

Available online at [www.sciencedirect.com](http://www.sciencedirect.com) ScienceDirectADVANCES IN  
Mathematics

Advances in Mathematics 217 (2008) 761–813

[www.elsevier.com/locate/aim](http://www.elsevier.com/locate/aim)

# Generalized Markoff maps and McShane's identity <sup>☆</sup>

Ser Peow Tan <sup>a,\*</sup>, Yan Loi Wong <sup>a</sup>, Ying Zhang <sup>b</sup><sup>a</sup> Department of Mathematics, National University of Singapore, 2 Science Drive 2,  
Singapore 117543, Republic of Singapore<sup>b</sup> School of Mathematical Sciences, Yangzhou University, Yangzhou, Jiangsu 225002, People's Republic of China

Received 4 October 2006; accepted 6 September 2007

Available online 1 November 2007

Communicated by Michael J. Hopkins

---

## Abstract

We study the (relative)  $SL(2, \mathbb{C})$  character varieties of the one-holed torus and the action of the mapping class group on the (relative) character variety. We show that the subset of characters satisfying two simple conditions called the Bowditch  $Q$ -conditions is open in the relative character variety and that the mapping class group acts properly discontinuously on this subset. Furthermore, this is the largest open subset for which this holds. We also show that a generalization of McShane's identity holds for all characters satisfying the Bowditch  $Q$ -conditions. Finally, we show that further variations of the McShane–Bowditch identity hold for characters which are fixed by an Anosov element of the mapping class group and which satisfy a relative version of the Bowditch  $Q$ -conditions, with applications to identities for incomplete hyperbolic structures on punctured torus bundles over the circle, and also for closed hyperbolic 3-manifolds which are obtained by hyperbolic Dehn surgery on such manifolds.

© 2007 Elsevier Inc. All rights reserved.

**Keywords:** One-holed torus; Character variety; Mapping class group; McShane's identity; Punctured torus bundle; Hyperbolic Dehn surgery

---

---

<sup>☆</sup> The authors are partially supported by the National University of Singapore academic research grant R-146-000-056-112. The third author is also supported partially by the NKBRF (China) grant G1999075104, NSF (China) grant 10671171 and NSF (Jiangsu) grant BK2007073.

\* Corresponding author.

*E-mail addresses:* [mattansp@nus.edu.sg](mailto:mattansp@nus.edu.sg) (S.P. Tan), [matwyl@nus.edu.sg](mailto:matwyl@nus.edu.sg) (Y.L. Wong), [yingzhang@yzu.edu.cn](mailto:yingzhang@yzu.edu.cn) (Y. Zhang).

**1. Introduction**

Let  $\mathbb{T}$  be the one-holed torus and  $\Gamma = \langle a, b \rangle$  its fundamental group, which is a free group on two generators  $a$  and  $b$ . In this paper, we study the relative  $SL(2, \mathbb{C})$  character varieties

$$\mathcal{X}_\tau := \{ \rho \in \text{Hom}(\Gamma, SL(2, \mathbb{C})) // SL(2, \mathbb{C}) \mid \text{tr } \rho(aba^{-1}b^{-1}) = \tau \}$$

of the one-holed torus, the action of the mapping class group

$$\mathcal{MCG} := \pi_0(\text{Homeo}(\mathbb{T}))$$

on  $\mathcal{X}_\tau$ , and generalizations and variations of McShane’s identity for characters  $\rho \in \mathcal{X}_\tau$ . In particular, we describe the (largest open) subset of  $\mathcal{X}_\tau$  for which the action of  $\mathcal{MCG}$  is properly discontinuous, and give generalizations of McShane’s identity to characters in this subset. We also give variations of McShane’s identity to certain characters which are fixed by an Anosov element of  $\mathcal{MCG}$ . These variations have interpretations as length series identities for certain closed hyperbolic 3-manifolds. Announcements of the results obtained in this paper, as well as [31] and [33] have appeared in [29].

The action of the mapping class group  $\mathcal{MCG}$  on the real  $SL(2)$  characters was studied by Goldman in [14], where he showed that the dynamics splits into two dramatically different types, either properly discontinuous or ergodic. As pointed out there, the dynamics of the action on the  $SL(2, \mathbb{C})$  characters is extremely mysterious and non-trivial, even for the type-preserving case. Part of this paper is an attempt to understand the dynamics of this action in the simplest instance, that is, when it is properly discontinuous, our approach here is very much based on that used by Bowditch in [5] (see also [33] for a discussion of the case where the dynamics of the action is more complicated).

To start with, the original motivation for this work came from the following remarkable identity of Greg McShane, proven in his PhD thesis in Warwick [18], concerning the lengths of simple closed geodesics on a once-punctured torus,  $\mathbb{T}$ , with a complete, finite area hyperbolic structure:

$$\sum_{\gamma} \frac{1}{1 + e^{l(\gamma)}} = \frac{1}{2}, \tag{1}$$

where  $\gamma$  ranges over all simple closed geodesics on  $\mathbb{T}$ , and  $l(\gamma)$  is the hyperbolic length of  $\gamma$  under the given hyperbolic structure on  $\mathbb{T}$ . This identity is independent of the hyperbolic structure on the torus, that is, it holds for all points in the Teichmüller space  $\mathcal{T}$  of the punctured torus. This result was later generalized to more general hyperbolic surfaces with cusps by McShane himself [19], to hyperbolic surfaces with cusps and/or geodesic boundary components by Mirzakhani [22], to hyperbolic surfaces with cusps, geodesic boundary and/or conical singularities, as well as to classical Schottky groups by the authors in [30,32].

On the other hand, Bowditch in [3] gave an alternative proof of (1) via Markoff maps, and extended it in [5] to type-preserving representations of the once-punctured torus group into  $SL(2, \mathbb{C})$  satisfying certain conditions which we call here the BQ-conditions (Bowditch’s Q-conditions). He also obtained in [4] a variation of (1) which applies to hyperbolic once-punctured torus bundles. Subsequently, Sakuma [26], and Akiyoshi, Miyachi and Sakuma [1,2] refined Bowditch’s results in [4] and generalized them to those which apply to hyperbolic punctured surface bundles.

This paper was first motivated by our attempt to understand the extent to which the generalized McShane identities for the one-hole/one-cone torus obtained in [30] hold for general representations of the torus group into  $SL(2, \mathbf{C})$ , and also the possible generalizations of the McShane–Bowditch identities in [4] for complete, hyperbolic punctured torus bundles over the circle to closed hyperbolic 3-manifolds obtained by hyperbolic Dehn surgery on such manifolds. In the process, we were also led to study the more general question of the  $SL(2, \mathbf{C})$  characters of the one-holed torus and the dynamics of the action of  $MCG$  on the relative character variety.

We give a brief description of our main results, see Section 2 for precise statements. Let  $\mathcal{C}$  be the set of free homotopy classes of essential simple closed curves on the torus. The first main result (Theorem 2.2) is that the generalized McShane identity obtained in [30] for the one-holed/one-coned torus holds for representations/characters  $\rho \in \mathcal{X}_\tau$  which satisfy the conditions (i)  $\text{tr } \rho(\gamma) \notin [-2, 2]$  for all  $\gamma \in \mathcal{C}$ ; and (ii)  $|\text{tr } \rho(\gamma)| \leq 2$  for only finitely many (possibly no)  $\gamma \in \mathcal{C}$ . Bowditch called these the Q-conditions in [5], we call them here the BQ-conditions. There are many interesting examples of representations/characters which satisfy these conditions arising from some basic and important geometric constructions. For example, representations arising from the one-coned/one-holed hyperbolic torus, the hyperbolic three-holed sphere (often called pair of pants), and more generally, classical Schottky groups with two generators all give rise to such representations. However, the class of representations satisfying the BQ-conditions is in general much larger, and an interesting problem is to classify this class geometrically. In the case of type-preserving representations ( $\rho \in \mathcal{X}_{-2}$ ) studied in [5], Bowditch has conjectured that the class coincides with the quasifuchsian representations (Conjecture A in [5]). The general case is less clear since the representations are in general not discrete, or not faithful. The basic strategy for proving Theorem 2.2 follows very closely that used in [5]. The problem is reformulated in terms of generalized Markoff maps and it is shown that if the BQ-conditions are satisfied, then the generalized Markoff map has Fibonacci growth. This is sufficient to obtain the absolute convergence of the series on the left-hand side of the identity. To obtain the actual value, in the type-preserving case, Bowditch made ingenious use of the quantities  $\frac{x}{yz}$ ,  $\frac{y}{zx}$  and  $\frac{z}{xy}$  associated to a Markoff triple  $(x, y, z)$  and certain properties of these quantities. By exploring the geometric interpretation of these quantities, we were able to find analogous quantities (see also [16]),  $\Psi(y, z, x)$ ,  $\Psi(z, x, y)$  and  $\Psi(x, y, z)$  (for explicit expressions see Section 4), associated to a generalized Markoff triple, and follow through the proof of Bowditch to complete the proof of Theorem 2.2.

The mapping class group  $MCG$  of the torus acts on the character variety  $\mathcal{X}$ , preserving the trace of the commutator of the generators, hence it also acts on the relative character varieties  $\mathcal{X}_\tau$ , see [14] for details. Our next result is that  $MCG$  acts properly discontinuously on the subset of characters satisfying the BQ-conditions (Theorem 2.3). Furthermore, this is the *largest open subset* for which this is true (Proposition 2.4). In particular, the set of characters in  $\mathcal{X}_\tau$  which satisfy the BQ-conditions is open in  $\mathcal{X}_\tau$ .

Bowditch also gave variations of McShane’s identity for complete, finite volume hyperbolic 3-manifolds which fiber over the circle, with fiber the once-punctured torus [4]. These complete structures can be deformed to incomplete structures, as shown by Thurston in [34], and in certain cases, one can perform hyperbolic Dehn surgery to obtain closed (complete) hyperbolic 3-manifolds. The next set of results (Theorems 5.3, 5.4) is that a further variation of the McShane–Bowditch identity holds for these deformations satisfying a *relative version* of the BQ-conditions, and hence for the closed hyperbolic 3-manifolds obtained by hyperbolic Dehn surgery on such manifolds. An interesting aspect of this result is that it seems to be easier to obtain generalizations of McShane’s identity for this class of closed (complete) hyperbolic 3-

manifolds than it is for (general) closed hyperbolic surfaces. The only example in the surface case which we know is an identity for the genus two surface (see [20] and [30]). Another interesting aspect is that the identity obtained for the incomplete structures gives a *quantitative relation* between the (complex) lengths of the meridian and longitude of the boundary torus.

All the results above are proven by transforming to equivalent statements for generalized Markoff maps and working in that framework. The generalized Markoff map corresponding to the character  $\rho$  is essentially the map  $\phi := \phi_\rho: \mathcal{C} \rightarrow \mathbf{C}$  given by  $\phi(\gamma) = \text{tr } \rho(\gamma)$ . We remark that besides these connections with hyperbolic geometry, generalized Markoff maps are also closely related to dynamical systems (see [8,14]), algebraic number theory (see [10], and [28]) and mathematical physics (see [25]). There also seem to be a close connection to the pair-of-pants complex introduced by Hatcher and Thurston in [15], and developed in connection with the Weil–Petersson metric on moduli space in the recent work by Brock in [7] (see also [22] for another connection of McShane’s identity with the Weil–Petersson volume). A further exploration of these ideas may lead to the correct framework of Markoff maps for general surfaces.

The rest of this paper is organized as follows. In Section 2 we set the notation, give the definitions and state the results in terms of representations/characters. In Section 3 we reformulate the results in terms of generalized Markoff maps and also state and prove some basic and fundamental results for generalized Markoff maps, including Theorem 2.3 and Proposition 2.4. In Section 4 we introduce the  $\Psi$  function and give the proof of Theorem 3.5, the reformulation of our first main theorem. In Section 5 we generalize Bowditch’s variations of McShane’s identity to identities for certain incomplete hyperbolic torus bundles over the circle (Theorems 5.3, 5.4), with applications to hyperbolic Dehn surgery. Finally, in Appendices A and B, we give the geometric meaning of two important functions  $\mathfrak{h}$  and  $\Psi$  used in this paper (Appendix A), and explain how to draw the gaps in the extended complex plane to visualize the generalized McShane’s identity (18) (Appendix B).

## 2. Notation and statements of results

In this section, we set some basic notation and definitions and give precise statements of our results in terms of representations of the one-holed torus group,  $\Gamma$ , into  $\text{SL}(2, \mathbf{C})$  (more succinctly,  $\text{SL}(2, \mathbf{C})$  characters of the one-holed torus). As much of the paper is influenced by [5] and [4], we will borrow the notation and definitions from them as much as possible to avoid confusion.

**The one-holed torus group.** Let  $\mathbb{T}$  be a (topological) one-holed torus, that is, a torus with a disk removed, and let  $\Gamma$  be its fundamental group. Note that algebraically,  $\Gamma \cong \mathbf{Z} * \mathbf{Z} = \langle a, b \rangle$ , the free group on two generators, where  $a, b$  correspond to simple closed curves on  $\mathbb{T}$  with geometric intersection number one.

We define an equivalence relation,  $\sim$ , on  $\Gamma$  such that  $g \sim h$  if and only if  $g$  is conjugate to  $h$  or  $h^{-1}$ . Note that  $\Gamma / \sim$  can be identified with the set of free homotopy classes of unoriented closed curves on  $\mathbb{T}$ .

**Simple closed curves on the torus.** Let  $\mathcal{C}$  be the set of free homotopy classes of *essential* (non-trivial, non-peripheral) *simple closed curves* on  $\mathbb{T}$  and let  $\hat{\Omega} \subset \Gamma / \sim$  be the subset corresponding to  $\mathcal{C}$ .

Note that  $\hat{\Omega}$  can be identified with  $\mathbf{Q} \cup \{\infty\}$  by considering the “slope” of  $[g] \in \hat{\Omega}$  as follows. Fix a pair of generators  $a$  and  $b$  of  $\Gamma$ . Then each class  $[g]$  of  $\hat{\Omega}$  has a representative

$g = W(a^{\pm 1}, b)$  which is a cyclically reduced word in  $\{a^{\pm 1}, b\}$ , and the exponents of  $a$  in each word are either all positive or all negative. The word is unique up to cyclic permutations, and the slope is the quotient of the sum of the exponents of  $a$  with the sum of the exponents of  $b$  in the word; see [27] for details. Hence,  $a$  is identified with  $\infty$ ,  $b$  with  $0$ ,  $ab$  with  $1$ , and so on. Note that  $\hat{\Omega}$  inherits a cyclic ordering from the cyclic ordering of  $\mathbf{Q} \cup \{\infty\}$  induced from the standard embedding into  $\mathbf{R} \cup \{\infty\} \cong S^1$ .

**The  $\tau$ -representations.** A representation  $\rho : \Gamma \rightarrow \mathrm{SL}(2, \mathbf{C})$  is said to be a  $\tau$ -representation, or  $\tau$ -character, where  $\tau \in \mathbf{C}$ , if for some (hence every) pair of free generators  $a, b \in \Gamma$ ,  $\mathrm{tr} \rho([a, b]) = \tau$ , where  $[a, b] = aba^{-1}b^{-1}$ . The space of equivalence classes of  $\tau$ -representations is denoted by  $\mathcal{X}_\tau$  and is called the  $\tau$ -relative character variety, where the equivalence classes are obtained by taking the closure of the orbit under the conjugation action by  $\mathrm{SL}(2, \mathbf{C})$ . The mapping class group of  $\mathbb{T}$ ,  $\mathcal{MCG} := \pi_0(\mathrm{Homeo}(\mathbb{T}))$  acts on  $\mathcal{X}_\tau$ , see [14]. For concreteness, we adopt the convention here that  $\mathcal{MCG}$  consists of orientation-preserving homeomorphisms fixing the boundary, there will be no essential difference to the ensuing discussion.

Note that we can also talk about  $\tau$ -representations for representations  $\rho$  into  $\mathrm{PSL}(2, \mathbf{C})$ , since the trace of the commutator is well defined. Most geometric constructions give rise to representations into  $\mathrm{PSL}(2, \mathbf{C})$ . It will, however, be more convenient for us to work with representations into  $\mathrm{SL}(2, \mathbf{C})$ , as the statements of the results are neater in this case. This will not affect the validity of our results since the results stated will be independent of the lift chosen, and the results could have been stated in terms of representations into  $\mathrm{PSL}(2, \mathbf{C})$  as well.

It is well known that the image of the representation  $\rho$  is non-elementary ( $\rho$  is irreducible) if and only if  $\tau \neq 2$ , we will only be interested in  $\tau$ -representations with  $\tau \neq 2$  in this paper, as the reducible characters  $\rho \in \mathcal{X}_2$  never satisfy the BQ-conditions (see [33]). The case where  $\tau = -2$  is particularly interesting. In this case, the representation  $\rho$  is *type-preserving*, that is,  $\rho(aba^{-1}b^{-1})$  is always a parabolic element for any generating pair  $\langle a, b \rangle$  of  $\Gamma$ , except when  $\mathrm{tr} \rho(a) = \mathrm{tr} \rho(b) = \mathrm{tr} \rho(ab) = 0$ . Bowditch calls  $\mathcal{X}_{-2}$  the space of type-preserving representations in [5] (strictly speaking, it is the closure of the space of type-preserving representations). This is the case usually considered in the study of Kleinian groups (see [21]), and is also the case extensively studied in [5].

**The BQ-conditions.** For a fixed  $\tau \neq 2$ , a  $\tau$ -representation  $\rho : \Gamma \rightarrow \mathrm{SL}(2, \mathbf{C})$  (or  $\mathrm{PSL}(2, \mathbf{C})$ ) is said to satisfy the *BQ-conditions* (Bowditch’s Q-conditions) if

- (BQ1)  $\mathrm{tr} \rho(g) \notin [-2, 2]$  for all  $[g] \in \hat{\Omega}$ ; and
- (BQ2)  $|\mathrm{tr} \rho(g)| \leq 2$  for only finitely many (possibly no)  $[g] \in \hat{\Omega}$ .

We also call such a representation  $\rho$  a *BQ-representation*, or Bowditch representation, and the space of equivalence classes of such representations the Bowditch representation space, denoted by  $(\mathcal{X}_\tau)_Q$ . If we replace (BQ1) by  $\mathrm{tr} \rho(g) \notin (-2, 2)$  for all  $[g] \in \hat{\Omega}$ , we call the resulting space the extended Bowditch representation space, representations in the extended space still satisfy the generalized McShane’s identity, indeed, together with BQ(2) these give necessary and sufficient conditions for the identity to hold (see [31]).

Note that  $\mathrm{tr} \rho(g_1) = \mathrm{tr} \rho(g_2)$  if  $[g_1] = [g_2]$  (since  $g_1$  is conjugate to  $g_2$  or its inverse by definition); so the conditions (BQ1) and (BQ2) make sense.

**Some conventions.** Throughout this paper we assume that

- the function  $\cosh^{-1}$  has images with non-negative real parts and with imaginary parts in  $(-\pi, \pi]$ ;
- the function  $\log$  has images with imaginary parts in  $(-\pi, \pi]$ ; while
- the function  $\tanh^{-1}$  has images with imaginary parts in  $(-\pi/2, \pi/2]$ ;
- the function  $\sqrt{*}$  has non-negative real part.

**The functions  $l/2$  and  $l$ .** For  $x \in \mathbf{C}$ , let  $l(x)/2 \in \mathbf{C}/2\pi i\mathbf{Z}$  be defined by

$$l(x)/2 = \cosh^{-1}(x/2). \tag{2}$$

Hence  $l(x) = 2 \cosh^{-1}(x/2) = \cosh^{-1}(x^2/2 - 1) \in \mathbf{C}/2\pi i\mathbf{Z}$ .

In particular,  $\Re(l(x)/2) \geq 0$  and if  $\Re(l(x)/2) = 0$  then  $\Im(l(x)/2) \geq 0$ .

Note that  $e^{l(x)/2} \in \mathbf{C}$  is well defined. It can be shown (see Lemma 4.1) that if  $x \notin [-2, 2]$  then  $e^{-l(x)/2} = xh(x)$  where  $h(x)$  is defined below. Hence

$$e^{l(x)} = x^{-2}h(x)^{-2} = h(x)^{-1} - 1. \tag{3}$$

For  $A \in \text{SL}(2, \mathbf{C})$ , we define its half translation length  $l(A)/2 \in \mathbf{C}/2\pi i\mathbf{Z}$  by

$$\frac{l(A)}{2} = \frac{l(\text{tr } A)}{2} = \cosh^{-1}\left(\frac{\text{tr } A}{2}\right). \tag{4}$$

**The function  $h$ .** We define an even function  $h : \mathbf{C} \setminus \{0\} \rightarrow \mathbf{C}$  by

$$h(x) = \frac{1}{2} \left( 1 - \sqrt{1 - \frac{4}{x^2}} \right). \tag{5}$$

It is easy to check that

$$h(x)^2 - h(x) + x^{-2} = 0. \tag{6}$$

In fact, if  $x \notin [-2, 2]$  then  $h(x)$  is the root of the quadratic equation which has smaller real part. By (3), we also have

$$h(x) = \frac{1}{1 + e^{l(x)}}. \tag{7}$$

**The function  $\mathfrak{h} = \mathfrak{h}_\tau$ .** For  $\tau \in \mathbf{C}$ , set  $\nu = \cosh^{-1}(-\tau/2)$ . We define a function  $\mathfrak{h} = \mathfrak{h}_\tau : \mathbf{C} \setminus \{\pm\sqrt{\tau + 2}\} \rightarrow \mathbf{C}$  by

$$\mathfrak{h}(x) = 2 \tanh^{-1} \left( \frac{\sinh \nu}{\cosh \nu + e^{l(x)}} \right) \tag{8}$$

$$= \log \frac{e^\nu + e^{l(x)}}{e^{-\nu} + e^{l(x)}} \tag{9}$$

$$= \log \frac{1 + (e^\nu - 1)h(x)}{1 + (e^{-\nu} - 1)h(x)}, \tag{10}$$

where (10) follows from (9) by (3). Note that  $h(0)$  is well defined if  $\tau \neq -2$ . In fact,

$$h(0) = \nu + \pi i.$$

Note also that  $e^{\pm\nu} + e^{l(x)} = 0$  if and only if

$$\pm\nu + \pi i = l(x),$$

which implies that  $x^2 = \tau + 2$ .

**Geometric interpretation of  $h$  and  $h$  in the real case.** When the representation arises from a real one-coned/cusped/holed torus, then  $h$  and  $h$  correspond to the gaps on the boundary of the torus arising from intersecting geodesics, see [19] or [30], we give a more detailed description in Appendix A.

**Bowditch’s extension of McShane’s identity.** With the above notation and definitions, Bowditch’s extension and reformulation of McShane’s identity (1) can be stated as follows.

**Theorem 2.1.** (See Theorem 3 in [5].) *Let  $\rho : \Gamma \rightarrow \text{SL}(2, \mathbf{C})$  be a type-preserving representation which satisfies the BQ-conditions. Then*

$$\sum_{[g] \in \hat{\Omega}} h(\text{tr } \rho(g)) = \frac{1}{2}, \tag{11}$$

where the sum converges absolutely.

**Remarks.**

- (i) The identity (11) is also true for BQ-representations into  $\text{PSL}(2, \mathbf{C})$  since  $h(x)$  is an even function and the trace of an element of  $\text{PSL}(2, \mathbf{C})$  is well defined up to sign.
- (ii) For an element  $A \in \text{PSL}(2, \mathbf{C})$ , its *complex translation length*

$$l(A) = l(\text{tr } A) = \cosh^{-1} \left( \frac{1}{2} \text{tr}^2 A - 1 \right) \in \mathbf{C}/2\pi i \mathbf{Z}$$

is well defined with  $\Re l(A) \geq 0$  and  $h(\text{tr } A) = 1/(1 + e^{l(A)})$ .

**Our first main theorem**, which is a generalization of Theorem 2.1 of Bowditch above, and Theorem 1.4 of [30], is the following

**Theorem 2.2.** *Let  $\rho : \Gamma \rightarrow \text{SL}(2, \mathbf{C})$  be a  $\tau$ -representation (where  $\tau \neq 2$ ) satisfying the BQ-conditions. Set  $\nu = \cosh^{-1}(-\tau/2)$ . Then*

$$\sum_{[g] \in \hat{\Omega}} h(\text{tr } \rho(g)) = \nu \pmod{2\pi i}, \tag{12}$$

where the sum converges absolutely.

**Remarks.**

- (i) Here  $\nu$  is a specific choice of *half* the complex translation length of the commutator  $[\rho(a), \rho(b)]$ .
- (ii)  $\mathfrak{h}$  is given by any one of the equivalent formulas (8), (9) or (10). Since  $\mathfrak{h}$  is an even function, Theorem 2.2 also holds for representations into  $\text{PSL}(2, \mathbf{C})$  satisfying the BQ-conditions.
- (iii) Note that for  $x \neq 0$ , when  $\tau \rightarrow -2$ , or equivalently,  $\nu \rightarrow 0$ , we have

$$\begin{aligned} \mathfrak{h}(x) &= \log \frac{1 + (e^\nu - 1)h(x)}{1 + (e^{-\nu} - 1)h(x)} \\ &= \log \left( 1 + \frac{2 \sinh \nu h(x)}{1 + (e^{-\nu} - 1)h(x)} \right) \\ &\sim 2\nu h(x). \end{aligned}$$

Hence the identity (11) can be ‘obtained’ by considering the first-order infinitesimal terms of (12).

- (iv) Theorem 2.2 has an equivalent formulation as Theorem 3.5 in terms of  $\mu$ -Markoff maps.

Next, consider the action of the mapping class group of  $\mathbb{T}$ ,  $\mathcal{MCG} \cong \text{SL}(2, \mathbf{Z})$ , and its induced action on  $\Gamma$  and on  $\mathcal{X}_\tau$ , the space of  $\tau$ -representations. (Recall that  $\mathcal{MCG}$  is the group of isotopy classes of diffeomorphisms of  $\mathbb{T}$  which fixes the boundary, that is, there exists a neighborhood  $N$  of the boundary of  $\mathbb{T}$  which is pointwise fixed by the elements of  $\mathcal{MCG}$ .) Any  $H \in \mathcal{MCG}$  induces an automorphism  $H_*$  of  $\Gamma$ , and  $H$  acts on  $\mathcal{X}_\tau$  by

$$H(\rho)(g) = \rho(H_*^{-1}(g)),$$

for any  $\rho \in \mathcal{X}_\tau$  and  $g \in \Gamma$ . We shall see in the next section that the set  $(\mathcal{X}_\tau)_Q$  of Bowditch representations is open in  $\mathcal{X}_\tau$  (Theorem 3.2). It is also clear that  $\mathcal{MCG}$  acts on  $(\mathcal{X}_\tau)_Q$ . We have the following result:

**Theorem 2.3.**  *$\mathcal{MCG}$  acts properly discontinuously on  $(\mathcal{X}_\tau)_Q$ .*

We also have a partial converse as follows.

