About semiclassical polynomials on the unit circle corresponding to the class (2,2)

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Abstract

In the present paper we pose the problem of characterizing the orthogonal polynomials related to the unit circle whose moment functional $\mathcal{L}$ verify a functional relation of the following type $D(\phi, \mathcal{L}) + \psi, \mathcal{L} = 0$, where $\phi$ and $\psi$ are polynomials with $\deg \phi = 2$ and $\deg \psi = 1$. Two different situations appear depending on whether the roots of $\phi$ are unimodular or not. Here, we solve the last case. Moreover, we analyse the definite positive character of the solutions.

Keywords: Orthogonal polynomials; Unit circle; Semiclassical polynomials; Regular functionals; Measures on the unit circle; Difference equations

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1. Introduction

In the real line the classical orthogonal polynomials are characterized by different equivalent properties among which is the following: Given the functional equation $D(\phi, \mathcal{L}) + \psi, \mathcal{L} = 0$ where $\phi$ and $\psi$ are polynomials with $\deg \phi \leq 2$ and $\deg \psi = 1$, the only regular solutions are the functionals $\mathcal{L}$ corresponding to the classical families. For these families, the explicit form of $\phi$ and $\psi$ is as follows:

<table>
<thead>
<tr>
<th></th>
<th>$D(\phi, \mathcal{L}) + \psi, \mathcal{L} = 0$</th>
<th>$\phi(x)$</th>
<th>$\psi(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hermite</td>
<td>$1$</td>
<td>$2x$</td>
<td></td>
</tr>
<tr>
<td>Laguerre</td>
<td>$x$</td>
<td>$x-a-1$</td>
<td></td>
</tr>
<tr>
<td>Bessel</td>
<td>$x^2$</td>
<td>$-\alpha x - 2$</td>
<td></td>
</tr>
<tr>
<td>Jacobi</td>
<td>$1-x^2$</td>
<td>$(\alpha + \beta + 2)x + \alpha - \beta$</td>
<td></td>
</tr>
</tbody>
</table>

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In the present paper we pose the same problem on the unit circle $T$ of characterizing the orthogonal polynomials whose moment functionals verify this type of functional equation. In this direction we want to point out the characterization of the class $(1,1)$ in $[11, 13]$, that is the study of the solutions of the above functional equation with $\deg \phi=1$. Therefore, we try to obtain the regular and hermitian functionals $\mathcal{L}$ which are solutions of the functional equation with $\deg \phi=2$:

$$D((z-\alpha)(z-\beta))\mathcal{L} = (b_1z + b_0)\mathcal{L} \quad \text{with} \ b_1 \neq 0. \hspace{1cm} (1.1)$$

The organization of the paper is as follows: Section 2 is devoted to some preliminary results and definitions. In Section 3 we obtain that Eq. (1.1) corresponds to one of the following cases:

$$D((z-\alpha)(\bar{z} - 1))\mathcal{L} = -i((1 + |z|^2)z - 2\alpha)\mathcal{L}, \hspace{1cm} (1.2)$$

$$D((z-\alpha)(z-\beta))\mathcal{L} = -i((\alpha + \beta)z - 2\alpha\beta)\mathcal{L}. \hspace{1cm} (1.3)$$

We restrict ourselves to the first case for which we obtain in Theorem 4 the hermitian solutions. Finally, Section 4 contains the statement of our main results concerning the regularity and positive definite character of these solutions.

