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An equation admitting infinite true contact transformations

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Abstract

The linear wave equation is shown to possess the unique property that if w_n is a true contact transformation admitted by the wave equation, i.e., w_n is not linear in the first derivatives of the dependent variable, then so is $\sum_n w_n$. We comment of the physical implications.

Keywords: True contact transformation; Linear wave equation

1. Introduction

In this paper we show that the linear wave equation

$$u_{tt} - u_{xx} = 0 \tag{1}$$

admits infinite contact transformations. By expanding the admitted contact transformations as a power series in the first derivatives u_t and u_x, we are able to determine particular forms of the contact transformations. These particular forms possess the unique property that if w_n is an admitted contact transformation of (1), then so is $\sum_n w_n$. This property is well known for ordinary differential equations [7]. As a result of this property, we show how the general solution

$$u = F(\psi) + G(\rho), \tag{2}$$

where

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$$\psi = t + x, \qquad \rho = t - x, \tag{3}$$

of (1) can be written in terms of the coefficients of the terms of the power series expansion of the contact transformations. Contact transformations and their applications are discussed in [6] and [2–4,11]. Pucci and Saccomandi [10] use contact transformations of second-order partial differential equations to obtain *pseudo-invariant* solutions of these second-order partial differential equations. Abraham-Shrauner et al. [1] investigate contact transformations admitted by third-order ordinary differential equations to obtain *hidden* transformations. Recently Ibragimov and Khabirov [5] have investigated a contact symmetry group classification of nonlinear wave equations. Momoniat and Mahomed [8] have shown that evolution type equations do not admit contact transformations. Momoniat has derived special classes of nonlinear wave equations [9] that do admit true contact transformations. The results obtained in this paper show that for linear equations, at least the linear wave equation does admit nontrivial contact transformations.

The transformations

$$\begin{aligned} &\bar{t} = \bar{t}(t, x, u, u_t, u_x, a), \qquad \bar{x} = \bar{x}(t, x, u, u_t, u_x, a), \qquad \bar{u} = \bar{u}(t, x, u, u_x, u_t, a), \\ &\bar{u}_t = \bar{u}_t(t, x, u, u_t, u_x, a), \qquad \bar{u}_x = \bar{u}_x(t, x, u, u_t, u_x, a), \end{aligned} \tag{4}$$

where a is a real parameter, form a one parameter group of contact transformations if they satisfy the group properties and

$$\bar{u}_t = \frac{\partial \bar{u}}{\partial \bar{t}}, \qquad \bar{u}_x = \frac{\partial \bar{u}}{\partial \bar{x}} \tag{5}$$

holds. The generator of a group of contact transformations can be given in terms of the Lie characteristic function $W(t, x, u, u_t, u_x)$ as follows:

$$\begin{aligned} X = {} & -W_{u_t}\partial_t - W_{u_x}\partial_x + (W - u_t W_{u_t} - u_x W_{u_x})\partial_u \\ & + (W_t + u_t W_u)\partial_{u_t} + (W_x + u_x W_u)\partial_{u_x}, \end{aligned} \tag{6}$$

where $\partial_t = \partial/\partial t$, $\partial_x = \partial/\partial x$, etc. The first prolongation of the generator (6) is given by

$$\tilde{X} = X + \zeta_{11}\partial_{u_{tt}} + \zeta_{12}\partial_{u_{tx}} + \zeta_{22}\partial_{u_{xx}}, \tag{7}$$

where the coefficients of (7) be calculated from the prolongation formulae

$$\zeta_{ij} = D_i D_j W - W_{u_k} u_{kij}, \tag{8}$$

with summation on k, where D_i is the operator of total differentiation given by

$$D_i = \partial_{x_i} + u_i \partial_u + u_{ij}\partial_{u_j} + \cdots. \tag{9}$$

2. Contact transformations admitted by the linear wave equation

To determine contact transformations of (1) we solve the determining equation

$$\tilde{X}(u_{tt} - u_{xx})|_{u_{tt}=u_{xx}} = 0, \tag{10}$$

which, from (7), can be written as

$$\zeta_{11} - \zeta_{22}|_{u_{tt}=u_{xx}} = 0. \tag{11}$$

Solving (11) we find that (1) admits the infinite contact transformation

$$\begin{aligned} X_\infty = {} & -(F_{1_\phi} + F_{2_\varphi})\partial_t - (F_{1_\phi} - F_{2_\varphi})\partial_x \\ & + \big(E_1 u + F_1 + F_2 + G_1(\psi) + G_2(\rho) \\ & \quad - u_x(F_{1_\phi} - F_{2_\varphi}) - u_t(F_{1_\phi} + F_{2_\varphi})\big)\partial_u, \end{aligned} \tag{12}$$

where

$$\phi = u_t + u_x, \qquad \varphi = u_t - u_x \tag{13}$$

and

$$F_1 = F_1(\psi, \phi), \qquad F_2 = F_2(\rho, \varphi), \qquad E_1 = \text{constant}. \tag{14}$$

Equation (12) is not very useful in applications as it contains arbitrary functions of the derivatives u_t and u_x. To determine a useful form of (12) we consider a power series expansion of Lie characteristic function W. Note that the contact transformation generator (12) reduces to the Lie point symmetry generator of (1) given in [4] if we assume $F_{1_\phi} = \kappa_1(\psi)$ and $F_{2_\varphi} = \kappa_2(\rho)$, where κ_1 is an arbitrary function of ψ and κ_2 is an arbitrary function of ρ.

