subseteq \Omega$ satisfying $Cov(P) \leq U/D$, if every element in P is an indispensable covering, i.e., for any $C_i \in P$, if $Cov(P - C_i) \nleq U/D$, then P is called a relative reduct of Ω relative to D. The disjunction of all the indispensable elements in Ω is called the core of Ω to D, denoted by $NCore_D(\Omega)$.The relative reduct of a consistent covering decision system is the subset of coverings to ensure the consistency of the decision information system.

When the attribute reduction of a neighborhood-based multigranultion rough set is to calculate, we will employ the discernibility matrix approach proposed by Chen et al. for this objective, which is as follows:

Definition 10 [1]. Let $(U, \Omega, D = \{d\})$ be a consistent covering decision system. Suppose $U = \{x_1, x_2, \ldots, x_n\}$, by $M(U, \Omega, D)$, we denote a $n \times n$ matrix (c_{ij}), called the discernibility matrix of $(U, \Omega, D = \{d\})$, defined as

$$c_{ij} = \begin{cases} \{C \in \Omega : (C_{x_i} \not\subset C_{x_j}) \wedge (C_{x_j} \not\subset C_{x_i})\} \cup \{C_s \wedge C_t : (C_{sx_i} \subset C_{x_j}) \wedge (C_{sx_j} \subset C_{x_i})\}, & d(\Omega_{x_i}) \neq d(\Omega_{x_j}), \\ \Omega, & d(\Omega_{x_i}) = d(\Omega_{x_j}). \end{cases}$$

In which $D = \{d\}$ and $d(x)$ is a decision function $d : U \rightarrow V_d$ of the universe U into value set V_d. For every $x_i, x_j \in U$, if $\Omega_{x_i} \subseteq \Omega_{x_j}$, then $d(x_i) = d([x_i]_D) = d(\Omega_{x_i}) = d(\Omega_{x_j}) = d(x_j) = d([x_j]_D)$. If $d(\Omega_{x_i}) \neq d(\Omega_{x_j})$, then $\Omega_{x_i} \cap \Omega_{x_j} = \emptyset$, i.e., $\Omega_{x_i} \not\subset \Omega_{x_j}$ and $\Omega_{x_j} \not\subset \Omega_{x_i}$. But if $\Omega_{x_i} \not\subset \Omega_{x_j}$ and $\Omega_{x_j} \not\subset \Omega_{x_i}$, then either $d(\Omega_{x_i}) = d(\Omega_{x_j})$ or $d(\Omega_{x_i}) \neq d(\Omega_{x_j})$ are possible. For this case, if $\Omega_{x_i} \cap \Omega_{x_j} \neq \emptyset$, we have $d(\Omega_{x_i}) = d(\Omega_{x_j})$. If $d(\Omega_{x_i}) = d(\Omega_{x_j})$, then both $\Omega_{x_i} \not\subset \Omega_{x_j}$ and $\Omega_{x_j} \not\subset \Omega_{x_i}$, or $\Omega_{x_i} \subseteq \Omega_{x_j}$ or $\Omega_{x_j} \subseteq \Omega_{x_i}$ are possible.

In the following, we give an example to illustrate the covering reduct of 1-type neighborhood multigranulation rough set through using the discernibility matrix approach proposed by Chen et al. The covering reduct of 2-type neighborhood multigranulation rough set can be done similarly.

Example 4. Table 2 depicts a neighborhood decision information system $NIS = (U, AT \cup \{d\}, N)$ in which $AT = \{$*outlook, temperature, windy*$\}$, $\{d\} = \{$*play*$\}$. The numerical attribute value of *temperature* is standardized into [0, 1] (see [6]) for computing and we suppose $\delta = 0.1$. By Definition 2, we have that:

Let $P_1 = \{O\}$, then $C_1 = \{\{x_1, x_2, x_8\}, \{x_3, x_7\}, \{x_4, x_5, x_6\}\}$.

Let $P_2 = \{T\}$, then $C_2 = \{\{x_1, x_2, x_3\}, \{x_2, x_1, x_3, x_4, x_8\}, \{x_3, x_1, x_2\}\}, \{x_4, x_2, x_5, x_6, x_7, x_8\}; \{x_5, x_4, x_6, x_7, x_8\}, \{x_6, x_4, x_5, x_7, x_8\}, \{x_7, x_4, x_5, x_6, x_8\}, \{x_8, x_2, x_4, x_5, x_6, x_7\}\}$.

Let $P_3 = \{W\}$, then $C_3 = \{\{x_1, x_3, x_4, x_5, x_8\}, \{x_2, x_7, x_6\}\}$.

Let $P_4 = \{O, T\}$, then $C_4 = \{\{x_1, x_2\}, \{x_2, x_1, x_8\}, \{x_3\}, \{x_4, x_5, x_6\}, \{x_7\}, \{x_8, x_2\}\}$.

Table 2
A playing tennis information system with mixed attributes.