**Proposition 2.4.** *Suppose that  $\rho \in \mathcal{X}_\tau$  does not satisfy the extended BQ-conditions, that is, either  $\text{tr } \rho(g) \in (-2, 2)$  for some  $[g] \in \hat{\Omega}$  or  $|\text{tr } \rho(g)| \leq 2$  for infinitely many  $[g] \in \hat{\Omega}$ . Then there exists a sequence of distinct elements  $H_i \in \mathcal{MCG}$  and  $\rho_0 \in \mathcal{X}_\tau$  such that  $H_i(\rho)$  converges to  $\rho_0 \in \mathcal{X}_\tau$ .*

From Proposition 2.4 it is easy to deduce that  $(\mathcal{X}_\tau)_Q$  is the largest open subset of  $\mathcal{X}_\tau$  on which  $\mathcal{MCG}$  acts properly discontinuously. Indeed, the subset  $(\mathcal{X})_Q \subset \mathcal{X}$  of Bowditch characters is open in  $\mathcal{X}$  and is the largest open subset on which  $\mathcal{MCG}$  acts properly discontinuously, by an easy extension of the proofs, and by noting that  $(\mathcal{X})_Q$  does not contain any reducible representations, see [33]. The proofs of Theorem 2.3 and Proposition 2.4 will be deferred to the end of Section 3.

Bowditch [4] studied representations  $\rho \in \mathcal{X}_{\text{tp}} = \mathcal{X}_{-2}$  stabilized by a cyclic subgroup  $\langle H \rangle < \mathcal{MCG}$  generated by an Anosov/hyperbolic element (that is,  $H([\rho]) = [\rho]$  where  $[\rho]$  is the equivalence class of  $\rho$ ), and proved a variation of the McShane’s identity. This result can be generalized



for  $\tau$ -representations as follows. The case  $\tau = -2$  with  $\mathfrak{h}$  replaced by  $h$  is Bowditch’s variation (Theorem A in [4]).

**Theorem 2.5.** *Suppose that a  $\tau$ -representation  $\rho : \Gamma \rightarrow \text{SL}(2, \mathbf{C})$ , where  $\tau \neq 2$ , is stabilized by a hyperbolic element  $H \in \text{MCG} \cong \text{SL}(2, \mathbf{Z})$  and  $\rho$  satisfies the BQ-conditions on  $\hat{\Omega}/\langle H_* \rangle$ , that is,*

- (i)  $\text{tr } \rho([g]) \notin [-2, 2]$  for all classes  $[g] \in \hat{\Omega}/\langle H_* \rangle$ , and
- (ii)  $|\text{tr } \rho([g])| \leq 2$  for only finitely many (possibly no) classes  $[g] \in \hat{\Omega}/\langle H_* \rangle$ .

Then

$$\sum_{[g] \in \hat{\Omega}/\langle H_* \rangle} \mathfrak{h}(\text{tr } \rho([g])) = 0 \pmod{2\pi i}, \tag{13}$$

where the sum converges absolutely.

Theorem 2.5 will be reformulated and proved as Theorem 5.7 in Section 5, together with a more interesting version where the sum is taken only over the left partition (Theorem 5.9), with interpretations in terms of (in)complete hyperbolic structures on punctured torus bundles (Theorems 5.3 and 5.4).

### 3. Generalized Markoff maps

In this section we study the basic geometry of *generalized Markoff maps*, the connection with the characters in the character variety  $\mathcal{X}$ , and show that many of the fundamental results obtained by Bowditch for Markoff maps extend to this case as well. In particular, we obtain the Fibonacci growth for generalized Markoff maps which satisfy Bowditch’s Q-conditions. We shall follow the notation and proofs of Bowditch [5] whenever possible. However, instead of working in the more general combinatorial set-up there, we will fix a concrete realization of the combinatorial structure involved to help with the visualization of the maps.

**The Farey triangulation  $\mathcal{F}$  and the dual binary tree  $\Sigma$ .** Let  $\mathcal{F}$  be the Farey triangulation of the hyperbolic plane  $\mathbb{H}^2$ , and  $\Sigma$  be the dual binary tree (also called an infinite trivalent tree). Recall that the Farey triangulation consists of edges which are complete hyperbolic geodesics joining all pairs  $\{\xi, \eta\} \subset \mathbf{Q} \cup \{\infty\}$  which are Farey neighbors, and that  $\xi = p/q$  and  $\eta = r/s$  (where we always assume that  $p, q, r, s \in \mathbf{Z}$  and  $(p, q) = (r, s) = 1$ ) are Farey neighbors if  $ps - rq = \pm 1$ . See Fig. 1.

**Complementary regions.** A *complementary region* of  $\Sigma$  is the closure of a connected component of the complement.

We denote by  $\Omega = \Omega(\Sigma)$  the set of complementary regions of  $\Sigma$ . Similarly, we use  $V(\Sigma)$ ,  $E(\Sigma)$  for the set of vertices and edges of  $\Sigma$  respectively.

**Action of  $\text{PSL}(2, \mathbf{Z})$ .** It is clear from the construction that there is a natural action of  $\text{PSL}(2, \mathbf{Z})$  (in fact,  $\text{PGL}(2, \mathbf{Z})$ ) on  $\Sigma$ , and there is a natural correspondence of  $\Omega(\Sigma)$  with  $\mathbf{Q} \cup \{\infty\}$  (the vertices of  $\mathcal{F}$ ), and hence  $\mathcal{C}$ , together with the induced cyclic ordering.

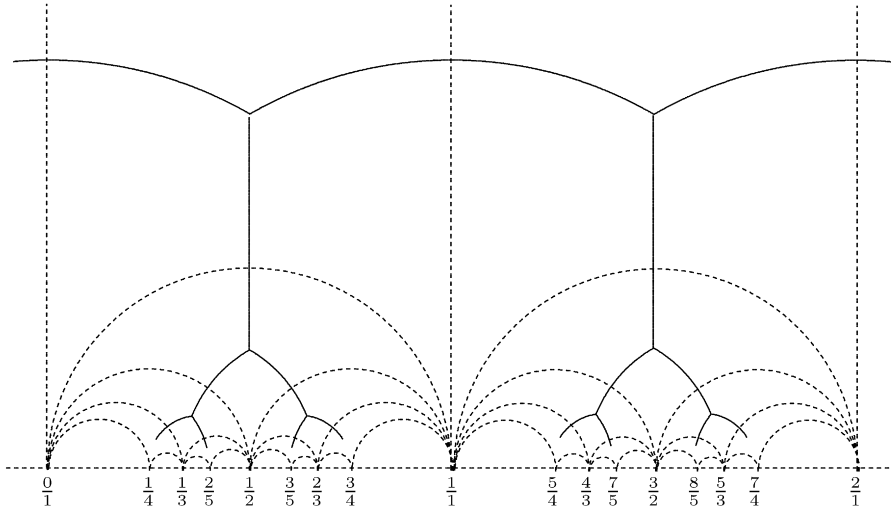


Fig. 1. Farey tessellation and the binary tree  $\Sigma$ .

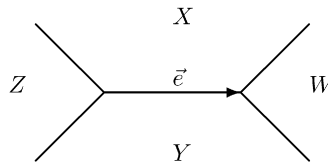


Fig. 2. The directed edge  $\vec{e} = (X, Y; Z \rightarrow W)$ .

We use the letters  $X, Y, Z, W, \dots$  to denote the elements of  $\Omega$ , and also introduce the notation  $X(p/q)$  to indicate that  $X \in \Omega$  corresponds to  $p/q \in \mathbf{Q} \cup \{\infty\}$ . For  $e \in E(\Sigma)$ , we also use the notation  $e \leftrightarrow (X, Y; Z, W)$  to indicate that  $e = X \cap Y$  and  $e \cap Z$  and  $e \cap W$  are the endpoints of  $e$ ; see Fig. 2, ignoring the direction of  $e$  there. Note also that there is a correspondence of the ends of  $\Sigma$  with  $\mathbf{R} \cup \{\infty\} \cong S^1$  which is one-to-one from ends to the irrationals and two-to-one from ends to the rational numbers, much like the decimal expansions for the real numbers although we shall not use it in this paper.

**Directed edges, the sets  $\Omega^0(\vec{e}), \Omega^\pm(\vec{e}), \Omega^{0\pm}(\vec{e})$  and  $C(T)$ .** Denote by  $\vec{E}(\Sigma)$  (or just  $\vec{E}$ ) the set of directed edges of  $\Sigma$  where the direction is always taken to be from the tail to the head (as in the direction of the arrow), and for a directed edge  $\vec{e} \in \vec{E}(\Sigma)$ , we use  $\vec{e} = (X, Y; Z \rightarrow W)$  (or just  $\vec{e} = (X, Y; \rightarrow W)$ ), to indicate that  $e \cap W$  is the head of  $\vec{e}$ , that is,  $\vec{e}$  is the directed edge from  $Z$  to  $W$ , see Fig. 2. We also use  $e$  to indicate the underlying undirected edge corresponding to the directed edge  $\vec{e}$ , and  $-\vec{e}$  to indicate the directed edge in the opposite direction of  $\vec{e}$ . Associated to each  $\vec{e} \in \vec{E}(\Sigma)$  is a partition of  $\Omega$  which we will use repeatedly later. For  $e \leftrightarrow (X, Y; Z, W) \in E$ , define  $\Omega^0(e) = \{X, Y\}$ . For  $\vec{e} = (X, Y; Z \rightarrow W) \in \vec{E}$ ,  $\Sigma \setminus \text{int}(e)$  consists of two components, denoted by  $\Sigma^+(\vec{e})$  and  $\Sigma^-(\vec{e})$ , where  $\Sigma^+(\vec{e})$  is the component containing the head of  $\vec{e}$  and  $\Sigma^-(\vec{e})$  is the component containing the tail of  $\vec{e}$  (so that  $Z$  meets  $\Sigma^-(\vec{e})$  and  $W$  meets  $\Sigma^+(\vec{e})$ ). Define  $\Omega^+(\vec{e})$  (resp.  $\Omega^-(\vec{e})$ ) to be the set of regions in  $\Omega$  whose boundaries lie in  $\Sigma^+(\vec{e})$  (resp.  $\Sigma^-(\vec{e})$ ). Hence  $\Omega = \Omega^+(\vec{e}) \sqcup \Omega^0(e) \sqcup \Omega^-(\vec{e})$ . Define  $\Omega^{0+}(\vec{e}) = \Omega^0(e) \cup \Omega^+(\vec{e})$

(resp.  $\Omega^{0-}(\vec{e}) = \Omega^0(e) \cup \Omega^-(\vec{e})$ ). For a finite subtree  $T \subset \Sigma$ , the circular set  $C(T)$  is the set of directed edges not in  $T$  whose heads lie in  $T$ . A set of directed edges is called a circular set if it is of the form  $C(T)$  for some finite subtree  $T \subset \Sigma$ .

**$\mu$ -Markoff triples.** For a complex number  $\mu$ , a  $\mu$ -Markoff triple is an ordered triple  $(x, y, z)$  of complex numbers satisfying the  $\mu$ -Markoff equation:

$$x^2 + y^2 + z^2 - xyz = \mu. \tag{14}$$

Thus Markoff triples are just 0-Markoff triples.

It is easily verified that if  $(x, y, z)$  is a  $\mu$ -Markoff triple, so are  $(x, y, xy - z)$ ,  $(x, xz - y, z)$ ,  $(yz - x, y, z)$  and the permutation triples of each of them.

**$\mu$ -Markoff maps.** A  $\mu$ -Markoff map is a function  $\phi : \Omega \rightarrow \mathbf{C}$  such that

- (i) for every vertex  $v \in V(\Sigma)$ , the triple  $(\phi(X), \phi(Y), \phi(Z))$  is a  $\mu$ -Markoff triple, where  $X, Y, Z \in \Omega$  are the three regions meeting  $v$ ; and
- (ii) for every edge  $e \in E(\Sigma)$  such that  $e \leftrightarrow (X, Y; Z, W)$ , we have

$$xy = z + w, \tag{15}$$

where  $x = \phi(X)$ ,  $y = \phi(Y)$ ,  $z = \phi(Z)$  and  $w = \phi(W)$ .

We shall use  $\Phi_\mu$  to denote the set of all  $\mu$ -Markoff maps and lower case letters to denote the  $\phi$  values of the regions, for example,  $\phi(X) = x$ ,  $\phi(Y) = y$ . Recall that in [5] the set of all Markoff maps is denoted by  $\Phi$ ; while here it is denoted by  $\Phi_0$  in our notation. We use  $\Phi$  to denote the set of all generalized Markoff maps.

As in the case of Markoff maps, if the edge relation (15) is satisfied along all edges, then it suffices that the vertex relation (14) be satisfied at a single vertex. In fact one may establish a bijective correspondence between  $\mu$ -Markoff maps and  $\mu$ -Markoff triples, by fixing three regions  $X, Y, Z$  which meet at some vertex  $v_0$ , say  $X(\infty), Y(0)$  and  $Z(1)$ . This process may be inverted by constructing a tree of  $\mu$ -Markoff triples as Bowditch did in [5] for Markoff triples: given a triple  $(x, y, z)$ , set  $\phi(X) = x$ ,  $\phi(Y) = y$ ,  $\phi(Z) = z$ , and extend over  $\Omega$  as dictated by the edge relations. In this way one obtains an identification of  $\Phi_\mu$  with the algebraic variety in  $\mathbf{C}^3$  given by the  $\mu$ -Markoff equation. In particular,  $\Phi_\mu$  gets an induced topology as a subset of  $\mathbf{C}^3$ .

The natural action of  $\text{PSL}(2, \mathbf{Z})$  on  $\Sigma$  induces an action on  $\Phi_\mu$ , given by

$$H(\phi)(X) = \phi(H_*^{-1}(X)),$$

where  $H \in \text{PSL}(2, \mathbf{Z})$ ,  $\phi \in \Phi_\mu$ ,  $X$  is any element of  $\Omega$  and  $H_*$  is the induced action of  $H$  on  $\Omega$ . There is also an action of the Klein-four group,  $\mathbf{Z}_2^2$ , on  $\Phi_\mu$  obtained by changing two of the signs in a  $\mu$ -Markoff triple, for example,  $(x, y, z) \mapsto (-x, -y, z)$ . (We get the same action, up to automorphisms of  $\mathbf{Z}_2^2$ , no matter at which vertex we choose to perform this operation.) This action is free and properly discontinuous on  $\Phi_\mu \setminus \{(\pm\sqrt{\mu}, 0, 0), (0, \pm\sqrt{\mu}, 0), (0, 0, \pm\sqrt{\mu})\}$ .

**Natural correspondence:**  $\mathcal{X}_{\mu-2} \equiv \Phi_\mu$ . There is also a natural correspondence between  $(\mu - 2)$ -characters  $\rho$  and  $\mu$ -Markoff maps, obtained by fixing a generating pair  $a, b$  (and  $ab$ ) for  $\Gamma$ . First note that by fixing  $a$  and  $b$ , we get a correspondence between  $\Omega$  and  $\hat{\Omega}$  via the

correspondence with  $\mathbf{Q} \cup \{\infty\}$  (see Section 2 and earlier discussion in this section). Note that  $X(p/q), Y(r/s) \in \Omega$  share an edge if and only if  $p/q$  and  $r/s$  are Farey neighbors, if and only if the corresponding elements  $[g_1], [g_2] \in \hat{\Omega}$  correspond to simple closed curves on  $\mathbb{T}$  which have geometric intersection number one, or equivalently, have representatives  $g_1$  and  $g_2$  which are a pair of free generators for  $\Gamma$ .

A  $(\mu - 2)$ -character  $\rho : \Gamma \rightarrow \text{SL}(2, \mathbf{C})$  naturally gives rise to a  $\mu$ -Markoff map  $\phi \in \Phi_\mu$  via the correspondence between  $\Omega$  and  $\hat{\Omega}$  by taking the traces of  $[g] \in \hat{\Omega}$ , that is, by  $\phi(X) = \text{tr } \rho(g)$  where  $[g] \in \hat{\Omega}$  represents the simple closed curve corresponding to  $X \in \Omega$ . The edge and vertex relations follow from the trace identities in  $\text{SL}(2, \mathbf{C})$ :

$$\text{tr } A \text{ tr } B = \text{tr } AB + \text{tr } AB^{-1}, \tag{16}$$

$$2 + \text{tr}[A, B] = (\text{tr } A)^2 + (\text{tr } B)^2 + (\text{tr } AB)^2 - \text{tr } A \text{ tr } B \text{ tr } AB. \tag{17}$$

Representations in the same equivalence class give rise to the same  $\mu$ -Markoff map, that is, the map is well defined on the  $(\mu - 2)$ -character variety.

Conversely, given any  $\mu$ -Markoff map  $\phi$ , we can recover the  $(\mu - 2)$ -character as follows. Consider the three regions  $X(\infty), Y(0)$  and  $Z(1)$  which meet at the vertex  $v_0$  and consider the  $\mu$ -Markoff triple  $(x, y, z)$ , where  $x = \phi(X), y = \phi(Y)$  and  $z = \phi(Z)$ . Then one can find  $A, B \in \text{SL}(2, \mathbf{C})$ , such that  $\text{tr } A = x, \text{tr } B = y$  and  $\text{tr } AB = z$ ; a specific choice is given by Bowditch in §4 of [5], see also [14] for a more natural choice. All choices of  $A, B$  correspond to the same character. This gives a  $(\mu - 2)$ -representation  $\rho$  with  $\rho(a) = A, \rho(b) = B$  and hence  $\text{tr } \rho(a) = x, \text{tr } \rho(b) = y, \text{tr } \rho(ab) = z$ .

In this way we can identify  $\Phi_\mu$  naturally with the relative character variety  $\mathcal{X}_{\mu-2}$ , and study the latter via the former. Also, there is a natural identification of the mapping class group  $\mathcal{MCG}$  with  $\text{SL}(2, \mathbf{Z})$ , the action of  $\mathcal{MCG}$  on  $\mathcal{X}_{\mu-2}$  is not effective since the elliptic involution fixes all characters  $\mathcal{X}_{\mu-2}$ . The effective action corresponds exactly to the action of  $\text{PSL}(2, \mathbf{Z})$  on  $\Phi_\mu$ , where  $\text{PSL}(2, \mathbf{Z})$  is isomorphic to the quotient of  $\mathcal{MCG}$  by the subgroup of order 2 generated by the elliptic involution.

A key observation is that a  $\mu$ -Markoff map gives a tremendous amount of information about the corresponding representation  $\rho$ , since it incorporates in its definition the action of the mapping class group on  $\rho$ , as well as the structure of the mapping class group itself (which is encoded in the tree  $\Sigma$ ).

**The subsets  $\Omega_\phi(k) \subset \Omega$ .** Given  $\phi \in \Phi_\mu$  and  $k \geq 0$ , the set  $\Omega_\phi(k) \subseteq \Omega$  is defined by

$$\Omega_\phi(k) = \{X \in \Omega \mid |\phi(X)| \leq k\}.$$

We now state Bowditch’s Q-conditions for  $\mu$ -Markoff maps.

**BQ-conditions for  $\mu$ -Markoff maps.** A generalized Markoff map  $\phi \in \Phi_\mu$  is said to satisfy the BQ-conditions if

- (BQ1)  $\phi^{-1}([-2, 2]) = \emptyset$ ; and
- (BQ2)  $\Omega_\phi(2)$  is finite (possibly empty).

We denote by  $(\Phi_\mu)_Q$  the set of all generalized  $\mu$ -Markoff maps which satisfy the BQ-conditions, and call the set of all such maps the Bowditch  $\mu$ -Markoff maps. It is clear that  $(\Phi_\mu)_Q$  corresponds to  $(\mathcal{X}_{\mu-2})_Q$ .

In the rest of this section we examine the results and arguments given in §§1–4 of [5] about Markoff maps and point out those which hold for generalized Markoff maps. A key observation here is that most of the basic results derived there only depend on the edge relation (15) and not on the vertex relation (14) and hence holds for generalized Markoff maps as well.

**Results for generalized Markoff maps.** We first state the results for generalized Markoff maps corresponding to and generalizing those in §§1–3 of [5] for Markoff maps.

**Theorem 3.1.** *If  $\phi \in \Phi_\mu$  then*

- (1) *there exists a constant  $m = m(\mu) > 0$  (depending on  $\mu$  but not on  $\phi$ ) such that  $\Omega_\phi(m)$  is non-empty; and*
- (2) *for any  $k \geq 2$ , the union  $\bigcup \Omega_\phi(k) := \bigcup_{X \in \Omega_\phi(k)} X$  is connected (as a subset of the hyperbolic plane). In particular,  $\bigcup \Omega_\phi(2)$  is connected.*

Part (2) of Theorem 3.1 is a fundamental result and is crucial to the study of the action of  $\text{PSL}(2, \mathbf{Z})$  on  $\Phi_\mu$ .

We have seen that  $\Phi_\mu$  admits a nice topology from its identification with the variety  $\{(x, y, z) \in \mathbf{C}^3 \mid x^2 + y^2 + z^2 - xyz = \mu\}$ . Using this topology, we have the following:

**Theorem 3.2.** *The set  $(\Phi_\mu)_Q$  is an open subset of  $\Phi_\mu$ .*

An important ingredient in [5] was to study the growth rate of a  $\mu$ -Markoff map  $\phi$  by comparing it to a simple function  $F_e : \Omega \mapsto \mathbf{N}$  called the Fibonacci function, defined for a fixed edge  $e \in E(\Sigma)$ . This is essentially the word length function for a fixed pair of generators  $\langle X, Y \rangle$  of  $\Gamma$ , where the elements of  $\Omega$  are identified with generators of  $\Gamma$ . Recall that Bowditch defined the term ‘ $\log^+|\phi|$  has Fibonacci growth’ in §2.1 [5], where  $\log^+(x) = \max\{0, \log(x)\}$  for  $x > 0$ . The definition of Fibonacci growth, lower and upper Fibonacci bound, and the Fibonacci function  $F_e$ , will be given in the later part of this section. The power of this notion lies in that if  $\log^+|\phi|$  has lower Fibonacci bound then  $\sum_{X \in \Omega} |\phi(X)|^{-t}$  converges for any  $t > 0$ ; in particular,  $\Omega_\phi(k)$  is finite for all  $k$  and so  $\phi(\Omega) \subseteq \mathbf{C}$  is discrete. The main thing is to find simple (effectively verifiable) conditions on  $\phi$  so that it has lower Fibonacci bound.

**Theorem 3.3.** *Suppose  $\phi \in (\Phi_\mu)_Q$  and  $\mu \neq 4$ . Then  $\log^+|\phi|$  has Fibonacci growth.*

Exactly as in [5], the argument for proving the lower Fibonacci bound in Theorem 3.3 can be applied to a single branch of the tree  $\Sigma$ .

**Proposition 3.4.** *(See Proposition 3.9 in [5].) Suppose  $\vec{e}$  is a directed edge of  $\Sigma$  such that  $\Omega^{0-}(\vec{e}) \cap \Omega_\phi(2)$  is finite and  $\Omega^{0-}(\vec{e}) \cap \phi^{-1}[-2, 2] = \emptyset$ . Then  $\log^+|\phi|$  has lower Fibonacci bound on  $\Omega^{0-}(\vec{e})$ .*

For  $\phi \in (\Phi_\mu)_Q$ , we have the following version of the generalized McShane’s identity, which is a reformulation of our first main result, Theorem 2.2 in terms of generalized Markoff maps.

**Theorem 3.5.** *If  $\phi \in (\Phi_\mu)_Q$  ( $\mu \neq 0, 4$ ) then*

$$\sum_{X \in \Omega} \mathfrak{h}(\phi(X)) = \nu \pmod{2\pi i}, \tag{18}$$

where  $\nu = \cosh^{-1}(1 - \mu/2)$  and  $\mathfrak{h} = \mathfrak{h}_\tau : \mathbb{C} \setminus \{\pm\sqrt{\mu}\} \rightarrow \mathbb{C}$  is defined as in Section 2 with  $\tau = \mu - 2$ . Moreover, the sum converges absolutely.

There is a version, Proposition 3.6, of Theorem 3.5 applicable to a single branch of  $\Sigma$ . We first need to define the edge weight  $\psi(\vec{e}) = \psi_\phi(\vec{e})$  and a specific half  $\hat{\mathfrak{h}}(x)$  of  $\mathfrak{h}(x)$  as follows.

**The function  $\hat{\mathfrak{h}}$ .** Given  $\mu \in \mathbb{C}$  with  $\mu \neq 0, 4$ , define  $\hat{\mathfrak{h}} : \mathbb{C} \setminus \{0, \pm\sqrt{\mu}\} \rightarrow \mathbb{C}$  by

$$\hat{\mathfrak{h}}(x) = \log \frac{1 + (e^\nu - 1)h(x)}{\sqrt{1 - \mu/x^2}} \in \mathbb{C}, \tag{19}$$

where  $\nu = \cosh^{-1}(1 - \mu/2)$ . It is easy to check that for all  $x \in \mathbb{C} \setminus \{0, \pm\sqrt{\mu}\}$ ,

$$2\hat{\mathfrak{h}}(x) = \mathfrak{h}(x) \pmod{2\pi i}. \tag{20}$$

**The edge weight  $\psi(\vec{e})$ .** Recall that we denote a directed edge  $\vec{e} \in \vec{E}(\Sigma)$  by  $\vec{e} = (X, Y; \rightarrow Z)$  if  $X, Y$  and  $Z$  are regions such that  $e = X \cap Y$  and  $\vec{e}$  points towards  $Z$ . For a fixed  $\phi \in \Phi_\mu$  and a directed edge  $\vec{e} = (X, Y; \rightarrow Z)$  with  $x, y \neq 0, \pm\sqrt{\mu}$ , we define the  $\phi$ -weight  $\psi(\vec{e})$  by

$$\psi(\vec{e}) = \psi_\phi(\vec{e}) = \log \frac{1 + (e^\nu - 1)(z/xy)}{\sqrt{1 - \mu/x^2}\sqrt{1 - \mu/y^2}}. \tag{21}$$

Note that  $\psi(\vec{e}) = \Psi(x, y, z)$  where the function  $\Psi$  is defined later in Section 4.

