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The Penultimate Rate of Growth for Graph Properties

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Given a property $\mathcal{P}$ of graphs, write $\mathcal{P}^n$ for the set of graphs with vertex set $[n]$ having property $\mathcal{P}$. We call $|\mathcal{P}^n|$ the *speed* of $\mathcal{P}$. Recent research has shown that the speed of a monotone or hereditary property $\mathcal{P}$ can be a constant, polynomial, or exponential function of n, and the structure of the graphs in $\mathcal{P}$ can then be well described. Similarly, $|\mathcal{P}^n|$ can be of the form $n^{(1-1/k+o(1))n}$ or $2^{(1-1/k+o(1))n^2/2}$ for some positive integer $k > 1$ and the properties can be described and have well-behaved speeds. In this paper, we discuss the behavior of properties with speeds between these latter bounds, i.e., between $n^{(1+o(1))n}$ and $2^{(1/2+o(1))n^2/2}$.

1. Introduction

A *graph property* is a (infinite) collection of (labeled) graphs closed under isomorphism. The property consisting of all finite graphs is the *trivial property*. A property is *hereditary* if it is closed under taking induced subgraphs, and it is *monotone* if it is closed under taking subgraphs. For example, being acyclic, planar, or perfect are hereditary properties, while only the first two are monotone. Rather trivially, every hereditary property can be defined in terms of forbidden induced subgraphs, and every monotone property can be defined in terms of forbidden subgraphs.

Given a property $\mathcal{P}$, write $\mathcal{P}^n$ for the set of all graphs in $\mathcal{P}$ with n vertices. We call this the *n-level* of $\mathcal{P}$. The number of graphs in the n-level, $|\mathcal{P}^n|$, is called the *speed* of a property. In recent years, there has been much research into the sequence $(|\mathcal{P}^n|)_{n=1}^{\infty}$ for hereditary properties (see, for example, [1, 5, 9]) and for monotone properties (see, for example, [2, 6]). The most natural questions about the speed of a hereditary property, which first appeared in [9], are as follows.

(1) Does $\lim_{n\to\infty} \frac{\log|\mathcal{P}^n|}{n}$ exist for all hereditary properties $\mathcal{P}$?
(2) Does $\lim_{n\to\infty} \frac{\log|\mathcal{P}^n|}{n\log n}$ exist for all hereditary properties $\mathcal{P}$?
(3) Does $\lim_{n\to\infty} \frac{\log\log|\mathcal{P}^n|}{\log n}$ exist for all hereditary properties $\mathcal{P}$?
(4) Does $\lim_{n\to\infty} \frac{\log|\mathcal{P}^n|}{n^2}$ exist for all hereditary properties $\mathcal{P}$?

The first question was answered affirmatively by Scheinerman and Zito in [9], and the others were left by them as open questions. The fourth question was answered affirmatively by Bollobás and Thomason in [5]. However, the second and third questions remained open. In this paper we answer both negatively, even under a strong condition on the structure of the property. In doing so, we shed some light on a gap in the existing research on speeds.

While investigating the questions above, Scheinerman and Zito [9] discovered that the speed sequence often has a well-defined behavior. They presented a rough hierarchy of speeds for hereditary properties, showing that the speed of a property must fall into certain ranges of growth. Earlier results by Bollobás and Thomason [5] had shown that the highest speed of growth is also highly constrained. The present authors provided a more detailed picture of the hierarchy of speeds for hereditary properties in [1] and furthermore described the structure of properties falling into each range of speed. Similar results for monotone properties were shown in [2].

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These results can be briefly summarized in the following theorem, which presents four functional ranges into which the speed of a hereditary property may fall. The first level of growth can be divided into three parts depending on whether $k = 0$, $k = 1$, or $k > 1$.

THEOREM 1. *Let $\mathcal{P}$ be a hereditary property of graphs. Then one of the following is true:*

(1) *there exists $N, k \in \mathbb{N}$ and a collection $\{p_i(n)\}_{i=0}^{k}$ of polynomials such that for all $n > N$, $|\mathcal{P}^n| = \sum_{i=0}^{k} p_i(n) i^n$;*
(2) *there exists $k \in \mathbb{N}$, $k > 1$ such that $|\mathcal{P}^n| = n^{(1-1/k+o(1))n}$;*
(3) $n^{(1+o(1))n} \leq |\mathcal{P}^n| \leq n^{o(n^2)}$;
(4) *there exists $k \in \mathbb{N}$, $k > 1$ such that $|\mathcal{P}^n| = 2^{(1-1/k+o(1))n^2/2}$.*

The first two cases and a jump to the third are described and proven by the authors in [1, 3]. The last case and the gap between case 3 and 4 are shown by Bollobás and Thomasson in [5, 6]. Specifically, Theorem 1 and the results of [1] imply that if the speed of a property falls into either the first two or the final range, the speed actually approaches a limiting function and furthermore the structure of graphs in the property can be described. Even from the form of the statement of Theorem 1, however, it is clear that the exact behavior of properties with speeds falling into the third range is not well understood.

For this range, even the bounds have not been fully described, although the lower bound is understood and can be approximated using results from [2]. In Section 2 of this paper, we explore the upper bound of this range and show, as in the other ranges (including within the first range), that there is a discontinuous jump in the actual speeds that may occur.

In Section 3 we describe a type of property, a *selectively restricted property*, which exists in the penultimate range of growth. We show that some selectively restricted properties have speeds towards the bottom of the third range of growth, while others have high speeds.

In Section 4, we demonstrate particular selectively restricted properties which provide an infinite collection of negative examples for the second and third questions of Scheinerman and Zito. Specifically, we describe a monotone (and hence hereditary) property with speed that oscillates infinitely often between two functions near the upper and lower bounds respectively on the penultimate range.

In the two subsequent sections, we discuss improvements on the construction given in Section 4. We close with a conjecture that the results presented here are nearly the best one could obtain.

2. BOUNDS ON THE FACTORIAL RANGE

What are the actual upper and lower limits on the penultimate range of growth? Theorem 1 implies that if $\mathcal{P}$ is hereditary and for some $\epsilon > 0$ we have $|\mathcal{P}^n| < n^{(1-\epsilon)n}$ for infinitely many values of n, then the property will fall into ranges 1 or 2. The same is true for monotone properties, as is shown in [2]. The theorem also says that if there is a c such that $|\mathcal{P}^n| > 2^{cn^2}$ for infinitely many values of n, then the speed of the property falls into range 4.

In fact, it is shown in [3] that the smallest property in the penultimate range is the property $\mathcal{P}_{\text{clique}} = \{G$: every component of G is complete$\}$ or the complementary property. These properties have speed $|\mathcal{P}^n_{\text{clique}}| = b(n)$, where $b(n)$ is the nth Bell number. Hence the lower bound on this speed range is

$$b(n) \sim n^n \left(\frac{n}{\log n}\right)^{1-1/\log n} (\log n)^{-n}.$$

With the lower bound known, we begin our investigation of the penultimate range at its upper bound. What speeds of the type $2^{o(n^2)}$ are possible for graph properties? This question may be answered more easily for monotone than for hereditary properties; in fact the question is open for the latter class. We shall show that given any monotone property $\mathcal{P}$, either there is an ϵ such that $|\mathcal{P}^n| < 2^{n^{2-\epsilon}}$ or else the speed is at least $2^{(1/4+o(1))n^2}$.

