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On the FEM solution of a coupled contact–two-phase Stefan problem in thermo-elasticity. Coercive case

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Abstract

In the paper a semi-implicit discretization in time of the weak formulation of the coupled signorini type contact–two-phase Stefan Problem is numerically analyzed. The problem leads to coupled elliptic variational inequalities, which are approximated by the FEM.

Keywords: Stefan Problem; Osteotomy; Contact problems; Variational inequality

1. Introduction

The problem investigated represents a simulation of geodynamic processes in subduction zones, based on plate tectonics, contact problems and two-phase Stefan problems. In this paper we shall consider a contact problem of the Signorini type without friction in thermo-elasticity, phase transitions and the case of uniformly moving colliding bodies. Thus the inertial forces are equal to zero and time can be taken as a parameter; the Signorini problem will depend on time through the boundary conditions. Therefore, the weak formulation of the contact part of our problem thus results in an elliptic variational inequality. A semi-implicit discretization in time of the weak formulation of the two-phase Stefan problem using the enthalpy method (cf. [3, 8]) results in an elliptic boundary-value problem and thus leads to an elliptic variational inequality. We shall consider a FEM approximation of the coupled problem and obtain the convergence of the FEM approximation of the solution to the weak solution, which we presume to exist.

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2. Problem definition

In this paper we shall deal with the following problem consisting of the equation of motion for the case of uniform motion in linear thermo-elasticity

$$\frac{\partial}{\partial x_j}(c_{ijkl}(x)e_{kl}(u(x,t)) + \beta_{ij}(x)(\Theta(x,t) - \Theta_0(x))) - f_i(x,t) = 0 \tag{2.1}$$

for almost every $(x,t) \in \Omega \times I$, $i,j,k,l = 1,2$,

and of the heat conduction equation

$$\rho(x)c_e(x)\frac{\partial \Theta(x,t)}{\partial t} + \rho(x)\beta_{ij}(x)\Theta_0(x)e_{ij}(\dot{u}(x,t)) - \frac{\partial}{\partial x_i}\left(\kappa_{ij}(x)\frac{\partial \Theta(x,t)}{\partial x_j}\right) = Q(x,t) \tag{2.2}$$

for almost every $(x,t) \subset \Omega \times I$, $i,j = 1,2$,

where $I = \langle t_0, t_1 \rangle$, $\Omega = \bigcup_{m=1}^{n} \Omega^m \subset \mathbb{R}^2$ is a convex polygonal (and therefore bounded) region occupied by colliding bodies Ω^m with boundaries $\partial\Omega^m = \bar{\Gamma}_\tau^m \cup \bar{\Gamma}_u^m \cup \bar{\Gamma}_c^m$, $\bar{\Gamma}_c^m = \bigcup_{l=1}^{n} \bar{\Gamma}_c^{ml}$, $\Gamma_c^{ml} = \bar{\Omega}^m \cap \bar{\Omega}^l$. Boundary $\partial\Omega$ consists of two parts, $\partial\Omega = \bar{\Gamma}_\tau \cup \bar{\Gamma}_u$, $\bar{\Gamma}_\tau = \bigcup_{m=1}^{n} \bar{\Gamma}_\tau^m$, $\bar{\Gamma}_u = \bigcup_{m=1}^{n} \bar{\Gamma}_u^m$. We also define $\Gamma_c = \bigcup_{k,l=1,k\neq l}^{n} \Gamma_c^{kl}$. On Γ_τ loading is prescribed, on Γ_u displacements are prescribed, Γ_c represents contact boundary. Einstein's summational convention is used.

We consider the following boundary conditions:

$$\tau_{ij}n_j = P_{0i}, \quad \Theta = 0, \quad (x,t) \in \Gamma_\tau \times I, \tag{2.3a, b}$$

$$u = u_0, \quad \kappa_{ij}\frac{\partial \Theta}{\partial x_j}n_i = q, \quad (x,t) \in \Gamma_u \times I, \tag{2.4a, b}$$

$$u_n^k - u_n^l \leqslant 0, \quad \tau_n^k = -\tau_n^l \leqslant 0, \quad \tau_n^k(u_n^k - u_n^l) = 0, \quad \tau_t^k = -\tau_t^l = 0,$$

$$\Theta^k = \Theta^l, \quad \kappa_{ij}\frac{\partial \Theta}{\partial x_j}n_i\bigg|_{(k)} = -\kappa_{ij}\frac{\partial \Theta}{\partial x_j}n_i\bigg|_{(l)}, \quad (x,t) \in \Gamma_c^{kl} \times I, \tag{2.5a–f}$$