**Proposition 3.6.** *Suppose  $\vec{e} = (X, Y; \rightarrow Z)$  is a directed edge of  $\Sigma$  such that  $\Omega^{0-}(\vec{e}) \cap \Omega_\phi(2)$  is finite and  $\Omega^{0-}(\vec{e}) \cap \phi^{-1}[-2, 2] = \emptyset$ . Then*

$$\psi(\vec{e}) = \sum_{X \in \Omega^0(e)} \hat{\mathfrak{h}}(\phi(X)) + \sum_{X \in \Omega^-(\vec{e})} \mathfrak{h}(\phi(X)) \pmod{2\pi i}, \tag{22}$$

where the infinite sum converges absolutely.

Some other dynamical properties of  $\Phi_\mu$  corresponding to those in §§4–5 of [5] for Markoff maps will be discussed in a future paper [33].

**Examination and extension of Bowditch’s arguments.** For the rest of this section, we fix on one  $\mu$ -Markoff map,  $\phi$ , where  $\mu \neq 0, 4$ .

We make the *assumption* that  $\phi^{-1}(0) = \emptyset$ , which is not essential, to simplify the exposition. Note that the assumption is true for  $\phi \in (\Phi_\mu)_Q$ .

We shall also adopt the following *convention* of Bowditch [5]: We use upper case Latin letters for elements of  $\Omega$ , and the corresponding lower case letters for the values assigned to them by  $\phi$ ; that is,  $x = \phi(X)$ ,  $y = \phi(Y)$ , etc.

The above convention and assumption will allow us to write the edge relation, (15), in the following convenient form:

$$\frac{z}{xy} + \frac{w}{xy} = 1. \tag{23}$$

**Arrows assigned by a  $\mu$ -Markoff map.** As Bowditch did in [5], we may use  $\phi \in \Phi_\mu$  to assign to each undirected edge,  $e$ , a particular directed edge,  $\alpha_\phi(e)$ , with underlying edge  $e$ . Suppose  $e \leftrightarrow (X, Y; Z, W)$ . If  $|z| > |w|$  then the arrow on  $e$  points towards  $W$ ; in other words,  $\alpha_\phi(e) = (X, Y; Z \rightarrow W)$ . Note that the statement is equivalent to  $\Re(\frac{z}{xy}) > \frac{1}{2}$ . In particular, it implies that  $2|z| > |xy|$ . If  $|z| < |w|$ , we put an arrow on  $e$  pointing towards  $Z$ , that is,  $\alpha_\phi(e) = (X, Y; W \rightarrow Z)$ . If it happens that  $|z| = |w|$  then we choose  $\alpha_\phi(e)$  arbitrarily.

The following elementary lemma, which is Lemma 3.2(3) of [5], is fundamental. It generalizes to  $\mu$ -Markoff maps as the proof only uses the edge relation (15).

**Lemma 3.7.** (See Lemma 3.2(3), [5].) *Suppose  $X, Y, Z \in \Omega$  meet at a vertex  $v \in V(\Sigma)$ , and that the arrows on the edges  $X \cap Y$  and  $X \cap Z$  both point away from  $v$ . Then either  $|x| \leq 2$ , or  $y = z = 0$ .*

**Proof.** Let  $y'$  be the value of  $\phi$  on the region opposite  $Y$ . Then from the direction of the arrow,  $2|y| \geq |y| + |y'| \geq |y + y'| = |xz|$ . Similarly,  $2|z| \geq |xy|$ , from which the conclusion follows by summing the two inequalities.  $\square$

**Proof of Theorem 3.1(2).** This is now a simple exercise. Assume that the result is false, choose a minimal path on  $\Sigma$  joining two connected components. If the path consists of only one edge, we get a contradiction using (15). If there is more than one edge on this connecting path, we observe that the two ends point outwards, and we derive a contradiction using Lemma 3.7, see [5] for details.  $\square$

Note that Lemma 3.2(1) [5] (which states that there are no sources for  $\phi \in \Phi_0 \setminus \{0\}$ ) no longer holds for arbitrary generalized Markoff maps. For example, if  $x \in [-2, 2]$ , then the generalized Markoff map which corresponds to the triple  $(x, x, x)$  at vertex  $v$  has a source at  $v$ . On the other hand, Lemma 3.2(2) of [5] can be modified as follows.

**Lemma 3.8.** (See Lemma 3.2(2) in [5].) *There is a constant  $m(\mu) > 0$  such that if three regions  $X, Y, Z$  meet at a sink, then*

$$\min\{|x|, |y|, |z|\} \leq m(\mu). \tag{24}$$

**Proof.** We show that if  $|x|, |y|, |z|$  are all sufficiently large then the vertex,  $v$ , that the regions  $X, Y, Z$  meet cannot be a sink. We may assume  $x, y, z \neq 0$  and  $|x| \leq |y| \leq |z|$ . We can rewrite (14) as:

$$\frac{z}{xy} + \frac{x}{yz} + \frac{y}{zx} = 1 + \frac{\mu}{xyz}, \tag{25}$$

hence

$$\Re\left(\frac{z}{xy}\right) + \Re\left(\frac{x}{yz}\right) + \Re\left(\frac{y}{zx}\right) = 1 + \Re\left(\frac{\mu}{xyz}\right). \tag{26}$$

Since  $|x|, |y|, |z|$  are all sufficiently large and  $\mu$  is fixed,  $|\frac{x}{yz}|, |\frac{y}{zx}|$  and  $|\frac{\mu}{xyz}|$  are all sufficiently small and so are  $|\Re(\frac{x}{yz})|, |\Re(\frac{y}{zx})|$  and  $|\Re(\frac{\mu}{xyz})|$ . It follows that  $\Re(\frac{z}{xy}) > \frac{1}{2}$ . Hence the arrow on the edge  $X \cap Y$  is directed away from  $v$  and therefore  $v$  is not a sink.  $\square$

It seems difficult to determine the optimal  $m(\mu)$  for general  $\mu$ , although it was proven in Lemma 3.8, [5] that the optimal  $m(0) = 3$ .

**Neighbors around a region.** For each  $X \in \Omega$ , its boundary  $\partial X$  is a bi-infinite path consisting of a sequence of edges of the form  $X \cap Y_n$ , where  $(Y_n)_{n \in \mathbf{Z}}$  is a bi-infinite sequence of complementary regions. Let  $x = \lambda + \lambda^{-1}$  where  $|\lambda| \geq 1$ . If  $x = 2$ , then the vertex relation tells us that  $y_{n+1} = y_n \pm \sqrt{\mu - 4}$ , and the edge relation tells us that  $y_{n+1} - y_n = y_n - y_{n-1}$ , hence the  $\pm$  sign is constant in  $n$ . Similarly, if  $x = -2$ , then  $y_{n+1} = -y_n \pm \sqrt{\mu - 4}$ , but this time,  $y_{n+1} - y_n = -(y_n - y_{n-1})$ , hence the  $\pm$  sign alternates in  $n$ . If  $x = \pm\sqrt{\mu}$ , then  $y_{n+1} = (1/2)(x \pm \sqrt{\mu - 4})y_n$  where the  $\pm$  sign is constant in  $n$ . If  $x \notin \{\pm 2, \pm\sqrt{\mu}\}$  then there are constants  $A, B \in \mathbf{C} \setminus \{0\}$  with  $AB = (x^2 - \mu)/(x^2 - 4)$  such that  $y_n = A\lambda^n + B\lambda^{-n}$ . Note that  $|\lambda| = 1$  if and only if  $x \in [-2, 2] \subseteq \mathbf{R}$ . Hence we deduce that the following holds. (This is Corollary 3.3 in [5] in the case  $\mu = 0$ .)

**Lemma 3.9.** *Suppose that  $X \in \Omega$  has neighboring regions  $Y_n, n \in \mathbf{Z}$ . Then*

- (1) *If  $x \notin [-2, 2] \cup \{\pm\sqrt{\mu}\}$ , then  $|y_n|$  grows exponentially as  $n \rightarrow \infty$  and as  $n \rightarrow -\infty$ .*
- (2) *If  $x \in (-2, 2)$ , then  $|y_n|$  remains bounded.*
- (3) *If  $x = 2$ , then either  $y_n = y_0 + n\sqrt{\mu - 4}$  for all  $n$ , or  $y_n = y_0 - n\sqrt{\mu - 4}$  for all  $n$ .*
- (4) *If  $x = -2$ , then either  $y_n = (-1)^n(y_0 + n\sqrt{\mu - 4})$  for all  $n$ , or  $y_n = (-1)^n(y_0 - n\sqrt{\mu - 4})$  for all  $n$ .*
- (5) *If  $x = \pm\sqrt{\mu}$ , then either  $y_n = y_0[(x + \sqrt{\mu - 4})/2]^n$  for all  $n$ , or  $y_n = y_0[(x - \sqrt{\mu - 4})/2]^n$  for all  $n$ .*

Using part (5) of Lemma 3.9, we can show:

**Lemma 3.10.** *If  $\phi \in \Phi_\mu$  and  $\phi(X) = \pm\sqrt{\mu}$  for some  $X \in \Omega$ , then  $\phi \notin (\Phi_\mu)_Q$ .*

**Proof.** Without loss of generality, we may assume that  $y_n = y_0[(x + \sqrt{\mu - 4})/2]^n$  for all  $n$ . If  $y_0 = 0$  then  $\phi \notin (\Phi_\mu)_Q$ . Suppose  $y_0 \neq 0$  and let  $\lambda = (x + \sqrt{\mu - 4})/2$ . Then it is easy to see that  $\lambda \neq 0$ . If  $|\lambda| = 1$ , then  $\lambda^{-1} = (x - \sqrt{\mu - 4})/2$ . Hence  $x = \lambda + \lambda^{-1} \in [-2, 2]$ , again  $\phi \notin (\Phi_\mu)_Q$ . Now suppose  $|\lambda| \neq 1$ . If  $|\lambda| < 1$  then  $|y_n| \rightarrow 0$  as  $n \rightarrow \infty$ , whereas if  $|\lambda| > 1$  then  $|y_n| \rightarrow 0$  as  $n \rightarrow -\infty$ . Hence  $\Omega_\phi(2)$  is infinite and  $\phi \notin (\Phi_\mu)_Q$ .  $\square$

The following lemma, together with Lemma 3.7, will imply that if  $\Omega_\phi(2) = \emptyset$  then there will be a sink.



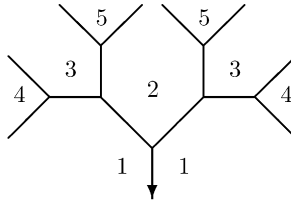


Fig. 3. Some values of the Fibonacci function  $F_e$ .

**Lemma 3.11.** (See Lemma 3.4 in [5].) Suppose  $\beta$  is an infinite ray in  $\Sigma$  consisting of a sequence,  $(e_n)_{n \in \mathbb{N}}$ , of edges of  $\Sigma$  such that the arrow on each  $e_n$  assigned by  $\phi$  is directed towards  $e_{n+1}$ . Then  $\beta$  meets at least one region  $X$  with  $|\phi(X)| < 2$ .

**Proof.** The same proof used for Lemma 3.4 of [5] works here with a slight refinement for its last part as follows. Suppose that all regions  $X$  which are incident to  $\beta$  have  $|\phi(X)| \geq 2$ . Then by the argument there, for  $n$  sufficiently large, there are directed edges  $\alpha_\phi(e_n) = (X, Y; Z \rightarrow W)$  in  $\beta$  with  $X \cap Z \subseteq \beta$  and  $Y \cap W \subseteq \beta$  so that (where  $\simeq$  means ‘arbitrarily close to’)  $z/xy \simeq 1/2$ ,  $x^2 \simeq 4$  and  $y^2 \simeq 4$ . Hence  $z^2 \simeq 4$  and  $xyz \simeq 2z^2 \simeq 8$ . Thus  $\mu = x^2 + y^2 + z^2 - xyz \simeq 4$  which is impossible since we have assumed that  $\mu \neq 4$ .  $\square$

Note that if the path  $\beta$  in Lemma 3.11 above does not eventually lie in the boundary of a fixed region, then  $\beta$  meets infinitely many regions  $X$  with  $|\phi(X)| < 2$ .

**Proof of Theorem 3.1(1).** If  $\Omega_\phi(2) = \emptyset$ , then Lemmas 3.7 and 3.11 tell us that there must be a sink. We now apply Lemma 3.8. This proves Theorem 3.1(1).  $\square$

**Fibonacci function.** We recall now the definition given in [5] for the Fibonacci function  $F_e$  associated to an edge  $e \in E(\Sigma)$ , and the definition for a function  $f: \Omega \rightarrow [0, \infty)$  to have an upper/lower Fibonacci bound.

Let  $\vec{e} \in \vec{E}(\Sigma)$  with underlying edge  $e$ . For  $X \in \Omega^{0-}(\vec{e})$  we define  $d(X) = d_{\vec{e}}(X)$  to be the number of edges in the shortest path joining the head of  $\vec{e}$  to  $X$ . Given any  $Z \in \Omega^-(\vec{e})$ , there are precisely two regions  $X, Y \in \Omega^{0-}(\vec{e})$  meeting  $Z$  and satisfying  $d(X) < d(Z)$  and  $d(Y) < d(Z)$ . Note that  $X, Y, Z$  all meet in a vertex.

Now we can define the *Fibonacci function*  $F_e: \Omega \rightarrow \mathbb{N}$  with respect to an edge  $e$  as follows. We orient  $e$  arbitrarily as  $\vec{e}$  and define  $F_{\vec{e}}: \Omega^{0-}(\vec{e}) \rightarrow \mathbb{N}$  by  $F_{\vec{e}}(W) = 1$  for  $W \in \Omega^0(e)$  and  $F_{\vec{e}}(Z) = F_{\vec{e}}(X) + F_{\vec{e}}(Y)$  for  $Z \in \Omega^-(\vec{e})$  where  $X, Y \in \Omega^{0-}(\vec{e})$  are the two regions meeting  $Z$  and satisfying  $d(X) < d(Z)$  and  $d(Y) < d(Z)$ . Now we define  $F_e(X) = F_{\vec{e}}(X)$  for  $X \in \Omega^{0-}(\vec{e})$  and  $F_e(X) = F_{-\vec{e}}(X)$  for  $X \in \Omega^+(\vec{e})$ . See Fig. 3 where the values  $F_e(X)$  are given for some  $X \in \Omega$ .

The functions  $F_e$  provide a means for measuring the growth rates of functions defined on subsets of  $\Omega$ . The following lemma can be easily proved by induction. Its corollary shows that the concept of upper and lower Fibonacci bound is independent of the edge  $e$  used.

**Lemma 3.12.** (See Lemma 2.1.1 in [5].) Suppose  $f: \Omega^{0-}(\vec{e}) \rightarrow [0, \infty)$  where  $\Omega^0(e) = \{X_1, X_2\}$ .

- (1) If  $f$  satisfies  $f(Z) \leq f(X) + f(Y) + c$  for some fixed constant  $c$  and arbitrary  $X, Y, Z \in \Omega^{0-}(\bar{e})$  meeting at a vertex and satisfying  $d(X) < d(Z)$  and  $d(Y) < d(Z)$ , then  $f(X) \leq (M + c)F_e(X) - c$  for all  $X \in \Omega^{0-}(\bar{e})$ , where  $M = \max\{f(X_1), f(X_2)\}$ .
- (2) If  $f$  satisfies  $f(Z) \geq f(X) + f(Y) - c$  for some fixed constant  $c$  where  $0 < c < m = \min\{f(X_1), f(X_2)\}$  and arbitrary  $X, Y, Z$  as in part (1), then  $f(X) \geq (m - c)F_e(X) + c$  for all  $X \in \Omega^{0-}(\bar{e})$ .

**Corollary 3.13.** (See Corollary 2.1.2 in [5].) Suppose  $f : \Omega \rightarrow [0, \infty)$  satisfies an inequality of the form  $f(Z) \leq f(X) + f(Y) + c$  for some fixed constant  $c$ , whenever  $X, Y, Z \in \Omega$  meet at a vertex. Then for any given edge  $e \in E(\Sigma)$ , there is a constant  $K > 0$ , such that  $f(X) \leq KF_e(X)$  for all  $X \in \Omega$ .

Note that for any edge  $e' \in E(\Sigma)$ ,  $f = F_{e'}$  satisfies the hypotheses of Corollary 3.13, with  $c = 0$ . Thus for any two edges  $e, e' \in E(\Sigma)$ , there is some constant  $K = K(e, e') > 0$  such that

$$K^{-1}F_e(X) \leq F_{e'}(X) \leq KF_e(X)$$

for all  $X \in \Omega$ . Hence the properties in the following definitions are independent of the choices the edge  $e \in E(\Sigma)$ .

**Fibonacci bounds.** Suppose  $f : \Omega \rightarrow [0, \infty)$ , and  $\Omega' \subseteq \Omega$ . We say that

$f$  has an upper Fibonacci bound on  $\Omega'$  if there is some constant  $\kappa > 0$  such that  $f(X) \leq \kappa F_e(X)$  for all  $X \in \Omega'$ ;

$f$  has a lower Fibonacci bound on  $\Omega'$  if there is some constant  $\kappa > 0$  such that  $f(X) \geq \kappa F_e(X)$  for all but finitely many  $X \in \Omega'$ ;

$f$  has Fibonacci growth on  $\Omega'$  if it has both upper and lower Fibonacci bounds on  $\Omega'$ ; and

$f$  has Fibonacci growth if  $f$  has Fibonacci growth on all of  $\Omega$ .

Note that if  $\Omega'$  is the union of a finite set of subsets  $\Omega_1, \dots, \Omega_m \subseteq \Omega$ , then  $f$  has an upper (lower) Fibonacci bound if and only if it has an upper (lower) Fibonacci bound on each of  $\Omega_i$ .

**Upper Fibonacci bounds.** The following lemma tells us that for an arbitrary  $\mu$ -Markoff map  $\phi$ , the function  $\log^+ |\phi|$  always has an upper Fibonacci bound on  $\Omega$ . Hence we only need to consider criteria for it to have a lower Fibonacci bound on certain branches of the binary tree  $\Sigma$  to determine if it has Fibonacci growth.

**Lemma 3.14.** If  $\phi \in \Phi_\mu$ , then  $\log^+ |\phi|$  has an upper Fibonacci bound on  $\Omega$ .

**Proof.** By Corollary 3.13 we only need to show that for an arbitrary  $\mu$ -Markoff map  $(x, y, z)$

$$\log^+ |z| \leq \log 4 + \log^+ |\mu| + \log^+ |x| + \log^+ |y|. \tag{27}$$

If  $|z| \leq 2|x|$  or  $|z| \leq 2|y|$  then (27) holds already. So we suppose  $|z| \geq 2|x|$  and  $|z| \geq 2|y|$ . Then since  $\mu + xyz = x^2 + y^2 + z^2$  we have

$$\begin{aligned} |\mu| + |xyz| &\geq |z|^2 - |x|^2 - |y|^2 \\ &= |z|^2/2 + (|z|^2/4 - |x|^2) + (|z|^2/4 - |y|^2) \\ &\geq |z|^2/2. \end{aligned}$$

Hence  $|z|^2 \leq 4|xyz|$  or  $|z|^2 \leq 4|\mu|$  according to whether  $|\mu| \leq |xyz|$  or  $|xyz| \leq |\mu|$ . Thus

$$|z| \leq 4|xy| \quad \text{or} \quad |z| \leq |z|^2 \leq 4|\mu|$$

(since we may assume  $|z| \geq 1$ ) from which (27) follows easily.  $\square$

**Lower Fibonacci bounds.** The lower Fibonacci bounds are more interesting since, as the following proposition shows, they give the convergence of certain series, in particular, the absolute convergence of series on the left-hand side of (18), since we have  $|h(x)| = O(|x|^{-2})$  as  $|x| \rightarrow \infty$ .

**Proposition 3.15.** (See Proposition 2.1.4 in [5].) *If  $f : \Omega \rightarrow [0, \infty)$  has a lower Fibonacci bound, then  $\sum_{X \in \Omega} f(X)^{-s}$  converges for all  $s > 2$  (after excluding a finite subset of  $\Omega$  on which  $f$  takes the value 0).*

Assuming Theorem 3.3, we have the following:

**Corollary 3.16.** *If  $\phi \in (\Phi_\mu)_Q$ , then for any  $t > 0$ , the series  $\sum_{X \in \Omega} |\phi(X)|^t$  converges absolutely.*

The following lemma and corollary of [5] hold with the same proofs there, giving a criterion for  $\log^+ |\phi|$  to have a lower Fibonacci bound on certain branches of  $\Sigma$ .

**Lemma 3.17.** (See Lemma 3.5 and Corollary 3.6 in [5].) *Suppose  $\vec{e} \in \vec{E}(\Sigma)$  is such that  $\alpha_\phi(e) = \vec{e}$  and  $\Omega^0(e) \cap \Omega_\phi(2) = \emptyset$ . Then  $\Omega^{0-}(\vec{e}) \cap \Omega_\phi(2) = \emptyset$  and the arrow on each edge of  $\Sigma^-$  is directed towards  $e$ .*

*Furthermore,  $\log |\phi(X)| \geq (m - \log 2)F_e(X)$  for all  $X \in \Omega^{0-}(\vec{e})$ , where  $m = \min\{\log |\phi(X)| \mid X \in \Omega^0(e)\} > \log 2$ .*

**Corollary 3.18.** (See Corollary 3.7 in [5].) *If  $\Omega_\phi(2) = \emptyset$ , then there is a unique sink, and  $\log^+ |\phi|$  has a lower Fibonacci bound.*

To prove Theorem 3.3 we expand somewhat on Lemma 3.17 and consider the case where  $\Omega^-(\vec{e}) \cap \Omega_\phi(2) = \emptyset$  and exactly one of the two regions in  $\Omega^0(e)$  has norm no greater than 2.

**Lemma 3.19.** (See Lemma 3.8 in [5].) *Suppose  $\vec{e} \in \vec{E}(\Sigma)$  is such that  $\alpha_\phi(e) = \vec{e}$  and  $\Omega^{0-}(\vec{e}) \cap \Omega_\phi(2) = \{X_0\}$  where  $X_0 \in \Omega^0(\vec{e})$  with  $x_0 \notin [-2, 2]$ . Then  $\log^+ |\phi|$  has a lower Fibonacci bound on  $\Omega^{0-}(\vec{e})$ .*

**Proof.** The proof of Lemma 3.8 in [5] applies. We repeat it here. Let  $(\vec{e}_n)_{n=0}^\infty$  be the sequence of directed edges lying in the boundary of  $X_0$  and in  $\Omega^{0-}(\vec{e})$  so that  $\vec{e}_0 = \vec{e}$  and  $\vec{e}_n$  is directed away from  $\vec{e}_{n+1}$ . For  $n \geq 1$ , let  $v_n$  be the vertex incident on both  $e_{n-1}$  and  $e_n$ , and let  $\vec{e}_n$  be the third edge (distinct from  $e_{n-1}$  and  $e_n$ ) incident on  $v_n$  and directed towards  $v_n$ . For  $n \geq 0$ , let  $Y_n$  be the region such that  $Y_n \cap X_0 = e_n$ . See Fig. 4 for an illustration. Thus  $\Omega^{0-}(\vec{e}) = \{X_0\} \cup \bigcup_{n=1}^\infty \Omega^{0-}(\vec{e}_n)$ .

By Lemma 3.9,  $|y_n|$  grows exponentially as  $n \rightarrow \infty$ , and so  $\log |y_n| \geq cn$  for some  $c > 0$ . Hence we have  $\log^+ |\phi(X)| \geq cnF_{\vec{e}_n}(X)$  for all  $n \geq 1$  and for all  $X \in \Omega^{0-}(\vec{e}_n)$ . Thus it follows easily (using Lemma 3.12(2)) that  $\log^+ |\phi|$  has a lower Fibonacci bound on  $\Omega^{0-}(\vec{e})$ .  $\square$

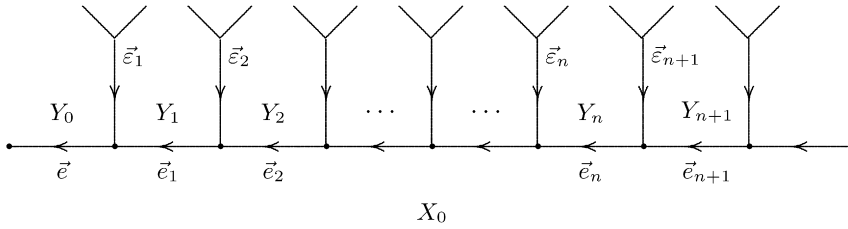


Fig. 4.

**Proof of Theorem 3.3.** The proof of Theorem 3.3 is then the same as that of Theorem 2 in [5]. We sketch it as follows. By Lemma 3.14, we only need to show that  $\log^+ |\phi|$  has a lower Fibonacci bound on  $\Omega$ . If  $\Omega_\phi(2)$  has at most one element, the conclusion follows easily by Corollary 3.18 and Lemma 3.19. Hence we suppose  $\Omega_\phi(2)$  has at least two elements.

Recall that  $\Omega_\phi(2) \subseteq \Omega$  is finite and  $\bigcup \Omega_\phi(2)$  is connected. Let  $T$  be the (finite) subtree of  $\Sigma$  spanned by the set of edges  $e$  such that  $\Omega^0(e) \subseteq \Omega_\phi(2)$ . Let  $C = C(T)$  be the circular set of directed edges given by  $T$ . Note that  $\Omega = \bigcup_{\vec{e} \in C} \Omega^{0-}(\vec{e})$ . Hence it suffices to show that  $\log^+ |\phi|$  has a lower Fibonacci bound on  $\Omega^{0-}(\vec{e})$  for every  $\vec{e} \in C$ . Then the conclusion of Theorem 3.3 follows by the following claim, Lemmas 3.17 and 3.19.

**Claim.** For each  $\vec{e} \in C$ , we have  $\vec{e} = \alpha_\phi(e)$ ,  $\Omega^-(\vec{e}) \cap \Omega_\phi(2) = \emptyset$  and  $\Omega^0(e) \cap \Omega_\phi(2)$  has at most one element.