Notation and definitions. For a graph G, we write $v(G) = |V(G)|$ and $e(G) = |E(G)|$. Further, we call a graph G an *n-graph* if $v(G) = n$, and $H \subseteq G$ a *k-subgraph* if $v(H) = k$. The collection of labeled n-graphs (on $[n]$) is denoted $\mathcal{G}^n$.

The following lemma is a simple observation which nonetheless provides strong information about large graph properties.

LEMMA 2. *Let $\epsilon > 0$ and $0 < c \leq 2$. There is an N such that for all $n > N$, if S is a set of graphs on n vertices and $|S| > 2^{n^{2-c+\epsilon}}$, then there is a graph $G \in S$ with $e(G) > n^{2-c}$.*

PROOF. Let $f_c(n)$ be the number of graphs on n vertices with at most n^{2-c} edges. Then,

$$f_c(n) \leq \sum_{j=0}^{n^{2-c}} \binom{\binom{n}{2}}{j} \leq n^{2-c}\left(\frac{en^2}{2n^{2-c}}\right)^{n^{2-c}}$$
$$= \left(\frac{e}{2}\right)^{n^{2-c}} n^{2-c} n^{cn^{2-c}} = 2^{n^{2-c}(\lg(e/2)+c\lg n)+(2-c)\lg n} = 2^{n^{2-c+o(1)}}.$$

Hence for all ϵ there is an N such that for all $n > N$, $f_c(n) < 2^{n^{2-c+\epsilon}}$. Thus if $n > N$ and there is a collection S of n-graphs with $|S| > 2^{n^{2-c+\epsilon}}$, then there is a graph $G \in S$ with $e(G) > n^{2-c}$. □

We are now ready to prove the main result of this section. Note that a property $\mathcal{P}$ is monotone if and only if there is a (possibly infinite) collection $\mathcal{H}$ of graphs such that $\mathcal{P} = \mathbf{Mon}(\mathcal{H})$, where $\mathbf{Mon}(\mathcal{H})$ is the set of all graphs G such that no subgraph of G is isomorphic to any $H \in \mathcal{H}$. We also define $\mathbf{Her}(\mathcal{H})$ as the set of all graphs G such that no induced subgraph of G is isomorphic to any $H \in \mathcal{H}$.

THEOREM 3. *Let $\mathcal{P}$ be a monotone property. If $|\mathcal{P}^n| = 2^{o(n^2)}$, then there is a $t \geq 1$ such that $|\mathcal{P}^n| \leq 2^{n^{2-1/t+o(1)}}$.*

PROOF. Let $\mathcal{P} = \mathbf{Mon}(\mathcal{H})$ be a monotone property with speed $|\mathcal{P}^n| = 2^{o(n^2)}$. If every graph $H \in \mathcal{H}$ has chromatic number at least 3, then $\{G : G \text{ is bipartite}\} \subseteq \mathbf{Mon}(\mathcal{H}) = \mathcal{P}$, in which case $|\mathcal{P}^n| \geq 2^{\binom{n}{2}/2}$. Hence $\mathcal{H}$ contains a bipartite graph H. Let t be the order of the larger set in the bipartition of H. Assume for the sake of contradiction that $|\mathcal{P}^n| > 2^{n^{2-1/t+o(1)}}$. By Lemma 2, there is then a $G \in \mathcal{P}$ such that $e(G) > n^{2-1/t}$. A result of Kővári *et al.* [8] says that G contains the graph $K_{t,t}$ as a subgraph. Thus G contains H as a subgraph, which is a contradiction. □

The result above is nearly the best result possible about the gap below 2^{cn^2}. In the proof, we showed that large properties must contain large complete bipartite graphs. On the other hand, we can construct properties that nearly reach the upper bound given for which a large complete bipartite graph is forbidden. To do so, we use the well-known fact that for any t there exists a bipartite graph on n vertices with at least $n^{2-2/t}$ edges that contains no subgraph isomorphic to $K_{t,t}$ (see, i.e., [4, p. 316, Theorem VI.2.10]). Hence, for example, if $\mathcal{P} = \mathbf{Mon}(\{K_{t,t}, K_3\})$, then $|\mathcal{P}^n| = 2^{o(n^2)}$ and $|\mathcal{P}^n| \geq 2^{n^{2-2/t}}$. Thus Theorem 3 in fact guarantees a t such that $2^{n^{2-2/t}} \leq |\mathcal{P}^n| < 2^{n^{2-1/t}}$. We conjecture that the same is true for hereditary properties, and again this would be the best possible, as $|\mathcal{P}^n| \geq 2^{n^{2-2/t}}$ for $\mathcal{P} = \mathbf{Her}(\{K_{t,t}, K_3\})$. The conjecture below differs from Theorem 3 only in considering hereditary rather than monotone properties.

CONJECTURE 4. *Let $\mathcal{P}$ be a hereditary property. If $|\mathcal{P}^n| = 2^{o(n^2)}$, then there is a $t \geq 1$ such that $|\mathcal{P}^n| \leq 2^{n^{2-1/t}}$.*

This conjecture is far from proven, however. While it would be surprising, it is not inconceivable that there could be properties with other speeds. A result of Ruzsa and Szemerédi about hypergraphs suggests that properties of hypergraphs do not behave so nicely, but the calculations and considerations are completely different.

3. SOME SPECIAL PROPERTIES AND THEIR GROWTH

The results of the previous section imply that if the speed of $\mathcal{P}$ is in the penultimate range, there is an integer $t \geq 1$ such that $n^{(1+o(1))n} < |\mathcal{P}^n| < 2^{n^{2-1/t}}$. This is proven for monotone $\mathcal{P}$ but only conjectured for hereditary $\mathcal{P}$. We now turn our attention to the question of what speeds are actually achieved in this range for graph properties, either monotone or hereditary. We present a collection of properties with speeds lying in the range.

Our first property is defined in terms of the density of subgraphs. We define the *c-dense property* $\mathcal{Q}_c$ by $\mathcal{Q}_c = \{G : e(H) \leq cv(H) \text{ for all } H \subseteq G\}$. The following assertion was proved by Scheinerman and Zito [9].

THEOREM 5. *For any $c > 1$, $|\mathcal{Q}_c^n| = n^{(c+o(1))n}$.*

The property $\mathcal{Q}_c$ is a monotone property, and this result shows that any speed of the type n^{cn}, $c > 1$ is achievable. The next property we describe is not monotone or hereditary, but its n-level contains the n-level of $\mathcal{Q}_c$. It will be useful in further proofs in this section. The *c-dense n-level* is simply $S_c^n = \{G \in \mathcal{G}^n : e(G) \leq cn\}$. The following lemma gives a bound on its size.