where n is the unit outward normal to Γ_c^l related to Ω^k, u_n is the normal component of the displacement vector and $\tau_n(u) = \tau_{ij}(u)n_i n_j$, $\tau_t = \tau - \tau_n n$ are the normal and tangential components of the stress vector. Since we consider the case without friction, condition (2.5d) holds. Let $R^s(t)$ denote the phase change boundaries at time t. These surfaces $R^s(t)$ divide Ω into regions Ω_S^s, Ω_L^s representing, respectively, the solid and the liquid phases as well as recrystallized phases. On the surface $R^s(t)$ the following conditions are given:

$$\Theta_S^s = \Theta_L^s = \Theta_R, \quad \left(\kappa_{ij}\frac{\partial \Theta}{\partial x_j}v_i\right)_S - \left(\kappa_{ij}\frac{\partial \Theta}{\partial x_j}v_i\right)_L = -\rho L^s v_v^s, \tag{2.6}$$

where ρ represents density, Θ_R the temperature of the phase transition, Θ_S^s, Θ_L^s are the temperatures at the phase change boundary $R^s(t)$ in the solid and liquid phases, respectively, v^s the unit normal to $R^s(t)$ pointing towards Ω_S^s, v_v^s is the speed of $R^s(t)$ along v^s and L^s is the latent heat of phase transition or recrystallization. Also let

$$\Theta(x, t_0) = \Theta_0(x). \tag{2.7}$$

Assume that c_S^s, c_L^s, ρ_S^s, ρ_L^s are the specific heats and densities in the solid and liquid phases, and $\rho_S^s = \rho_L^s$ on $R^s(t)$. Following [3, 5], we define the generalized enthalpy $H(\Theta)$ as the subdifferential of $\Phi(\Theta)$ — defined in the usual way as in the convex analysis (cf [4]) — by a monotonically increasing multivalued function of temperature with a jump discontinuity at the phase change temperature Θ_R of the form

$$\partial\Phi(\Theta) \equiv H(\Theta) = \begin{cases} \rho_L^s c_L^s(\Theta - \Theta_R) + \rho L & \text{for } \Theta > \Theta_R, \\ \rho_S^s c_S^s(\Theta - \Theta_R) & \text{for } \Theta < \Theta_R, \\ [0, \rho L] & \text{for } \Theta = \Theta_R, \end{cases} \tag{2.8}$$

where L is the latent heat, $[0, \rho L]$ is the enthalpy interval and $\Phi: \mathbb{R} \to \mathbb{R}$ is a piecewise quadratic convex function. Thus Φ is a continuous convex and locally Lipschitz continuous function, i.e.,

$$|\Phi(\Theta_1) - \Phi(\Theta_2)| \leqslant \max\{|\partial\Phi(\Theta_1)|, |\partial\Phi(\Theta_2)|\}|\Theta_1 - \Theta_2| \leqslant c_k|\Theta_1 - \Theta_2|, \tag{2.8a}$$

when $|\Theta_1|, |\Theta_2| \leqslant K$. Since $\partial H/\partial t = \rho c\, \partial\Theta/\partial t$, (2.2) becomes

$$\frac{\partial H(\Theta)}{\partial t} + \rho\beta_{ij}\Theta_0 e_{ij}(\dot{u}) = \frac{\partial}{\partial x_i}\left(\kappa_{ij}\frac{\partial\Theta}{\partial x_j}\right) + Q, \quad (x, t) \in \Omega \times I. \tag{2.2$'$}$$

Let also, without loss of generality, (2.3b) be rewritten to

$$H(x, t) = 0. \tag{2.3b$'$}$$

Moreover, (2.7) becomes

$$H(x, t_0) = H_0(x). \tag{2.7$'$}$$

Let the problem (2.1)–(2.7) be denoted by (P). We shall assume that the solution of the problem (P) exists and is unique. The semi-implicit discretization of the two-phase Stefan problem (2.2′), (2.3b′), (2.4b), (2.5e, f), (2.6) and (2.7′) leads at every time t to the boundary-value problem for nonlinear partial differential equations of elliptic type. This, together with contact problem (2.2), (2.3a), (2.4a) and (2.5a–d), gives a system of equations of elliptic type for the coupled contact–two-phase Stefan problem. On discretizing

$$\frac{\partial H(\Theta)}{\partial t} = \frac{H(\Theta^{m+1}) - H(\Theta^m)}{\Delta t}, \quad \Theta^m = \Theta(x, m\Delta t),$$