To prove the claim, let  $\vec{e} = (X, Y; Z \rightarrow W) \in C(T)$ . If one of  $X$  and  $Y$ , say  $X$ , is in  $\Omega_\phi(2)$  then  $Y, Z \notin \Omega_\phi(2)$  and  $W \in \Omega_\phi(2)$  by the definition of  $T$ . Hence in this case  $\vec{e} = \alpha_\phi(e)$ ,  $\Omega^-(\vec{e}) \cap \Omega_\phi(2) = \emptyset$  and  $\Omega^0(e) \cap \Omega_\phi(2)$  has one element,  $X$ . Now suppose neither  $X$  nor  $Y$  is in  $\Omega_\phi(2)$  then  $W \in \Omega_\phi(2)$  and  $Z \notin \Omega_\phi(2)$  since  $\bigcup \Omega_\phi(2)$  is connected. Thus in this case  $\vec{e} = \alpha_\phi(e)$ ,  $\Omega^{0-}(\vec{e}) \cap \Omega_\phi(2) = \emptyset$ . This proves the claim, completing the proof of the theorem.  $\square$

**The definition of  $H(x)$ .** To prove Theorem 3.2, that is,  $(\Phi_\mu)_Q$  is an open subset of  $\Phi_\mu$ , we introduce a continuous function  $H : \mathbb{C} \setminus ([-2, 2] \cup \{\pm\sqrt{\mu}\}) \rightarrow \mathbf{R}_{>0}$  so that for any  $\phi \in \Phi_\mu$  and  $X \in \Omega$ , if  $(Y_n)_{n \in \mathbf{Z}}$  is the bi-infinite sequence of regions meeting  $X$ , then there are integers  $n_1 \leq n_2$  such that  $|y_n| \leq H(x)$  if and only if  $n_1 \leq n \leq n_2$ , and  $|y_n|$  is monotonically decreasing for  $n \in (-\infty, n_1]$  and monotonically increasing for  $n \in [n_2, \infty)$ .

**Lemma 3.20.** For  $x \neq \pm\sqrt{\mu}$  and  $x \notin [-2, 2]$ ,  $H(x)$  can be chosen as

$$H(x) = \sqrt{\left| \frac{x^2 - \mu}{x^2 - 4} \right|} \frac{2|\lambda(x)|^2}{|\lambda(x)| - 1}, \tag{28}$$

where

$$\lambda(x) = \frac{x}{2} \left( 1 + \sqrt{1 - \frac{4}{x^2}} \right) = e^{l(x)/2}.$$

**Proof.** Let us write  $\lambda = \lambda(x)$ . Then  $|\lambda| > 1$  since  $x \notin [-2, 2]$ . As explained before Lemma 3.9, there exist  $A, B$  with  $AB = (x^2 - \mu)/(x^2 - 4)$  such that  $y_n = A\lambda^n + B\lambda^{-n}$ . Note that since we

can replace  $A, B$  respectively by  $A\lambda^m, B\lambda^{-m}$  for  $m \in \mathbf{Z}$ , we may assume that  $\sqrt{|AB|}|\lambda|^{-1} \leq |A|, |B| \leq \sqrt{|AB|}|\lambda|$ , that is,

$$\sqrt{\left| \frac{x^2 - \mu}{x^2 - 4} \right|} |\lambda|^{-1} \leq |A|, |B| \leq \sqrt{\left| \frac{x^2 - \mu}{x^2 - 4} \right|} |\lambda|. \tag{29}$$

(Note that this argument fails for  $x = \pm\sqrt{\mu}$  since in that case we have  $AB = 0$ .)

In the rest of the proof we assume that  $n > 0$ .

**Claim.** If

$$|\lambda|^{2n-1} \geq \frac{|\lambda| + 1}{|\lambda| - 1}, \tag{30}$$

then  $|y_{n+1}| \geq |y_n|$  and  $|y_{-n-1}| \geq |y_{-n}|$ .

To prove the claim, note that

$$|y_{n+1}| \geq |A||\lambda|^{n+1} - |B||\lambda|^{-n-1} \quad \text{and} \quad |y_n| \leq |A||\lambda|^n + |B||\lambda|^{-n}.$$

Thus  $|y_{n+1}| \geq |y_n|$  if

$$|A||\lambda|^{n+1} - |B||\lambda|^{-n-1} \geq |A||\lambda|^n + |B||\lambda|^{-n}$$

or equivalently

$$|\lambda|^{2n+1} \geq \frac{|B|}{|A|} \frac{|\lambda| + 1}{|\lambda| - 1}.$$

Since  $|B|/|A| \leq |\lambda|^2$ , we have  $|y_{n+1}| \geq |y_n|$  if  $|\lambda|^{2n-1} \geq (|\lambda| + 1)/(|\lambda| - 1)$ . The other inequality  $|y_{-n-1}| \geq |y_{-n}|$  can be similarly proved.

We continue the proof of Lemma 3.20. Note that

$$\begin{aligned} |y_n| &\leq |A||\lambda|^n + |B||\lambda|^{-n} \\ &\leq \sqrt{\left| \frac{x^2 - \mu}{x^2 - 4} \right|} |\lambda| (|\lambda|^n + |\lambda|^{-n}) \\ &\leq \sqrt{\left| \frac{x^2 - \mu}{x^2 - 4} \right|} |\lambda| (|\lambda|^n + 1). \end{aligned}$$

Hence if

$$|y_n| \geq H(x) = \sqrt{\left| \frac{x^2 - \mu}{x^2 - 4} \right|} |\lambda| \left( \frac{|\lambda| + 1}{|\lambda| - 1} + 1 \right)$$

then  $|\lambda|^n \geq \frac{|\lambda| + 1}{|\lambda| - 1}$ . Hence (30) holds. It follows from the claim that  $|y_{n+1}| \geq |y_n|$ . Similarly, if  $|y_{-n}| \geq H(x)$  then  $|y_{-n-1}| \geq |y_{-n}|$ .  $\square$

**Remark.** We observe from the proof that we always have  $|y_0| < H(x)$ .

On the other hand, for  $x \in [-2, 2]$  or  $x = \pm\sqrt{\mu}$ , we set  $H(x) = \infty$ . By replacing  $H(x)$  by  $\max\{H(x), 2\}$ , we may assume that  $H(x) \geq 2$ . Hence  $H$  is continuous on  $\mathbb{C} \setminus \{\pm\sqrt{\mu}\}$  (but is not continuous at  $\pm\sqrt{\mu}$  if  $\mu \notin [0, 4]$ ).

**The definition of  $T(t)$ .** Now for each  $\phi \in \Phi_\mu$  and  $t \geq 0$ , we define a subtree  $T(t)$  of  $\Sigma$  as follows. First, for each  $X \in \Omega$  with  $x = \phi(X) \notin [-2, 2]$  and  $x \neq \pm\sqrt{\mu}$ , and for  $r \geq H(x) \geq 2$ , we set

$$J_r(X) = \bigcup \{X \cap Y_n \mid |y_n| \leq r\}.$$

It is a subarc of  $\partial X$  determined by  $\phi$  with the property that if  $e$  is any edge in  $\partial X$  not lying in  $J_r(X)$  then the arrow on  $e$  assigned by  $\phi$  points towards  $J_r(X)$ . Note that  $J_r(X)$  contains at least one edge by the observation we have just made after the proof of Lemma 3.20.

Now, for given  $t \geq 0$ , let  $T(t) = T_\phi(t)$  be the union of all the arcs  $J_{H(x)+t}(X)$  as  $X$  varies in  $\Omega(2+t) = \Omega_\phi(2+t)$ .

It can be shown (by the argument of Lemma 3.11, [5]) that  $T(t)$  is connected, hence a subtree of  $\Sigma$ . It follows that  $T(t)$  has the following nice property.

**Lemma 3.21.** *If  $T(t) \neq \emptyset$ , then the arrow on any edge not in the subtree  $T(t)$  points towards  $T(t)$ .*

Actually, by Theorem 3.1(1), we have

**Lemma 3.22.**  *$T(t) \neq \emptyset$  for any  $\phi \in \Phi_\mu$  and for any  $t \geq m(\mu) - 2$ , where  $m(\mu) > 2$  is a constant such that  $\Omega_\phi(m(\mu)) \neq \emptyset$  for all  $\phi \in \Phi_\mu$ .*

Alternatively, we can describe  $T(t)$  directly in terms of its edges.

**Lemma 3.23.** *An edge  $e = X \cap Y$  is an edge of  $T(t)$  if and only if either  $|x| \leq 2 + t$  and  $|y| \leq H(x) + t$  or  $|y| \leq 2 + t$  and  $|x| \leq H(y) + t$ .*

Thus we have the following lemma which gives a finite criterion for recognizing that a given  $\mu$ -Markoff map  $\phi$  lies in  $(\Phi_\mu)_Q$ . This generalizes Lemma 3.15 in [5], and the proof there also works here.

**Lemma 3.24.** *For any fixed  $t \geq 0$ ,  $\phi \in (\Phi_\mu)_Q$  if and only if  $T_\phi(t)$  is finite.*

Now we can give the proof of Theorem 3.2 by showing that this criterion is an open property.

**Proof of Theorem 3.2.** The proof is essentially the same as that of Theorem 3.16, [5], which states that  $(\Phi_0)_Q$  is an open subset of  $\Phi_0$ . Here we supply a bit more details.

Fix any  $t_1 > m(\mu) - 2$ . Suppose  $\phi \in (\Phi_\mu)_Q$  and write  $T(t)$  for  $T_\phi(t)$ . By Lemma 3.24,  $T(t_1)$  is a finite subtree of  $\Sigma$ . We may choose  $t_2 > t_1$  large enough so that  $T(t_2)$  contains  $T(t_1)$  in its interior, that is, it contains  $T(t_1)$ , together with all the edges of the circular set  $C(T(t_1))$ . Note that  $T(t_2)$  is also a finite subtree of  $\Sigma$ .

For any given  $\phi' \in \Phi_\mu$ , we write  $T'(t)$  for  $T_{\phi'}(t)$ .

**Claim.** If  $\phi'$  is sufficiently close to  $\phi$ , then  $T'(t_1) \cap T(t_2) \subseteq T(t_1)$ .

To prove the claim, choose  $e = X \cap Y \in T(t_2) \setminus T(t_1)$ . We may assume  $|\phi(X)| \leq 2 + t_2$  and  $|\phi(Y)| \leq H(x) + t_2$  since  $e \in T(t_2)$ . Then either  $|\phi(X)| > 2 + t_1$  or  $|\phi(Y)| > H(x) + t_1 \geq 2 + t_1$  since  $e \notin T(t_1)$ . Thus if  $\phi'$  is sufficiently close to  $\phi$ , we have either  $|\phi'(X)| > 2 + t_1$  or  $|\phi'(Y)| > 2 + t_1$ , hence  $e \notin T'(t_1)$ . This proves the claim since there are only finitely many edges  $e$  in  $T(t_2) \setminus T(t_1)$ .

Since  $T(m(\mu) - 2)$  is a non-empty subtree of  $T(t_2)$  and  $t_1 > m(\mu) - 2$ , it follows that  $T'(t_1) \cap T(t_2) \supseteq T(m(\mu) - 2) \neq \emptyset$ , provided that  $\phi'$  is sufficiently close to  $\phi$ . Since  $T'(t_1) \cap T(t_2) \subseteq T(t_1)$  and  $T(t_1)$  is contained in the interior of  $T(t_2)$ , we know that  $T(t_2)$  contains a connected component of  $T'(t_1)$ . Since  $T'(t_1)$  is connected, we must have  $T'(t_1) \subseteq T(t_2)$ . Therefore  $T'(t_1)$  is finite, and so  $\phi' \in (\Phi_\mu)_Q$ . This proves Theorem 3.2.  $\square$

**Proof of Theorem 2.3.** This is equivalent to showing that  $\text{PSL}(2, \mathbf{Z})$  acts properly discontinuously on  $(\Phi_\mu)_Q$ , that is, for any compact subset  $K$  of  $(\Phi_\mu)_Q$ , the set  $\{H \in \text{PSL}(2, \mathbf{Z}) \mid HK \cap K \neq \emptyset\}$  is finite. Suppose not. Then there exists a sequence of distinct  $H_i \in \text{PSL}(2, \mathbf{Z})$  and  $\phi_i \in K$  such that  $H_i(\phi_i) \in K$ . Passing to a subsequence, by the compactness of  $K$ , we may assume that  $\phi_i \rightarrow \phi \in K$ ,  $H_i \rightarrow \infty$ , and  $H_i(\phi_i) \rightarrow \phi' \in K$ . Now, as in the proof of Theorem 3.2, we have that the tree  $T_\phi(t_1)$  of  $\phi$  is finite for some  $t_1 > 0$ , and that  $T_{\phi_i}(t_1)$  is contained in the finite tree  $T_\phi(t_2)$ , for some  $t_2 > t_1$  and for all  $i$  sufficiently large. This implies that the same constant  $\kappa$  can be used in the lower Fibonacci bound for all  $\phi_i$  for  $i$  sufficiently large, and hence  $H_i(\phi_i) \rightarrow \infty$  as  $i \rightarrow \infty$  (where we use the identification of  $\Phi_\mu$  with the character variety  $\{(x, y, z) \in \mathbf{C}^3 \mid x^2 + y^2 + z^2 - xyz = \mu\}$  to make sense of  $\phi \rightarrow \infty$ ). This contradicts  $H_i(\phi_i) \rightarrow \phi' \in K$ .  $\square$

**Proof of Proposition 2.4.** Again we prove the equivalent statement for the action of  $\text{PSL}(2, \mathbf{Z})$  on  $\Phi_\mu$ . Suppose that  $\phi \in \Phi_\mu$  does not satisfy the extended BQ-conditions, then either  $\phi(X) \in (-2, 2)$  for some  $X \in \Omega$  or  $\Omega_\phi(2)$  is infinite. In the first case, since the values of  $\phi$  on the neighbors of  $X$  are bounded by Lemma 3.9(2), we can find an infinite sequence  $H_i$  of parabolic elements in  $\text{PSL}(2, \mathbf{Z})$  all fixing  $X$  such that  $H_i(\phi)$  converges to  $\phi_0 \in \Phi_\mu$  (here it is possible that  $H_i(\phi) = \phi_0$  for all  $i$ ). In the latter case, there are infinitely many edges  $e \in E(\Sigma)$  such that  $\Omega^0(e) \subset \Omega_\phi(2)$ . Passing to a subsequence if necessary, this implies that there is an infinite sequence of vertices  $v_i \in V(\Sigma)$  such that if  $X_i, Y_i$  and  $Z_i$  are the regions in  $\Omega$  meeting at  $v_i$ , then  $(x_i, y_i, z_i)$  converges to  $(x_0, y_0, z_0)$  satisfying (14), hence there exists an infinite sequence of distinct  $H_i$  such that again  $H_i(\phi)$  converges.  $\square$

**Remark.** Note that for  $\phi$  satisfying the extended BQ-conditions, for any sequence of distinct elements  $H_i$  in  $\text{PSL}(2, \mathbf{Z})$ ,  $H_i(\phi) \rightarrow \infty$ .

#### 4. Proof of Theorem 3.5

In this section we prove Theorem 3.5 by defining the  $\phi$ -weight  $\psi(\vec{e})$  for each directed edge  $\vec{e} \in \vec{E}(\Sigma)$  and then following through the proof of Theorem 2 in [5]. The geometric meaning of  $\psi(\vec{e})$  is given in Appendix A, the reader interested in the geometric motivation for the definition of  $\psi$  is advised to look there first.

**The functions  $l$  and  $h$  again.** For  $x \in \mathbf{C}$ , recall that  $l(x)/2 \in \mathbf{C}/2\pi i\mathbf{Z}$  is defined by  $l(x)/2 = \cosh^{-1}(x/2)$ . In particular,  $\Re(l(x)/2) \geq 0$ .

**Lemma 4.1.** *If  $x \notin [-2, 2]$  then*

$$e^{-l(x)/2} = xh(x). \tag{31}$$

**Proof.** Note that  $\Re(l(x)/2) > 0$  since  $x \notin [-2, 2]$ . Hence  $|e^{l(x)/2}| > |e^{-l(x)/2}|$  and

$$|x^{-1}e^{l(x)/2}| > |x^{-1}e^{-l(x)/2}|.$$

On the other hand,  $\cosh(l(x)/2) = x/2$  implies that

$$e^{l(x)/2} + e^{-l(x)/2} = x.$$

Hence

$$\left(\frac{e^{l(x)/2}}{x}\right)^2 - \left(\frac{e^{l(x)/2}}{x}\right) + \frac{1}{x^2} = 0$$

and

$$\left(\frac{e^{-l(x)/2}}{x}\right)^2 - \left(\frac{e^{-l(x)/2}}{x}\right) + \frac{1}{x^2} = 0.$$

Then  $|x^{-1}e^{-l(x)/2}| < |x^{-1}e^{l(x)/2}|$  implies that

$$x^{-1}e^{-l(x)/2} = \frac{1}{2}\left(1 - \sqrt{1 - \frac{4}{x^2}}\right) = h(x).$$

This proves Lemma 4.1.  $\square$

**The function  $\Psi$ .** We define a function

$$\Psi : \mathbf{C}^3 \rightarrow \mathbf{C}$$

as follows. Given  $x, y, z \in \mathbf{C}$ , set

$$\mu = x^2 + y^2 + z^2 - xyz \quad \text{and} \quad \nu = \cosh^{-1}(1 - \mu/2).$$

Then  $\Psi(x, y, z) \in \mathbf{C}$  is defined by

$$\Psi(x, y, z) = \log \frac{xy + (e^\nu - 1)z}{(x^2 - \mu)^{1/2}(y^2 - \mu)^{1/2}}, \tag{32}$$

or equivalently, by the following two equations:



$$\cosh \Psi(x, y, z) = \frac{xy - (\mu/2)z}{(x^2 - \mu)^{1/2}(y^2 - \mu)^{1/2}}, \tag{33}$$

$$\sinh \Psi(x, y, z) = \frac{(\sinh v)z}{(x^2 - \mu)^{1/2}(y^2 - \mu)^{1/2}}. \tag{34}$$

**Remarks.**

(i) **Notational convention.** In this paper, for a complex number  $u \in \mathbf{C}$ , we use the notation  $u^{1/2}$  (as opposed to  $\sqrt{u}$ ) to mean either (once and for all) choice of one of the two square roots of  $u$ .

(ii) Note that if  $x^2, y^2 \neq \mu$  then  $xy + (e^v - 1)z = [xy - (\mu/2)z] + [(\sinh v)z] \neq 0$  since

$$[xy - (\mu/2)z]^2 = (x^2 - \mu)(y^2 - \mu) + [(\sinh v)z]^2. \tag{35}$$

Hence  $\Psi(x, y, z)$  is defined if  $x^2, y^2 \neq \mu$ , or equivalently,  $y^2 + z^2 \neq xyz$  and  $x^2 + z^2 \neq xyz$ .

(iii) Note that the value  $\Psi(x, y, z)$  depends on the choices of the square roots in its expression, thus it is only well-defined modulo  $\pi i$  without specifying the choices of the square roots. However, in Proposition 4.2 below, the appropriate sums there do not depend on the choices of square roots and hence are well-defined modulo  $2\pi i$  with arbitrary (once and for all) choices of the square roots.

(iv) It can be checked that

$$\frac{\partial}{\partial v} \Big|_{v=0} 2\Psi(x, y, z) = \frac{z}{xy}. \tag{36}$$

**Some properties of the function  $\Psi$ .** The function  $\Psi$  defined above has the following nice properties.

**Proposition 4.2.** Let  $\mu = x^2 + y^2 + z^2 - xyz$ ,  $v = \cosh^{-1}(1 - \mu/2)$  and suppose  $\mu \neq 0, 4$  in each of the following cases.

(i) For  $x, y, z \in \mathbf{C}$  with  $x^2, y^2, z^2 \neq \mu$ ,

$$\Psi(y, z, x) + \Psi(z, x, y) + \Psi(x, y, z) = v \pmod{2\pi i}. \tag{37}$$

(ii) For  $x, y, z, w \in \mathbf{C}$  with  $z + w = xy$  and  $x^2, y^2 \neq \mu$ ,

$$\Psi(x, y, z) + \Psi(x, y, w) = v \pmod{2\pi i}. \tag{38}$$

(iii) For  $x, y \in \mathbf{C} \setminus \{0\}$ , and

$$z = \frac{xy}{2} \left( 1 - \sqrt{1 - 4 \left( \frac{1}{x^2} + \frac{1}{y^2} - \frac{\mu}{x^2 y^2} \right)} \right), \tag{39}$$

we have

$$\lim_{y \rightarrow \infty} 2\Psi(x, y, z) = \mathfrak{h}(x) \pmod{2\pi i}. \tag{40}$$

(iv) If  $\phi \in (\Phi_\mu)_Q$  then there is a constant  $C = C(\phi) > 0$  such that

$$|2\Psi(x, y, z) - \mathfrak{h}(x)| \leq C|y|^{-2}, \tag{41}$$

for all  $x = \phi(X)$ ,  $X \in \Omega$ ;  $|y|$  sufficiently large; and  $z$  given by (39).

**Proof.** One can prove (i)–(iii) by direct calculations. For details, see §2.6 of [36]. The estimate (41) in (iv) follows from the fact that

$$|\log(1 + u)| \leq 2|u| \quad \text{for } u \in \mathbf{C} \text{ with } |u| \leq 1/2$$

and the following calculations:

$$\begin{aligned} & 2\Psi(x, y, z) - \mathfrak{h}(x) \\ &= \log \frac{x^2 y^2 [1 + (e^\nu - 1)(z/xy)]^2}{(x^2 - \mu)(y^2 - \mu)} - \log \frac{x^2 [1 + (e^\nu - 1)h(x)]^2}{x^2 - \mu} \\ &= \log \left( \frac{y^2}{y^2 - \mu} \left[ \frac{1 + (e^\nu - 1)(z/xy)}{1 + (e^\nu - 1)h(x)} \right]^2 \right) \\ &= \log \left( 1 + \frac{\mu}{y^2 - \mu} \right) + 2 \log \left( 1 + \frac{(e^\nu - 1)[z/xy - h(x)]}{1 + (e^\nu - 1)h(x)} \right) \end{aligned}$$

(note that the above are true equalities in  $\mathbf{C}$  without modulo  $2\pi i$ : the first one is because of (39) and the other two because each expression after the log symbol is sufficiently close to 1 provided that  $|y|$  is sufficiently large),

$$\begin{aligned} |z/xy - h(x)| &= \left| \sqrt{1 - 4/x^2} - \sqrt{1 - 4/x^2 - 4/y^2 + 4\mu/x^2 y^2} \right| / 2 \\ &= \frac{|(4/y^2)(1 - \mu/x^2)|}{2|\sqrt{1 - 4/x^2} + \sqrt{1 - 4/x^2 - 4/y^2 + 4\mu/x^2 y^2}|} \\ &\leq 2|y|^{-2} |1 - \mu/x^2| / \sqrt{|1 - 4/x^2|}, \end{aligned}$$

since by our convention the square roots here all have non-negative real parts, and for  $\phi \in (\Phi_\mu)_Q$ , the image  $\phi(\Omega)$  is discrete.  $\square$

**Proof of Theorem 3.5.** The proof now follows essentially the same line as that of Theorem 3, [5], with  $\psi(\vec{e}) = z/xy$  there now replaced by  $\Psi(x, y, z) \in \mathbf{C}$ . We write it out with details as follows, taking care of multiples of  $2\pi i$ . Note that this difficulty does not occur in [5].

Suppose  $\phi \in (\Phi_\mu)_Q$  where  $\mu \neq 0, 4$ . We write  $\mathfrak{h} = \mathfrak{h}_\tau$  where  $\tau = \mu - 2$ . Then  $\log^+ |\phi|$  has Fibonacci growth and the sum in (18) converges absolutely since  $\mathfrak{h}(x) = O(|x|^{-2})$  as  $|x| \rightarrow \infty$  and since  $\sum_{X \in \Omega} |\phi(X)|^{-2}$  converges absolutely by Corollary 3.16.

On the other hand, since  $\Omega_\phi(2)$  is finite, as in the proof of Theorem 3.3, there is a finite subtree  $T$  of  $\Sigma$  with the property that for each edge not in  $T$ , its arrow assigned by  $\phi$  points towards  $T$ .

Let  $C_n$ ,  $n \geq 0$ , be the set of directed edges  $\vec{e} = \alpha_\phi(e)$  at a distance  $n$  away from  $T$  (that is,  $e \notin T$ , and the minimal arc in  $\Sigma$  from the head of  $\vec{e}$  to  $T$  consists of  $n$  edges). It is easy to see that for each  $n \geq 0$ ,  $C_n$  is a circular set.

Now let  $\Omega_n, n \geq 0$ , be the set of regions  $X$  such that  $X \in \Omega^0(e)$  for some edge  $\vec{e} \in C_n$ . (Actually, each such region  $X$  intersects  $C_n$  in exactly two edges.) Note that  $\Omega_n$  forms a nested sequence with respect to inclusion, that is,  $\Omega_n \subset \Omega_{n+1}$  for all  $n \geq 0$  and furthermore,  $\Omega = \bigcup_{n=0}^\infty \Omega_n$ . On the other hand, all the  $C_n, n \geq 0$ , are disjoint. Note also that

$$\sum_{X \in \Omega} \mathfrak{h}(\phi(X)) = \lim_{n \rightarrow \infty} \sum_{X \in \Omega_n} \mathfrak{h}(\phi(X)).$$

For each  $\vec{e} = (X, Y; \rightarrow Z) \in C_n$ , we have defined

$$\psi(\vec{e}) = \Psi(x, y, z) = \log \frac{1 + (e^\nu - 1)(z/xy)}{\sqrt{1 - \mu/x^2}\sqrt{1 - \mu/y^2}} \in \mathbf{C}.$$

It is easy to see from (37) and (38) that  $\sum_{\vec{e} \in C_n} \psi(\vec{e}) = \nu \pmod{2\pi i}$ . Furthermore, we have:

**Claim.**  $\tilde{\nu} := \lim_{n \rightarrow \infty} \sum_{\vec{e} \in C_n} \psi(\vec{e}) \in \mathbf{C}$  exists.

To prove the claim, it suffices to show that  $\sum_{\vec{e} \in C_n} \psi(\vec{e}) = \sum_{\vec{e} \in C_{n+1}} \psi(\vec{e})$  for sufficiently large  $n$ . Consider any  $\vec{e} = (X, Y; W \rightarrow Z) \in C_n$ . Let  $\vec{e}_X = (Y, W; \rightarrow X)$  and  $\vec{e}_Y = (X, W; \rightarrow Y)$ . Then  $\vec{e}_X, \vec{e}_Y \in C_{n+1}$ . It is easy to see that when  $n$  is sufficiently large,  $|w|$  and one of  $|x|, |y|$ , say  $|x|$ , are sufficiently large. It follows from the definition of the edge weight  $\psi(\vec{e}_Y)$  that  $|\psi(\vec{e}_Y)|$  is sufficiently small. Since  $\psi(\vec{e}) = \psi(\vec{e}_X) + \psi(\vec{e}_Y) \pmod{2\pi i}$  by Proposition 4.2, we know that in fact  $\psi(\vec{e}) = \psi(\vec{e}_X) + \psi(\vec{e}_Y)$  without the modulo  $2\pi i$  restriction, for each  $\vec{e} \in C_n$ . This proves the claim.