2. Preliminary results

Let $A = \text{span}\{z^k, k \in \mathbb{Z}\}$ be the space of the Laurent polynomials and $\mathcal{L}: A \to \mathbb{C}$ be a regular and hermitian functional. If we denote the moments by $\mathcal{L}(z^n) = c_n$ for $n \in \mathbb{Z}$ we say

**Definition 1.** $\mathcal{L}$ is hermitian if $\forall n \geq 0 \ c_{-n} = \overline{c_n}$. $\mathcal{L}$ is regular (positive definite) if the principal submatrices of the moment matrix are nonsingular (positive), i.e.,

$$\forall n \geq 0 \ \Delta_n = \det (\mathcal{L}(z^{i-j}))_{i=0,\ldots,n;\ j=0,\ldots,n} \neq 0(>0).$$

In any case we denote $\forall n \geq 0 \ E_n = \Delta_n/\Delta_{n-1}$ with $\Delta_{-1} = 1$ and by $\{\Phi_n\}_0^\infty$ the monic orthogonal polynomial sequence (MOPS) related to $\mathcal{L}$. It is well known that $\{\Phi_n\}_0^\infty$ satisfies for $n \geq 1$ the following recurrence relations:

$$\Phi_n(z) = z\Phi_{n-1}(z) + \Phi_{n-2}(0)\Phi_{n-1}^*(z), \hspace{1cm} (2.1)$$

$$\Phi_n^*(z) = \Phi_{n-1}^*(z) + \overline{\Phi_{n-1}(0)}z\Phi_{n-1}(z), \hspace{1cm} (2.2)$$

$$\Phi_n(z) = (1 - |\Phi_n(0)|^2)z\Phi_{n-1}(z) + \Phi_{n-1}(0)\Phi_n^*(z), \hspace{1cm} (2.3)$$

$$\Phi_n^*(z) = (1 - |\Phi_n(0)|^2)\Phi_{n-1}^*(z) + \overline{\Phi_{n-1}(0)}\Phi_n(z), \hspace{1cm} (2.4)$$

where $\Phi_n^*(z) = z^n\overline{\Phi_n(1/z)}$ is the reversed polynomial of $\Phi_n(z)$.

Next we recall some definitions and results concerning semiclassical functionals. For more details see [10, 8, 2, 13].
Definition 2. Given a regular and hermitian functional $\mathcal{L}$, we say that $\mathcal{L}$ is semiclassical if there exist polynomials $\phi(z) \neq 0$ and $\psi(z)$ such that the following functional equation holds:
\[
D(\phi(z)\mathcal{L}) = \psi(z)\mathcal{L},
\]
where the derivative operator is defined by
\[
\forall P \in \Lambda \quad D\mathcal{L}(P(z)) = -i\mathcal{L}(zP'(z)).
\]
This definition is motivated by the differential behaviour of positive measures on the unit circle with respect to the integration on the unit circle. If $\deg \phi(z) = p'$ and $\max \{ p' - 1, \deg [(p' - 1)\phi(z) + i\psi(z)] \} = q$ we say that $\mathcal{L}$ belongs to the class $(p', q)$.

It is obvious that if $\mathcal{L}$ belongs to the class $(p', q)$, it also belongs to the class $(p' + 1, q + 1)$.

Theorem 1. If $\mathcal{L}$ is a semiclassical functional verifying $D(\phi(z)\mathcal{L}) = \psi(z)\mathcal{L}$ then it holds the following relation:
\[
z^{p'} \phi(z)(\psi(z) - iz\phi'(z))^* = z^p(\psi(z) - iz\phi'(z))\phi^*(z)
\]
with $p' = \deg \phi(z)$ and $p = \deg (\psi(z) - iz\phi'(z))$.

Theorem 2. Let $\mathcal{L}$ be a regular functional verifying $D(\phi(z)\mathcal{L}) = \psi(z)\mathcal{L}$. Then the series $S(z) = \sum_{k=0}^{\infty} \bar{c}_k z^k$ satisfies the following differential equation:
\[
z\phi(z)S'(z) + i(\psi(z) - iz\phi'(z))S(z) = \zeta(z),
\]
where $\zeta(z)$ is a polynomial such that $\deg \zeta(z) \leq \max \{ p', q \}$.