Let

$$W = \sum_{i,j=0}^{n} \phi_{i,j}(t, x, u) u_t^i u_x^j, \tag{15}$$

where $\phi_{i,j}$ is an arbitrary function of t, x and u. The coefficient of u_{xx}^2 from the determining equation (11) is

$$W_{u_t u_t} - W_{u_x u_x} = 0. \tag{16}$$

We find that the functions w_n as defined by

$$w_n = \begin{cases} \phi_{n,0} \sum_{i=0}^{n/2} \binom{n}{2i} u_t^{n-2i} u_x^{2i} + \phi_{n,1} \sum_{i=0}^{n/2} \binom{n}{2i-1} u_t^{n-(2i-1)} u_x^{2i-1}, \\ \quad n \text{ even}, \\ \phi_{n,0} \sum_{i=0}^{(n-1)/2} \binom{n}{2i} u_t^{n-2i} u_x^{2i} + \phi_{n,1} \sum_{i=0}^{(n+1)/2} \binom{n}{2i-1} u_t^{n-(2i-1)} u_x^{2i-1}, \\ \quad n \text{ odd}, \end{cases} \tag{17}$$

are solutions of (16). The summation

$$W = \sum_n w_n \tag{18}$$

is also a solution of (16). The remaining terms from (11) simplify (17) to

$$w_n = \begin{cases} (f_n(\psi) + g_n(\rho)) \sum_{i=0}^{n/2} \binom{n}{2i} u_t^{n-2i} u_x^{2i} + (f_n(\psi) - g_n(\rho)) \\ \quad \times \sum_{i=0}^{n/2} \binom{n}{2i-1} u_t^{n-(2i-1)} u_x^{2i-1}, \quad n \text{ even}, \\ (f_n(\psi) + g_n(\rho)) \sum_{i=0}^{(n-1)/2} \binom{n}{2i} u_t^{n-2i} u_x^{2i} + (f_n(\psi) - g_n(\rho)) \\ \quad \times \sum_{i=0}^{(n+1)/2} \binom{n}{2i-1} u_t^{n-(2i-1)} u_x^{2i-1}, \quad n \text{ odd}. \end{cases} \tag{19}$$

We can use (19) to obtain solutions to (1) by adopting the contact conditional symmetry approach, i.e., from (19) the invariant surface condition, $w_n = 0$, is solved. By defining

$$F^*(\psi) = \int \left(\frac{\lambda_n}{f_n(\psi)} \right)^{1/n} d\psi, \qquad G^*(\rho) = \int \left(\frac{\lambda_n}{g_n(\rho)} \right)^{1/n} d\rho, \tag{20}$$

where λ_n is constant, the general solution (2) of (1) can be written as

$$u = F^*(\psi) + G^*(\rho). \tag{21}$$

The ancillary condition

$$w_n = 0 \tag{22}$$

is also satisfied by (21).

3. Discussion

In this paper we have derived true contact transformations for the linear wave equation. We observe that the wave equation admits an infinite number of contact transformations. By imposing $w_n = 0$ on the general solution of the linear wave equation, we are able to write the general solution in terms of coefficients of the contact transformations. This result is physically relevant if one can attach a physical significance to any of the w_n. For example, if w_p represents the conserved form of the kinetic energy, then imposing w_p on the general solution of (1) allows one to obtain solutions to the wave equation which conserves kinetic energy. We conclude this paper with an example. Let

$$f_n(\psi) = \sin(\psi) = \sin(t + x) \tag{23}$$

be a solution of the linear wave equation (1). Then using (20) we can obtain new solutions from (23). Substituting (23) into (20) and using Mathematica [12] we find that

$$F(\psi) = -\frac{1}{2}\lambda_n^{1/n} \beta\left[\cos^2(\psi), \frac{1}{2}, \frac{n-1}{2n}\right], \tag{24}$$

where β is defined as

$$\beta(z, a, b) = \int_0^z t^{a-1}(1-t)^{b-1}\, dt. \tag{25}$$

By considering $n = 1, 2, \ldots$ we can derive many solution using (24) which have all been obtained from (23).

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