(2.2′) can be expressed in the following time-discrete form:

$$H(\Theta^{m+1}) - \Delta t \frac{\partial}{\partial x_i}\left(\kappa_{ij}\frac{\partial\Theta^{m+1}}{\partial x_j}\right) + \Delta t \rho\Theta_0\beta_{ij}e_{ij}(\dot{u}) = H(\Theta^m) + \Delta t Q^{m+1}, \tag{2.2$''$}$$

where

$$Q^{m+1} = \int_{m\Delta t}^{(m+1)\Delta t} Q(\tau)\,d\tau.$$

For simplicity let us put $\Theta \equiv \Theta^{m+1}$, $u \equiv u^{m+1}$, $\bar{Q} = H(\Theta^m) + \mu Q^{m+1} + \alpha\Theta_R$, $\alpha\Theta + \partial\Phi(\Theta) \equiv H(\Theta) + \alpha\Theta_R$, where $0 < \alpha \leqslant \min\{\rho_S^s c_S^s, \rho_L^s c_L^s\}$ and $\mu = \Delta t$ (see [3]).

Then Eq. (2.2″) yields

$$\alpha\Theta - \mu\frac{\partial}{\partial x_i}\left(\kappa_{ij}\frac{\partial\Theta}{\partial x_j}\right) + \partial\Phi(\Theta) + \mu\rho\Theta_0\beta_{ij}e_{ij}(\dot{u}) = \bar{Q}, \tag{2.2‴}$$

which corresponds to the generalized form of [4]. Then our problem for the fixed time t leads to a coupled elliptic variational problem and the techniques of [3, 4] for the thermal part and of [6] for the elastic part can be used.

Let $V = \{v \mid v \in [H^1(\Omega)]^2,\ v = 0$ on Γ_u in the sense of traces$\}$ be the space of virtual displacements, ${}^{\wedge}V = \{z \mid z \in H^1(\Omega),\ z = 0$ on Γ_τ in the sense of traces$\}$ be the space of virtual temperatures and $K = \{v \mid v \in V,\ v_n^k - v_n^l \leqslant 0$ on $\Gamma_c\}$ be the set of admissible displacements. On V we define bilinear form $a(.,.)$ and functionals $S(.)$, $b_s(\Theta,.)$ by

$$a(u, v) = \int_\Omega c_{ijkl}e_{ij}(u)e_{kl}(v)\,dx,$$

$$S(v) = \int_\Omega f_i v_i\,dx + \int_{\Gamma_\tau} P_{0i}v_i\,ds, \quad b_s(\Theta, v) = \int_\Omega \frac{\partial}{\partial x_j}(\beta_{ij}\Theta)\,v_i\,dx,$$

and on ${}^{\wedge}V$ bilinear form $a_\Theta(.,.)$, scalar product $(.,.)$ and functionals $s(.), j(.), b_p(.,z)$ by

$$a_\Theta(\Theta, z) = \int_\Omega \kappa_{ij}\frac{\partial\Theta}{\partial x_i}\frac{\partial z}{\partial x_j}\,dx, \quad (\Theta, z) = \int_\Omega \Theta z\,dx,$$

$$s(z) = -\mu\int_{\Gamma_u} \kappa_{ij}\frac{\partial\Theta}{\partial x_j}n_i z\,dx + \mu\int_\Omega Qz\,dx, \quad j(z) = \int_\Omega \Phi(z(x))\,dx,$$

$$b_p(v, z) = \mu\int_\Omega \rho\Theta_0\beta_{ij}\frac{\partial v_i}{\partial x_j}z\,dx.$$

Now let us formulate our problem in a variational form:

Definition 2.1. By *variational solution* of the problem (P) we understand a pair of functions (Θ, u), $\Theta \in {}^{\wedge}V$, $u \in K$ such that

$$a(u, v - u) + b_s(\Theta, v - u) \geqslant S(v - u) \quad \forall v \in K, \tag{2.9a}$$

$$\alpha(\Theta, z - \Theta) + \mu a_\Theta(\Theta, z - \Theta) + b_p(u, z - \Theta) + j(z) - j(\Theta) \geqslant s(z - \Theta) \quad \forall z \in {}^{\wedge}V. \tag{2.9b}$$