Theorem 3. Let $\mathcal{L}$ be a regular functional verifying $D(\phi(z)\mathcal{L}) = \psi(z)\mathcal{L}$. If $\mathcal{L}$ is positive definite then the series $S(z) = \sum_{k=0}^{\infty} \bar{c}_k z^k$ converges for $z$ such that $|z| < 1$ and the series $\bar{S}(z^{-1})$ converges for $z$ such that $|z| > 1$.

3. The functional equation: Hermitian solutions

The aim of this section is to obtain that Eq. (1.1) corresponds to two different cases depending on whether the roots of $\phi$ are unimodular or not. Here we restrict ourselves to the last case for which we obtain in Theorem 4 the semiclassical solutions. Besides in Theorem 5 and Corollary 1, we prove that these hermitian solutions verify another functional equation which is more convenient for the study of the regularity in the next section.

Lemma 1. If the functional equation given by (1.1) has a regular and hermitian solution then one of the following conditions must verify:

(i) $b_0 = 1$ with $|x| \neq 1$, $b_0 = 2i\frac{x}{\bar{x}}$ and $b_1 = -\frac{1 + |x|^2}{\bar{x}}$, or

(ii) $|x| = |\beta| = 1$, $b_0 = 2i\alpha \beta$ and $b_1 = -i(\alpha + \beta)$ with $\alpha + \beta \neq 0$. 

Proof. We denote by $\mathcal{L}$ a regular and hermitian solution of Eq. (1.1) and by $c_n$ its moments. Without loss of any generality we assume that $\mathcal{L}$ is normalized, i.e., $c_0 = 1$.

Since $\mathcal{L}$ is a semiclassical functional, by applying Theorem 1 we get that

$$
(z - \alpha)(z - \beta)(b_0z^2 + (b_1 - i(\alpha + \beta))z + 2i)
= (\alpha\beta z^2 - (\alpha + \beta)z + 1)(-2iz^2 + (i(\alpha + \beta) + b_1)z + b_0)
$$

(3.1)

from which it follows by identifying coefficients that

$$
b_0 = 2i\alpha\beta
$$

and

$$
\alpha\beta b_1 - b_1 = 3i(\alpha + \beta) - i\beta(\alpha + \beta).
$$

(3.2)

Computing $D((z - \alpha)(z - \beta)\mathcal{L})(z^n) = (b_1z + b_0)\mathcal{L}(z^n)$ we obtain that the moments verify the difference equation

$$
-in\alpha + 2 + (in(\alpha + \beta) - b_1)c_{n+1} - (in(\alpha + \beta) + b_1)c_n = 0.
$$

(3.3)

Taking $n = 0$ and $n = -1$ in (3.3) we get the system of two linear equations in two unknowns $b_0$ and $b_1$

$$
b_0 + b_1c_1 = 0,
$$

(3.4)

$$
\overline{c_1}b_0 + b_1 = i(c_1 - (\alpha + \beta) + \alpha\beta c_1)
$$

(3.5)

that has a unique solution

$$
b_1 = \frac{(c_1 - (\alpha + \beta) + \alpha\beta c_1)}{1 - |c_1|^2}
$$

and

$$
b_0 = -c_1b_1,
$$

i.e.,

$$
2\alpha\beta - \alpha\beta|c_1|^2 = -c_1^2 + c_1(\alpha + \beta).
$$

(3.6)

Since $b_1 \neq 0$ then $c_1 - (\alpha + \beta) + \alpha\beta c_1 \neq 0$. Besides by taking $n = -2$ in (3.3) it is easily verified that $c_1 \neq 0$ and this implies $b_0 \neq 0$.