Now for each  $\vec{e} \in C_{n+1}$ , we write  $\vec{e} = (X, Y; \rightarrow Z)$ , where we assume  $X \in \Omega_n$  and  $Y \in \Omega_{n+1} \setminus \Omega_n$ . Here  $z = \phi(Z)$  satisfies (39) since  $\vec{e}$  points towards  $Z$ . Note that as  $\vec{e}$  ranges over  $C_{n+1}$ , each  $X \in \Omega_n$  gets counted twice. Since  $\phi \in (\Phi_\mu)_Q, |y| = |\phi(Y)|$  will be sufficiently large if  $n$  is. Thus by (41) we have, as  $n \rightarrow \infty$ ,

$$\begin{aligned} 2 \left| \sum_{X \in \Omega_n} \mathfrak{h}(\phi(X)) - \tilde{\nu} \right| &= \left| \sum_{X \in \Omega_n} 2\mathfrak{h}(x) - \sum_{\vec{e} \in C_{n+1}} 2\psi(\vec{e}) \right| \\ &= \left| \sum_{\vec{e} \in C_{n+1}} (\mathfrak{h}(x) - 2\psi(\vec{e})) \right| \\ &\quad (\text{where } \vec{e} = (X, Y; \rightarrow Z) \text{ with } X \in \Omega_n, Y \in \Omega_{n+1} \setminus \Omega_n) \\ &\leq \sum_{\vec{e} \in C_{n+1}} |\mathfrak{h}(x) - 2\psi(\vec{e})| \\ &\leq \sum_{Y \in \Omega_{n+1} \setminus \Omega_n} \text{constant} \cdot |\phi(Y)|^{-2} \rightarrow 0, \end{aligned}$$

since  $\sum_{Y \in \Omega_{n+1} \setminus \Omega_n} |\phi(Y)|^{-2} \rightarrow 0$  (as  $n \rightarrow \infty$ ) by the convergence of

$$\sum_{Y \in \Omega} |\phi(Y)|^{-2} = \lim_{n \rightarrow \infty} \sum_{Y \in \Omega_n} |\phi(Y)|^{-2}.$$

Hence

$$\sum_{X \in \Omega} \mathfrak{h}(\phi(X)) = \lim_{n \rightarrow \infty} \sum_{X \in \Omega_n} \mathfrak{h}(\phi(X)) = \tilde{v}. \tag{42}$$

This proves Theorem 3.5.  $\square$

Similar to the estimate (41), we have (with the same assumption there)

$$|\psi(\vec{e}) - \hat{\mathfrak{h}}(x)| \leq C|y|^{-2}. \tag{43}$$

Then Proposition 3.6 can be similarly proved using (43).

**Proof of Proposition 3.6.** By Proposition 3.4, we know that  $\sum_{X \in \Omega^-(\vec{e})} \mathfrak{h}(\phi(X))$  and  $\sum_{X \in \Omega^-(\vec{e})} 2\hat{\mathfrak{h}}(\phi(X))$  both converge absolutely, and they have difference a multiple of  $2\pi i$  since  $\mathfrak{h} = 2\hat{\mathfrak{h}} \pmod{2\pi i}$ . Thus we only need to evaluate

$$\sum_{X \in \Omega^0(\vec{e})} \hat{\mathfrak{h}}(\phi(X)) + \sum_{X \in \Omega^-(\vec{e})} 2\hat{\mathfrak{h}}(\phi(X)).$$

We apply the same arguments as in the proof of Theorem 3.5.

Let  $C'_n$  be the set of directed edges  $e'$  in  $\Sigma^-(\vec{e})$  at a distance  $n$  from  $\vec{e}$  (that is,  $e' \subset \Sigma^-(\vec{e})$  and the minimal arc in  $\Sigma$  from the head of  $e'$  to the tail of  $\vec{e}$  has exactly  $n$  edges). Then  $C_n = C'_n \cup \{-\vec{e}\}$  is a circular set. By the assumption on  $\vec{e}$ , we know that, when  $n$  is sufficiently large,  $e' = \alpha_\phi(e')$  for every  $e' \in C'_n$ .

Let  $\Omega_n := \Omega_n^{0-}(\vec{e})$  be the set of regions  $X$  in  $\Omega^{0-}(\vec{e})$  such that  $X \in \Omega^0(e')$  for some  $e' \in C'_n$ . Then  $\Omega_n \subset \Omega_{n+1}$  and  $\Omega^{0-}(\vec{e}) = \bigcup_{n=0}^\infty \Omega_n$ . Thus

$$\sum_{X \in \Omega^{0-}(\vec{e})} \mathfrak{h}(\phi(X)) = \lim_{n \rightarrow \infty} \sum_{X \in \Omega_n} \mathfrak{h}(\phi(X)).$$

By (37) and (38), we have  $\psi(\vec{e}) = \sum_{\vec{e}' \in C'_n} \psi(\vec{e}') \pmod{2\pi i}$  for all  $n \geq 0$ . Similarly to the claim in the proof of Theorem 3.5, we can prove that

**Claim.**  $\tilde{\psi}(\vec{e}) := \lim_{n \rightarrow \infty} \sum_{\vec{e}' \in C'_n} \psi(\vec{e}') \in \mathbf{C}$  exists.

Thus for  $n$  sufficiently large we have from (43)

$$\begin{aligned} & \left| \tilde{\psi}(\vec{e}) - \sum_{X \in \Omega^0(\vec{e})} \hat{\mathfrak{h}}(\phi(X)) - \sum_{X \in \Omega^-(\vec{e})} 2\hat{\mathfrak{h}}(\phi(X)) \right| \\ &= \left| \sum_{\vec{e}' \in C'_{n+1}} \psi(\vec{e}') - \sum_{X \in \Omega^0(\vec{e})} \hat{\mathfrak{h}}(\phi(X)) - \sum_{X \in \Omega^-(\vec{e})} 2\hat{\mathfrak{h}}(\phi(X)) \right| \\ &= \left| \sum_{\vec{e}' \in C'_{n+1}} [\psi(\vec{e}') - \hat{\mathfrak{h}}(\phi(X))] \right| \end{aligned}$$

$$\begin{aligned}
 & \text{(where } \vec{e}' = (X, Y; \rightarrow Z) \text{ with } X \in \Omega_n \text{ and } Y \in \Omega_{n+1} \setminus \Omega_n) \\
 & \leq \sum_{\vec{e}' \in C_{n+1}} |\psi(\vec{e}') - \hat{h}(x)| \\
 & \leq \sum_{Y \in \Omega_{n+1} \setminus \Omega_n} \text{constant} \cdot |\phi(Y)|^{-2} \longrightarrow 0,
 \end{aligned}$$

since  $\sum_{Y \in \Omega_{n+1} \setminus \Omega_n} |\phi(Y)|^{-2} \rightarrow 0$  (as  $n \rightarrow \infty$ ) by the convergence of

$$\sum_{Y \in \Omega^{0-}(\vec{e})} |\phi(Y)|^{-2} = \lim_{n \rightarrow \infty} \sum_{Y \in \Omega_n} |\phi(Y)|^{-2}.$$

Hence

$$\begin{aligned}
 \tilde{\psi}(\vec{e}) &= \sum_{X \in \Omega^0(\vec{e})} \hat{h}(\phi(X)) + \lim_{n \rightarrow \infty} \sum_{X \in \Omega_n} 2\hat{h}(\phi(X)) \\
 &= \sum_{X \in \Omega^0(\vec{e})} \hat{h}(\phi(X)) + \sum_{X \in \Omega^-(\vec{e})} 2\hat{h}(\phi(X)). \tag{44}
 \end{aligned}$$

This proves Proposition 3.6 since  $\tilde{\psi}(\vec{e}) = \psi(\vec{e}) \pmod{2\pi i}$ .  $\square$

### 5. Variations to once-punctured torus bundles

In this section we generalize Bowditch’s variations in [4] of McShane’s identity. Bowditch’s variations in [4] of McShane’s identity are for complete, finite volume hyperbolic 3-manifolds fibering over the circle, with fiber the once-punctured torus. These complete structures can be deformed to incomplete structures, as shown by Thurston in [34], and in certain cases, one can perform hyperbolic Dehn surgery to obtain closed (complete) hyperbolic 3-manifolds. The main result of this section is that a further variation of the Bowditch’s identities holds for these deformations satisfying the relative BQ-conditions (to be defined later), and hence for the closed hyperbolic 3-manifolds obtained by hyperbolic Dehn surgery on such manifolds (Theorems 5.3, 5.4 and Corollary 5.5). Briefly, this is done as follows. Let  $M$  be a once-punctured torus bundle over the circle with non-trivial monodromy  $H$  which is an hyperbolic element in  $SL(2, \mathbf{Z})$  identified with the mapping class group of the once-punctured torus. Starting with a complete, finite volume hyperbolic structure on  $M$ , and by considering the holonomy representation restricted to the fundamental group of the fiber, Bowditch [4] constructed a Markoff map which is stabilized by the cyclic subgroup  $\langle H \rangle < SL(2, \mathbf{Z})$ . His variations of McShane’s identity (Theorems 5.1 and 5.2) are obtained by studying the Markoff map modulo the action of  $\langle H \rangle$ . If we deform the complete hyperbolic structure on  $M$  to an incomplete one and look again at the holonomy representation restricted to the fiber, we get a generalized Markoff map, with the same stabilizer group  $\langle H \rangle$ . We then obtain further variations of the McShane–Bowditch identities (Theorems 5.3 and 5.4) for these deformations, which also hold for almost all of the closed hyperbolic 3-manifolds obtained by hyperbolic Dehn surgery on these manifolds (Corollary 5.5).

### 5.1. Bowditch's settings for torus bundles

Let  $M$  be an orientable 3-manifold which fibers over the circle, with the fiber a once-punctured torus,  $\mathbb{T}$ . Suppose that  $M$  has a complete finite-volume hyperbolic structure.

Let  $\mathcal{S}$  be the set of closed geodesics in  $M$  which are homotopic in  $M$  to essential simple closed curves in the fiber. Geometrically, to each  $\sigma \in \mathcal{S}$ , we associate its **complex length**,  $l(\sigma) \in \mathbf{C}/2\pi i\mathbf{Z}$ , where  $\Re l(\sigma)$  is the (real) hyperbolic length of  $\sigma$  and  $\Im l(\sigma)$  is the rotational component, i.e. the angle through which a normal vector turns when parallel transported once around the closed geodesic  $\gamma$ . In particular,  $e^{l(\sigma)} \in \mathbf{C}$  is well defined. Then Bowditch obtained

**Theorem 5.1.** (See Theorem A in [4].)

$$\sum_{\sigma \in \mathcal{S}} \frac{1}{1 + e^{l(\sigma)}} = 0,$$

where the sum converges absolutely.

The curves in  $\mathcal{S}$  fall naturally into two classes,  $\mathcal{S}_L$  and  $\mathcal{S}_R$ , as follows. Recall that  $\mathcal{C}$  is the set of free homotopy classes of non-trivial non-peripheral simple closed curves on  $\mathbb{T}$  and can be thought of as the set of rational points in the projective lamination space,  $\mathcal{P}$ , of  $\mathbb{T}$ , which in this case is a circle. The mapping class group  $\mathcal{MCG}$  of  $\mathbb{T}$  acts on  $\mathcal{P}$  preserving the set  $\mathcal{C}$ . The monodromy  $H \in \mathcal{MCG}$  of  $M$  generates an infinite cyclic subgroup,  $\langle H \rangle$ , of the mapping class group. This subgroup, or  $H$ , has two fixed points in  $\mathcal{P}$ , namely the stable and unstable laminations,  $\mu_s$  and  $\mu_u$ , of the monodromy. These two points separate  $\mathcal{P}$  into two open intervals. Since  $\mu_s$  and  $\mu_u$  are irrational points, this gives a natural partition of  $\mathcal{C}$  into two subsets,  $\mathcal{C}_L$  and  $\mathcal{C}_R$ , which in turn partitions  $\mathcal{S}$  into two subsets,  $\mathcal{S}_L$  and  $\mathcal{S}_R$ .

If we restrict the sum appearing in Theorem 5.1 to one or other of  $\mathcal{S}_L$  and  $\mathcal{S}_R$ , we will get the same answer up to change of signs. Bowditch showed (see Theorem 5.2) that this number turns out essentially to be the modulus of the cusp of  $M$ .

More precisely, since  $M$  has a single parabolic cusp,  $M$  is homeomorphic to the interior of a compact manifold  $M \cup \partial M$ , with one toroidal boundary component,  $\partial M$ . Then  $\partial M$  carries a natural Euclidean structure, well defined up to similarity, which arises from identifying  $\partial M$  with a horocycle. If  $M$  has positive monodromy, we may represent  $\partial M$  as the quotient of  $\mathbf{C}$  with the Euclidean metric by the lattice  $\mathbf{Z} \oplus \lambda\mathbf{Z}$ , generated by the translations  $\zeta \mapsto \zeta + 1$  and  $\zeta \mapsto \zeta + \lambda$  corresponding to the meridian and longitude respectively, where the **longitude** is defined as the intersection of two leaves, one from each of the two foliations of  $M$  determined by the stable and unstable laminations. We call  $\lambda = \lambda(M)$  the **modulus** of the cusp. We can suppose that  $\Im[\lambda(M)] > 0$ . If  $M$  has negative monodromy, we define  $\lambda(M)$  as the modulus of the cusp of the sister of  $M$ . In this setting Bowditch showed

**Theorem 5.2.** (See Theorem B in [4].)

$$\sum_{\sigma \in \mathcal{S}_L} \frac{1}{1 + e^{l(\sigma)}} = \pm \lambda(M),$$

where the sign depends only on our conventions of orientation.

**Natural ideal triangulations.** Recall the setting of Section 3. Let us fix an ordered pair of free generators  $a, b$  of  $\Gamma$  so that the algebraic intersection number of the corresponding ordered pair of oriented simple closed curves  $C_a, C_b$  on  $\mathbb{T}$  is equal to  $+1$ . Note that the commutator  $[a, b] = aba^{-1}b^{-1}$  is peripheral.

The mapping class group  $\mathcal{MCG}$  of  $\mathbb{T}$  may be identified as  $SL(2, \mathbf{Z})$  which acts naturally on  $\mathbb{H}^2, \Sigma, \Omega, \mathcal{P}$  and  $\mathcal{C}$ . In each case, the kernel is given by the elliptic involution, so the induced action of  $PSL(2, \mathbf{Z})$  is faithful. An element  $H \in SL(2, \mathbf{Z})$  is *hyperbolic* if it has two fixed points  $\mu_u$  and  $\mu_s$  in  $\mathcal{P}$ , namely the stable and unstable laminations. The points  $\mu_u$  and  $\mu_s$  are joined by a bi-infinite arc  $\beta \subseteq \Sigma$ , which is translated by  $H$  in the direction of  $\mu_s$ . The path  $\beta$  can be described combinatorially in terms of the “right–left” decomposition of the matrix  $H$ . Note that some conjugate of  $H$  can be written as a product of the matrices

$$L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

which correspond to Dehn twists about  $C_b$  and  $C_a$  respectively. This decomposition is well defined up to cyclic reordering, and the sequence of  $L$ s and  $R$ s is the same as the periodic sequence of left and right turns of  $\beta$  in  $\Sigma$ . For more details, see [6]. Note that  $\beta$  partitions  $\Omega$  into two subsets,  $\Omega_L$  and  $\Omega_R$ , which lie on, respectively, the left and right of  $\beta$ . These correspond to the subsets,  $\mathcal{C}_L$  and  $\mathcal{C}_R$  of  $\mathcal{C}$  described earlier.

Now, if we take a homeomorphism,  $\eta$ , of  $\mathbb{T}$  representing the mapping class  $H$ , we may form the mapping torus,  $M_H$ , which is given by  $(\mathbb{T} \times [0, 1]) / \sim$ , where  $\sim$  identifies  $(x, 1)$  with  $(\eta(x), 0)$  for all  $x \in \mathbb{T}$ . The manifold  $M_H$  has a natural compactification by adjoining a toroidal boundary,  $\partial M_H$ . The compactified manifold,  $M_H \cup \partial M_H$ , as described in [13], has a natural ideal triangulation arising from the left–right decomposition of  $H$ . This can briefly be summarized as follows. Associated with each vertex of  $\Sigma$  is an ideal triangulation of  $\mathbb{T}$  (the edges of which are dual to the simple closed curves corresponding to the three complementary regions incident to the vertex). Moving along an edge in the tree corresponds to performing a (dual) Whitehead move. If we traverse a period of the path  $\beta$ , we get a sequence of Whitehead moves which take us from a given triangulation to its image under  $\eta$ . Each Whitehead move gives rise to an ideal simplex in  $\mathbb{T} \times [0, 1]$ , and so, after identifying  $\mathbb{T} \times \{0\}$  with  $\mathbb{T} \times \{1\}$  via the relation  $\sim$ , we obtain an ideal triangulation of  $M_H$ . This ideal triangulation gives us, in particular, a triangulation of the boundary,  $\partial M_H$ . As in [13], we may describe the combinatorial structure of this triangulation lifted to the universal cover,  $\mathbf{R}^2$ , of  $\partial M_H$ . First, we describe the case of positive monodromy. To do this, consider the bi-infinite sequence of vertices of  $\Sigma$  lying along the arc  $\beta \subseteq \Sigma$ . This sequence is dual to a sequence of ideal triangles in our tessellation of  $\mathbb{H}^2$ . The union of these triangles gives a bi-infinite strip invariant under the transformation  $H$ . We transfer this strip homeomorphically to the vertical strip  $[0, 1] \times \mathbf{R}$  in  $\mathbf{R}^2$ , so that the transformation  $H$  is conjugated to the map  $[(x, y) \mapsto (x, y + 1)]$ . We extend this to a triangulation of  $\mathbf{R}^2$  by a process of repeated reflection in the pair of vertical lines which form the boundary of this strip. The triangulation of  $\partial M_H$  is given by the quotient by the group generated by  $[(x, y) \mapsto (x, y + 1)]$  and  $[(x, y) \mapsto (x + 4, y)]$ . It is not hard to see that these transformations describe, respectively, the longitude and meridian of  $\partial M_H$ . Note that the triangulation is in fact invariant under the map  $[(x, y) \mapsto (x + 2, y)]$ . Up to this symmetry, there are two “vertical” lines in the triangulation. The bi-infinite sequence of vertices along one of these lines corresponds to sequence of regions of  $\Omega$  which meet  $\beta$  and all lie either in  $\Omega_L$ , or in  $\Omega_R$ . Two vertices are joined by an edge in this triangulation if and only if the corresponding regions are adjacent. Thus the “vertical” edges correspond to regions

meeting on the same side of  $\beta$ , whereas all the other edges correspond to regions meeting on opposite sides of  $\beta$ . The picture where the monodromy is negative is similar, except that in this case,  $\partial M_H$  is given as a quotient of  $\mathbf{R}^2$  by the group generated by  $[(x, y) \mapsto (x + 2, y + 1)]$  and  $[(x, y) \mapsto (x + 4, y)]$ . The “longitude” might be thought of as a “half of” the curve given by  $[(x, y) \mapsto (x, y + 2)]$ . Now, it follows from the work of Thurston [35] that  $M = M_H$  admits a complete finite-volume hyperbolic structure. See also [23] for an alternative proof and exposition. This structure is unique by Mostow rigidity. Note that  $\partial M_H$  carries a Euclidean structure, well defined up to similarity, obtained for example by identifying it with a horocycle in  $M_H$ . In this hyperbolic structure, we may realize each tetrahedron in our ideal triangulation of  $M$  as a hyperbolic ideal tetrahedron. In this way we get a “hyperbolic ideal triangulation” of  $M$ . This gives rise to a Euclidean realization of the combinatorial triangulation of  $\partial M_H$  (compare with [11]). Note that Parker [24] has prove that this hyperbolic ideal triangulation of  $M$  obtained above is indeed positively oriented, hence is a genuine hyperbolic ideal triangulation. Thus we have a resulting geometric triangulation of  $\partial M_H$  which will be used to prove the variations of McShane’s identity.

Regarding  $\mathbb{T}$  as a fibre of  $M$ , we get an identification of  $\Gamma = \pi_1(\mathbb{T})$  as a normal subgroup of  $\pi_1(M)$ . In fact,  $\pi_1(M)$  is an HNN extension of  $\Gamma$  with stable letter  $t$  so that  $tgt^{-1} = H_*(g)$  for all  $g \in \Gamma$ , where  $H_*$  is the automorphism of  $\Gamma$  induced by the monodromy  $H$  (see for example [17]). We also get an identification of  $\mathcal{S}$  with the quotient,  $\Omega/\langle H \rangle$ , of  $\Omega$  under the cyclic group,  $\langle H \rangle$ , generated by  $H$ . Clearly,  $H$  respects the partition of  $\Omega$  as  $\Omega_L \sqcup \Omega_R$ , and we may identify  $\mathcal{S}_L$  with  $\Omega_L/\langle H \rangle$  and  $\mathcal{S}_R$  with  $\Omega_R/\langle H \rangle$ . The hyperbolic structure on  $M$  may be described by a representation,  $\hat{\rho} : \pi_1(M) \rightarrow \text{PSL}(2, \mathbf{C})$ . It follows from [9] that  $\hat{\rho}$  lifts to a representation  $\rho : \pi_1(M) \rightarrow \text{SL}(2, \mathbf{C})$ .

Restricting our attention to the fibre subgroup  $\Gamma < \pi_1(M)$ , we define a Markoff map  $\phi : \Omega \rightarrow \mathbf{C}$  by  $\phi(X) = \text{tr } \rho(g)$ , where  $g \in \Gamma$  represents the simple closed curve on  $\mathbb{T}$  corresponding to the region  $X \in \Omega$ . Clearly,  $\phi$  is invariant under the  $\langle H \rangle$ -action, and so gives rise to a well-defined map  $\Omega/\langle H \rangle \rightarrow \mathbf{C}$  which we also denote by  $\phi$ . We write  $[X]$  for the orbit of  $X$  under  $\langle H \rangle$ . If  $\sigma \in \mathcal{S}$  corresponds to  $[X] \in \Omega/\langle H \rangle$ , then the complex length,  $l(\sigma)$ , of  $\sigma$  is determined by the formula  $l(\sigma) = l(\phi([X])) = 2 \cosh^{-1}(\phi([X])/2)$ . Thus  $h(\phi([X])) = 1/(1 + e^{l(\sigma)})$ , where  $h : \mathbf{C} \setminus [-2, 2] \rightarrow \mathbf{C}$  is defined by  $h(x) = (1 - \sqrt{1 - 4/x^2})/2$  as in Section 3. Here we take the square root with positive real part, corresponding to the fact that  $\Re l(\sigma) > 0$ . In these terms Bowditch’s Theorems 5.1 and 5.2 can be respectively expressed as the identities:

$$\sum_{[X] \in \Omega/\langle H \rangle} h(\phi([X])) = 0, \tag{45}$$

$$\sum_{[X] \in \Omega_L/\langle H \rangle} h(\phi([X])) = \lambda(\partial M). \tag{46}$$

### 5.2. Incomplete hyperbolic torus bundles

In this subsection we consider once-punctured torus bundles over the circle, but now with incomplete hyperbolic structures.

Let  $M$  be a once-punctured torus bundle over the circle such that the monodromy  $H \in \mathcal{MCG}$  is hyperbolic/Anosov. Then as described in the previous subsection,  $M$  can be given a complete, finite volume hyperbolic structure, and in fact,  $M$  can be decomposed into a collection of ideal hyperbolic tetrahedra. Hence we may obtain  $M$  by gluing a collection of ideal hyperbolic



tetrahedra with suitable edge invariants. Now we consider an incomplete hyperbolic structure on  $M$  obtained by changing the edge invariants while keeping the consistency conditions as described by Thurston [34]. We can also regard this incomplete structure as being obtained by deforming the complete hyperbolic structure slightly in the representation space of  $\pi_1(M)$ .

The developing image of  $M$ , with this incomplete hyperbolic structure, in  $\mathbb{H}^3$  misses some lines in  $\mathbb{H}^3$ . We may assume the  $z$ -axis  $[0, \infty]$  is among them. Consider, for  $\epsilon > 0$  sufficiently small, an  $\epsilon$ -neighborhood  $N$  of  $[0, \infty]$  in  $\mathbb{H}^3$ , i.e. a solid cone around the  $z$ -axis. Its boundary  $\partial N$  is the  $\epsilon$ -equidistant surface with center the  $z$ -axis, hence has a similarity structure. Actually, we can identify this similarity surface  $\partial N$  with  $\mathbb{C} \setminus \{0\}$ , where the identification is given by orthogonal projection in  $\mathbb{H}^3$  from  $\mathbb{C} \setminus \{0\}$  onto the  $z$ -axis via lines normal to it. Note that  $N \setminus [0, \infty]$  projects onto a neighborhood of  $\partial M$  in  $M$ , hence  $\partial N$  projects onto  $\partial M$ . In this way  $\partial M$  gets a similarity structure from the (incomplete) hyperbolic structure of  $M$ .

Recall the combinatorial triangulation of  $M$  described in Section 5.1. Now this triangulation of  $M$  is realized similarly by a “hyperbolic ideal triangulation” and  $\partial M$  gets an induced conformal triangulation. This triangulation can be thought of as being deformed from the triangulation in the complete case when we deform the complete structure on  $M$  into an incomplete one.

This incomplete hyperbolic structure on  $M$  is given by a representation  $\hat{\rho}: \pi_1(M) \rightarrow \text{PSL}(2, \mathbb{C})$  which lifts to a representation  $\rho: \pi_1(M) \rightarrow \text{SL}(2, \mathbb{C})$ . Restricted to  $\Gamma = \pi_1(\mathbb{T}) \triangleleft \pi_1(M)$ , we obtain a  $\mu$ -representation  $\rho: \Gamma \rightarrow \text{SL}(2, \mathbb{C})$  for some  $\mu \in \mathbb{C}$  with  $\mu \neq 4$ . Let  $\nu = \cosh^{-1}(1 - \mu/2)$  as in Section 3.