3. FEM approximation

Assume $\Omega \subset \mathbb{R}^2$ to be polygonal, $h_0 > 0$. Let the domain Ω be triangulated by a triangulation $\boldsymbol{T}^h$ for $h \in (0, h_0]$. Let $\{\boldsymbol{T}^h\}$ be a system of regular triangulations defined as in [6], with the end points $\bar{\Gamma}_u \cap \bar{\Gamma}_\tau$, $\bar{\Gamma}_u \cap \bar{\Gamma}_c$, $\bar{\Gamma}_c \cap \bar{\Gamma}_\tau$ coinciding with the vertices of the triangles T^h. Let $^\wedge V_h$, V_h be the spaces of linear finite elements

$$^\wedge V_h = \{z \mid z \in C(\bar{\Omega}), |z/T^h \in P_1, z = 0 \text{ on } \Gamma_\tau, \forall T^h \in \boldsymbol{T}^h\},$$

$$V_h = \{v \mid v \in [C(\bar{\Omega})]^2, |v/T^h \in [P_1]^2, v = 0 \text{ on } \Gamma_u, \forall T^h \in \boldsymbol{T}^h\},$$

where P_1 is the space of linear polynomials and let $K_h = \{v \mid v \in V_h, v_n^k - v_n^l \leqslant 0 \text{ on } \Gamma_c\}$ be the set of finite element approximations of the set of admissible displacements.

Definition 3.1. The pair of functions (Θ_h, u_h) is said to be a *finite element approximation* of the problem (P) if

$$a(u_h, v_h - u_h) + b_s(\Theta_h, v_h - u_h) \geqslant S(v_h - u_h) \quad \forall v_h \in K_h,$$

$$\alpha(\Theta_h, z_h - \Theta_h) + \mu a_\Theta(\Theta_h, z_h - \Theta_h) + b_p(u_h, z_h - \Theta_h) + j(z_h) - j(\Theta_h) \geqslant s(z_h - \Theta_h)$$
$$\forall z_h \in {}^\wedge V_h. \tag{3.1a, b}$$

Put

$$\Omega_+ = \{x \mid \Theta(x) > \Theta_R\}, \quad \Omega_{+h} = \{T^h \in \boldsymbol{T}^h \mid \Theta > \Theta_R \text{ on } T^h\},$$
$$\Omega_- = \{x \mid \Theta(x) < \Theta_R\}, \quad \Omega_{-h} = \{T^h \in \boldsymbol{T}^h \mid \Theta < \Theta_R \text{ on } T^h\}, \tag{3.2a, f}$$
$$\Omega_0 = \{x \mid \Theta(x) = \Theta_R\}, \quad \Omega_{0h} = \{T^h \in \boldsymbol{T}^h \mid \Theta = \Theta_R \text{ on } T^h\}.$$

Let $\Omega_{*h} = \Omega - (\Omega_{+h}\Omega_{-h} \cup \Omega_{0h})$. Since the free boundary is contained in some triangles, Ω_{*h} is not empty. The length of the free boundary $\bigcup R^s(t)$ satisfies meas $\{\bigcup R^s(t)\}$ = meas $\{\partial\Omega_+ \cap \partial\Omega_-\}$ + meas$\{\partial\Omega_- \cap \partial\Omega_0\}$ + meas$\{\partial\Omega_0 \cap \partial\Omega_+\} < M$, i.e., it is finite. Then there exists a constant c, independent of h, such that the number of triangles N_{*h} in Ω_{*h} is bounded by $N_{*h} < c/h$. Denote by $\|\cdot\|_{0,1}$ the norm in $H^0(\Omega) = L_2(\Omega)$, $\|\cdot\|_{1,1}$ in $H^1(\Omega)$, $\|\cdot\|_{1,2}$ in $[H^1(\Omega)]^2$.

For any pair of functions (z, v) on Ω we define its Lagrange interpolate $(z_{\mathrm{LI}}, v_{\mathrm{LI}})$, $z_{\mathrm{LI}} \in {}^\wedge V_h$, $v_{\mathrm{LI}} \in V_h$ and $(z_{\mathrm{LI}}, v_{\mathrm{LI}}) = (z, v)$ at each vertex of triangulation $\boldsymbol{T}^h$. Then due to the well-known interpolate estimate (cf. [1]) it holds also for Lagrange interpolate

$$\|v - v_{\mathrm{LI}}\|_{0,2} + h\|v - v_{\mathrm{LI}}\|_{1,2} \leqslant ch^2, \quad \forall v \in [H_0^1(\Omega)]^2 \cap [H^2(\Omega)]^2,$$

$$\|z - z_{\mathrm{LI}}\|_{0,1} + h\|z - z_{\mathrm{LI}}\|_{1,1} \leqslant ch^2, \quad \forall z \in H_0^1(\Omega) \cap H^2(\Omega).$$

For $\Theta, z \in {}^\wedge V_h$ put

$$(\Theta, z)_h = \sum_{T^h \in \boldsymbol{T}^h} \tfrac{1}{3}\,\mathrm{meas}(T^h)\Theta_i^T z_i^T, \quad j_h(z) = \int_\Omega (\Phi(z))_{\mathrm{LI}}\,\mathrm{d}x = \sum_{T^h \in \boldsymbol{T}^h} \tfrac{1}{3}\mathrm{meas}\,(T^h) \sum_{i=1}^{3} \Phi(z_i^T),$$

where meas(T^h) represents the area of triangle T^h, z_i represents the value of z at node i of triangle T^h.