Computing $D((z - \alpha)(z - \beta)\mathcal{L})(z^{-n}) = (b_1z + b_0)\mathcal{L}(z^{-n})$ and combining the hermitian character of $\mathcal{L}$ and the expression of $b_0$ we find

$$
i(n - 2)\alpha\beta c_n + (\overline{b_1} - ni(\alpha + \beta))c_{n-1} + inc_{n-2} = 0
$$

(3.7)

which implies

$$
in\alpha\beta c_{n+2} + (\overline{b_1} - (n + 2)i(\alpha + \beta))c_{n+1} + i(n + 2)c_n = 0.
$$

(3.8)

Eliminating $c_{n+2}$ between this last relation and (3.3) we get

$$
(\overline{b_1} - i(n + 2)(\alpha + \beta) + in\alpha\beta(\alpha + \beta) - b_1\alpha\beta)c_{n+1} + i(n + 2)(1 - |\alpha\beta|^2)c_n = 0
$$

(3.9)
and combining relations (3.2) and (3.9) it holds
\[(n - 1)(|x|^2\bar{\beta} + |\beta|^2 \bar{x} - \bar{\beta})c_{n+1} + (n + 2)(1 - |x\beta|^2)c_n = 0.\] (3.10)

Putting \(n = 1\) in (3.10) and applying that \(c_1 \neq 0\) we get
\[|x\beta| = 1.\] (3.11)

On the other hand, taking \(n = 0\) in (3.10) and using (3.11) we obtain
\[(\bar{\beta}(|x|^2 - 1) + \bar{\alpha}(|\beta|^2 - 1))c_1 = 0,
\]
which is equivalent to
\[\bar{\beta}(|x|^2 - 1) = \frac{1}{\alpha}(|x|^2 - 1).\]

Therefore we find either \(\bar{\beta}x = 1\) with \(|x| \neq 1\) or \(|x| = 1\).

Next we obtain \(b_0\) and \(b_1\) in each of the two previous cases:
(i) If \(\bar{\beta}x = 1\) with \(|x| \neq 1\), we substitute in relation (3.2) \(b_1 = -b_0/c_1 = \frac{2i\alpha/\bar{\alpha}\Phi(0)}{-\bar{\alpha}(\Phi(0))^2} = |\Phi(0)|^2(\alpha + \Phi(0)(1 + |x|^2)).\) (3.12)

Since \(c_1 = -\Phi(0)\), Eq. (3.6) becomes in this case in
\[-\bar{\alpha}(\Phi(0))^2 = -\alpha|\Phi(0)|^2 + \Phi(0)(1 + |x|^2) + 2\alpha.\] (3.13)

Eliminating \(|\Phi(0)|^2\) between (3.12) and (3.13) it holds that \(\Phi(0)\) satisfies the following equation:
\[\bar{\alpha}(1 + |x|^2)(\Phi(0))^3 + (2|x|^2 + (1 + |x|^2)^2)(\Phi(0))^2 + 3\alpha(1 + |x|^2)\Phi(0) + 2\alpha^2 = 0,
\]
from which we conclude that \(\Phi(0)\) must be \(-2\alpha/(1 + |x|^2)\) or \(-1/\alpha\) or \(-\alpha\). For \(\Phi(0) = -1/\alpha\) or \(\Phi(0) = -\alpha\), by substituting in (3.13), we get to a contradiction \(|x| = 1\).

(ii) If \(|x| = |\beta| = 1\) proceeding in the same way as in the previous case we obtain
\[-(\Phi(0))^2 = |\Phi(0)|^2(\alpha + \beta + \Phi(0)(\alpha + \beta))\] (3.14)

by substituting \(b_1 = 2i\alpha\beta/\Phi(0)\) in (3.2). Since \(c_1 = -\Phi(0)\), eliminating \(|\Phi(0)|^2\) between (3.6) after substitution of \(c_1\) and (3.14) we find
\[(\alpha + \beta)(\Phi(0))^3 + (2\alpha\beta + (\alpha + \beta)^2)(\Phi(0))^2 + 3\alpha\beta(\alpha + \beta)\Phi(0) + 2(\alpha\beta)^2 = 0.
\]

Solving the equation we get for \(\Phi(0)\) the following values: \(-2\alpha\beta/(\alpha + \beta)\) or \(-\beta\) or \(-\alpha\). Since the two last values get to a contradiction because \(|\Phi(0)| \neq 1\) we conclude the result. \(\square\)

**Theorem 4.** Let the functional equation
\[D \left( (z-\alpha) \left( z - \frac{1}{\bar{\alpha}} \right) \right) L = (-i) \left( \frac{(1 + |x|^2)}{\bar{\alpha}} z - \frac{2\alpha}{\bar{\alpha}} \right) L\] (3.15)
with $|x| \neq 1$, and let $L$ be a regular and hermitian solution verifying the initial conditions $L(1)=1$ and $L(z) = 2x/(1 + |x|^2)$.