Lifting the similarity structure on  $\partial M$  to  $\mathbb{C} \setminus \{0\}$ , we get a triangulation of  $\mathbb{C} \setminus \{0\}$  which is invariant under the transformations  $\zeta \mapsto e^\nu \zeta$  and  $\zeta \mapsto e^\lambda \zeta$  if  $M$  has positive monodromy. These two transformations give respectively half the meridian and the longitude of  $\partial M$ , similar to the complete case in Section 5. If  $M$  has negative monodromy then the two transformations are given by  $\zeta \mapsto e^\nu \zeta$  and  $\zeta \mapsto e^{\nu+\lambda} \zeta$ .

Note that  $\nu \in \mathbb{C}/2\pi i\mathbb{Z}$  is half of the complex length of the  $\rho$ -image of a peripheral simple closed curve on the punctured torus  $\mathbb{T}$ , the fiber of  $M$ .

We assume the notation introduced in the beginning of this section. In particular, for each closed geodesic  $\sigma \in \mathcal{S}$ , its complex length  $l(\sigma) \in \mathbb{C}/2\pi i\mathbb{Z}$  is defined now under the given incomplete hyperbolic structure.

We say that  $M$  (with the incomplete hyperbolic structure) has *discrete length spectrum* on the punctured-torus fiber  $\mathbb{T}$  if for each  $\kappa > 0$  there are only finitely many free homotopy classes of simple closed curves on  $\mathbb{T}$  such that their geodesic realizations in  $M$  have length  $\leq \kappa$ . We will see later that  $M$  has this property if it is obtained from the unique complete finite volume hyperbolic structure by a sufficiently small deformation (Theorem 5.8). Then we have the following results corresponding to Bowditch’s Theorems 5.1 and 5.2.

**Theorem 5.3.** *Suppose  $M$  (with incomplete hyperbolic structure) has discrete length spectrum on its torus fiber. Then*

$$\sum_{\sigma \in \mathcal{S}} \log \left( \frac{e^\nu + e^{l(\sigma)}}{e^{-\nu} + e^{l(\sigma)}} \right) = 0 \pmod{2\pi i}, \tag{47}$$

where the sum converges absolutely.

**Theorem 5.4.** *Suppose  $M$  (with incomplete hyperbolic structure) has discrete length spectrum on its torus fiber. Then*

$$\sum_{\sigma \in \mathcal{S}_L} \log \left( \frac{e^v + e^{l(\sigma)}}{e^{-v} + e^{l(\sigma)}} \right) = \pm \lambda \pmod{2\pi i}, \tag{48}$$

where the sign depends only on our conventions of orientation and where  $\lambda$  is the complex length of the chosen longitude.

As a corollary we have

**Corollary 5.5.** *Let  $M$  be a once-punctured torus bundle over the circle with Anosov monodromy and let  $\bar{M}(p/q)$  be the closed 3-manifold obtained from  $M$  by performing Dehn surgery on  $\partial M$  with surgery slope  $p/q \in \mathbf{Q}$ . Then, except for a finite number of surgery slopes,  $\bar{M}(p/q)$  has a hyperbolic structure and, moreover, the following identity holds:*

$$\sum_{\sigma \in \mathcal{S}_L} \log \left( \frac{e^v + e^{l(\sigma)}}{e^{-v} + e^{l(\sigma)}} \right) = \pm \lambda \pmod{2\pi i}, \tag{49}$$

where the meanings of  $\lambda$ ,  $v$  and  $\mathcal{S}_L$ , etc. are defined as before for the (incomplete) hyperbolic structure on  $M$  induced from that of  $\bar{M}(p/q)$  and the sign depends only on our conventions of orientation.

**Remark.** We may weaken the condition of discrete length spectrum in the above results to the condition that the set  $\{\gamma \in \mathcal{S} \mid \Re l(\gamma) \leq \log(3 + 2\sqrt{2})\}$  is finite, by a simple calculation.

### 5.3. Periodic generalized Markoff maps

In this subsection we consider generalized Markoff maps which are invariant under a hyperbolic element of the mapping class group  $\mathcal{MCG}$  of  $\mathbb{T}$ .

Consider the action of the mapping class group of  $\mathbb{T}$ ,  $\mathcal{MCG} \cong \mathrm{SL}(2, \mathbf{Z})$ , and its induced action on  $\Gamma$  and on the space  $\mathcal{X}_\tau$  of  $\tau$ -representations  $\rho: \Gamma \rightarrow \mathrm{SL}(2, \mathbf{C})$  modulo conjugation. Any  $H \in \mathcal{MCG}$  induces an automorphism  $H_*$  of  $\Gamma$ , and  $H$  acts on  $\mathcal{X}_\tau$  by

$$H(\rho)(g) = \rho(H_*^{-1}(g)),$$

for any  $\rho \in \mathcal{X}_\tau$  and  $g \in \Gamma$ . Bowditch [4] studied representations  $\rho \in \mathcal{X}_{\mathrm{tp}} = \mathcal{X}_{-2}$  stabilized by a cyclic subgroup  $\langle H \rangle < \mathcal{MCG}$  generated by a hyperbolic element and proved a variation of the McShane’s identity for such representations. This result can be generalized for  $\tau$ -representations as follows, and is equivalent to Theorem 5.3, with slightly weaker assumptions. The case  $\tau = -2$  with  $\mathfrak{h}$  replaced by  $h$  is Bowditch’s variation (Theorem A in [4]).

**Theorem 5.6.** *Suppose that a  $\tau$ -representation  $\rho: \Gamma \rightarrow \mathrm{SL}(2, \mathbf{C})$ , where  $\tau \neq 2$ , is stabilized by a hyperbolic element  $H \in \mathrm{SL}(2, \mathbf{Z}) \cong \mathcal{MCG}$  and  $\rho$  satisfies the BQ-conditions on  $\hat{\mathcal{S}}/\langle H_* \rangle$ , that is,*

- (i)  $\mathrm{tr} \rho(g) \notin [-2, 2]$  for all classes  $[g] \in \hat{\mathcal{S}}/\langle H_* \rangle$ , and
- (ii)  $|\mathrm{tr} \rho(g)| \leq 2$  for only finitely many classes  $[g] \in \hat{\mathcal{S}}/\langle H_* \rangle$ .

Then we have

$$\sum_{[g] \in \hat{\Omega} / \langle H_* \rangle} \mathfrak{h}(\text{tr } \rho(g)) = 0, \tag{50}$$

where the function  $\mathfrak{h} = \mathfrak{h}_\tau$  is defined by (9) and the sum converges absolutely.

We prove Theorem 5.6 by reformulating it in terms of generalized Markoff maps as follows.

**Theorem 5.7.** *Suppose that  $\phi \in \Phi_\mu$  ( $\mu \neq 4$ ) is invariant under the action of a hyperbolic element  $H \in \text{SL}(2, \mathbf{Z}) \cong \mathcal{MCG}$  and  $\phi$  satisfies the BQ-conditions on  $\Omega / \langle H \rangle$ , that is, (i)  $\phi^{-1}([-2, 2]) = \emptyset$ , and (ii)  $|\phi([X])| \leq 2$  for only finitely many classes  $[X] \in \Omega / \langle H \rangle$ . Then*

$$\sum_{[X] \in \Omega / \langle H \rangle} \mathfrak{h}(\phi([X])) = 0 \pmod{2\pi i}, \tag{51}$$

where the sum converges absolutely and the function  $\mathfrak{h} = \mathfrak{h}_\tau$  is defined as in (9) with  $\tau = \mu - 2$  and  $v = \cosh^{-1}(-\tau/2)$ .

**Proof.** The proof is essentially the same as that given by Bowditch in [4], except here we use the generalized function  $\mathfrak{h}$  and the generalized  $\phi$ -weight  $\psi(\vec{e})$  (this follows along the same lines as the proof for Theorem 3.5).  $\square$

We also have the following openness result for  $\mu$ -Markoff maps invariant under a fixed hyperbolic element  $H \in \text{SL}(2, \mathbf{Z})$  and satisfying the relative BQ-conditions.

**Theorem 5.8.** *Let  $\Phi^H$  be the set of generalized Markoff maps invariant under a fixed hyperbolic element  $H \in \text{SL}(2, \mathbf{Z})$  and  $\Phi_Q^H$  be the subset satisfying the BQ-conditions on  $\Omega / \langle H \rangle$ . Then  $\Phi_Q^H$  is open in  $\Phi^H$ .*

**Proof.** This can be proved similarly as in Theorem 3.2. For  $\phi \in \Phi_Q^H$  we construct the tree  $T_\phi(t_1)$  which is invariant under  $H$ .  $T_\phi(t_1)$  is not finite but  $T_\phi(t_1) / \langle H \rangle$  is a finite graph in the graph  $\Sigma / \langle H \rangle$ . Note that  $T_\phi(t_1) / \langle H \rangle$  contains the spine of  $\Sigma / \langle H \rangle$  which is the projection of the invariant axis of  $H$  in  $\Sigma$ . Now if  $\phi' \in \Phi_Q^H$  is sufficiently close to  $\phi$ , then  $T_{\phi'}(t_1) / \langle H \rangle \subseteq T_\phi(t_2) / \langle H \rangle$  for some given  $t_2 > t_1$ , hence,  $\phi' \in \Phi_Q^H$ .  $\square$

The above result tells us that  $M$  (with the incomplete hyperbolic structure) would continue to have discrete length spectrum on its torus fiber for sufficiently small deformations from the complete structure.

We may also reformulate Theorem 5.4 in terms of generalized Markoff maps.

Let  $H \in \text{SL}(2, \mathbf{Z}) \cong \mathcal{MCG}$  be the monodromy of the once-punctured torus bundle  $M$  over the circle. Let  $\hat{\rho} : \pi_1(M) \rightarrow \text{PSL}(2, \mathbf{C})$  be the representation for the hyperbolic structure on  $M$  and  $\rho : \pi_1(M) \rightarrow \text{SL}(2, \mathbf{C})$  be a lift of the representation to  $\text{SL}(2, \mathbf{C})$ . Also denote by  $\rho$  its restriction to  $\Gamma = \pi_1(\mathbb{T})$ . Recall that we regard  $\mathbb{T}$  as the fiber of  $M$ . Suppose this  $\rho : \Gamma \rightarrow \text{SL}(2, \mathbf{C})$  is a  $\tau$ -representation and let  $\phi \in \Phi_\mu$  be the corresponding  $\mu$ -Markoff map, where  $\mu = \tau + 2$ . Then  $\phi$  is invariant under  $H$  as defined in the statement of Theorem 5.7. With the notation above and in Section 3, Theorem 5.4 can be reformulated as follows.

**Theorem 5.9.** *If  $M$  has discrete length spectrum on its torus fiber then*

$$\sum_{[X] \in \Omega_L / \langle H \rangle} \mathfrak{h}(\phi([X])) = \lambda \pmod{2\pi i}, \tag{52}$$

where the sum converges absolutely.

5.4. Proof of Theorem 5.4

In this subsection we give the proof of Theorem 5.4.

In order to prove Theorem 5.4, we need to compute the sum on the left-hand side of (52). We shall assume that the monodromy is positive, the other case is similar. Let us fix an orientation on the meridian of  $\partial M$  consistent with the orientation of the fibre. We shall use the upper half-space model of  $\mathbb{H}^3$ , so that its ideal boundary is identified with the extended complex plane,  $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ , which has  $\text{PSL}(2, \mathbb{C})$  acting in the usual way. We can normalize our representation  $\rho : \pi_1(M) \rightarrow \text{SL}(2, \mathbb{C})$  so that 0 and  $\infty$  are the fixed points of the images of the meridian and longitude. Then the developing map of the similarity structure on  $\partial M$  maps to  $\widetilde{\mathbb{C} \setminus \{0\}}$ , the holonomy representation is given by the subgroup of  $\rho(\pi_1(M))$  generated by the images of the meridian and the longitude.

Suppose  $a, b$  are free generators of the fibre group  $\Gamma$  such that the ordered pair of simple closed curves on  $\mathbb{T}$  that they represent have algebraic intersection number  $+1$ . The commutator,  $[a, b] = aba^{-1}b^{-1}$  is peripheral in  $\mathbb{T}$ , and so represents a meridian of  $\partial M$ . Then  $\rho([a, b])$  describes the translation  $\zeta \mapsto e^{2\nu}\zeta$ .

Now, to each region  $X \in \Omega$ , we shall associate an incomplete semi-infinite geodesic  $\Delta(X)$  in the developing image of  $M$  in  $\mathbb{H}^3$  so that  $\Delta(X)$  is normal to the  $z$ -axis and has one end,  $q(X)$ , lying in the  $z$ -axis. Note that  $\Delta(X)$  will be well defined up to the action of  $\zeta \mapsto e^\nu\zeta$ , indeed all the geometric constructions are really defined up to the action of  $\zeta \mapsto e^\nu\zeta$ . Let  $p(X) \in \mathbb{C} \setminus \{0\}$  be the ideal point which is the orthogonal projection of  $q(X)$  on  $\mathbb{C} \setminus \{0\}$  along the direction of  $\Delta(X)$ , we want to determine the position of  $p(X)$ , which will then determine  $\Delta(X)$ . This is done as follows. We know that  $X$  corresponds to some simple closed curve  $\gamma(X)$  on  $\mathbb{T}$ . Let  $\delta(X)$  be an arc on  $\mathbb{T}$  with both endpoints at the puncture such that  $\gamma(X) \cap \delta(X) = \emptyset$ . The homotopy class of  $\delta(X)$  relative to its endpoints is well defined. We are identifying  $\mathbb{T}$  with a fibre of  $M$ , so  $\mathbb{T}$  is naturally homotopy equivalent to the infinite cyclic cover of  $M$ . Under this equivalence,  $\delta(X)$  has a unique realization as a geodesic  $\Delta(X)$  in  $M$  whose developing images to  $\mathbb{H}^3$  are normal to the missing lines. Choose a lift of  $\Delta(X)$  to  $\mathbb{H}^3$  which is normal to the  $z$ -axis. Any other choice of lift would give us an image of  $\Delta(X)$  (hence an image of  $p(X)$ ) under the cyclic action generated by  $\zeta \mapsto e^\nu\zeta$ .

Consider the bi-infinite sequence,  $(X_j)_{j \in \mathbb{Z}}$ , of all regions of  $\Omega_L$  adjacent to the invariant path in  $\Sigma$  under  $H$ . We may choose  $\Delta(X_j)$  so that the transformation  $\zeta \mapsto e^\lambda\zeta$  acts on them as a shift. This is guaranteed by the geometric triangulation of  $\partial M$  described above.

It is not hard to see that in the geometric triangulation of  $\mathbb{C} \setminus \{0\}$ , the vertex corresponding to  $X_j$  is given by one of the images of  $p(X_j)$  under the action generated by  $\zeta \mapsto e^\nu\zeta$ , so we may as well assume that it actually equals  $p_j = p(X_j)$ . The choice of  $p_j$  naturally determines that of  $p_{j-1}$  and  $p_{j+1}$ , and so, inductively,  $p_j$  for all  $j \in \mathbb{Z}$ . Now the sequence  $(p_j)$  is periodic under the transformation corresponding to the longitude of  $\partial M$ . This transformation is given by  $\zeta \mapsto e^\lambda\zeta$ , where  $\lambda = \lambda(\partial M)$  is the length of the longitude. This corresponds to the action of  $\langle H \rangle$  on  $\Omega$  which has the effect of shifting the sequence  $(X_j)$ . Let  $m$  be the number of steps through which this sequence is shifted. Thus,  $\lambda = \log p_m - \log p_0 = \sum_{j=1}^m (\log p_j - \log p_{j-1}) \pmod{2\pi i}$ .

We thus want to compute the numbers  $\log p_j - \log p_{j-1} \pmod{2\pi i}$ . Let  $\vec{e}_j$  be the directed edge given by  $X_j \cap X_{j-1}$ , whose head lies in  $\beta$  and let  $C_L$  be the set  $\{\vec{e}_1, \dots, \vec{e}_m\}$  of directed edges.

Fix some  $j \in \{1, \dots, m\}$ . Let  $X = X_{j-1}$ ,  $Y = X_j$ , and let  $Z$  be the region at the head of  $\vec{e}_j$ . As described earlier, we can find free generators  $a, b$  for  $\Gamma$  which correspond, respectively, to the regions  $X$  and  $Y$ . Moreover, we can suppose that  $a$  and  $b$  are as described earlier and  $Z$  is represented by  $ab$ . By our discussion of the geometric meaning of  $\Psi$  in Appendix A, especially, by Lemma A.10, we have

$$\log p_j - \log p_{j-1} = \Psi(x, y, z) \pmod{2\pi i},$$

where  $x = \text{tr } \rho(a) = \phi(X)$ ,  $y = \text{tr } \rho(b) = \phi(Y)$  and  $z = \text{tr } \rho(ab) = \phi(Z)$ . Thus

$$\log p_j - \log p_{j-1} = \psi(\vec{e}_j) \pmod{2\pi i}.$$

It follows that in  $\mathbf{C}/2\pi i\mathbf{Z}$ , combining with Proposition 3.6, we have

$$\begin{aligned} \lambda(\partial M) &= \log p_m - \log p_0 = \sum_{j=1}^m (\log p_j - \log p_{j-1}) \\ &= \sum_{\vec{e} \in C_L} \psi(\vec{e}) = \sum_{[X] \in \Omega_L / \langle H \rangle} \mathfrak{h}(\phi([X])) \\ &= \sum_{\sigma \in \mathcal{A}_L} \log \left( \frac{e^v + e^{l(\sigma)}}{e^{-v} + e^{l(\sigma)}} \right). \end{aligned}$$

This proves Theorem 5.4 under certain choices of the orientations involved.  $\square$

**Proof of Corollary 5.5.** By Thurston’s Hyperbolic Dehn Surgery Theorem (see [34]),  $\overline{M}(p/q)$  has a complete hyperbolic structure for all but a finite number of surgery slopes  $p/q \in \mathbf{Q}$ . By further excluding a finite number of slopes, we may assume that the induced incomplete hyperbolic structure on  $M$  is obtained by a slight deformation from the unique complete hyperbolic structure on  $M$ . By Theorem 5.8,  $M$  then has discrete length spectrum on its torus fiber  $\mathbb{T}$  and the conclusion follows from Theorem 5.4.  $\square$

**Acknowledgments**

We would like to thank Bill Goldman, Caroline Series and Makoto Sakuma for their encouragement, helpful conversations/correspondence and comments. Part of this work was carried out while Tan and Zhang were visiting the University of Maryland, College Park; they would like to thank their host, Bill Goldman, and the Department of Mathematics there for their hospitality and warm welcome.

**Appendix A. Geometric meanings of the functions  $\mathfrak{h}$  and  $\Psi$**

In this appendix we explore the geometric meaning of the gap function  $\mathfrak{h}(x) = \mathfrak{h}_\tau(x)$  used earlier as well as the geometric meaning of the function  $\Psi(x, y, z)$  defined and used in Section 4. In particular, the geometric meaning of  $\Psi(x, y, z)$  was used in an essential way in Section 5.4.

**Naturally oriented axes.** To each  $A \in \text{SL}(2, \mathbf{C}) \setminus \{\pm I\}$ , we associate its *naturally oriented axis*  $\mathbf{a}(A)$  as follows:

- (i) when  $A$  represents a loxodromic (including hyperbolic) isometry of  $\mathbb{H}^3$ , the orientation of  $\mathbf{a}(A)$  is directed from its repelling fixed ideal point to its attracting fixed ideal point;
- (ii) when  $A$  represents an elliptic isometry of  $\mathbb{H}^3$  but not an involution, the orientation of  $\mathbf{a}(A)$  is defined so that it has rotation angle in  $(0, \pi)$  with respect to  $\mathbf{a}(A)$ ; and
- (iii) when  $A = (A_{ij})_{2 \times 2}$  is an involution, the orientation of  $\mathbf{a}(A)$  is the same as Fenchel defined in [12], that is,  $\mathbf{a}(A)$  is directed from  $\text{Fix}^-(A)$  to  $\text{Fix}^+(A)$ , where the ideal fixed points  $\text{Fix}^\mp(A)$  of  $A$  are distinguished as follows:
  - $\text{Fix}^-(A) := (A_{11} - i)/A_{21}$  and  $\text{Fix}^+(A) := (A_{11} + i)/A_{21}$  if  $A_{21} \neq 0$ ;
  - $\text{Fix}^-(A) := A_{12}i/2$  and  $\text{Fix}^+(A) := \infty$  if  $A_{21} = 0, A_{11} = i$ ; and
  - $\text{Fix}^-(A) := \infty$  and  $\text{Fix}^+(A) := -A_{12}i/2$  if  $A_{21} = 0, A_{11} = -i$ .

**Remark.** If  $A$  is an involution, then  $\mathbf{a}(-A) = \mathbf{a}(A^{-1})$  has the opposite orientation as  $\mathbf{a}(A)$ . If  $A$  is not an involution, then  $\mathbf{a}(-A) = \mathbf{a}(A)$  while  $\mathbf{a}(A^{-1})$  has the opposite orientation as  $\mathbf{a}(A)$ .

**Complex length  $\Delta_n(\mathbf{l}, \mathbf{m})$ .** Let us recall the complex length between oriented lines in  $\mathbb{H}^3$  as defined by Fenchel in §V.3, [12]. Let  $\mathbf{l}, \mathbf{m}, \mathbf{n} \in \text{SL}(2, \mathbf{C})$  be involutions (that is,  $\mathbf{l}^2 = \mathbf{m}^2 = \mathbf{n}^2 = -I$ ) representing oriented lines in  $\mathbb{H}^3$  such that  $\mathbf{n}$  is an oriented common normal to  $\mathbf{l}$  and  $\mathbf{m}$ . (We say two oriented lines in  $\mathbb{H}^3$  are *normal* to each other if they intersect orthogonally.) Then  $\Delta = \Delta_n(\mathbf{l}, \mathbf{m}) \in \mathbf{C}/2\pi i\mathbf{Z}$ , the *complex length* from  $\mathbf{l}$  to  $\mathbf{m}$  along  $\mathbf{n}$ , is defined by

$$\cosh \Delta = -(1/2) \text{tr}(\mathbf{ml}), \quad \sinh \Delta = -(i/2) \text{tr}(\mathbf{mnl}). \tag{A.1}$$

Geometrically,  $\Re \Delta \in \mathbf{R}$  is the signed hyperbolic length from the point  $\mathbf{l} \cap \mathbf{n}$  to the point  $\mathbf{m} \cap \mathbf{n}$  measured in the direction of  $\mathbf{n}$ , and  $\Im \Delta \in (-\pi, \pi]$  is the angle rotated from the direction of  $\mathbf{l}$  to that of  $\mathbf{m}$  measured along the direction of  $\mathbf{n}$ .

It is easy to see that, modulo  $2\pi i$ ,  $\Delta_n(-\mathbf{l}, -\mathbf{m}) = \Delta_n(\mathbf{l}, \mathbf{m})$  and  $\Delta_n(-\mathbf{l}, \mathbf{m}) = \Delta_n(\mathbf{l}, -\mathbf{m}) = \Delta_n(\mathbf{l}, \mathbf{m}) + \pi i$ .

**Geometric meaning of the gap function  $\mathfrak{h} = \mathfrak{h}_\tau$ .** The function  $\mathfrak{h} = \mathfrak{h}_\tau$  has the following geometric interpretation as the gap function as used in [30].

Let  $\phi \in \Phi_\mu$  be a  $\mu$ -Markoff map, where  $\mu \neq 0, 4$ , and  $\rho : \Gamma \rightarrow \text{SL}(2, \mathbf{C})$  be the corresponding  $\tau$ -representation, where  $\tau = \mu - 2$ .

Let  $a, b$  be an arbitrary pair of generators of  $\Gamma$  and let  $X, Y \in \Omega$  be the two regions which correspond to  $a, b$  respectively. Let  $A = \rho(a), B = \rho(b)$ . Then by our convention,  $x = \phi(X) = \text{tr} A, y = \phi(Y) = \text{tr} B$ . Let  $z = \text{tr} BA$ .

Consider the commutator  $[B^{-1}, A^{-1}] = (B^{-1}A^{-1}B)A$ . It follows from the trace identity (17) that  $\text{tr}[B^{-1}, A^{-1}] = x^2 + y^2 + z^2 - xyz - 2 = \mu - 2 = \tau$ .

If  $A$  is loxodromic (including hyperbolic), so is  $B^{-1}A^{-1}B$ . Let the attracting and repelling fixed points of  $A$  be respectively denoted as  $\text{Fix}^+(A)$  and  $\text{Fix}^-(A)$ . Similarly we have  $\text{Fix}^+(B^{-1}A^{-1}B)$  and  $\text{Fix}^-(B^{-1}A^{-1}B)$ .

We denote the oriented line in  $\mathbb{H}^3$  which is normal to the axis  $\mathbf{a}(B^{-1}A^{-1}BA)$  and has  $\text{Fix}^+(A)$  as its ending ideal point by  $[\mathbf{a}(B^{-1}A^{-1}BA), \text{Fix}^+(A)]$ . Similarly, we have another oriented line  $[\mathbf{a}(B^{-1}A^{-1}BA), \text{Fix}^-(B^{-1}A^{-1}B)]$ .

Then the complex length from the oriented line  $[\mathbf{a}(B^{-1}A^{-1}BA), \text{Fix}^+(A)]$  to  $[\mathbf{a}(B^{-1}A^{-1}BA), \text{Fix}^-(B^{-1}A^{-1}B)]$  along  $\mathbf{a}(B^{-1}A^{-1}BA)$  is exactly given by  $\mathfrak{h}(x) = \mathfrak{h}_\tau(x)$ .

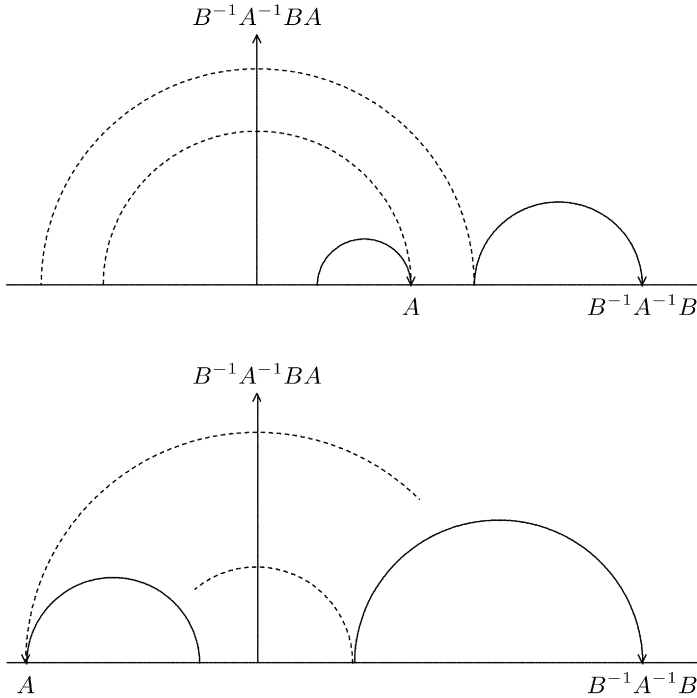


Fig. 5. The complex gap.