Lemma 3.2. (Ciavaldini). *The following estimates hold*:

(a) $|(\Theta, z)_h - (\Theta, z)| \leqslant ch^2 \|\Theta\|_{1,1} \|z\|_{1,1} \quad \forall \Theta, z \in {}^\wedge V_h$,
(b) $\|z\|_{0,1} \leqslant (z, z)_h^{1/2} \leqslant c\|z\|_{0,1} \quad \forall z \in {}^\wedge V_h$.

For the proof see [2].

Lemma 3.3. *For* $\Theta \in {}^\wedge V$, $u \in K$, $\Theta_h \in {}^\wedge V_h$, $u_h \in K_h$ *we have*

$$\alpha \|\Theta - \Theta_h\|_{0,1}^2 + \mu \|\Theta - \Theta_h\|_{1,1}^2 \leqslant c[\alpha(\Theta_h - \Theta, z_h - \Theta) + \mu a_\Theta(\Theta_h - \Theta, z_h - \Theta)$$
$$+ \alpha(\Theta, z - \Theta_h) + \mu a_\Theta(\Theta, z - \Theta_h) + \alpha(\Theta, z_h - \Theta) + \mu a_\Theta(\Theta, z_h - \Theta) + b_p(u, z_h - \Theta)$$
$$+ b_p(u, z - \Theta_h) + j(z) - j(\Theta_h) + j(z_h) - j(\Theta) - s(z - \Theta_h) - s(z_h - \Theta)]^{1/2}$$
$$\forall z \in {}^\wedge V, \quad z_h \in {}^\wedge V_h, \tag{3.3a}$$

$$\|u - u_h\|_1 \leqslant c_0[a(u_h - u, v_h - u) + a(u, v - u_h) + a(u, v_h - u) - S(v - u_h) - S(v_h - u)$$
$$+ b_s(\Theta, v - u_h) + b_s(\Theta, v_h - u)]^{1/2} \quad \forall v \in K, \quad v_h \in K_h, \quad c_0 = const. > 0. \tag{3.3b}$$

Proof. Let $\Theta \in {}^\wedge V$, $\Theta_h \in {}^\wedge V_h$, $u \in K$, $u_h \in K_h$. Using (2.9b), (3.1b) and adding and subtracting terms $\alpha(\Theta, \Theta - \Theta_h) - \alpha(\Theta_h, \Theta) + \mu a_\Theta(\Theta, \Theta - \Theta_h) - \mu a_\Theta(\Theta_h, \Theta)$ and similarly using (2.9a), (3.1a) and adding and subtracting $a(u, u - u_h) - a(u_h, u)$ leads directly to:

(a) Firstly,

$$\alpha(\Theta, \Theta - \Theta_h) - \alpha(\Theta_h, \Theta) - \alpha(\Theta, \Theta - \Theta_h) + \alpha(\Theta_h, \Theta) + \alpha(\Theta, z - \Theta) + \alpha(\Theta_h, z_h - \Theta_h)$$
$$+ \mu a_\Theta(\Theta, z - \Theta) + \mu a_\Theta(\Theta_h, z_h - \Theta_h) + \mu a_\Theta(\Theta, \Theta - \Theta_h) - \mu a_\Theta(\Theta_h, \Theta) - \mu a_\Theta(\Theta, \Theta - \Theta_h)$$
$$+ \mu a_\Theta(\Theta_h, \Theta) + b_p(u, z - \Theta) + b_p(u_h, z_h - \Theta_h) + j(z) - j(\Theta) + j(z_h) - j(\Theta_h) - s(z - \Theta)$$
$$- s(z_h - \Theta_h) \geqslant 0.$$

From here it follows that

$$\alpha(\Theta, \Theta - \Theta_h) - \alpha(\Theta_h, \Theta - \Theta_h) + \mu a_\Theta(\Theta, \Theta - \Theta_h) - \mu a_\Theta(\Theta_h, \Theta - \Theta_h)$$
$$= \alpha(\Theta - \Theta_h, \Theta - \Theta_h) + \mu a_\Theta(\Theta - \Theta_h, \Theta - \Theta_h)$$
$$\leqslant \alpha(\Theta_h - \Theta, z_h - \Theta) + \mu a_\Theta(\Theta_h - \Theta, z_h - \Theta) + \alpha(\Theta, z - \Theta_h)$$
$$+ \mu a_\Theta(\Theta, z - \Theta_h) + \alpha(\Theta, z_h - \Theta) + \mu a_\Theta(\Theta, z_h - \Theta) + b_p(u, z - \Theta)$$
$$+ b_p(u_h, z_h - \Theta_h) + j(z) - j(\Theta) + j(z_h) - j(\Theta_h) + s(\Theta_h - z) + s(\Theta - z_h).$$

Hence (3.3a) follows.