If we denote $L(z^n) = c_n$, then for $n \geq 0$

$$c_n = Au_n + (1 - A)v_n,$$

where

$$u_n = \frac{x^n}{1 + |x|^2}[(n + 1) + (1 - n)|x|^2], \quad v_n = \frac{1}{x^2(1 + |x|^2)}[(1 - n) + (n + 1)|x|^2],$$

$$A = 1 - \frac{x^2 \Phi_2(0)}{(1 - |x|^4)} \quad \text{and for} \quad n < 0 \quad \text{we define} \quad c_n = \bar{c}_{-n}.$$

**Proof.** Applying (3.15) to $z^n$ we find the difference equation satisfied by $c_n$:

$$n\bar{x}c_{n+2} - (n + 1)(1 + |x|^2)c_{n+1} + \alpha(n + 2)c_n = 0 \quad (3.16)$$

with $c_0 = 1$ and $c_1 = 2x/(1 + |x|^2)$.

To solve the equation we use the method of the generating function [1]. First we obtain the corresponding generating function $U(s) = \sum_{n=0}^{\infty} c_n s^n$ by solving the following differential equation:

$$\alpha(s - \bar{x}) \left( s - \frac{1}{x} \right) U'(s) - 2(\bar{x} - \alpha s^2)U(s) = -2\bar{x} - \frac{2|x|^2}{1 + |x|^2}s.$$

Then the solution is given by

$$U(s) = (-2|x|^2s - C\alpha(1 + |x|^2)s^2 - \frac{2\bar{x}}{\alpha}(1 + |x|^2 + |x|^4)s + \frac{\bar{x}^2}{\alpha}(1 + |x|^2))$$

$$\times \frac{1}{\alpha(1 + |x|^2)(s - \bar{x})^2(s - \frac{1}{x})^{2}}.$$

In order that the series converges, we need, depending on the case, $\bar{x}$ or $1/\alpha$ be a root of the numerator $U(s)$.

In the first situation we conclude $C = (1 + |x|^2 + 4|x|^4)/\alpha^2(1 + |x|^2)$ and the corresponding moments are given by

$$u_n = \frac{x^n}{1 + |x|^2}[(n + 1) + (1 - n)|x|^2].$$

Then the functional is regular and the MOPS is $\Phi_n(z) = z^{n-2}(z - \alpha)^2$ for $n \geq 2$ (see [3]).

In the second case $C = (|x|^2(4 + |x|^2 + |x|^4))/\alpha^2(1 + |x|^2)$ and the moments are

$$v_n = \frac{1}{\bar{x}^2(1 + |x|^2)}[(1 - n) + (n + 1)|x|^2].$$

Then the functional is regular and the MOPS is $\Psi_n(z) = z^{n-2}(z - 1/\bar{x})^2$ for $n \geq 2$ (see [3]).
Since both solutions are linearly independent we obtain the general solution of (3.16) is
\[ c_n = A u_n + B v_n \]
and the solutions verifying the initial conditions \( c_0 \) and \( c_1 \) are
\[ c_n = A u_n + (1 - A) v_n. \]
In order to compute the complex number \( A \) we use the regularity of \( \mathcal{L} \) which implies that \( c_2 = c_2^2 - (1 - |c_1|^2) \Phi_2(0) \). Then if we substitute in the preceding expression of \( c_n \) for \( n = 2 \) we obtain that
\[ A = \frac{1 - \bar{c}_2 \Phi_2(0)}{1 - |c_1|^4}. \]