See Fig. 5 for an illustration in the cases where the isometries  $A$ ,  $B^{-1}A^{-1}B$  and  $B^{-1}A^{-1}BA$  have coplanar non-intersecting axes.

**Lemma A.1.** *With the above notation, we have*

$$\begin{aligned} \mathfrak{h}_\tau(x) = \Delta_{\mathbf{a}(B^{-1}A^{-1}BA)}([\mathbf{a}(B^{-1}A^{-1}BA), \text{Fix}^+(A)], \\ [\mathbf{a}(B^{-1}A^{-1}BA), \text{Fix}^-(B^{-1}A^{-1}B)]). \end{aligned} \tag{A.2}$$

**Remark.** Here it is important to note that the gap value  $\mathfrak{h}(x)$  is independent of the choice of  $b$  in the generating pair  $a, b$  of  $\Gamma$ . Note also that  $\text{Fix}^-(B^{-1}A^{-1}B) = \text{Fix}^+(B^{-1}AB)$  and that  $B^{-1}AB$  is a conjugate of  $A$ .

Lemma A.1 is in fact a special case of the following general lemma.

**Lemma A.2.** *Let  $A, B \in \text{SL}(2, \mathbf{C})$ . Then the complex distance from the oriented line  $[\mathbf{a}(BA), \text{Fix}^+(A)]$  to the oriented line  $[\mathbf{a}(BA), \text{Fix}^-(B)]$  measured along the oriented line  $\mathbf{a}(BA)$  is given by*

$$\begin{aligned} 2 \tanh^{-1} \left( \frac{\sinh \frac{l(-BA)}{2}}{\cosh \frac{l(-BA)}{2} + \exp(\frac{l(A)}{2} + \frac{l(B)}{2})} \right) \\ = \log \left( \frac{\exp \frac{l(-BA)}{2} + \exp(\frac{l(A)}{2} + \frac{l(B)}{2})}{\exp(-\frac{l(-BA)}{2}) + \exp(\frac{l(A)}{2} + \frac{l(B)}{2})} \right), \end{aligned} \tag{A.3}$$

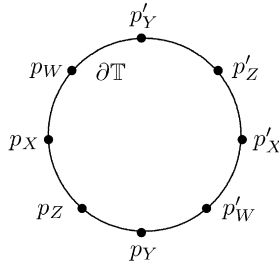


Fig. 6. Intersections of  $\delta_X, \delta_Y, \delta_Z, \delta_W$  with  $\partial\mathbb{T}$ .

where  $l(A)/2, l(B)/2, l(-BA)/2$  are the half translation lengths of  $A, B, -BA$  as defined by (4) in Section 2.

We omit the proof here. A detailed proof by cross-ratio calculations can be found in [36]. For another proof, one may adapt Mirzakhani’s proof in the real case as given in [22] to let it work for the general case here by using Fenchel’s cosine rule for oriented right-angled hexagons given in §VI.2 of [12].

**Geometric meaning of the function  $\Psi$ .** We first show the picture and give an informal treatment in the case where  $\phi$  corresponds to the holonomy representation  $\rho$  of a real hyperbolic one-holed torus  $\mathbb{T}$  with geodesic boundary  $\partial\mathbb{T}$ . For the limiting case where the boundary is a cusp, see [5] or [1]. Let  $e \leftrightarrow (X, Y; Z, W)$ , and let  $\gamma_X, \gamma_Y, \gamma_Z, \gamma_W$  be the simple closed geodesics on  $\mathbb{T}$  corresponding to  $X, Y, Z$  and  $W$  respectively.  $\gamma_X$  and  $\gamma_Y$  intersect each other once, at a Weierstrass point of  $\mathbb{T}$ , and  $\gamma_Z$  and  $\gamma_W$  are the two unique simple closed geodesics which intersect both  $\gamma_X$  and  $\gamma_Y$  exactly once. The points of intersection of these geodesics are the Weierstrass points, there are exactly three, and each geodesic passes through two and misses the third. Corresponding to  $\gamma_X$  is a unique geodesic segment  $\delta_X$  disjoint from  $\gamma_X$ , passing through the third Weierstrass point and intersecting  $\partial\mathbb{T}$  at both ends normally. Denote these points of intersection on  $\partial\mathbb{T}$  by  $p_X$  and  $p'_X$ , they are necessarily on opposite ends of  $\partial\mathbb{T}$ . Similarly, we have  $\delta_Y, \delta_Z$  and  $\delta_W$ , and the corresponding points of intersection with  $\partial\mathbb{T}$ . The eight points of intersection on  $\partial\mathbb{T}$  are shown in Fig. 6. When we consider the intervals  $[p_X, p_Y]$  and  $[p_X, p'_Y]$  in  $\partial\mathbb{T}$ , one of them contains one of the points  $p_Z$  or  $p'_Z$ , the other contains one of the points  $p_W$  or  $p'_W$ . Then  $\Psi(x, y, z)$  is the length of the interval which contains  $p_W$  or  $p'_W$ . (Note that we get the same answer if we used  $p'_X$  instead of  $p_X$ .)  $\Psi(x, y, w)$  is then the length of the other interval. It is not difficult to see from this description that  $\Psi$  satisfies Eqs. (37) and (38).

We now consider the more general case. Given  $x, y, z \in \mathbf{C}$ , set

$$\mu := x^2 + y^2 + z^2 - xyz$$

and assume  $\mu \neq 0, 4$ . As explained in Section 3, there exist  $A, B \in \text{SL}(2, \mathbf{C})$ , unique up to conjugation in  $\text{SL}(2, \mathbf{C})$ , so that the traces of  $A, B$  and  $C := (BA)^{-1}$  are respectively  $x, y, z$ , that is,

$$\text{tr } A = x, \quad \text{tr } B = y, \quad \text{tr } AB = \text{tr } BA = \text{tr } C = z.$$

Note that  $CBA = I$ . Denote  $[B^{-1}, A^{-1}] = B^{-1}A^{-1}BA = B^{-1}A^{-1}C^{-1}$  and let  $\tau := \text{tr}[B^{-1}, A^{-1}] = \text{tr } B^{-1}A^{-1}BA$ . Then by the trace identity (17) in  $\text{SL}(2, \mathbf{C})$ , we have  $\tau = \mu - 2 \neq \pm 2$ . Let  $v = \cosh^{-1}(-\tau/2) = \cosh^{-1}(1 - \mu/2)$ .



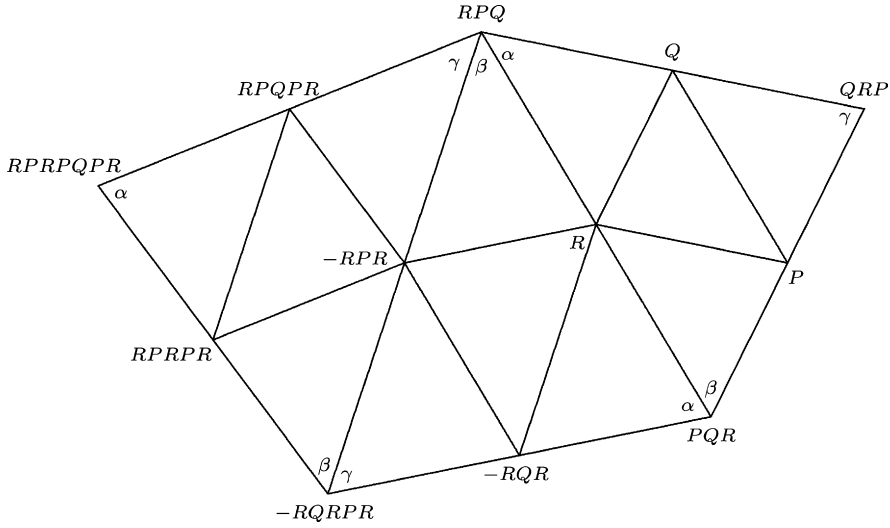


Fig. 7.

There are matrices  $Q, R, P \in SL(2, \mathbf{C})$  such that  $Q^2 = R^2 = P^2 = -I$  and  $A = -RQ, B = -PR, C = -QP$ . Hence  $[B^{-1}, A^{-1}] = -(RPQ)^2$ . Here matrices in  $SL(2, \mathbf{C})$  are considered as orientation-preserving isometries of  $\mathbb{H}^3$ , the upper half-space model of the hyperbolic 3-space. In fact,  $Q, R, P$  are involutions representing appropriately oriented lines in  $\mathbb{H}^3$  which are common normals to  $\mathbf{a}(C)$  and  $\mathbf{a}(A)$ , to  $\mathbf{a}(A)$  and  $\mathbf{a}(B)$ , and to  $\mathbf{a}(B)$  and  $\mathbf{a}(C)$ , respectively, as explained by Fenchel [12]. These oriented lines represented by  $Q, R, P$  will play the role of the Weierstrass points used in the real case. Note that the negative signs in this paragraph are important for later use of Fenchel’s cosine and sine rules for right-angled hexagons in this section.

Consider the conjugation  $A \mapsto KAK^{-1}$  by a matrix  $K \in SL(2, \mathbf{C})$ . It preserves the natural orientations of axes defined above in the following sense.

**Lemma A.3.** For  $A \in SL(2, \mathbf{C})$ , we have  $\mathbf{a}(KAK^{-1}) = K\mathbf{a}(A)K^{-1}$ . In particular, when  $K$  is an involution, that is  $K^2 = -I$ , the conjugation is given by  $A \mapsto -KAK$ .

Now consider the schematic figure as illustrated in Fig. 7. It is easy to check that  $RPQ \leftrightarrow QRP \leftrightarrow PQR \leftrightarrow RPQ$  by conjugation by  $Q, P, R$  respectively. Hence  $\mathbf{a}(RPQ) \leftrightarrow \mathbf{a}(QRP) \leftrightarrow \mathbf{a}(PQR) \leftrightarrow \mathbf{a}(RPQ)$  by conjugation by  $Q, P, R$  respectively.

**Right-angled hexagons  $H(\mathbf{l}, \mathbf{m}, \mathbf{n})$ .** We shall use the following notation for oriented common normals and right-angled hexagons in  $\mathbb{H}^3$ .

- (i) For each ordered pair of oriented lines  $\mathbf{l}$  and  $\mathbf{m}$  in  $\mathbb{H}^3$ , let  $[\mathbf{l}, \mathbf{m}]$  denote a definitely chosen oriented common normal to them, so that  $[\mathbf{l}, \mathbf{m}]$  and  $[\mathbf{m}, \mathbf{l}]$  always have opposite directions. For example, we may assume  $[\mathbf{l}, \mathbf{m}]$  is directed from  $\mathbf{l}$  to  $\mathbf{m}$  when  $\mathbf{l}$  and  $\mathbf{m}$  are disjoint.

(ii) Given oriented lines  $\mathbf{l}, \mathbf{m}, \mathbf{n}$  in  $\mathbb{H}^3$ , we use  $\mathbf{H}(\mathbf{l}, \mathbf{m}, \mathbf{n})$  to denote the right-angled hexagon in the sense of Fenchel [12] formed by the oriented lines

$$\mathbf{l}; [\mathbf{l}, \mathbf{m}]; \mathbf{m}; [\mathbf{m}, \mathbf{n}]; \mathbf{n}; [\mathbf{n}, \mathbf{l}]$$

in this cyclic order.

Thus the oriented lines

$$\begin{aligned} &\mathbf{a}(RPQ); [\mathbf{a}(RPQ), \mathbf{a}(PQR)]; \\ &\mathbf{a}(PQR); [\mathbf{a}(PQR), \mathbf{a}(QRP)]; \\ &\mathbf{a}(QRP); [\mathbf{a}(QRP), \mathbf{a}(RPQ)], \end{aligned}$$

in this cyclic order, form the right-angled hexagon  $\mathbf{H}(\mathbf{a}(RPQ), \mathbf{a}(PQR), \mathbf{a}(QRP))$ . Since, as we have observed above,  $\mathbf{a}(RPQ) \leftrightarrow \mathbf{a}(QRP) \leftrightarrow \mathbf{a}(PQR) \leftrightarrow \mathbf{a}(RPQ)$  by conjugation by  $Q, P, R$  respectively, we know  $\mathbf{H}(\mathbf{a}(RPQ), \mathbf{a}(PQR), \mathbf{a}(QRP))$  has the oriented lines  $\mathbf{a}(R), \mathbf{a}(P), \mathbf{a}(Q)$  as the ‘midpoints’ of its three sides, that is,

$$\begin{aligned} \Delta_{[\mathbf{a}(RPQ), \mathbf{a}(PQR)]}(\mathbf{a}(RPQ), \mathbf{a}(R)) &= \Delta_{[\mathbf{a}(RPQ), \mathbf{a}(PQR)]}(\mathbf{a}(R), \mathbf{a}(PQR)) =: \tilde{c}, \\ \Delta_{[\mathbf{a}(PQR), \mathbf{a}(QRP)]}(\mathbf{a}(PQR), \mathbf{a}(P)) &= \Delta_{[\mathbf{a}(PQR), \mathbf{a}(QRP)]}(\mathbf{a}(P), \mathbf{a}(QRP)) =: \tilde{a}, \\ \Delta_{[\mathbf{a}(QRP), \mathbf{a}(RPQ)]}(\mathbf{a}(QRP), \mathbf{a}(Q)) &= \Delta_{[\mathbf{a}(QRP), \mathbf{a}(RPQ)]}(\mathbf{a}(Q), \mathbf{a}(RPQ)) =: \tilde{b}. \end{aligned}$$

Let the complex lengths of the other three sides of the right-angled hexagon  $\mathbf{H}(\mathbf{a}(RPQ), \mathbf{a}(PQR), \mathbf{a}(QRP))$  be denoted as

$$\begin{aligned} \Delta_{\mathbf{a}(RPQ)}([\mathbf{a}(QRP), \mathbf{a}(RPQ)], [\mathbf{a}(RPQ), \mathbf{a}(PQR)]) &=: \alpha, \\ \Delta_{\mathbf{a}(PQR)}([\mathbf{a}(RPQ), \mathbf{a}(PQR)], [\mathbf{a}(PQR), \mathbf{a}(QRP)]) &=: \beta, \\ \Delta_{\mathbf{a}(QRP)}([\mathbf{a}(PQR), \mathbf{a}(QRP)], [\mathbf{a}(QRP), \mathbf{a}(RPQ)]) &=: \gamma. \end{aligned}$$

Also let

$$\begin{aligned} \Delta_{\mathbf{a}(RQ)}(\mathbf{a}(Q), \mathbf{a}(R)) &=: \tilde{p}, \\ \Delta_{\mathbf{a}(PR)}(\mathbf{a}(R), \mathbf{a}(P)) &=: \tilde{q}, \\ \Delta_{\mathbf{a}(QP)}(\mathbf{a}(P), \mathbf{a}(Q)) &=: \tilde{r}. \end{aligned}$$

Then

$$\begin{aligned} \cosh \tilde{p} &= (1/2) \operatorname{tr}(-RQ) = (1/2) \operatorname{tr} A = x/2, \\ \cosh \tilde{q} &= (1/2) \operatorname{tr}(-PR) = (1/2) \operatorname{tr} B = y/2, \\ \cosh \tilde{r} &= (1/2) \operatorname{tr}(-QP) = (1/2) \operatorname{tr} C = z/2. \end{aligned}$$

Now applying Fenchel’s cosine rules as in §VI.2 of [12] to the right-angled hexagons

$$\mathbf{H}(\mathbf{a}(Q), \mathbf{a}(P), \mathbf{a}(QRP)) \quad \text{and} \quad \mathbf{H}(\mathbf{a}(RPQ), \mathbf{a}(PQR), \mathbf{a}(QRP))$$

respectively we have

$$\cosh \tilde{r} = \cosh \tilde{a} \cosh \tilde{b} + \sinh \tilde{a} \sinh \tilde{b} \cosh \gamma, \tag{A.4}$$

$$\cosh 2\tilde{c} = \cosh 2\tilde{a} \cosh 2\tilde{b} + \sinh 2\tilde{a} \sinh 2\tilde{b} \cosh \gamma. \tag{A.5}$$

**Claim.**

$$\frac{\cosh \tilde{a}}{\cosh \tilde{p}} = \frac{\cosh \tilde{b}}{\cosh \tilde{q}} = \frac{\cosh \tilde{c}}{\cosh \tilde{r}} = \kappa, \tag{A.6}$$

where  $\kappa^2 = 4/\mu$ .

To prove the claim, multiplying  $4 \cosh \tilde{a} \cosh \tilde{b}$  to both sides of (A.4) gives

$$4 \cosh \tilde{a} \cosh \tilde{b} \cosh \tilde{r} = 4 \cosh^2 \tilde{a} \cosh^2 \tilde{b} + \sinh 2\tilde{a} \sinh 2\tilde{b} \cosh \gamma.$$

Comparing with (A.5) gives

$$4 \cosh \tilde{a} \cosh \tilde{b} \cosh \tilde{r} - \cosh 2\tilde{c} = 4 \cosh^2 \tilde{a} \cosh^2 \tilde{b} - \cosh 2\tilde{a} \cosh 2\tilde{b}.$$

After simplification we have

$$(2 \cosh \tilde{a})(2 \cosh \tilde{b})(2 \cosh \tilde{r}) = (2 \cosh \tilde{a})^2 + (2 \cosh \tilde{b})^2 + (2 \cosh \tilde{c})^2 - 4. \tag{A.7}$$

Similarly we have

$$(2 \cosh \tilde{a})(2 \cosh \tilde{q})(2 \cosh \tilde{c}) = (2 \cosh \tilde{a})^2 + (2 \cosh \tilde{b})^2 + (2 \cosh \tilde{c})^2 - 4, \tag{A.8}$$

$$(2 \cosh \tilde{p})(2 \cosh \tilde{b})(2 \cosh \tilde{c}) = (2 \cosh \tilde{a})^2 + (2 \cosh \tilde{b})^2 + (2 \cosh \tilde{c})^2 - 4. \tag{A.9}$$

Hence from (A.7)–(A.9) we have

$$\frac{\cosh \tilde{a}}{\cosh \tilde{p}} = \frac{\cosh \tilde{b}}{\cosh \tilde{q}} = \frac{\cosh \tilde{c}}{\cosh \tilde{r}}. \tag{A.10}$$

Let the common value in (A.10) be denoted by  $\kappa$ . Then

$$2 \cosh \tilde{a} = \kappa 2 \cosh \tilde{p} = \kappa x,$$

$$2 \cosh \tilde{b} = \kappa 2 \cosh \tilde{q} = \kappa y,$$

$$2 \cosh \tilde{c} = \kappa 2 \cosh \tilde{r} = \kappa z.$$

Now from (A.7) we have

$$\kappa^2xyz = \kappa^2(x^2 + y^2 + z^2) - 4,$$

and hence (recalling that  $\mu = x^2 + y^2 + z^2 - xyz$ )

$$\kappa^2 = 4/\mu. \tag{A.11}$$

This proves the claim.

It follows from the above proof that

**Lemma A.4.**

$$\cosh \gamma = \frac{z/2 - xy/\mu}{(x^2/\mu - 1)^{1/2}(y^2/\mu - 1)^{1/2}}, \tag{A.12}$$

where the square roots do not necessarily have non-negative real parts.

Indeed, from (A.4), we have (since  $\kappa^2 = 4/\mu$ )

$$\begin{aligned} \cosh \gamma &= \frac{\cosh \tilde{r} - \cosh \tilde{a} \cosh \tilde{b}}{\sinh \tilde{a} \sinh \tilde{b}} \\ &= \frac{z/2 - \kappa(x/2)\kappa(y/2)}{[(\kappa x/2)^2 - 1]^{1/2}[(\kappa y/2)^2 - 1]^{1/2}} \\ &= \frac{z/2 - xy/\mu}{(x^2/\mu - 1)^{1/2}(y^2/\mu - 1)^{1/2}}. \end{aligned}$$

As an immediate corollary, we have

**Corollary A.5.**

$$\sinh \gamma = \pm \frac{(\sinh v)z/\mu}{(x^2/\mu - 1)^{1/2}(y^2/\mu - 1)^{1/2}}. \tag{A.13}$$

The rest of this section is devoted to the determination of the sign in (A.13). Geometrically, we have

**Lemma A.6.**  $(\alpha + \pi i) + (\beta + \pi i) + (\gamma + \pi i) = v \pmod{2\pi i}$ .

**Proof.** We refer to Fig. 7. Consider the following right-angled hexagons

$$\begin{aligned} \mathcal{H}_1 &:= \mathbf{H}(\mathbf{a}(RPQ), \mathbf{a}(PQR), \mathbf{a}(QRP)), \\ \mathcal{H}_2 &:= \mathbf{H}(\mathbf{a}(PQR), \mathbf{a}(RPQ), \mathbf{a}(-RQRPR)), \\ \mathcal{H}_3 &:= \mathbf{H}(\mathbf{a}(RPRPQPR), \mathbf{a}(-RQRPR), \mathbf{a}(RPQ)). \end{aligned}$$

It is easy to check that the conjugation by  $R$  maps  $\mathcal{H}_1$  to  $\mathcal{H}_2$ , and the conjugation by  $RPR$  maps  $\mathcal{H}_2$  to  $\mathcal{H}_3$ ; thus the conjugation by  $PR$  maps  $\mathcal{H}_1$  to  $\mathcal{H}_3$ . Since conjugations preserve the relevant complex lengths, we have

$$\begin{aligned} \Delta_{\mathbf{a}(RPQ)}([\mathbf{a}(PQR), \mathbf{a}(RPQ)], [\mathbf{a}(RPQ), \mathbf{a}(-RQRPR)]) &= \beta, \\ \Delta_{\mathbf{a}(RPQ)}([\mathbf{a}(-RQRPR), \mathbf{a}(RPQ)], [\mathbf{a}(RPQ), \mathbf{a}(RPRPQPR)]) &= \gamma. \end{aligned}$$

Thus

$$\begin{aligned} &(\alpha + \pi i) + (\beta + \pi i) + (\gamma + \pi i) \\ &= \Delta_{\mathbf{a}(RPQ)}([\mathbf{a}(QRP), \mathbf{a}(RPQ)], [\mathbf{a}(RPQ), \mathbf{a}(PQR)]) \\ &\quad + \Delta_{\mathbf{a}(RPQ)}([\mathbf{a}(PQR), \mathbf{a}(RPQ)], [\mathbf{a}(RPQ), \mathbf{a}(-RQRPR)]) \\ &\quad + \Delta_{\mathbf{a}(RPQ)}([\mathbf{a}(-RQRPR), \mathbf{a}(RPQ)], [\mathbf{a}(RPQ), \mathbf{a}(RPRPQPR)]) + \pi i \\ &= \Delta_{\mathbf{a}(RPQ)}([\mathbf{a}(QRP), \mathbf{a}(RPQ)], [\mathbf{a}(RPQ), \mathbf{a}(PQR)]) \\ &\quad + \Delta_{\mathbf{a}(RPQ)}([\mathbf{a}(RPQ), \mathbf{a}(PQR)], [\mathbf{a}(-RQRPR), \mathbf{a}(RPQ)]) \\ &\quad + \Delta_{\mathbf{a}(RPQ)}([\mathbf{a}(-RQRPR), \mathbf{a}(RPQ)], [\mathbf{a}(RPQ), \mathbf{a}(RPRPQPR)]) + \pi i \\ &= \Delta_{\mathbf{a}(RPQ)}([\mathbf{a}(QRP), \mathbf{a}(RPQ)], [\mathbf{a}(RPQ), \mathbf{a}(RPQPRPR)]) + \pi i \\ &= \Delta_{\mathbf{a}(RPQ)}([\mathbf{a}(QRP), \mathbf{a}(RPQ)], [\mathbf{a}(RPRPQPR), \mathbf{a}(RPQ)]) \\ &= v \pmod{2\pi i}. \end{aligned}$$

The last equality follows from Lemmas A.7 and A.8 below, since it is easy to see that  $v = \cosh^{-1}(-\tau/2)$  is the complex translation length of  $RPQ$  (recall that  $[B^{-1}, A^{-1}] = -(RPQ)^2$  and  $\tau = \text{tr}[B^{-1}, A^{-1}]$ ) and that the conjugation by  $RPQ$  maps  $[\mathbf{a}(QRP), \mathbf{a}(RPQ)]$  to  $[\mathbf{a}(RPRPQPR), \mathbf{a}(RPQ)]$ .  $\square$

**Lemma A.7.** *The complex translation length of  $K \in \text{SL}(2, \mathbf{C})$  is given by*

$$l(K) = \cosh^{-1}\left(\frac{1}{2} \text{tr}(K^2)\right). \tag{A.14}$$

**Proof.** Recall that  $l(K)/2 = \cosh^{-1}(\frac{1}{2} \text{tr}K)$ . Hence

$$\begin{aligned} \cosh l(K) &= 2 \cosh^2(l(K)/2) - 1 \\ &= 2\left(\frac{1}{2} \text{tr} K\right)^2 - 1 \\ &= \frac{1}{2}(\text{tr}^2 K - 2) \\ &= \frac{1}{2} \text{tr}(K^2), \end{aligned}$$

from which (A.14) follows since  $\Re l(K) \geq 0$ .  $\square$

**Lemma A.8.** *If  $K \in \text{SL}(2, \mathbf{C})$  is non-parabolic and  $L \in \text{SL}(2, \mathbf{C})$  is a line matrix such that  $\mathbf{a}(K) \perp \mathbf{a}(L)$ , then  $KLK^{-1}$  is also a line matrix and the complex translation length  $l(K)$  of  $K$  is given by*

$$l(K) = \Delta_{\mathbf{a}(K)}(\mathbf{a}(L), \mathbf{a}(KLK^{-1})). \tag{A.15}$$

This is Lemma 2.17 in [36], with a proof given there.

Now we can determine the signs in the expressions (A.13) etc. as follows.

