Decomposing Finitely Generated Integral Monoids by Elimination

Jennifer Ryan
Mathematics Department
University of Colorado at Denver
Campus Box 170
P.O. Box 173364
Denver, Colorado 80127-3364

Submitted by Richard A. Brualdi

ABSTRACT

A finitely generated integral monoid can always be decomposed into a finite (although generally exponential) number of integral monoids having special structure, in that their related integer programming feasibility problems can be solved in polynomial time. The decomposition can be constructed using an elimination scheme due to Presburger and Williams that generalizes Fourier-Motzkin elimination.

1. INTRODUCTION

A finitely generated integral monoid can be expressed as the set of right hand sides for which a related integer programming problem is feasible. In this paper we show that an integral monoid can always be decomposed into a finite set of integral monoids having special structure in that their related integer programming feasibility problems can be solved in polynomial time.

DEFINITION. An integral monoid \( Y \) is set of integral vectors containing 0 that is closed under addition and, redundantly, scalar multiplication by a positive integer. An integral monoid \( Y \) is finitely generated by \( \{a_1, \ldots, a_m\} \subseteq \mathbb{Z}^n \) if \( Y = Y(A) = \{yA | y \in \mathbb{Z}_+^m\} \), where \( A \) is the \( m \times n \) matrix whose rows are the vectors \( a_i, i = 1, \ldots, m \).

The Chvátal functions (defined by Blair and Jeroslow [1, p. 239]) are obtained by starting with linear functions from \( \mathbb{Q}^n \) to \( \mathbb{Q} \), taking nonnegative
rational combinations, and rounding down to the nearest integer. Note that
the set of Chvátal functions properly contains the linear functions. \(L_n\) will
denote the set of \(n\)-dimensional linear functions.

**Definition.** The set \(C_n\) of \(n\)-dimensional functions from \(Q^n\) to \(Q\) is
the smallest class \(K\) with the following properties:

(i) \(f \in K\) if \(f \in L_n\);
(ii) \(f, g \in K\) and \(\alpha, \beta \in Q^+\) implies \(\alpha f + \beta g \in K\); and
(iii) \(f \in K\) implies \(|f| \in K\), where \(|f|(x) = \lfloor f(x) \rfloor\) for all \(x \in Q^n\).

An integral monoid \(Y\) is said to be finitely constrained by Chvátal
functions if it can be expressed as \(Y = \{x \in Z^n \mid f_i(x) > 0, i = 1, \ldots, p\}\), where
\(f_i \in C_n, i = 1, \ldots, p\). The following analogue of the theorems of Weyl and
Minkowski for cones states that any finitely generated integral monoid is
finitely constrained by Chvátal functions. The “only if” in the following
theorem is from Blair and Jeroslow [1]. The “if” can be found in [8].

**Theorem 1.** Let \(Y\) be an integral monoid. Then \(Y\) is finitely generated if
and only if there exist \(n\)-dimensional Chvátal functions \(f_1, \ldots, f_p\) such that

\[ Y = \{x \in Z^n \mid f_i(x) > 0, i = 1, \ldots, p\}. \]

Chvátal functions are superadditive, i.e., \(f(a + b) \geq f(a) + f(b)\) for all
rational vectors \(a\) and \(b\). In particular, the functions of Theorem 1 can be
related to the superadditive dual of an integer programming problem (see
e.g. [1] and [7]). There is no known algorithm to construct the functions of
Theorem 1. We will investigate a broader (although no longer superadditive)
class of functions below that also can be used to constrain an integral monoid,
and that can be constructed using an elimination scheme. The construction
will give rise to a decomposition of integral monoids.

Note that a vector \(b \in Z^n\) belongs to the finitely generated integral
monoid \(Y = \{yA \mid y \in Z^m_+\}\), where \(A \in Z^{m \times n}\), if and only if the integer
programming problem

\[
\max cx \\
Ax = b \\
x \geq 0 \\
x \text{ integer}
\]  

(IP)

is feasible, where \(c \in Z^n\). There are two special cases of finitely generated
integral monoids where the Chvátal functions constraining them are of a
particular form, and the associated integer programming feasibility problem can be solved in polynomial time.

Let $G(Y)$ denote the $\mathbb{Z}$-module generated by an integral monoid $Y$ in $\mathbb{Z}^n$; i.e., $G(Y)$ is the set of integer (not necessarily nonnegative) linear combination of elements of $Y$. Let $K(Y)$ denote the cone generated by $Y$, i.e., $K(Y)$ is the set of nonnegative rational linear combinations of elements of $Y$. Note that if $A$ is an integer $m \times n$ matrix, and if $Y \equiv Y(A)$, then $G(Y) = \{ yA \mid y \in \mathbb{Z}^n \}$ and $K(Y) = \{ yA \mid y \in \mathbb{Q}^n \}$.