**Theorem 5.** Let \( \mathcal{L} \) be a linear hermitian functional verifying the functional equation (3.15) with the initial conditions \( \mathcal{L}(1) = 1 \) and \( \mathcal{L}(z) = 2\alpha/(1 + |\alpha|^2) \). Then \( \mathcal{L} \) also verifies the following functional equation:
\[ D(z - \alpha)^2 \left( \frac{z - 1}{\bar{z}} \right)^2 \mathcal{L} = 2i(z - \alpha)^2 \left( \frac{z - 1}{\bar{z}} \right)^2 \mathcal{L}. \]  
(3.17)

**Proof.** To obtain the functional equation (3.17) we take into account some results in [3]. Indeed, if \( \mathcal{L} \) is a linear hermitian functional verifying (3.17) with the initial conditions \( \mathcal{L}(1) = 1 \) and \( \mathcal{L}(z) = 2\alpha/(1 + |\alpha|^2) \), it is easy to check that the corresponding moments \( \mathcal{L}(z^n) = c_n \) verify the following difference equation:
\[ (n + 2)|z|^2 c_{n+4} - 2\bar{\alpha}(1 + |\alpha|^2) c_{n+3} + (1 + |\alpha|^4 + 4|\alpha|^2) c_{n+2} - 2\alpha(1 + |\alpha|^2) c_{n+1} + \alpha^2 c_n = 0. \]  
(3.18)
On the other hand, it is easy to verify that the moments
\[ c_n = A u_n + (1 - A) v_n \]
given in Theorem 4 before satisfy Eq. (3.18), and this yields the result. \( \square \)

**Corollary 1.** Let \( \mathcal{L} \) be a linear hermitian functional verifying the functional equation (3.15) with the initial conditions \( \mathcal{L}(1) = 1 \) and \( \mathcal{L}(z) = 2\alpha/(1 + |\alpha|^2) \) and let \( \mathcal{L}_A \) be the linear functional given by
\[ \mathcal{L}_A(z^n) = \begin{cases} 0 & \text{for } n \neq 0, \\
(1 - 2\Re A)(|\alpha|^2 - 1)^3 \\
1 + |\alpha|^2 & \text{for } n = 0. \end{cases} \]
Then \( \mathcal{L} \) also verifies the following functional equation:
\[ (z - \alpha)^2 \left( \frac{1}{z - \bar{\alpha}} \right)^2 \mathcal{L} = \mathcal{L}_A, \]
with initial conditions \( \mathcal{L}(1) = 1, \mathcal{L}(z) = 2\alpha/(1 + |\alpha|^2) \) and \( \mathcal{L}(z^2) = (-A(|\alpha|^2 - 1)^3 + 3|\alpha|^2 - 1)/(1 + |\alpha|^2)\bar{\alpha}. \)
Proof. Taking into account Theorem 5 we obtain that the hermitian solutions of (3.15) with the initial conditions \( \mathcal{L}(1) = 1 \) and \( \mathcal{L}(z) = 2\alpha/(1 + |\alpha|^2) \) verify

\[
\begin{align*}
\bar{x}^2 c_{n+2} - 2\bar{x}(1 + |\alpha|^2)c_{n+1} + (1 + |\alpha|^4 + 4|\alpha|^2)c_n &= 0, \\
-2\alpha(1 + |\alpha|^2)c_{n-1} + \alpha^2 c_{n-2} &= \left\{ \\
&\begin{cases}
0 & \text{for } n \neq 0, \\
(1 - 2\Re A)(|\alpha|^2 - 1)^2 & \text{for } n = 0,
\end{cases}
\end{align*}
\]

and this implies the result. \( \square \)

Therefore, in order to get the regular solutions of our problem (Eq. (3.15)), we study in the next section, the regularity of the hermitian solutions \( \mathcal{L} \) of the functional equation \((z - \alpha)^2(1/z - \alpha)^2 \mathcal{L} = \mathcal{L}_A \) with prescribed initial conditions, i.e., we solve what is usually called an inverse problem.