LEMMA 6. *If $c > 1$, then $|S_c^n| < f(n)n^{cn}$, with $f(n) = O(1.5^n)$.*

PROOF. Note simply that

$$|S_c^n| \leq \sum_{j=0}^{cn} \binom{\binom{n}{2}}{j} \leq \sum_{j=0}^{cn} (en^2/2j)^j < cn(en^2/2cn)^{cn} = cn(en/2)^{cn} = f(n)n^{cn},$$

where, easily, $f(n) = O(1.5^n)$. □

We showed in the previous section that there are properties with speeds at the top of the third range, that is, with $2^{n^{2-2/t}} \leq |\mathcal{P}^n| < 2^{n^{2-1/t}}$. Hence, there are properties with speeds throughout the range of the third case of Theorem 1. This does not necessarily mean that any speed can be achieved, however. We examine constraints on demonstrably achievable functions later in this paper.

As mentioned in the Introduction, if the speed of a property falls into any but the third range, $|\mathcal{P}^n|$ can be described with a 'nice' function. However, we shall show that this is not the case for the third range. In this range, it is possible for the speed to oscillate between two different functions infinitely often.

More precisely, the question we examine is as follows. Is there a property $\mathcal{P}$ so that for functions $f(n) < g(n)$, $|\mathcal{P}^n|$ oscillates infinitely often between them? Clearly there are choices of $f(n)$ and $g(n)$ for which it is not possible to construct such a hereditary property. In particular, Theorem 1 implies that if $g(n) \leq n^{(1-1/k+o(1))n}$ for some k or $f(n) \geq n^{(1-1/k+o(1))n^2/2}$ for some k, then $|\mathcal{P}^n|$ cannot oscillate.

However, we shall show that for many choices of $f(n)$ and $g(n)$ in a subrange of the third range of Theorem 1, we can construct such a property. Furthermore, this property is monotone (and therefore also hereditary). Our methods are probabilistic in nature, and we proceed in steps, first demonstrating a property with fixed upper and lower bounds on the oscillation of its speed, and then by adjusting the upper and lower bounds.

We begin with a technical probabilistic lemma regarding sets of graphs and $G_{n,p}$ (i.e., a random graph of order n in which edges are selected independently at random with probability p).

LEMMA 7. *Let $\epsilon > 0$ be fixed and let $p = p(n) \leq 1/2$. If n is sufficiently large and a set of graphs $\mathcal{T} \subset \mathcal{G}^n$ satisfies $\mathbb{P}(G_{n,p} \in \mathcal{T}) \geq 1/2 + 2\epsilon$, then $|\mathcal{T}| \geq \epsilon\sqrt{pqN}\binom{N}{pN}$, where $N = \binom{n}{2}$ and $q = 1 - p$.*

PROOF. If n is sufficiently large, $\mathbb{P}(e(G_{n,p}) \leq pN) \leq 1/2 + \epsilon$. Hence $\mathbb{P}(G_{n,p} \in \mathcal{T}$ and $e(G_{n,p}) > pN) \geq 1/2 + 2\epsilon - (1/2 + \epsilon) = \epsilon$. Note that for any $H \in \mathcal{G}^n$ the probability $\mathbb{P}(G_{n,p} = H)$ depends only on the number of edges in H. Hence, if $H_0, H_1 \in \mathcal{G}^n$ with $e(H_0) < e(H_1)$, then $\mathbb{P}(G_{n,p} = H_0) \geq \mathbb{P}(G_{n,p} = H_1)$. Thus, if $e(H) > pN$, then for any H' with $e(H') = pN$, we have

$$\mathbb{P}(G_{n,p} = H) < \mathbb{P}(G_{n,p} = H') < \left(\sqrt{pqN}\binom{N}{pN}\right)^{-1}.$$

Then

$$\mathbb{P}(G_{n,p} \in \mathcal{T} \text{ and } e(G_{n,p}) > pN) \leq |\{H \in \mathcal{T}, e(H) > pN\}| \left(\sqrt{pqN}\binom{N}{pN}\right)^{-1},$$

so $|\mathcal{T}| \geq \epsilon\sqrt{pqN}\binom{N}{pN}$. □

We will be applying this lemma in a specific form, expressed in the following corollary.

COROLLARY 8. *Suppose $p = p(n) < 1/2$, $p\binom{n}{2} \to \infty$. If n is sufficiently large and the set $\mathcal{T} \subset \mathcal{G}^n$ satisfies $\mathbb{P}(G_{n,p} \in T) \geq 2/3$, then*

$$|T| \geq \binom{\binom{n}{2}}{p\binom{n}{2}} \geq (1/p)^{p\binom{n}{2}}. \tag{1}$$

This corollary will be applied to show that the oscillating properties we construct grow as desired. The following basic construction will be used as a starting point in each of the theorems that are to come. Let $c > 1$ and $\nu = (\nu_1, \nu_2, \dots)$ be an increasing (possibly finite) sequence of natural numbers. We define a *selectively restricted property* $\mathcal{P}_{\nu,c}$ as $\{G :$ if $H \subseteq G$ and $v(H) = \nu_i$ for some i, then $e(H) \leq c\nu_i\}$. Note that $\mathcal{P}_{\nu,c}$ is monotone and therefore also hereditary.

The property $\mathcal{P}_{\nu,c}$ has a speed which grows in a predictable way. □

LEMMA 9. *Let* $c > 1$, $\epsilon > 1/c$, $\nu = (\nu_i)_{i=1}^{\infty}$ *be a sequence of natural numbers and* $k = \sup\{\nu_i \in \nu\}$. *Then:*

(1) $|\mathcal{P}_{\nu,c}^n| \geq n^{(c+o(1))n}$ *and* $|\mathcal{P}_{\nu,c}^n| = n^{(c+o(1))n}$ *whenever* $n = \nu_i$,
(2) *if* $k < \infty$ *and* n *is sufficiently large,* $|\mathcal{P}_{\nu,c}^n| \geq 2^{n^{2-\epsilon}}$.

PROOF. For any sequence ν and $c > 1$, we have $\mathcal{Q}_c \subset \mathcal{P}_{\nu,c}$ and, for any i, the ν_i-level $\mathcal{P}_{\nu,c}^{\nu_i} \subseteq S_c^{\nu_i}$. Hence we have $|\mathcal{P}_{\nu,c}^n| \geq n^{(c+o(1))n}$ for all n and $|\mathcal{P}^{\nu_i}| \leq |S_c^{\nu_i}| \leq \nu_i^{(c+o(1))\nu_i}$ on the set $\{\nu_i\}$ by Lemma 6. Using Theorem 5 for a lower bound, we obtain $|\mathcal{P}_{\nu,c}^n| = n^{(c+o(1))n}$ whenever $n = \nu_i$.