**Lemma A.9.**

$$\sinh \gamma = -\frac{(\sinh v)z}{(x^2/\mu - 1)^{1/2}(y^2/\mu - 1)^{1/2}} \tag{A.16}$$

and similarly for  $\sinh \alpha$  and  $\sinh \beta$ .

**Proof.** Let  $\Psi(x, y, z) \in \mathbf{C}$  be defined as in Section 4 by:

$$\Psi(x, y, z) = \log \frac{[xy + (e^v - 1)z]/\mu}{(x^2/\mu - 1)^{1/2}(y^2/\mu - 1)^{1/2}} \tag{A.17}$$

or equivalently, by the following two equations:

$$\cosh \Psi(x, y, z) = \frac{[xy - (\mu/2)z]/\mu}{(x^2/\mu - 1)^{1/2}(y^2/\mu - 1)^{1/2}}, \tag{A.18}$$

$$\sinh \Psi(x, y, z) = \frac{(\sinh v)z/\mu}{(x^2/\mu - 1)^{1/2}(y^2/\mu - 1)^{1/2}}. \tag{A.19}$$

Similarly for  $\Psi(y, z, x)$ ,  $\Psi(z, x, y) \in \mathbf{C}$ . We have from Proposition 4.2(i) that

$$\Psi(y, z, x) + \Psi(z, x, y) + \Psi(x, y, z) = v \pmod{2\pi i}.$$

On the other hand, it follows from Lemma A.4 that  $-\cosh \gamma = \cosh \Psi(x, y, z)$  etc. and hence, modulo  $2\pi i$ ,

$$\alpha + \pi i = \pm \Psi(y, z, x), \quad \beta + \pi i = \pm \Psi(z, x, y), \quad \gamma + \pi i = \pm \Psi(x, y, z). \tag{A.20}$$

Now applying Fenchel’s sine rule as in §VI.2 of [12] to the right-angled hexagon

$$\mathbf{H}(\mathbf{a}(RPQ), \mathbf{a}(PQR), \mathbf{a}(QRP))$$

gives

$$\frac{\sinh \alpha}{\sinh 2\tilde{a}} = \frac{\sinh \beta}{\sinh 2\tilde{b}} = \frac{\sinh \gamma}{\sinh 2\tilde{c}}.$$

It follows from (A.13) together with  $\kappa z = 2 \cosh \tilde{c}$  that the above common value is given by

$$\frac{\sinh \gamma}{\sinh 2\tilde{c}} = \pm \frac{(\sinh \nu)\kappa z}{\kappa \sinh \tilde{a} \sinh \tilde{b} \sinh 2\tilde{c}} = \pm \frac{\sinh \nu}{\kappa \sinh \tilde{a} \sinh \tilde{b} \sinh \tilde{c}} \tag{A.21}$$

(note that in (A.21) the  $\pm$  is the same as that in (A.13) etc.) and hence the  $\pm$  in expressions (A.13) etc. of  $\sinh \gamma, \sinh \alpha, \sinh \beta$  are constant. So are the signs in (A.20), that is, we have either, modulo  $2\pi i$ ,

$$\alpha + \pi i = \Psi(y, z, x), \quad \beta + \pi i = \Psi(z, x, y), \quad \gamma + \pi i = \Psi(x, y, z) \tag{A.22}$$

or, modulo  $2\pi i$ ,

$$\alpha + \pi i = -\Psi(y, z, x), \quad \beta + \pi i = -\Psi(z, x, y), \quad \gamma + \pi i = -\Psi(x, y, z). \tag{A.23}$$

Since we also have  $(\alpha + \pi i) + (\beta + \pi i) + (\gamma + \pi i) = \nu \pmod{2\pi i}$  (Lemma A.6) and  $\nu \neq 0$  we may conclude that (A.22) must hold. This proves Lemma A.9.  $\square$

Note that

$$\begin{aligned} \alpha + \pi i &= \Delta_{\mathbf{a}(RPQ)}([\mathbf{a}(RPQ), \mathbf{a}(Q)], [\mathbf{a}(R), \mathbf{a}(RPQ)]) + \pi i \\ &= \Delta_{\mathbf{a}(RPQ)}([\mathbf{a}(RPQ), \mathbf{a}(Q)], [\mathbf{a}(RPQ), \mathbf{a}(R)]) \end{aligned}$$

and  $\mathbf{a}(RPQ) = \mathbf{a}(-B^{-1}A^{-1}C^{-1})$ . Then the above argument gives us the following geometric interpretation of  $\Psi(y, z, x)$  etc.

**Lemma A.10.** *Given  $x, y, z \in \mathbf{C}$  so that  $\mu := x^2 + y^2 + z^2 - xyz \neq 0, 4$ , there exist line matrices  $Q, R, P \in \text{SL}(2, \mathbf{C})$  (that is,  $Q^2 = R^2 = P^2 = -I$ ) such that  $A := -RQ, B := -PR, C := -QP \in \text{SL}(2, \mathbf{C})$  satisfy  $\text{tr } A = x, \text{tr } B = y$  and  $\text{tr } C = z$ . Let  $A' := [B^{-1}, A^{-1}], B' := [C^{-1}, B^{-1}], C' := [A^{-1}, C^{-1}]$ . Then*

$$\Psi(y, z, x) = \Delta_{\mathbf{a}(-A')}([\mathbf{a}(-A'), \mathbf{a}(Q)], [\mathbf{a}(-A'), \mathbf{a}(R)]) \pmod{2\pi i}, \tag{A.24}$$

$$\Psi(z, x, y) = \Delta_{\mathbf{a}(-B')}([\mathbf{a}(-B'), \mathbf{a}(R)], [\mathbf{a}(-B'), \mathbf{a}(P)]) \pmod{2\pi i}, \tag{A.25}$$

$$\Psi(x, y, z) = \Delta_{\mathbf{a}(-C')}([\mathbf{a}(-C'), \mathbf{a}(P)], [\mathbf{a}(-C'), \mathbf{a}(Q)]) \pmod{2\pi i}, \tag{A.26}$$

where  $\Psi(y, z, x), \Psi(z, x, y), \Psi(x, y, z) \in \mathbf{C}$  are defined by (32).

### Appendix B. Drawing the gaps

Here we explain how to visualize and draw correctly, for a given representation  $\rho : \Gamma \rightarrow \text{SL}(2, \mathbf{C})$  satisfying the BQ-conditions, the gaps in the extended complex plane which is the ideal boundary of  $\mathbb{H}^3$  in the upper half-space model.

Fix a pair of generators  $a, b \in \Gamma$  and let (in this section only)  $c := [b^{-1}, a^{-1}] = b^{-1}a^{-1}ba$ . Then each element of  $\Gamma$  can be written uniquely as a reduced word in the letters  $a, b, a^{-1}, b^{-1}$ . For each  $[g] \in \hat{\Omega}$ , we wish to draw the gaps for an infinite set of representatives  $g_i$  in the class  $[g]$ , with respect to the axis  $\mathbf{a}(C)$  of the image  $C = \rho(c)$  of the commutator  $c$ . These representatives are essentially the cyclically reduced words, and conjugates of the cyclically reduced word by the elements of the commutator subgroup. As discussed in the previous section, for each  $g_i$ , we

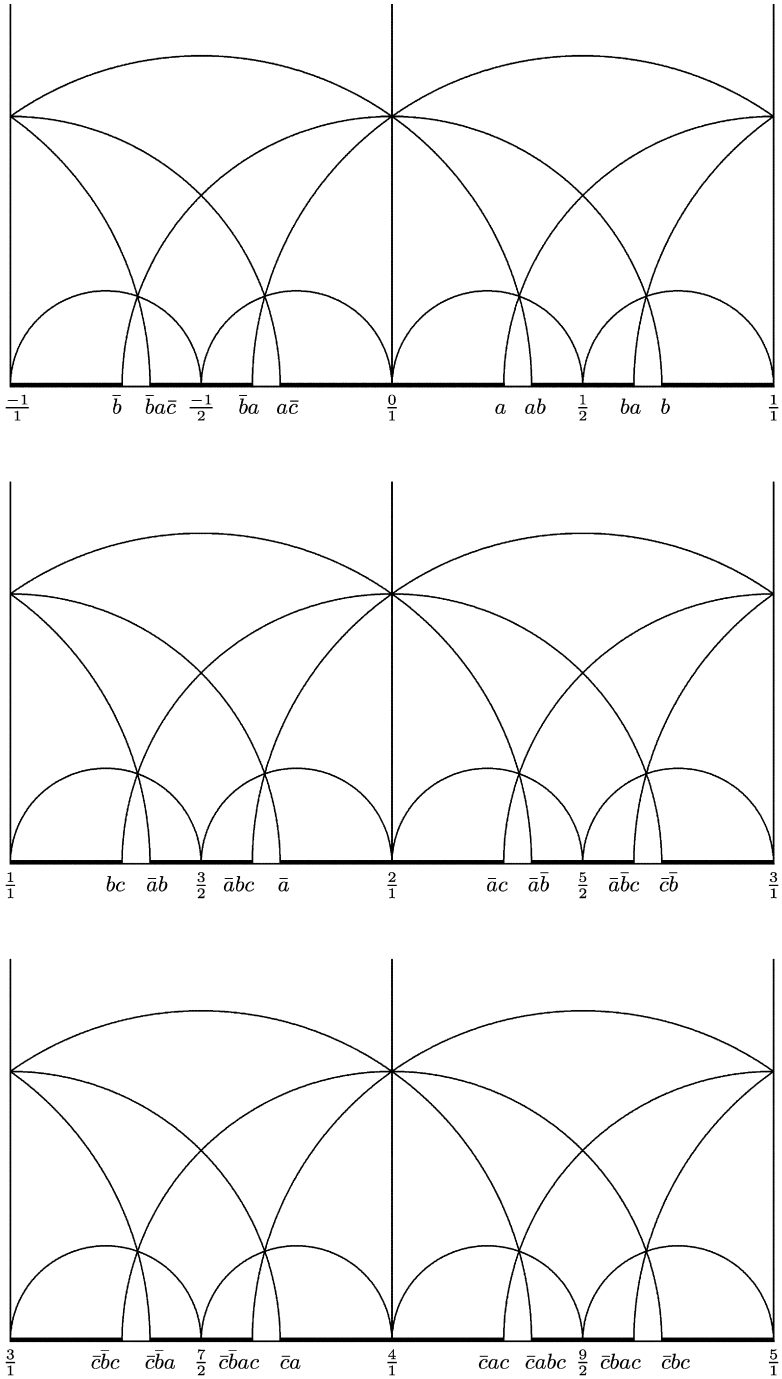


Fig. 8. The sequence of pairs of words and corresponding gaps.



need to produce a word  $g'_i \in \Gamma$  which is a conjugate of  $g_i$  so that  $(g'_i)^{-1}g_i = c$  in  $\Gamma$ . Hence we must have  $g'_i = g_i c^{-1}$  after cancellation. We need to do this consistently so that all gaps are in their correct places viewed in the extended complex plane.

**Constructing the ‘universal cover’ of  $\hat{\Omega}$ .** From the above discussion, to draw the gaps with respect to a fixed commutator, we really need to construct the ‘universal cover’ of the rational projective lamination space  $\hat{\Omega}$  as follows. We shall construct pairs of words  $(L_{\frac{p}{q}}, R_{\frac{p}{q}})$  in  $\Gamma$ , parametrized by  $\frac{p}{q} \in \mathbf{Q}$ , where  $\mathbf{Q}$  is regarded as the “universal cover” of  $\mathbf{Q} \cup \infty$ . The particular representation we will draw will be that corresponding to the once-punctured hyperbolic torus which has lifted holonomy representation  $\rho : \Gamma \rightarrow \text{SL}(2, \mathbf{C})$  such that

$$A = \rho(a) = \begin{pmatrix} 0 & \sqrt{2}/2 \\ -\sqrt{2} & 2\sqrt{2} \end{pmatrix} \quad \text{and} \quad B = \rho(b) = \begin{pmatrix} \sqrt{2} & \sqrt{2}/2 \\ \sqrt{2} & \sqrt{2} \end{pmatrix}. \tag{B.1}$$

Note that  $C := \rho(c) = B^{-1}A^{-1}BA = \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix}$ . See Fig. 8, where we use  $\bar{a}, \bar{b}, \bar{c}$  to denote  $a^{-1}, b^{-1}, c^{-1}$  respectively and we label the attracting fixed points of  $A, A^{-1}$  by  $a, \bar{a}$  respectively, and similarly for the other words.

First, we construct a bi-infinite sequence of pairs of reduced words  $(L_n, R_n)$ ,  $n = \frac{n}{1} \in \mathbf{Z}$ , inductively as follows.

- (i) Set  $R_0 = a$  and  $L_1 = b$ .
- (ii) We require

$$(L_n)^{-1}R_n = c$$

for all  $n \in \mathbf{Z}$ . Hence, for example,  $L_0 = b^{-1}ab = ac^{-1}$  and  $R_1 = a^{-1}ba = bc$ .

- (iii) Define

$$L_{n+2} = (R_n)^{-1}$$

or equivalently

$$R_{n-2} = (L_n)^{-1}$$

for  $n \in \mathbf{Z}$ . Hence  $R_{n+2} = (R_n)^{-1}c$ .

It is easy to check that the following relations hold for all  $n \in \mathbf{Z}$ :

$$[(L_{n+1})^{-1}, (R_n)^{-1}] = c,$$

and

$$L_{n+4} = c^{-1}L_n c.$$

Next, we generate the general  $(L_{\frac{p}{q}}, R_{\frac{p}{q}})$  for all  $\frac{p}{q} \in \mathbf{Q}$  by the following rule:

(iv) For  $\frac{p+r}{q+s} \in \mathbf{Q}$ , constructed from Farey neighbors  $\frac{p}{q}, \frac{r}{s} \in \mathbf{Q}$ , where  $\frac{p}{q} < \frac{r}{s}$ , we define

$$L_{\frac{p+r}{q+s}} = R_{\frac{p}{q}} L_{\frac{r}{s}}, \quad R_{\frac{p+r}{q+s}} = L_{\frac{r}{s}} R_{\frac{p}{q}}. \tag{B.2}$$

**Remark.** We note that here  $\frac{p}{q}$  in the index is *not* the slope of the free homotopy class  $[L_{\frac{p}{q}}] = [R_{\frac{p}{q}}] \in \hat{\mathcal{Q}}$ . In fact the rationals in the interval  $[0, 2)$  in our index set corresponds to the set  $\mathbf{Q} \cup \infty$  of the slopes, via the Farey triangulation described, and the full set of rationals  $\mathbf{Q}$  in our index set corresponds to the universal cover. The reason we need to do this is because we need to consider the inner automorphisms by the commutator subgroup, for a careful treatment, see [14].

We can prove by induction that

(I) for any Farey neighbors  $\frac{p}{q}, \frac{r}{s} \in \mathbf{Q}$ , where  $\frac{p}{q} < \frac{r}{s}$ , there is the commutator identity:

$$[(L_{\frac{r}{s}})^{-1}, (R_{\frac{p}{q}})^{-1}] = c; \tag{B.3}$$

(II) for all  $\frac{p}{q} \in \mathbf{Q}$ , there is the conjugation relation:

$$L_{\frac{p}{q}+2} = (R_{\frac{p}{q}})^{-1} \tag{B.4}$$

and

$$L_{\frac{p}{q}+4} = c^{-1} L_{\frac{p}{q}} c. \tag{B.5}$$

Thus the pair of words  $(L_{\frac{p}{q}}, R_{\frac{p}{q}})$  we constructed above are indeed conjugate to each other as the following lemma shows.

**Lemma B.1.** For any Farey neighbors  $\frac{p}{q}, \frac{r}{s} \in \mathbf{Q}$ , where  $\frac{p}{q} < \frac{r}{s}$ , we have

$$L_{\frac{p}{q}} = (L_{\frac{r}{s}})^{-1} R_{\frac{p}{q}} L_{\frac{r}{s}}. \tag{B.6}$$

**Drawing the gaps.** Now for each pair of words  $(L_{\frac{p}{q}}, R_{\frac{p}{q}})$ ,  $\frac{p}{q} \in \mathbf{Q}$ , constructed as above, we draw a line segment in the extended complex plane connecting the attracting fixed points of  $\rho(L_{\frac{p}{q}})$  and  $\rho(R_{\frac{p}{q}})$  to indicate the gap corresponding to the pair. Of course these gaps should be measured against the commutator, in the case where the commutator is parabolic, we may normalize the picture so that the commutator fixes  $\infty$ , see Fig. 8. When the commutator is loxodromic we may normalize so that the fixed points of the commutator  $C$  are 0 and  $\infty$ . The picture obtained for a particular generalized Markoff map  $\phi \in (\Phi_\mu)_Q$  is given in Fig. 9 (with this normalization), and we have drawn some of the gaps for  $\frac{p}{q}$  between 0 and 4. The complete picture for  $\frac{p}{q} \in \mathbf{Q}$  can be obtained by applying the Möbius transformation  $C = \rho(c)$  and its inverse on the given picture repeatedly.

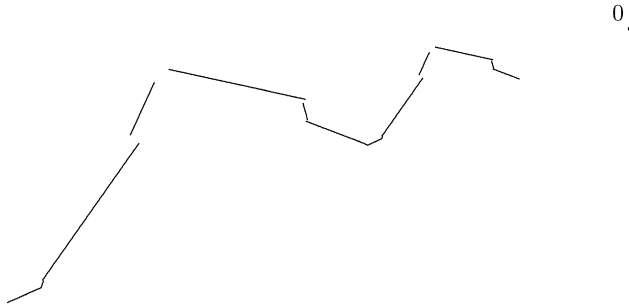


Fig. 9. Gaps for a  $(-0.8)$ -Markoff map with  $x = 6, y = 3 + i$ .

**The attracting fixed points.** Finally we explain the formula to find the attracting fixed point for a given loxodromic (hyperbolic)  $A \in \text{SL}(2, \mathbf{C})$ .

Let  $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ . The two fixed points of  $A$  are the roots of quadratic equation  $A_{21}z^2 + (A_{22} - A_{11})z - A_{12} = 0$ , hence given by

$$z = [(A_{11} - A_{22}) \pm \sqrt{(A_{11} + A_{22})^2 - 4}]/2.$$

Recall that the square root here has been assumed to have positive real part.

In this form, however, it is *not* true that one sign gives always the attracting or always the repelling fixed point. We can rewrite it in the following form:

$$z = [(A_{11} - A_{22}) \pm (A_{11} + A_{22})\sqrt{1 - 4(A_{11} + A_{22})^{-2}}]/2.$$

Then corresponding to the plus and minus signs are respectively the attracting and repelling fixed points.

**Lemma B.2.** *Suppose  $A \in \text{SL}(2, \mathbf{C})$  is loxodromic/hyperbolic. Then*

$$\text{Fix}^+(A) = [(A_{11} - A_{22}) + (A_{11} + A_{22})\sqrt{1 - 4(A_{11} + A_{22})^{-2}}]/2, \tag{B.7}$$

$$\text{Fix}^-(A) = [(A_{11} - A_{22}) - (A_{11} + A_{22})\sqrt{1 - 4(A_{11} + A_{22})^{-2}}]/2. \tag{B.8}$$

**Proof.** This is true since we know that they correspond respectively the two eigenvalues of the matrix  $A$ :

$$\lambda^+ = (A_{11} + A_{22})[1 + \sqrt{1 - 4(A_{11} + A_{22})^{-2}}]/2, \tag{B.9}$$

$$\lambda^- = (A_{11} + A_{22})[1 - \sqrt{1 - 4(A_{11} + A_{22})^{-2}}]/2 \tag{B.10}$$

which have respectively norms greater and less than 1.  $\square$

## References

- [1] Hirotaka Akiyoshi, Hideki Miyachi, Makoto Sakuma, A refinement of McShane's identity for quasifuchsian punctured torus groups, in: W. Abikoff, A. Haas (Eds.), *Tradition of Ahlfors and Bers, III, The Ahlfors–Bers Colloquium, October 2001, Univ. of Connecticut at Storrs*, in: *Contemp. Math.*, vol. 355, American Mathematical Society, 2004, pp. 21–40.
- [2] Hirotaka Akiyoshi, Hideki Miyachi, Makoto Sakuma, Variations of McShane's identity for punctured surface groups, in: *Spaces of Kleinian Groups*, in: *London Math. Soc. Lecture Note Ser.*, vol. 329, Cambridge Univ. Press, Cambridge, 2006, pp. 151–185.
- [3] Brian H. Bowditch, A proof of McShane's identity via Markoff triples, *Bull. London Math. Soc.* 28 (1) (1996) 73–78.
- [4] Brian H. Bowditch, A variation of McShane's identity for once-punctured torus bundles, *Topology* 36 (2) (1997) 325–334.
- [5] Brian H. Bowditch, Markoff triples and quasi-Fuchsian groups, *Proc. London Math. Soc.* (3) 77 (3) (1998) 697–736.
- [6] Brian H. Bowditch, Colin Maclachlan, Alan W. Reid, Arithmetic hyperbolic surface bundles, *Math. Ann.* 302 (1) (1995) 31–60.
- [7] Jeff Brock, The Weil–Petersson metric and volumes of 3-dimensional hyperbolic convex cores, *J. Amer. Math. Soc.* 16 (2003) 495–535.
- [8] Robert Brooks, J. Peter Matelski, The dynamics of 2-generator subgroups of  $\mathrm{PSL}(2, \mathbb{C})$ , in: *Riemann Surfaces and Related Topics: Proceedings of the 1978 Stony Brook Conference*, State Univ. New York, Stony Brook, NY, 1978, in: *Ann. of Math. Stud.*, vol. 97, Princeton Univ. Press, Princeton, NJ, 1981, pp. 65–71.
- [9] Marc Culler, Lifting representations to covering groups, *Adv. Math.* 59 (1) (1986) 64–70.
- [10] Thomas W. Cusick, Mary E. Flahive, The Markoff and Lagrange spectra, *Math. Surveys Monogr.*, vol. 30, American Mathematical Society, Providence, RI, 1989.
- [11] David B.A. Epstein, Robert C. Penner, Euclidean decompositions of non-compact hyperbolic manifolds, *J. Differential Geom.* 27 (1) (1988) 67–80.
- [12] Werner Fenchel, *Elementary Geometry in Hyperbolic Space*, Walter de Gruyter & Co., Berlin, 1989.
- [13] William Floyd, Allen Hatcher, Incompressible surfaces in punctured torus bundles, *Topology Appl.* 13 (3) (1982) 263–282.
- [14] William M. Goldman, The modular group action on real  $\mathrm{SL}(2)$ -characters of a one-holed torus, *Geom. Topol.* 7 (2003) 443–486.
- [15] Allen Hatcher, William Thurston, A presentation for the mapping class group, *Topology* 19 (1980) 221–237.
- [16] Troels Jørgensen, On pairs of once-punctured tori, in: *Kleinian Groups and Hyperbolic 3-Manifolds*, in: *London Math. Soc. Lecture Note Ser.*, vol. 299, Cambridge Univ. Press, 2003.
- [17] Colin Maclachlan, Alan W. Reid, *The Arithmetic of Hyperbolic 3-Manifolds*, *Grad. Texts in Math.*, vol. 219, Springer-Verlag, New York, 2003.
- [18] Greg McShane, A remarkable identity for lengths of curves, PhD thesis, University of Warwick, 1991.
- [19] Greg McShane, Simple geodesics and a series constant over Teichmüller space, *Invent. Math.* 132 (3) (1998) 607–632.
- [20] Greg McShane, Simple geodesics on surfaces of genus 2, *Ann. Acad. Sci. Fenn. Math.* 31 (1) (2006) 31–38.
- [21] Yair N. Minsky, The classification of punctured-torus groups, *Ann. of Math.* (2) 149 (2) (1999) 559–626.
- [22] Maryam Mirzakhani, Simple geodesics and Weil–Petersson volumes of moduli spaces of bordered Riemann surfaces, *Invent. Math.* 167 (1) (2007) 179–222.
- [23] Jean-Pierre Otal, Le théorème d'hyperbolisation pour les variétés fibrées de dimension trois, *Astérisque* 235 (1996).
- [24] John R. Parker, Tetrahedral decomposition of punctured torus bundles, in: *Kleinian Groups and Hyperbolic 3-Manifolds*, Warwick, 2001, in: *London Math. Soc. Lecture Note Ser.*, vol. 299, Cambridge Univ. Press, Cambridge, 2003, pp. 275–291.
- [25] John A.G. Roberts, Escaping orbits in trace maps, *Phys. A* 228 (1–4) (1996) 295–325.
- [26] Makoto Sakuma, Variations of McShane's identity for the Riley slice and 2-bridge links, in: *Hyperbolic Spaces and Related Topics*, Kyoto, 1998, in: *RIMS Kôkyûroku*, vol. 1104, 1999, pp. 103–108.
- [27] Caroline Series, The geometry of Markoff numbers, *Math. Intelligencer* 7 (3) (1985) 20–29.
- [28] Joseph H. Silverman, The Markoff equation  $X^2 + Y^2 + Z^2 = aXYZ$  over quadratic imaginary fields, *J. Number Theory* 35 (1) (1990) 72–104.
- [29] Ser Peow Tan, Yan Loi Wong, Ying Zhang, The  $\mathrm{SL}(2, \mathbb{C})$  character variety of a one-holed torus, *Electron. Res. Announc. Amer. Math. Soc.* 11 (2005) 103–110.

- [30] Ser Peow Tan, Yan Loi Wong, Ying Zhang, Generalizations of McShane's identity to hyperbolic cone-surfaces, *J. Differential Geom.* 72 (1) (2006) 73–112.
- [31] Ser Peow Tan, Yan Loi Wong, Ying Zhang, Necessary and sufficient conditions for McShane's identity and variations, *Geom. Dedicata* 119 (1) (2006) 199–217.
- [32] Ser Peow Tan, Yan Loi Wong, Ying Zhang, McShane's identity for classical Schottky groups, preprint, arXiv: math.GT/0411628.
- [33] Ser Peow Tan, Yan Loi Wong, Ying Zhang, End invariants of  $SL(2, \mathbb{C})$  characters of the one-holed torus, *Amer. J. Math.*, in press.
- [34] William P. Thurston, *The Geometry and Topology of 3-Manifolds*, Lecture Notes, Princeton University, 1977/78.
- [35] William P. Thurston, *Hyperbolic structures on 3-manifolds II: Surface groups and 3-manifolds that fiber over the circle*, arXiv: math.GT/9801045.
- [36] Ying Zhang, *Hyperbolic cone-surfaces, generalized Markoff maps, Schottky groups and McShane's identity*, PhD thesis, National University of Singapore, 2004.