	Outlook	Temperature	Windy	Play
x_1	Sunny	85	False	No
x_2	Sunny	80	True	No
x_3	Overcast	83	False	Yes
x_4	Rainy	70	False	Yes
x_5	Rainy	68	False	Yes
x_6	Rainy	65	True	No
x_7	Overcast	64	True	Yes
x_8	Sunny	72	False	No

Let $P_5 = \{O, W\}$, then $C_5 = \{\{x_1, x_8\}, \{x_2\}, \{x_3\}, \{x_4, x_5\}, \{x_6\}, \{x_7\}\}$.
Let $P_6 = \{W, T\}$, then $C_6 = \{\{x_1, x_3\}, \{x_2\}, \{x_1, x_3\}, \{x_4, x_5, x_8\}, \{x_6, x_7\}, \{x_8, x_4, x_5\}\}$.
Let $P_7 = \{O, T, W\}$, then $C_7 = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4, x_5\}, \{x_6\}, \{x_7\}, \{x_8\}\}$.
And $U/D = \{\{x_1, x_2, x_6, x_8\}, \{x_3, x_4, x_5, x_7\}\}$. From Definition 7, we have that $\Omega_1 = \{x_1\}, \Omega_2 = \{x_2\}, \Omega_3 = \{x_3\}, \Omega_4 = \{x_4, x_5\}, \Omega_5 = \{x_4, x_5\}, \Omega_6 = \{x_6\}, \Omega_7 = \{x_7\}, \Omega_8 = \{x_8\}$.
Obviously, $Cov(\Omega) = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4, x_5\}, \{x_5, x_4\}, \{x_6\}, \{x_7\}, \{x_8\}\}$ is a covering on the universe U induced by Ω.
Note that the discernibility matrix is a symmetric, we only consider its lower triangular matrix of the discernibility matrix as the following:

$$\begin{bmatrix} \Omega & & & & & & & \\ \Omega & \Omega & & & & & & \\ \Omega_{31} & \Omega_{32} & \Omega & & & & & \\ \Omega_{41} & \Omega & \Omega & \Omega & & & & \\ \Omega_{51} & \Omega & \Omega & \Omega & \Omega & & & \\ \Omega & \Omega_{62} & \Omega & \Omega_{64} & \Omega_{65} & \Omega & & \\ \Omega & \Omega_{72} & \Omega & \Omega & \Omega & \Omega_{76} & \Omega & \\ \Omega & \Omega & \Omega_{83} & \Omega_{84} & \Omega_{85} & \Omega & \Omega_{87} & \Omega \end{bmatrix}$$

where $\Omega_{31} = \Omega_{76} = \Omega_{84} = \Omega_{85} = \{C_1, C_4, C_5, C_7\}, \Omega_{32} = \Omega_{87} = \{C_1, C_3, C_4, C_5, C_6, C_7\}, \Omega_{41} = \Omega_{51} = \Omega_{62} = \Omega_{72} = \Omega_{83} = \{C_1, C_2, C_4, C_5, C_6, C_7\}$, and $\Omega_{64} = \Omega_{65} = \{C_3, C_5, C_6, C_7\}$.

$$\begin{aligned} f(U, \Omega)(\overline{C_1}, \overline{C_2}, \ldots, \overline{C_7}) &= \{C_1 \vee C_4 \vee C_5 \vee C_7\} \wedge \{C_1 \vee C_3 \vee C_4 \vee C_5 \vee C_6 \vee C_7\} \\ &\wedge \{C_1 \vee C_2 \vee C_4 \vee C_5 \vee C_6 \vee C_7\} \wedge \{C_3 \vee C_5 \vee C_6 \vee C_7\} \\ &= \{C_1 \vee C_4 \vee C_5 \vee C_7\} \wedge \{C_3 \vee C_5 \vee C_6 \vee C_7\} \\ &= (C_1 \wedge C_3) \vee (C_1 \wedge C_6) \vee (C_4 \wedge C_3) \vee (C_4 \wedge C_6) \vee C_5 \vee C_7. \end{aligned}$$

Finally, all reducts of this neighborhood decision information system are $\{C_1, C_3\}, \{C_1, C_6\}, \{C_4, C_3\}, \{C_4, C_6\}, \{C_5\}$, and $\{C_7\}$.

Remark: If we consider a simple case, that is each attribute induces a covering (i.e., neighborhood granular structure), we draw some interesting conclusions. For example, through calculating the reducts of coverings in the condition part, we also can obtain the corresponding attribute reduct. In the last example, from the above reduct of coverings, we can know that attribute reducts of this neighborhood information system are $\{O, W\}$ and $\{O, T\}$, and $Ncore_D(U) = \{O\}$ is their core attribute.

5. Conclusions

To extend the applicable area of MGRS, in this paper, we have proposed 1-type neighborhood-based multigranulation rough sets and 2-type neighborhood-based multigranulation rough sets, which can be used to deal with the data sets with hybrid attributes. The theoretical analysis shows that the proposed neighborhood multigranulation rough sets are generalized versions of original MGRS, in which each of NMGRS will degenerate into the corresponding version of classical MGRS. To extract simple decision rules, a concept of covering reduct has also been introduced to describe the smallest attribute subset that preserves the lower and upper approximations of all decision classes in NMGRS. These results will enrich the multigranulation rough set theory and be very helpful for knowledge discovery from various data sets in the context of multiple granulations.

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