For the second part, assume $k < \infty$. Consider the property $\mathcal{P}_{(k),c}$, where (k) is the sequence $(1, \ldots, k)$. We have $|\mathcal{P}_{(k),c}^n| \leq |\mathcal{P}_{\nu,c}^n|$ for all n. Hence we would have the result if, for sufficiently large n, $|\mathcal{P}_{(k),c}^n| \geq n^{n^{2-\epsilon}}$.

Choose δ so that $\epsilon > \delta > 1/c$. Let $p = n^{-\delta}$ and consider $G_{n,p}$. We consider the probability that $G_{n,p} \notin \mathcal{P}_{(k),c}^n$. This is the probability that $G_{n,p}$ has a 'bad' subgraph, that is,

$$\begin{aligned}\mathbb{P}(G_{n,p} \notin \mathcal{P}_{(k),c}^n) &= \mathbb{P}(G \in G_{n,p} : \text{ there is } H \subseteq G \text{ with } v(H) \leq k \text{ and } e(H) > cv(H)) \\ &\leq \sum_{j=1}^{k} \mathbb{E}\,(\text{number of } j\text{-subgraphs of } G_{n,p} \text{ with more than } cj \text{ edges}) \\ &\leq \sum_{j=1}^{k} \binom{n}{j} \binom{\binom{j}{2}}{cj} p^{cj} \leq \sum_{j=1}^{k} \left(\frac{en}{j}\right)^j \left(\frac{ej^2}{2cj}\right)^{cj} n^{-\delta cj} \\ &\leq \sum_{j=1}^{k} \left(\frac{e}{j}\left(\frac{ej}{2c}\right)^c n^{1-\delta c}\right)^j = \sum_{j=1}^{k} (C_j n^{1-\delta c})^j,\end{aligned}$$

where C_j ($\sim j^{c-1}$) is a constant depending on j and c. Since $\delta c > 1$, we have $1 - \delta c < 0$, so this probability goes to zero as n goes to infinity. Choose n_0 minimal so that the probability that G_{n_0,n_0^δ} has a 'bad' subgraph is less than $1/3$. Note that this probability is monotone decreasing in n. That is, if $\mathbb{P}(G \in \mathcal{G}_{n_0,n_0^\delta} : G \text{ has a bad subgraph}) < 1/3$, then $\mathbb{P}(G \in \mathcal{G}_{n,n^\delta} : G \text{ has a bad subgraph}) < 1/3$ for all $n > n_0$.

Now we can apply Corollary 8 to the set $T = \mathcal{P}_{(k),c}$ to obtain the result

$$|\mathcal{P}_{(k),c}^{n_0}| \geq (n_0^\delta)^{n_0^{-\delta}\binom{n_0}{2}} > 2^{{n_0}^{2-\delta}/2}.$$

All of the inequalities above hold whenever $\mathbb{P}(G_{n,p} \in T) \geq 2/3$, so we have $|\mathcal{P}_{(k),c}^n| > 2^{n^{2-\delta}/2}$ for all $n > n_0$. Thus we can choose n large enough to ensure $2^{n^{2-\delta}/2} > 2^{n^{2-\epsilon}}$ and obtain our result. □

4. OSCILLATING PROPERTIES: THE SECOND AND THIRD QUESTIONS

Having done the preliminary calculations in Section 3, we are now ready to prove the first of three theorems regarding properties that oscillate. We first construct a property with a large range of oscillation. The oscillation of this speed provides a negative answer to the third question of Scheinerman and Zito; we shall answer the second question with Theorem 11.

Lemma 9 gives a property with the proper bounds on its speed, so all that remains is to choose a sequence so that the property grows as desired. We shall use sequences and their elements significantly in the rest of the paper and shall abuse notation slightly. For a sequence

N, we shall say $n \in N$ if the value n appears somewhere in the sequence. If N is a subsequence of sequence M, we shall write $N \subset M$. In other words, we shall use set notation with sequences to mean that the relations hold for the set of elements in the sequence.

THEOREM 10. *Let $c > 1$ and $\epsilon > 1/c$. There exist sequences $\nu = (\nu_i)_{i=1}^{\infty}$ and $\mu = (\mu_i)_{i=1}^{\infty}$, where $\mu_i = \nu_i - 1$ for all i, such that:*

(1) $|\mathcal{P}^n_{\nu,c}| = n^{(c+o(1))n}$ *whenever* $n = \nu_i$,
(2) $|\mathcal{P}^n_{\nu,c}| \geq 2^{n^{2-\epsilon}}$ *whenever* $n = \mu_i$,
(3) $n^{(c+o(1))n} \leq |\mathcal{P}^n_{\nu,c}| < 2^{n^{2-\epsilon}}$ *if* $n \neq \mu_i$.

PROOF. We choose $\nu_1, \nu_2, \ldots$, one by one, starting with $\nu_1 = 3$. Having chosen $\nu_1, \ldots, \nu_k$, we set $\nu = (\nu_1, \ldots, \nu_k)$ and note that by Lemma 9, $|\mathcal{P}^n_{\nu,c}| \geq 2^{n^{2-\epsilon}}$ for sufficiently large n. Choose $\mu_{k+1} > \nu_k$ minimal such that $|\mathcal{P}^{\mu_{k+1}}_{\nu,c}| \geq {\mu_{k+1}}^{{\mu_{k+1}}^{2-\epsilon}}$. Set $\nu_{k+1} = \mu_{k+1} + 1$.

Continuing in this way, we obtain an infinite sequence and the required property. □

The results in Theorems 3 and 10 suggest that $n^{cn} \leftrightsquigarrow 2^{n^{2-1/c}}$ is a natural range of oscillation that may occur in the penultimate range. However, there are many other types of oscillation possible. We first show that the upper bound of the oscillation can be any function in the range that we choose, and, further, that the oscillation can be constrained to remain very close to the upper bound. Choosing $f(n) = n^{(d+o(1))n}$ for some $d > c$ then gives a negative answer to the second question of Scheinerman and Zito.

With Theorem 12, we shall show a similar, though slightly weaker, result for the lower bound.

Given a function $f(n) \leq 2^{n^{2-\epsilon}}$, Theorem 10 gives a sequence ν which guarantees that $|\mathcal{P}^n_{\nu,c}|$ oscillates between $n^{(c+o(1))n}$ and some value above $f(n)$ infinitely often. Clearly, we can choose ν so that the speed only goes above $f(n)$ when $n = \nu_i - 1$ for some i. However, we can do better than this by carefully truncating our properties at level $\mu_i = \nu_i - 1$ and showing that this will not affect any aspect of the construction we perform subsequently. This is precisely the method of the following theorem.

Note that we constrain $f(n) > n^{c'n} > n^{(c+o(1))n}$ so that oscillation will actually occur.