(b) Secondly,

$$a(u, v-u) + a(u_h, v_h - u_h) + a(u, u-u_h) - a(u_h, u) - a(u, u-u_h) + a(u_h, u) + b_s(\Theta, v-u)$$
$$+ b_s(\Theta_h, v_h - u_h) - S(v-u) - S(v_h - u_h) \geqslant 0.$$

From here it follows that

$$a(u, u-u_h) - a(u_h, u-u_h)$$
$$= a(u-u_h, u-u_h) \leqslant a(u_h - u, v_h - u) + a(u, v-u_h) + a(u, v_h - u)$$
$$+ b_s(\Theta, v-u) + b_s(\Theta_h, v_h - u_h) + S(u_h - v) + S(u - v_h).$$

Hence (3.3b) follows. □

Corollary. *Let* ${}^{\wedge}V_h \subset {}^{\wedge}V$, $K_h \subset K$. *Then by substituting* $z = \Theta_h$ in (3.3a) *and* $v = u_h$ *in* (3.3b) *and adding the resulting inequalities, we obtain*

$$\alpha \| \Theta - \Theta_h \|_{0,1}^2 + \mu \| \Theta - \Theta_h \|_{1,1}^2 + \| u - u_h \|_{1,2}^2$$
$$\leqslant C[\alpha(\Theta_h - \Theta, z_h - \Theta) + \mu a_\Theta(\Theta_h - \Theta, z_h - \Theta) + a(u_h - u, v_h - u) + \alpha(\Theta, z_h - \Theta)$$
$$+ \mu a_\Theta(\Theta, z_h - \Theta) + a(u, v_h - u) + b_p(u, \Theta_h - \Theta) + b_p(u, z_h - \Theta_h) + b_s(\Theta, u_h - u)$$
$$+ b_s(\Theta, v_h - u_h) + j(z_h) - j(\Theta) + s(\Theta - z_h) + S(u - v_h)] \quad \forall z_h \in {}^{\wedge}V, v_h \in K_h. \tag{3.4}$$

Lemma 3.4. *It holds that*

(a) $j(v) \leqslant j_h(v) \quad \forall v \in V_h$,
(b) $|j(u_{\mathrm{LI}}) - j(u)| \leqslant c \| u - u_{\mathrm{LI}} \|_{0,1}$,
(c) $|j_h(u_{\mathrm{LI}}) - j(u_{\mathrm{LI}}) \leqslant ch^2$.

For the proof see [3].

Lemma 3.5. *Let* $\partial \beta_{ij}/\partial x_j \in L^\infty(\Omega)\ \forall i,j$. *Then*

$$|b_s(\Theta - \Theta_0, v) + b_p(v, \Theta)| \leqslant c((1 + \| \Theta(t) \|_{1,1}) \| v(t) \|_{0,2} + \| \Theta(t) \|_{0,1} \| v(t) \|_{1,2}).$$

The proof follows from the above assumption and the definitions of b_s and b_p (also see [8]).

Theorem 3.6. *Let* Γ_c *be polygonal and let free boundary* $\bigcup_s R^s(t)$ *be finite. Let* $\Theta \in W^{1,\infty}(\Omega) \cap H^2(\Omega)$, $u \in K \cap [H^2(\Omega)]^2$, $u/\Gamma_c \in H^2(\Gamma_c)^2$. *Let* $\Omega_+, \Omega_-, \Omega_0, \Omega_{+h}, \Omega_{-h}, \Omega_{0h}$ *be defined by* (3.2a–f) *and let* $N_{*h} < c/h$. *Then*

$$\alpha \| \Theta - \Theta_h \|_{0,1}^2 + \mu \| \Theta - \Theta_h \|_{1,1}^2 + \| u - u_h \|_{1,2}^2 \leqslant ch. \tag{3.5}$$

