Some Oscillation Theorems for Second Order Differential Equations

NAM P. BHATIA*

Western Reserve University, Cleveland, Ohio
Submitted by Richard Bellman

INTRODUCTION

Waltman [1] has demonstrated that all solutions of the differential equation

\[ x'' + m(t) x^{2n-1} = 0, \quad n \text{ a positive integer}, \]  

(0.1)

where \( m(t) \in C(0, +\infty) \), are oscillatory if

\[ \int_0^\infty m(\tau) \, d\tau = +\infty. \]  

(0.2)

The importance of this condition lies in the fact that \( m(t) \geq 0 \) is not assumed as is the case in earlier works [2, 3].

The purpose of this note is to develop some oscillation theorems for differential equations of the type

\[ \ell(t) x' + m(t) g(x) = 0, \]  

(0.3)

where \( \ell(t), m(t) \in C(0, +\infty), \ell(t) > 0, \) and \( g(x) \in C(-\infty, +\infty) \). Our Theorem 1.1 contains as special cases the results of Leighton [4] and Wintner [5] for the linear case, and the result of Waltman for the nonlinear case. Our Theorem 1.2 is a sort of a comparison theorem and allows us to establish Theorem 1.3, which may be considered as a generalization of the following well-known result.

**Theorem 0.1.** If \( \int_0^\infty g(\xi) \, d\xi \to +\infty \) as \( x \to \pm \infty \), and \( xg(x) > 0 \) for \( x \neq 0 \), then all solutions of the equation

\[ x'' + g(x) = 0, \]  

(0.4)

are oscillatory (indeed all solutions are periodic in this case).

*The author acknowledges partial support of the National Science Foundation under grant GP-4582.
1. Oscillation Theorems

We shall assume throughout that all solutions of the differential equations considered below are defined in the future; i.e., they are functions $u(t) \in C^\infty([t_0, + \infty))$ for some $t_0 \geq 0$. A solution will be said to be oscillatory if it has an infinity of zeros on an interval $[t_0, + \infty)$ on which it is defined. An equation will be called oscillatory if all of its solutions are oscillatory. We remark that uniqueness of solutions satisfying some given initial conditions is not assumed.

We are now ready for our first theorem.

**Theorem 1.1.** Let

\[ \int_0^\infty \frac{1}{\ell(t)} \, dt = + \infty, \quad \text{(1.1)} \]

\[ \int_0^\infty m(t) \, dt = + \infty. \quad \text{(1.2)} \]

Let the function $g(x)$ be such that

\[ xg(x) > 0, \quad x \neq 0, \quad \text{(1.3)} \]

and

\[ g'(x) = \frac{dg(x)}{dx} \geq 0 \quad \text{for all } x. \quad \text{(1.4)} \]

Then all solutions of (0, 3) are oscillatory.

**Proof.** Assume the theorem is false. Then there is a nonoscillatory solution $u(t)$ of (0,3). Assume that $u(t) > 0$ for $t > t_0$ (the case $u(t) < 0$ can be treated similarly). $u(t)$ satisfies the identity

\[ \ell(t) u'(t)' + m(t) g(u(t)) = 0, \quad \text{(1.5)} \]

which for $t \geq t_0$ may be written as

\[ \frac{\ell(t) u'(t)'}{g(u(t))} + m(t) = 0. \quad \text{(1.6)} \]

Integration of (1.6) yields

\[ \ell(t) \frac{u'(t)}{g(u(t))} - \ell(t_0) \frac{u'(t_0)}{g(u(t_0))} + \int_{t_0}^t \ell(\tau) \, d\tau + \int_{t_0}^t m(\tau) \, d\tau = 0, \quad \text{(1.7)} \]

where

\[ \ell(t) = \ell(t) \left[ \frac{u'(t)}{g(u(t))} \right]^2 g'(u(t)). \]
Since $p(t) \geq 0$, we conclude because of (1.2) that
\[ \frac{\ell(t) u'(t)}{g(u(t))} \to -\infty \quad \text{as} \quad t \to +\infty. \] (1.8)

Consequently $u'(t) < 0$ for all large $t$, say $t \geq T_0 (\geq t_0)$. Indeed (1.5) yields on integration
\[ \ell(t) u'(t) - \ell(T) u'(T) + \int_T^t m(\tau) g(u(\tau)) \, d\tau = 0. \] (1.9)

Now consider the integral in (1.9). We have
\[ \int_T^t m(\tau) g(u(\tau)) \, d\tau = g(u(t)) \int_T^t m(\tau) \, d\tau - g(u(T)) \int_T^t m(\tau) \, d\tau \]
\[ - \int_T^t g'(u(\tau)) u'(\tau) \left[ \int_{T_0}^{\tau} m(s) \, ds \right] \, d\tau. \] (1.10)