THEOREM 11. *Let $c > 1$, $c' > c$, and $\epsilon > 1/c$. Let $f(n)$ be a function such that $n^{c'n} < f(n) \leq 2^{n^{2-\epsilon}}$ for all n. There exist sequences $\nu = (\nu_i)_{n=1}^{\infty}$ and $\mu = (\mu_i)_{n=1}^{\infty}$ and a monotone property $\mathcal{P}$ such that:*

(1) $|\mathcal{P}^n| = n^{(c+o(1))n}$ *whenever* $n = \nu_i$,
(2) $|\mathcal{P}^n| > f(n) - n!$ *whenever* $n = \mu_i$,
(3) $|\mathcal{P}^n| \leq f(n)$ *for all* n,
(4) $|\mathcal{P}^n| \geq n^{(c+o(1))n}$.

PROOF. Choose sequences ν and μ as in Theorem 10, selecting values of μ_i according to $|\mathcal{P}^n_{\nu,c}| \geq f(n)$ rather than $|\mathcal{P}^n_{\nu,c}| \geq 2^{n^{2-\epsilon}}$.

$|\mathcal{P}^{\mu_i}_{\nu,c}| \geq f(\mu_i)$ and $|\mathcal{P}^n_{\nu,c}| < f(n)$ for all sufficiently large $n \neq \mu_i$, since $n^{(c+o(1))n} < f(n)$.

We will use the fact that if $\mathcal{P}$ is monotone and G is an n-graph in $\mathcal{P}$ such that $G \not\subseteq H$ for any other graph on n or more vertices in $\mathcal{P}$, then $\mathcal{P} \setminus \{H : H \cong G\}$ is still a monotone property. That is, removing graphs from $\mathcal{P}^k$ has no effect on $\mathcal{P}^n$ and does not depend on the graphs in $\mathcal{P}^n$ for any $n < k$. Furthermore, removing a graph and all graphs isomorphic to it from a property reduces $|\mathcal{P}^n|$ by at most $n!$. If we choose graphs to remove carefully, this property will remain monotone. (N.B.: the same is true for hereditary properties.)

We shall call a graph $G \in \mathcal{P}$ *eligible in* $\mathcal{P}$ if $e(G) > cv(G)$ and there are no graphs $H \in \mathcal{P}$ with $G \subsetneq H$. For the property $\mathcal{P}_{\nu,c}$, if $\nu_{i-1} \leq v(G) < \nu_i - 1\ (= \mu_i)$, G is eligible if and only if there are no μ_i-graphs H with $G \subset H$. If $v(G) = \mu_i$, then we need the further condition that no ν_i-graphs contain G as a subgraph.

To construct a property satisfying the theorem, we remove μ_i-graphs from $\mathcal{P}_{\nu,c}$ to obtain $\mathcal{P}$. We only need to show that there is a set $\mathcal{F}$, closed under isomorphism, consisting of eligible μ_i-graphs in $\mathcal{P}_{\nu,c}^{\mu_i}$ such that $f(n) - n! < |\mathcal{P}_{\nu,c}^{\mu_i} - \mathcal{F}| \leq f(n)$. By the comment in the previous paragraph, changing a property at the μ_i-level affects other levels if and only if it affects the ν_i-level.

How many graphs in $\mathcal{P}_{\nu,c}^{\mu_i}$ are subgraphs of graphs in $\mathcal{P}_{\nu,c}^{\nu_i}$? For any monotone property $\mathcal{P}$, if $D^k = \{G : v(G) = k - 1$ and there is an $H \in \mathcal{P}^k$ such that $G \subset H\}$, then $D^k = \{G : G \cong H - v,\ H \in \mathcal{P}^k,\ v \in V(H)\}$. Since $\mathcal{P}$ is monotone the fact that $\mathcal{P}^k$ is closed under taking subgraphs ensures that we get all possible subgraphs. Hence $|D^k| \leq k \cdot |\mathcal{P}^k|$. Thus, there are at most $\nu_i \cdot \nu_i^{(c+o(1))\nu_i}$ graphs in $\mathcal{P}_{\nu,c}^{\mu_i}$ that are subgraphs of those in $\mathcal{P}_{\nu,c}^{\nu_i}$. Hence $|D^{\nu_i}| \leq \mu_i^{(c+o(1))\mu_i}$ for sufficiently large i.

Given a collection of graphs $\{G_j\}_{j\in A}$, let $\mathcal{F}(\{G_j\}_{j\in A})$ be the set of all graphs isomorphic to G_j for some $j \in A$ and let $\mathcal{P}_k^i = \mathcal{P}_{\nu,c}^{\nu_i - 1} \setminus \mathcal{F}(\{G_j\}\{j = 1\}^k)$. We wish to build a collection of eligible graphs so that $\mathcal{P}_k^i$ will be monotone and $f(\mu_i) \geq |\mathcal{P}_k^i| > f(\mu_i) - \mu_i!$

As $|D^{\nu_i}| \leq \mu_i^{(c+o(1))\mu_i} < f(\mu_i) \leq |\mathcal{P}_{\nu,c}^{\mu_i}|$, there are eligible graphs in $\mathcal{P}_{\nu,c}^{\mu_i}$. Let G_1 be an eligible graph in $\mathcal{P}_{\nu,c}^{\mu_i}$. The property $\mathcal{P}_1^i$ is monotone since G_1 eligible implies $G_1 \nsubseteq H$ for any $H \in \mathcal{P}_{\nu,c} - G_1$. Further $|\mathcal{P}_{\nu,c}^{\mu_i}| - |\mathcal{P}_1^i| \leq \mu_i!$, so $|\mathcal{P}_1^i| > f(\mu_i) - \mu_i!$ We proceed by picking eligible graphs in order, stopping at the first point when $|\mathcal{P}_k^i| \leq f(\mu_i)$. Clearly, if we have picked $\{G_i\}_{i=1}^k$ and $|\mathcal{P}_k^i| > f(\mu_i)$, the counting argument above guarantees that $\mathcal{P}_k^i$ still has an eligible graph G_{k+1}, so this process can continue, and $|\mathcal{P}_k^i| - |\mathcal{P}_{k+1}^i| \leq \mu_i!$ Thus, if when considering μ_i we stop with a set of l_i graphs, $|\mathcal{P}_{l_i}^i| > f(\mu_i)!$

Let $\mathcal{P}^n = \mathcal{P}_{\nu,c}^n$ for all $n \notin \mu$ and $\mathcal{P}^{\mu_i} = \mathcal{P}_{l_i}^i$ for all i. As noted above, $\mathcal{P}$ is a monotone property. Clearly $|\mathcal{P}^{\nu_i}| = \nu_i^{(c+o(1))\nu_i}$ and $f(\mu_i) \geq |\mathcal{P}^{\mu_i}| > f(\mu_i) - \mu_i!$ Also, by our choice of ν_i, $|\mathcal{P}^n| < f(n)$ for all $n \notin \mu$. □

5. Oscillation from Below

Can we produce oscillation similar to that in Section 4, but which has a function other than n^{cn} as its lower bound? That is, given a function $f(n)$, is there a property with speed that oscillates from just below $f(n)$ to just above $2^{n^{2-\epsilon}}$ infinitely often? A modification of the property in Theorem 10 again provides a candidate for the oscillation. However, we must relax the condition that the oscillation stay close to the upper bound in order to make the proof work easily. In particular, there is a range of levels for which we cannot say whether $|P^n| < 2^{n^{2-\epsilon}}$.