Proof. To prove this theorem we use a technique similar to that of [6]. Using Lemma 3.3, we estimate (3.5). To achieve this, we shall assume that Θ and u are sufficiently regular. Terms $\alpha(\Theta, z-\Theta_h)+\mu a_\Theta(\Theta, z-\Theta_h)-s(z-\Theta_h)$, $\alpha(\Theta, z_h-\Theta)+\mu a_\Theta(\Theta, z_h-\Theta)-s(z_h-\Theta)$ and $a(u, v-u_h)-S(v-u_h)$, $a(u, v_h-u)-S(v_h-u)$ are estimated by applying Green's lemma and later by using a suitable choice of $\Theta_h \in {}^\wedge V_h$, $\Theta \in {}^\wedge V$, $u_h \in K_h$, $u \in K$. We then obtain

$$
\begin{aligned}
&\alpha\|\Theta-\Theta_h\|_{0,1}^2+\mu\|\Theta-\Theta_h\|_{1,1}^2+\|u-u_h\|_{1,2}^2 \\
&\quad\leqslant c\Big[\alpha(\Theta_h-\Theta, z_h-\Theta)+\mu a_\Theta(\Theta_h-\Theta, z_h-\Theta)+a(u_h-u, v_h-u)+\int_{\Gamma_u} q(z_h-\Theta)\,\mathrm{d}s \\
&\qquad+\int_{\partial\Omega}\tau_{ij}(u)n_j(v_h-u)_i\,\mathrm{d}s-\int_{\Gamma_\tau}P_i(v_h-u)_i\,\mathrm{d}s+j(z_h)-j(\Theta)+j(z)-j(\Theta_h)+b_p(u, z_h-\Theta) \\
&\qquad+b_s(\Theta-\Theta_0, v_h-u)+b_p(u, z-\Theta_h)+b_s(\Theta-\Theta_0, v-u_h)\Big] \\
&\quad\leqslant c\Big[\alpha\|\Theta_h-\Theta\|_{0,1}\|z_h-\Theta\|_{0,1}+\mu\|\Theta_h-\Theta\|_{1,1}\|z_h-\Theta\|_{1,1}+\|u_h-u\|_{1,2}\|v_h-u\|_{1,2} \\
&\qquad+\int_{\bigcup_{kl}\Gamma_c^{kl}}\tau_n^k(u)((v_h^k-v_h^l)_n-(u_n^k-u_n^l))\,\mathrm{d}s+\int_{\Gamma_u}\Theta_{,n}(z-\Theta_h)\,\mathrm{d}s \\
&\qquad+j(z_h)-j(\Theta)+j(z)-j(\Theta_h) \\
&\qquad+b_p(u, z_n-\Theta)+b_s(\Theta-\Theta_0, v_h-u)+b_p(u, z-\Theta_h)+b_s(\Theta-\Theta_0, v-u_h)\Big] \\
&\quad\leqslant c\Big[\tfrac{1}{2}\varepsilon\|\Theta-\Theta_h\|_{0,1}^2+\tfrac{1}{2}\varepsilon^{-1}\|\Theta-z_h\|_{0,1}^2+\tfrac{1}{2}\varepsilon\|\Theta-\Theta_h\|_{1,1}^2 \\
&\qquad+\tfrac{1}{2}\varepsilon^{-1}\|\Theta-z_h\|_{1,1}^2+\tfrac{1}{2}\varepsilon\|u_h-u\|_{1,2}^2 \\
&\qquad+\tfrac{1}{2}\varepsilon^{-1}\|v_h-u\|_{1,2}^2+\int_{\bigcup_{kl}\Gamma_c^{kl}}\tau_{ij}(u)n_j((v_h^k-v_h^l)_n-(u_n^k-u_n^l))\,\mathrm{d}s \\
&\qquad+\int_{\Gamma_u}\Theta_{,n}(z-\Theta_h)\,\mathrm{d}s+j(z_h)-j(\Theta) \\
&\qquad+j(z)-j(\Theta_h)+b_p(u, z_h-\Theta)+b_s(\Theta-\Theta_0, v_h-u) \\
&\qquad+b_p(u, z-\Theta_h)+b_s(\Theta-\Theta_0, v-u_h)\Big].
\end{aligned}
$$

To estimate the last inequality, we put $z_h=\Theta_{\mathrm{LI}}$, $v_h=u_{\mathrm{LI}}$, where $\Theta_{\mathrm{LI}}\in {}^\wedge V_h$ is the Lagrange interpolation of Θ on triangulation $\boldsymbol{T}^h$ and $u_{\mathrm{LI}}\in V_h$ is the Lagrange interpolation of u on triangulation $\boldsymbol{T}^h$. Following [7], we have $(u_{\mathrm{LI}}^k-u_{\mathrm{LI}}^l)_n\leqslant 0$ on Γ_c, thus $u_{\mathrm{LI}}\in K$. Since $u_{\mathrm{LI}}\in V_h$, then $u_{\mathrm{LI}}\in K_h$. Furthermore, we have

$$\|u_{\mathrm{LI}}-u\|_{1,2}\leqslant c_r h\|u\|_{1,2},$$

$$\|(u_{\mathrm{LI}}^k-u_{\mathrm{LI}}^l)_n-(u_n^k-u_n^l)\|_{[L^2(\Gamma_c)]^2}\leqslant c_s h^2\sum_{\Gamma_c^{kl}}\|u_n^k-u_n^l\|_{[H^2(\Gamma_c)]^2}.$$