Since $\int_{T_0}^t m(\tau) \, d\tau \to \infty$ as $t \to +\infty$, we may choose $T \geq T_0$ such that $\int_{T_0}^T m(s) \, ds = 0$, and $\int_{T_0}^t m(s) \, ds \geq 0$ for $t \geq T$. With such a choice of $T$ we get from (1.9)
\[ \ell(t) u'(t) - \ell(T) u'(T) + g(u(t)) \int_{T_0}^{t} m(\tau) \, d\tau \]
\[ - \int_T^t g'(u(\tau)) u'(\tau) \left[ \int_{T_0}^{\tau} m(s) \, ds \right] \, d\tau = 0. \] (1.11)

(1.11) now gives
\[ \ell(t) u'(t) \leq \ell(T) u'(T), \] (1.12)
and this yields
\[ u(t) - u(T) \leq \ell(T) u'(T) \int_{T_0}^{t} \frac{1}{\ell(\tau)} \, d\tau. \] (1.13)

Since $\ell(T) u'(T) < 0$, we conclude that $u(t) \to -\infty$ as $t \to +\infty$. A contradiction. The theorem is proved.

We now enquire how far the restriction $g'(x) \geq 0$ imposed in the above theorem can be relaxed. Towards this end we consider equations of the type
\[ (\ell(t)x')' + m(t) g(x) + f(t, x) = 0. \] (1.14)
where \( f(t, x) \) is continuous and defined for \( t \in [0, \infty) \), \( x \in (-\infty, +\infty) \). We have

**Theorem 1.2.** Let \( \ell(t), m(t), g(x) \) satisfy conditions of Theorem 1.1. Let \( f(t, x) \) satisfy

\[
x f(t, x) \geq 0 \quad \text{for all } x \text{ and for all } t \in [t_0, \infty).
\]

Then (1.14) is oscillatory.

**Proof.** The proof follows the same lines as that of the last theorem and is omitted.

We have now the following theorem, which may be derived as a corollary of the above theorem.

**Theorem 1.3.** Let

\[
\ell(t) \geq 0 \quad \text{for } t \geq t_0, \quad \text{and} \quad \int_{t_0}^{\infty} m(t) \, dt = +\infty,
\]

\[
x g(x) > 0, \quad x \neq 0,
\]

and

\[
\liminf_{x \to +\infty} g(x) > 0, \quad \limsup_{x \to -\infty} g(x) < 0.
\]

Then the equation (0.3) is oscillatory.

**Proof.** Observe that if (1.18), (1.19) hold, \( g(x) \) may be written in the form

\[
g(x) = h(x) + f(x),
\]

where \( f(x) \) and \( h(x) \) satisfy the conditions \( x h(x) > 0 \), \( x f(x) \geq 0 \), \( h'(x) > 0 \). Thus (0.3) may be written as

\[
[\ell(t) x']' + m(t) h(x) + m(t) f(x) = 0.
\]

Since the conditions of Theorem 1.2 hold for (1.21), we conclude the desired result.

Indeed it is possible to give an independent proof of this last theorem.

**Alternate Proof of Theorem 1.3.** Let the theorem be false. Then there is a solution \( u(t) \) of (0.3) such that \( u(t) \neq 0 \) for \( t \geq t_0 \). Assume that \( u(t) > 0 \) for \( t \geq t_0 \). Then the identity (1.5) shows that

\[
[\ell(t) u'(t)]' < 0 \quad \text{for } t \geq t_0.
\]
Consequently $\ell(t) u'(t)$ is decreasing and has a limit $C$, where $C$ may be positive, zero, negative, or $-\infty$. The last two cases are impossible as they imply that $u'(t) < 0$ for large $t$, so that if we consider (1.9) with $u'(T) < 0$, divide by $\ell(t)$ and integrate, we can conclude $u(t) \to -\infty$ as $t \to +\infty$, which is a contradiction. This shows that $u'(t) \geq 0$, and so $u(t)$ is increasing and tends to a limit, which may be positive or $+\infty$. In either case the integral in (1.9) diverges to $+\infty$, showing that $\ell(t) u'(t) \to -\infty$, which has already been ruled out. This establishes Theorem 1.3.

Finally, we give an example to show that conditions (1.19) in Theorem 1.3 cannot easily be relaxed.

**Example.** Consider the differential equation

$$\left[ \frac{1}{t} x' \right]' + \frac{1}{t} g(x) = 0,$$

where $g(x) = x$ for $|x| \leq 1$, and $g(x) = 1/x$ for $|x| \geq 1$. This differential equation has a solution $x = t$, for $t \geq 1$. This solution is not oscillatory. Notice that conditions (1.16)-(1.18) of the above theorem are satisfied, but not the condition (1.19).

Thus, in the general situation considered in Theorem 1.3, the condition (1.19) cannot be replaced by $\int_0^{+\infty} g(\xi) \, d\xi = +\infty$, as is the case in Theorem 0.1.

**Remark.** If in the above theorem the condition (1.19) is not assumed, then following the steps in the alternative proof of Theorem 1.3, one can establish that all bounded solutions of (0.3) are oscillatory.

**References**

7. M. Rab. Kriterien für die Oszillation der Lösungen der Differentialgleichung $[p(x) y']' + q(x) y = 0$. *Časopis pro pěstování matematiky* 84 (1959), 335-370.