The following relations always hold:

$$Y \subseteq K(Y) \cap G(Y) \subseteq K(Y) \cap \mathbb{Z}^m.$$

Theorem 1 says that any finitely generated integral monoid is finitely constrained by Chvátal functions. Theorem 2 below (from [8]) states that the finitely generated integral monoids of the form $Y = K(Y) \cap \mathbb{Z}^m$ or $Y = K(Y) \cap G(Y)$ can be characterized as those integral monoids finitely constrained by certain subclasses of the Chvátal functions; these simple constraints can be constructed using elimination schemes. We need to define a new subclass of the Chvátal functions.

**Definition.** For each $n \in \mathbb{Z}_+$, let $C_n^o$ denote the set of functions $f \in C_n$ which either are linear or can be written in the form

$$[g] - g$$

where $g$ is a linear function.

**Theorem 2** [8]. *Let $A$ be an integer matrix, and let $Y \equiv M(A)$. Then,*

(i) $Y$ is finitely linearly constrained $\iff Y = K(Y) \cap \mathbb{Z}^m$;

(ii) $Y$ is finitely $C^o$ constrained $\iff Y = K(Y) \cap G(Y)$.

If an integral monoid is of the form (i) or (ii) in Theorem 2, then the related integer programming feasibility problem can be solved in polynomial time, using Karmarkar's algorithm [5] for (i), and Karmarkar's algorithm together with a polynomial time unimodular elimination scheme (see e.g. [4]) for (ii).
2. A DECOMPOSITION OF INTEGRAL MONOIDS

The set of Chvátal-Gomory functions, CG, introduced by Blair and Jeroslow [1], is constructed in the same way as the Chvátal functions, with one additional operation, “min.” The formal definition follows.

**Definition.** The set of $n$-dimensional Chvátal-Gomory functions, $\text{CG}_n$, is the smallest class $K$ of functions from $Q^n$ to $Q$ having the following properties:

(i) $f \in K$ if $f \in L_n$;
(ii) $f, g \in K$ and $\alpha, \beta \in Q_+$ implies $\alpha f + \beta g \in K$;
(iii) $f \in K$ implies $[f] \in K$;
(iv) $f, g \in K$ implies $\min\{f, g\} \in K$, where $\min\{f, g\}(x) = \min\{f(x), g(x)\}$ $\forall x \in Q^n$.

Since the min function is superadditive, we have, as for $C$, that every function in $\text{CG}_n$ is superadditive.

A function in $\text{CG}$ may be visualized as a Chvátal function with imbedded min’s. The following proposition asserts that one may assume there is only one min, occurring at the outermost level of nesting.

**Proposition 1 [1].** A function $f \in \text{CG}_n$ is the minimum of finitely many functions in $C_n$.

Given Proposition 1, it is clear that Theorem 1 could be restated with $\text{CG}$ replacing $C$. That is, the class of finitely generated integral monoids is exactly the class of sets of integer vectors that can be constrained by a finite set of CG functions.

We now introduce a new class of functions.

**Definition.** The set of $n$-dimensional disjunctive Chvátal-Gomory functions, $\text{DCG}_n$, is the smallest class $K$ of functions from $Q^n$ to $Q$, having the following properties:

(i) $f \in K$ if $f \in L_n$;
(ii) $f, g \in K$ and $\alpha, \beta \in Q_+$ implies $\alpha f + \beta g \in K$;
(iii) $f \in K$ implies $[f] \in K$;
(iv) $f, g \in K$ implies $\min\{f, g\} \in K$;
(v) $f, g \in K$ implies $\max\{f, g\} \in K$, where $\max\{f, g\}(x) = \max\{f(x), g(x)\}$ $\forall x \in Q^n$. 
Since the max function is not superadditive, a function \( f \) in DCG is not necessarily superadditive. Note that if \( f = \max\{f_1, \ldots, f_p\} \), then \( f(x) \geq 0 \) if and only if \( f_1(x) \geq 0 \) or \( f_2(x) \geq 0 \) or \( \ldots \) or \( f_p(x) \geq 0 \); hence the use of the term *disjunctive* in the definition above.

Just as any Chvátal-Gomory function is the minimum of finitely many Chvátal functions, it can be shown by induction that any disjunctive Chvátal-Gomory function is the maximum of finitely many Chvátal-Gomory functions. The proof of the following proposition is straightforward, and is left to the reader (or see [7]).