Due to Lemmas 3.4 and 3.2,

$$j(\Theta_h) - j(\Theta) + j_h(\Theta_{LI}) - j_h(\Theta_h) \leqslant j_h(\Theta_{LI}) - j(\Theta) = j_h(\Theta_{LI}) - j(\Theta_{LI}) + j(\Theta_{LI}) - j(\Theta)$$
$$\leqslant ch^2 + c\,\|\Theta - \Theta_{LI}\|_{0,1},$$

$$\alpha[(\Theta_h, \Theta_h - \Theta_{LI}) - (\Theta_h, \Theta_h - \Theta_{LI})_h] \leqslant ch^2,$$

$$\|\Theta_{LI} - \Theta\|_{1,1} \leqslant c_r h^2 \|\Theta\|_{2,1}, \quad \|\Theta_{LI} - \Theta\|_{0,1} \leqslant c_p h^2 \|\Theta\|_{2,1},$$

$$\alpha(\Theta_h, \Theta_{LI} - \Theta) + \mu a_\Theta(\Theta_h - \Theta, \Theta_{LI} - \Theta) - s(\Theta_{LI} - \Theta)$$
$$\leqslant \alpha\|\Theta_h\|_{0,1}\|\Theta_{LI} - \Theta\|_{0,1} + \mu\|\Theta_h - \Theta\|_{1,1}\|\Theta_{LI} - \Theta\|_{1,1} + \|Q\|_{0,1}\|\Theta_{LI} - \Theta\|_{0,1}$$
$$\leqslant c\|\Theta_{LI} - \Theta\|_{0,1} + \tfrac{1}{2}\varepsilon\mu\|\Theta_h - \Theta\|_{1,1}^2 + \tfrac{1}{2}\varepsilon^{-1}\mu\|\Theta_{LI} - \Theta\|_{1,1}^2.$$

Due to Lemma 3.5 ($v_h = u_{LI}$ and $z_h = \Theta_{LI}$)

$$|b_s(\Theta - \Theta_0, v_h - u) + b_p(u, z_h - \Theta)| \leqslant c(1 + \|\Theta\|_{1,1}\|u_{LI} - u\|_{0,2} + \|u\|_{1,2}\|\Theta_{LI} - \Theta\|_{0,1})$$

and ($v = u_{LI} \in V_h$ and $z = \Theta_{LI} \in {}^\wedge V_h$)

$$|b_s(\Theta - \Theta_0, u_{LI} - u_h) + b_p(u, \Theta_{LI} - \Theta_h)| \leqslant c(1 + \|\Theta\|_{1,1}\|u_{LI} - u_h\|_{0,2}$$
$$+ \|u\|_{1,2}\|\Theta_{LI} - \Theta_h\|_{0,1}).$$

Hence

$$\alpha\|\Theta - \Theta_h\|_{0,1}^2 + \mu\|\Theta - \Theta_h\|_{1,1}^2 + \|u - u_h\|_{1,2}^2 \leqslant ch,$$

which completes the proof. □

4. Algorithm

The algorithm is based on the semi-implicit scheme. Since $(\beta_{ij}(T - T_0)_{,j} \in [L_2(\Omega)]^2$ and $\rho\beta_{ij}T_0 e_{ij}(\dot{u}) \in L_2(\Omega)$ holds for the coupled terms, they have the meaning of body forces and thermal sources, and therefore on every time level we solve the following problems: Problem (3.1a) leads to minimization

$$J(v) = \tfrac{1}{2} v^{T} V v - b^{T} v \quad \Bigg| \quad J(u_h) = \inf_{v \in K_h} J(v)$$

$$Av \leqslant 0$$

and problem (3.1b) leads to minimization

$$J_\Theta(z) = \tfrac{1}{2} z^{T} B z - d^{T} z + \Phi(z) \quad \Bigg| \quad J_\Theta(T) = \inf_{z \in {}^\wedge V_h} J_\Theta(z).$$

These problems represent optimization problems with constraints in the first case and the nonlinear minimization problem in the second case.

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