THEOREM 12. *Let $c > 1$ and $\epsilon > 1/c$. Let $f(n)$ be a function such that: $n^{(c+o(1))n} \leq f(n) < 2^{n^{2-\epsilon}}$ for all n. There exists a pair of sequences $R = (\rho_i)_{i=1}^\infty$ and $M = (\mu_i)_{i=1}^\infty$ and a monotone property $\mathcal{P}$ such that:*

(1) $f(\rho_i) - \rho_i! < |\mathcal{P}^{\rho_i}| \leq f(\rho_i)$ *for all i,*
(2) $|\mathcal{P}^{\mu_i}| > 2^{\mu_i^{2-\epsilon}}$ *for all i,*
(3) $|\mathcal{P}^n| \geq f(n)$ *for all $n \notin R$,*
(4) $|\mathcal{P}^n| < 2^{n^{2-\epsilon}}$ *for $n \in \bigcup_i [\rho_i, \mu_{i+1} - 1]$.*

PROOF. The proof follows along the same lines as the proofs of Theorems 10 and 11, only this time we construct two sequences, R and ν. We first build a sequence $\nu = (\nu_i)_{i=1}^\infty$ as in

Theorem 10. Again let $\mu_i = \nu_i - 1$ for all i, and consider $\mathcal{P}_{\nu,c}$. This satisfies conditions 2 and 4 of the theorem (for any sequence R which does not intersect $M = (\mu_i)$). Hence we need to modify $\mathcal{P}_{\nu,c}$ to obtain conditions 1 and 3. However, in doing so, we need to be sure we do not create a property contradicting conditions 2 or 4.

We choose the sequence R as follows. For all i, let ρ_i be the maximal n such that $\nu_i \leq n < \nu_{i+1}$ and $\mathcal{P}^n_{\nu,c} \leq f(n)$. Since $|\mathcal{P}^{\nu_i}_{\nu,c}| = \nu_i^{(c+o(1))\nu_i}$, $|\mathcal{P}^{\nu_{i+1}}_{\nu,c}| > 2^{n^{2-\epsilon}}$, and $n^{(c+o(1))n} \leq f(n) < 2^{n^{2-\epsilon}}$, there always will be such an n.

We shall add graphs to $\mathcal{P}^{\rho_i}_{\nu,c}$ so that its speed is close to $f(n)$. We know that this will not affect the n-levels of our property for $n > \rho_i$. If we can pick these graphs so that every μ_i-subgraph is in $\mathcal{P}^{\mu_i}_{\nu,c}$, we will not affect any n-level for $n \leq \mu_i$ either. However, adding such a graph to $\mathcal{P}^{\rho_i}_{\nu,c}$ will enlarge the n-levels for $\mu_i < n < \rho_i$.

If $|\mathcal{P}^{\rho_i}_{\nu,c}| > f(\rho_i) - \rho_i!$, we need not modify $|\mathcal{P}^{\rho_i}_{\nu,c}|$. Otherwise, consider the sequence $N' = (\nu_1, \ldots, \nu_{\{i-1\}})$. Then $\mathcal{P}_{\nu,c} \subseteq \mathcal{P}_{N',c}$. In particular, $\mathcal{P}^{\rho_i}_{\nu,c} \subset \mathcal{P}^{\rho_i}_{N',c}$. Since $|\mathcal{P}^{\rho_i}_{\nu,c}| < f(\rho_i) < 2^{n^{2-\epsilon}} < |\mathcal{P}^{\rho_i}_{N',c}|$, there is a graph $G \in (\mathcal{P}^{\rho_i}_{N',c} - \mathcal{P}^{\rho_i}_{\nu,c})$ such that every $H \subseteq G$ with $v(H) \leq \mu_i$ is in $\mathcal{P}^{v(H)}_{\nu,c}$. We call such a graph *insertable*. Let G_1 be an insertable graph with a minimal number of edges. Then every proper ρ_i-subgraph of G is in $\mathcal{P}^{\rho_i}_{\nu,c}$, so $|\mathcal{P}^{\rho_i}_{\nu,c} \cup \mathcal{F}(\{G_1\})| \leq |\mathcal{P}^{\rho_i}_{\nu,c}| + \rho_i!$ Also, if $\mathcal{P}_1$ is a minimal property containing $\mathcal{P}_{\nu,c} \cup \mathcal{F}(\{G_1\})$, then $\mathcal{P}^n_1 = \mathcal{P}^n_{\nu,c}$ for $n > \rho_i$ and $n < \nu_i$. For $\nu_i \leq n \leq \rho_i$, the speed $|\mathcal{P}^n_1| \leq |\mathcal{P}^n_{\nu,c}| + (\rho_i)!\binom{\rho_i}{n}$. We continue choosing ρ_i-graphs in this way until we have a collection $\{G_1, \ldots, G_{l_i}\}$ so that $f(\rho_i) - \rho_i! < |\mathcal{P}^{\rho_i}_{l_i}| \leq f(\rho_i)$. As the only condition we needed to guarantee an insertable graph was that the property had speed below $f(n)$, it is clear in that case we can always find an insertable graph. If we consider each i in turn and construct the property $\mathcal{P}' = \mathcal{P}_{\{l_j\}}$ in the obvious way, we obtain a monotone property satisfying conditions 1 and 2.

However, condition 3 does not necessarily hold for $\mathcal{P}'$ on the intervals $\{[\nu_i, \rho_i)\}$. Consider each value of i in turn and examine the interval $[\nu_i, \rho_i)$ from the right. If, for $t = \rho_i - 1$, the speed $|(\mathcal{P}')^t| < f(t)$, we can proceed as we did for $(\mathcal{P}^{\rho_i}_{\nu,c})$: add a finite collection graphs to $(\mathcal{P}')^t$ to obtain a new property with speed above $f(t)$. It is clear that we only affect the n-levels for $n \in [\nu_i, t]$. So continuing for each smaller value in the interval, we obtain a property $\mathcal{P}$ satisfying all of the conditions of the theorem. □

Ideally, given any two functions in the proper range with positive difference ($\neq o(1)$), we would like to construct a property with speed that oscillates infinitely often between the two functions. However, this is clearly not possible, as for any monotone or hereditary property, $|\mathcal{P}^{n+1}|/|\mathcal{P}^n| \leq 2^n$. Thus, for example, choosing functions that increase together by more than a factor of 2^n would make it impossible to keep the speed between the bounds. With a restriction to 'smooth' functions avoiding this problem, it seems that oscillation is possible. However, as we have seen in the proof of Theorem 12, even with a 'smooth' function the proof would be cumbersome. In fact, even a proper definition of 'smooth' would be unappealing.