**Proposition 2.** Let \( f \in \text{DCG}_n \). Then there exist \( f_1, \ldots, f_p \in \text{CG}_n \) such that

\[
f = \max\{f_1, \ldots, f_p\}.
\]

Thus any disjunctive Chvátal-Gomory function is the maximum of finitely many Chvátal-Gomory functions.

The following corollaries relate sets of integers finitely constrained by functions in \( C \) and those finitely constrained by functions in DCG.

**Corollary 1.** A finite set of DCG constraints is equivalent to a disjunction of a finite number of finite sets of C constraints, and conversely. That is, \( f_1, \ldots, f_p \) are DCG constraints if and only if there exist Chvátal functions \( g_{11}, g_{12}, \ldots, g_{1r_1}, g_{21}, \ldots, g_{s r_s} \) for some positive integers \( s \) and \( r_1, \ldots, r_s \), such that

\[
\{x \in \mathbb{Z}^m | f_i(x) \geq 0, \ i = 1, \ldots, p\} \equiv \left\{x \in \mathbb{Z}^m | \bigvee_k g_{kj} \geq 0, \ j = 1, \ldots, r_k\right\}.
\]

**Proof.** Let \( Y = \{x \in \mathbb{Z}^n | f_i(x) \geq 0, \ i = 1, \ldots, p\} \), where \( f_i \in \text{DCG}_n \) for \( i = 1, \ldots, p \). Then

\[
Y = \left\{x \in \mathbb{Z}^n | \min_i f_i(x) \geq 0\right\}.
\]

By Proposition 2, there exist \( g_1, \ldots, g_s \in \text{CG}_n \) such that

\[
\min_i f_i(x) = \max_j g_j(x) \quad \forall x \in \mathbb{Q}^n.
\]
Thus

\[
Y = \left\{ x \in \mathbb{Z}^n \mid \max_j g_j(x) \geq 0 \right\} = \left\{ x \in \mathbb{Z}^n \mid \bigvee_j [g_j(x) \geq 0] \right\},
\]

where \( \bigvee \) denotes the logical "or." Each \( g_j(x) \geq 0 \) is equivalent to a finite set of Chvátal constraints (Proposition 1).

The converse follows from the definition of DCG.

The following corollary is an easy consequence of the above.

**Corollary 2.** Any finitely DCG-constrained set in \( \mathbb{Z}^n \) is the union of finitely many finitely \( C \)-constrained sets.

Clearly, as with \( C \) and \( CG \), any finitely generated integral monoid can be expressed as the set of integer vectors constrained by a finite set of DCG constraints. However, because of the loss of superadditivity, we now have only a partial converse, given by Corollary 3.

**Corollary 3.** Let \( Y \) be a finitely DCG-constrained set,

\[
Y = \left\{ x \in \mathbb{Z}^n \mid f_i(x) \geq 0, \ i = 1, \ldots, p \right\},
\]

where each \( f_i \in DCG_n \). Then there exists a positive integer \( q \) such that

\[
Y = \bigcup_{j=1}^q Y_j,
\]

where each \( Y_j \) is a finitely generated integral monoid.

**Proof.** By Corollary 2, \( Y = \bigcup_{j=1}^q Y_j \) where each \( Y_i \) is finitely constrained by functions in \( C \). By Theorem 1, each \( Y_i \) is a finitely generated integral monoid.

Functions in \( C \) and \( DCG \) are, by definition, homogeneous. However, in the discussion to follow, constants arise, which will correspond to coefficients of an \((n + 1)\)st variable that is set to 1.

In [10] (see also [11]), Williams presents an elimination scheme which can be used to solve integer programming problems. It is shown below that
this elimination scheme is in fact forming DCG constraints for a finitely generated integral monoid, which are disjunctions (or maximums) of functions in $C^n$.

Given a disjunction

$$R_1 \lor R_2 \lor \cdots \lor R_n,$$

where each $R_i$ is a conjunction of inequalities and congruence relations on $\mathbb{Z}^n$, and $\lor$ denotes the logical "or," Williams [10] shows how to eliminate a variable from any $R_i$ and end up with a system of the same form. The reader interested in the details of Williams's elimination scheme is referred to [10]. It is important to note that in general the number of conjunctions will increase exponentially throughout the procedure. It will be seen here that Williams's scheme is in fact eliminating a variable from a particular set of DCG constraints that may involve a constant. Suppose the variable to be eliminated from the disjunction is $x$. In Williams's scheme, each $R_i$ is made up of inequalities and congruences of the form

(E) \quad p_1 x = s,

(L) \quad p_2 x \leq t,

(G) \quad p_3 x \geq u,

(M) \quad p_4 x \equiv v \pmod{k},

where the $p_i$'s and $k$ are positive integers. Each $s$, $t$, $u$, and $v$ is a linear function of the original right hand side, uneliminated variables, and possibly a constant. Note that (M) holds if and only if

$$\left[ \frac{p_4 x - v}{k} \right] - \frac{p_4 x - v}{k} \geq 0.$$

Thus at every stage of Williams's elimination scheme each $R_i$ can be considered to be a conjunction of $C^n$ constraints. Then the disjunction of the $R_i$'s would be a DCG constraint (see Corollary 1). So Williams's elimination can be viewed as an elimination scheme for DCG constraints that are disjunctions of sets of $C^n$ constraints possibly involving constants.

If (M) is empty in each $R_i$, then Williams's procedure is the same as Fourier-Motzkin elimination. Williams also points out, in [10], that this
elimination scheme is a special case of a decision procedure of Presburger [6] for arithmetic without multiplication.

Note that Williams’s elimination scheme can be applied without knowing the right hand sides explicitly. It can thus be used to find DCG constraints for a finitely generated integral monoid

\[ Y = \{ yA | y \in \mathbb{Z}^m \}, \quad A \in \mathbb{Z}^{m \times n}. \]

For \( b \in \mathbb{Z}^n \) consider the set of inequalities and congruences

\[ y_i a_{ij} = b_j - (y_2 a_{2j} + \cdots + y_m a_{mj}), \quad j = 1, \ldots, n \]
\[ y_i \geq 0, \quad i = 1, \ldots, m \]
\[ y_i \equiv 0 \pmod{1}, \quad i = 1, \ldots, m. \]

These have a solution if and only if \( b \in Y \), and they are in the correct form to apply Williams’s elimination scheme. Suppose the elimination scheme is applied \( m \) times, successively eliminating each \( y_i \). At the end of this process the constraints will involve only the vector \( b \) and perhaps a constant. There will be sets \( R_1, \ldots, R_r \), where each \( R_i \) is a conjunction of nonhomogeneous \( C^o \) constraints such that

\[ Y = \{ b \in \mathbb{Z}^n | b \text{ satisfies } R_1 \vee \cdots \vee R_r \} \]
\[ = \bigcup_{i=1}^{r} \{ b \in \mathbb{Z}^n | b \text{ satisfies } R_i \}. \]

As noted above, the number of conjunctions will generally increase exponentially through the application of Williams’s elimination procedure. So \( r \) may be exponential in \( m \) and \( n \).

If \( Y_i \) is defined to be

\[ \{ b \in \mathbb{Z}^n | b \text{ satisfies } R_i \} \]

then \( Y = \bigcup_{i=1}^{r} Y_i \). If, in each \( R_i \), we make each constant the coefficient of an \((n + 1)\)st term \( b_{n+1} \) of \( b \), then \( Y_i \) is equal to \( \{ (b, 1) | (b, 1) \text{satisfies } R_i \} = \{ b | (b, 1) \in Y_i' \} \), where \( Y_i' = \{ (b, b_{n+1}) | (b, b_{n+1}) \text{satisfies } R_i \} \).

Since each \( R_i \) now has only homogeneous \( C_{n+1}^o \) constraints, by Theorem 2 each \( Y_i' = K(Y_i') \cap G(Y_i') \). Thus membership in each \( Y_i' \) can be determined
in polynomial time. We have thus derived a decomposition of an arbitrary finitely generated integral monoid.

**Theorem 3.** Let $Y = \{yA \mid y \in \mathbb{Z}^n\}$, where $A \in \mathbb{Z}^{m \times n}$. Then there exist finitely generated integral monoids $Y'_i$, $i = 1, \ldots, r$, each of the form $Y'_i = K(Y'_i) \cap G(Y'_i)$, such that $Y = \bigcup_{i=1}^r Y'_i$ and $Y_i = \{b \mid (b, 1) \in Y'_i\}$.

If $Y$ is a finitely generated integral monoid of the form $Y = K(Y) \cap G(Y)$ and its Chvátal constraints are not known, they may be obtained as follows. Fourier-Motzkin elimination may be used to obtain the constraints of $K(Y)$ (see [9]), and unimodular elimination may be used to obtain the integrality and subspace constraints of $G(Y)$ (see [3]). Then the integrality constraints may be converted to Chvátal constraints as in (2) above. It is an interesting open problem to find an elimination scheme to construct the Chvátal constraints for an arbitrary finitely generated integral monoid.

**References**

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Received 17 April 1989, final manuscript accepted 28 August 1990