However, an outline of the approach we would take to prove the desired statement is as follows. Given two such functions $f(n) < g(n)$, we wish to obtain a property which achieves speeds close $f(n)$ for infinitely many n and close to $g(n)$ for infinitely many n. Rather than finding the sequence ν from Theorem 10, we would start with the sequence from Theorem 11. In the final step, when we add or remove graphs according to whether the property's speed is too high or too low, we need to take care that in removing graphs we do not alter later properties. This may require adjusting our sequence so that the level for which the speed is above $g(n)$ is in the interval between μ_i and ρ_i rather than at μ_i. The condition $n^{(c'-c)n} f(n) < g(n)$ would ensure the conditions of Theorem 11 and the positive difference between the functions.

This, however, does not solve the problem we have discussed regarding condition 4 of Theorem 12. We believe that it is not worth the effort to describe in more detail what needs to be done. Nevertheless, we believe the following statement to be true, and would be happy to see an elegant proof.

Let $c > 1$, $c' > c$, and $\epsilon > 1/c$. Let $f(n)$, $g(n)$ be 'smooth' functions such that

$$n^{(c+o(1))n} \leq f(n) < n^{(c'-c)n} f(n) \leq g(n) \leq 2^{n^{2-\epsilon}}$$

for all n. There exists a pair of sequences $R = (\rho_i)_{i=1}^{\infty}$ and $S = (\sigma_i)_{i=1}^{\infty}$ and a monotone property $\mathcal{P}$ such that:

(1) $|\mathcal{P}^n| \geq f(n)$ *and* $|\mathcal{P}^n| \leq g(n)$ *for all* $n \notin R \cup S$,
(2) $f(\rho_i) > |\mathcal{P}^{\rho_i}| > f(\rho_i) - \rho_i!$ *for all* $\rho_i \in R$,
(3) $g(\sigma_i) < |\mathcal{P}^{\sigma_i}| < g(\sigma_i) + \sigma_i!$ *for all* $\sigma_i \in S$.

6. A More Natural Oscillating Property

The aim of this section is to 'sharpen' our results from a different point of view. The properties given in Theorems 10 and 11 are useful for our purposes. In particular they neatly answer the questions of [9] mentioned in the Introduction. However, the properties we describe are extremely artificial, their oscillation coming, to a large degree, from 'unnecessary' graphs. In particular, there are many (isomorphism classes of) graphs in $\mathcal{P}_{\nu,c}$ that may be removed without affecting the hereditary nature of the property. In fact, we have used this fact rather heavily in the proofs of Theorems 11 and 12. However, while the removal of the graphs would not affect the hereditary nature of the properties in question, it would affect their speed. It would be nice, therefore, to know if there is a property for which each isomorphism class is necessary and for which the speed still oscillates.

Given a property $\mathcal{P}$, we define the *limit* of $\mathcal{P}$ as $\mathcal{P}^* = \{G :$ for all $n > v(G)$ there is an n-graph $H \in \mathcal{P}$ with $G \leq H\}$. Then every graph in $\mathcal{P}$ is necessary if and only if $\mathcal{P} = \mathcal{P}^*$. In this case, we say that $\mathcal{P}$ is a *limit property*. Note that the limit of a property is a limit property, that is $(\mathcal{P}^*)^* = \mathcal{P}^*$.

In [7], Bollobás and Thomason show that if $|\mathcal{P}^n| = 2^{(c+o(1))\binom{n}{2}}$ and $|\mathcal{P}^{*n}| = 2^{(c'+o(1))\binom{n}{2}}$, then $c = c'$. Hence for properties in the highest range of speeds, where $c > 0$, a property and its limit have the same speed. However, this is clearly not true for all properties, as $\mathcal{P}^*_{\nu,c} = \mathcal{Q}_c$ for all infinite increasing sequences ν, while $|\mathcal{P}^n_{\nu,c}|$ may oscillate but $|\mathcal{Q}^n_c|$ does not. Hence we would like to demonstrate a property that has a limit whose speed oscillates. The following theorem provides a limit property with the same type of oscillation as that in Theorem 10.

Theorem 13. *Let $c > 1, \epsilon > 1/c$. There is a monotone limit property $\mathcal{P}$ and two sequences $R = (\rho_i)_{i=1}^{\infty}$ and $S = (\sigma_i)_{i=1}^{\infty}$ with $\sigma_i < \rho_i < \sigma_{i+1}$ such that:*

(1) $|\mathcal{P}^n| = n^{(c+o(1))n}$ *whenever* $n = \rho_i$ *for some* i,
(2) $|\mathcal{P}^n| = 2^{(1+o(1))n^{2-\epsilon}}$ *whenever* $n = \sigma_i$ *for some* i,
(3) $n^{(c+o(1))n} \leq |\mathcal{P}^n| \leq 2^{(1+o(1))n^{2-\epsilon}}$ *for all* n.

Proof. For two sequences R, S and a property $\mathcal{P}$, consider the properties $\mathcal{A}_{R,S}$ and $\mathcal{B}_{R,S}$ defined by levels as follows. $\mathcal{A}^n_{R,S} = \{G : v(G) = n$ and for all i and for all $\sigma_i < l \leq \rho_i$, every l-subgraph $H \subseteq G$ has $e(H) \leq cl\}$, and $\mathcal{B}^n_{R,S} = \{G : v(G) = n$ and $G = H \cup \overline{K_l}$ where $H \in \mathcal{P}^{\sigma_i}$ and $l = n - \sigma_i$ for $\sigma_i = \max\{s : n > s \in S\}\}$. Note that $\mathcal{A}_{R,S}$ is a property of the type $\mathcal{P}_{\nu,c}$ for some $\nu \supset R$. We will construct a property $\mathcal{P} \subseteq \mathcal{A}_{R,S} \cup \mathcal{B}_{R,S}$ which is monotone, limit, and has the proper speeds.

As in the proof of the previous theorems, we proceed by constructing sequences R and S so that $\mathcal{P}$ is as described. We shall calculate values of ρ_i, σ_i based on those of ρ_{i-1}, σ_{i-1}, and describe $\mathcal{P}$ incrementally by levels.

Let $\rho_0 = 2$ and let $\sigma_1 > \rho_0$ be the smallest value such that $|\mathcal{T}^{\sigma_i}| > 2^{\sigma_1^{2-\epsilon}}$, where $\mathcal{T}$ is the trivial property. As in the proof of Theorem 11, we can remove graphs from $\mathcal{T}^{\sigma_1}$ so that $|\mathcal{T}^{\sigma_1}| \leq 2^{\sigma_1^{2-\epsilon}} + n!$ Let $\mathcal{P}^{\sigma_i}$ be the collection of graphs which remain, and for $n < \sigma_1$, let $\mathcal{P}^n = \{G : v(G) = n \text{ and there is } H \in \mathcal{P}^{\sigma_i} \text{ with } G \subseteq H\}$.

Assume we have chosen sequences R_{i-1}, S_i where $R_i = (\rho_1, \ldots, \rho_{i-1})$, $S_i = (\sigma_1, \ldots, \sigma_i)$ and we have defined the n-level of $\mathcal{P}$ for $n \leq \sigma_i$. We wish to find ρ_i so that $|\mathcal{A}^{\rho_i}_{R_{i-1},S_i} \cup \mathcal{B}^{\rho_i}_{R_{i-1},S_i}| = \rho_i^{(c+o(1))\rho_i}$. By Lemma 6, we know that for any choice of ρ_i, the speed $|\mathcal{A}^{\rho_i}_{R_{i-1},S_i}| = \rho_i^{(c+o(1))\rho_i}$. So if we choose ρ_i (minimal) so that $|\mathcal{B}^{\rho_i}_{R_{i-1},S_i}| < \rho_i^{c\rho_i}$ the desired relation will hold. There is such a ρ_i, since for all $n > \sigma_i$, $|\mathcal{B}^n_{R_{i-1},S_i}| \leq \binom{n}{\sigma_i}|\mathcal{P}^{\sigma_i}_{R_{i-1},S_i}| \leq n^{\sigma_i} 2^{\sigma_i^2}$, where the last estimate comes from all graphs being in the σ_i-level of $\mathcal{P}$. Hence $\rho_i = 2^{\sigma_i}$ would be more than sufficient. For $\sigma_i < n \leq \rho_i$, let $\mathcal{P}^n = \mathcal{A}^n_{R_i,S_i} \cup \mathcal{B}^n_{R_i,S_i}$.

Given ρ_i, let $\sigma_{i+1} > \rho_i$ be the smallest number such that $|\mathcal{A}^{\sigma_{i+1}}_{R_i,S_i} \cup \mathcal{B}^{\sigma_{i+1}}_{R_i,S_i}| > 2^{\sigma_{i+1}^{2-\epsilon}}$. The existence of such a number is guaranteed by Lemma 9. As in the proof of Theorem 11, we can remove eligible graphs, one isomorphism class at a time, from $\mathcal{A}^{\sigma_{i+1}}_{R_i,S_i} \cup \mathcal{B}^{\sigma_{i+1}}_{R_i,S_i}$ to obtain $\mathcal{P}^{\sigma_i+1}$ with $|\mathcal{P}^{\sigma_i+1}| < 2^{\sigma_{i+1}^{2-\epsilon}} + \sigma_{i+1}!$ As we want to create a limit property, we will then remove graphs from $\mathcal{P}^n$ for $n < \sigma_{i+1}$, keeping only those graphs which appear as subgraphs of those in $\mathcal{P}^{\sigma_{i+1}}$. However, we want to be sure that $\mathcal{P}$ remains at the proper speed. In particular, we will remove no graphs from $\mathcal{B}^{\sigma_{i+1}}_{R_i,S_i}$ and no graphs in $\mathcal{Q}_c^{\sigma_{i+1}}$ (noting that $\mathcal{Q}_c^n \subseteq \mathcal{A}^n_{R,S}$ for all n and any sequences R, S). Clearly there are enough eligible graphs avoiding these collections, as $|\mathcal{B}^{\sigma_{i+1}}_{R_i,S_i}| + |\mathcal{Q}_c^{\sigma_{i+1}}| < \sigma_{i+1}^{(c+o(1))\sigma_{i+1}}$. Note that with this restriction, we will not remove any graphs from $\mathcal{P}^n$ for $n \leq \rho_i$.

In this way we construct infinite sequences R and S. It is clear that $\mathcal{P}$ is a monotone property, and the construction guarantees that $\mathcal{P}$ is limit property, since we remove all graphs that are not contained in arbitrarily large graphs. The speeds given in conditions 1 and 2 are correct on the elements of R and S, respectively, by the construction. Furthermore, $\mathcal{Q}_c \subseteq \mathcal{P}$, so the lower bound given in condition 3 is correct.

For the upper bound, we split the interval (σ_i, σ_{i+1}) into two parts. Our choice of the sequence S guarantees that for $\rho_i < n < \sigma_{i+1}$, $|\mathcal{P}^n| < 2^{n^{2-\epsilon}}$. For $\sigma_i < n \leq \rho_i$, we note $|\mathcal{P}^n| \leq |\mathcal{A}^n_{R,S}| + |\mathcal{B}^n_{R,S}|$. Hence $|\mathcal{P}^n_{R.S}| < n^{(c+o(1))n} + \binom{n}{\sigma_i}|\mathcal{P}^{\sigma_i}_{R,S}| < n^{\sigma_i} 2^{n^{2-\epsilon}} < 2^{(1+o(1))n^{2-\epsilon}}$. □

Thus we have presented a 'sensible' property for which the speed oscillates over nearly the whole interval from $n^{(1+o(1))n}$ to $2^{n^{2-\epsilon}}$. This property, as is true of all of the properties presented in the paper, has an infinite class of forbidden subgraphs corresponding to the infinite sequences constructed. That is, if $\mathcal{P}$ is one of our oscillating properties and $\mathcal{F}$ is a minimal class of graphs such that $\mathcal{P} = \mathbf{Mon}(\mathcal{F})$, then $\mathcal{F}$ is infinite. Is this a necessary condition for oscillation to occur? We believe that it is: if a monotone property has a finite class of forbidden subgraphs, then all of the limits presented in the Introduction should exist. So far, however, a proof of such a result is elusive.

7. Tight Bounds on the Penultimate Range

The results of Sections 4–6 demonstrate that the penultimate range differs significantly from the other ranges of speed. In fact, it is unclear that properties in this range need to satisfy any well-defined behavior other than the broad bounds given in Section 2. Nevertheless, based on results involving a different measure of properties in [2], we believe that the range of oscillation demonstrated in the properties presented here is the maximum possible. The converse of the conjecture is true for monotone properties, as shown by Theorem 3 and in [2]. However, the first part of the conjecture is open even for monotone properties.

Conjecture 14. *For all $c > 1$, there exists an $\epsilon > 0$ such that if $\mathcal{P}$ is a hereditary property and $|\mathcal{P}^n| \geq 2^{n^{2-\epsilon}}$ holds infinitely often, then $|\mathcal{P}^n| \geq n^{(c+o(1))n}$. Conversely, for all $d > 1$ there exists a $\delta > 0$ such that if $|\mathcal{P}^n| \leq n^{(d+o(1))n}$ infinitely often, then $|\mathcal{P}^n| \leq 2^{n^{2-\delta+o(1)}}$.*

It is clear from Lemma 9 that, if Conjecture 14 is true, $\delta \leq 1/d$. Perhaps Conjecture 14 even holds with $\epsilon = 1/c$ and $\delta = 1/d$. However, there are no results of this type known. Thus the penultimate region of speeds remains a fertile area for further research.

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