4. An inverse problem. Regular solutions

This section contains the statement of our main results: Theorems 6 and 7, in which we analyse the regularity and the positive-definite character of the solutions of the posed problem.

Lemma 2. If \( \mathcal{L} \) is a regular and hermitian solution of \((z - \alpha)^2(1/z - \alpha)^2 \mathcal{L} = \mathcal{L}_A \), with \( |\alpha| \neq 1 \) and \( \mathcal{L}(1) = 1 \), \( \mathcal{L}(z) = 2\alpha/(1 + |\alpha|^2) \) and \( \mathcal{L}(z^2) = (-A(|\alpha|^2 - 1)^3 + 3|\alpha|^2 - 1)/(1 + |\alpha|^2)\bar{\alpha}^2 \), then the following relations hold:

\[
\begin{align*}
\mathcal{L} \left( \frac{1}{z - \alpha} \right)^2 (z - \alpha) \frac{1}{z^j} &= -\frac{1}{\alpha^{j-1}}(\bar{x}^2 - \Phi_2(0)) \left( \frac{1 - |\alpha|^2}{1 + |\alpha|^2} \right)^2, & \forall j \geq 0, \\
\mathcal{L} \left( \frac{1}{z - \alpha} \right)^2 \frac{1}{z^j} &= \frac{1}{\alpha^j(j + 1)}(\bar{x}^2 - \Phi_2(0)) \left( \frac{1 - |\alpha|^2}{1 + |\alpha|^2} \right)^2, & \forall j \geq 0, \\
\mathcal{L} \left( \frac{1}{z - \alpha} \right)^2 (z - \alpha)z^{j+1} &= \alpha^j(1 - \bar{x}^2\Phi_2(0)) \left( \frac{1 - |\alpha|^2}{1 + |\alpha|^2} \right)^2, & \forall j \geq 0, \\
\mathcal{L} \left( \frac{1}{z - \alpha} \right)^2 z^{j+2} &= \alpha^j(j + 1)(1 - \bar{x}^2\Phi_2(0)) \left( \frac{1 - |\alpha|^2}{1 + |\alpha|^2} \right)^2, & \forall j \geq 0.
\end{align*}
\]

Proof. We denote by \( K(A) \) the number \( \mathcal{L}_A(1) \) and by \( \mathcal{L}_0 \) the Lebesgue functional. From Corollary 1 we have the following relation:

\[
(z - \alpha)^2 \left( \frac{1}{z - \alpha} \right)^2 \mathcal{L} = K(A) \mathcal{L}_0.
\]
(1) For $j \geq 0$ we have
\[ L \left( \left( \frac{1}{z} - \alpha \right)^2 \left( z - \alpha \right) \frac{1}{z^j} \right) = L \left( \left( \frac{1}{z} - \alpha \right)^2 \left( z - \alpha \right) \left( \frac{1}{z^j} - \frac{1}{\alpha^j} + \frac{1}{\alpha^j} \right) \right) \]
\[ = L \left( \left( \frac{1}{z} - \alpha \right)^2 \left( z - \alpha \right) \left( \frac{\alpha^j - z^j}{\alpha z^j} \right) \right) + \frac{1}{\alpha^j} L \left( \left( \frac{1}{z} - \alpha \right)^2 \left( z - \alpha \right) \right). \]
Since for $j \geq 2$ $z^j - \alpha^j = (z - \alpha) \sum_{k=0}^{j-1} \alpha^k z^{j-k-1}$, then
\[ L \left( \left( \frac{1}{z} - \alpha \right)^2 \left( z - \alpha \right) \frac{z^j - \alpha^j}{z^j} \right) = K(A) L_0 \left( \sum_{k=0}^{j-1} \alpha^k z^{j-k-1} \right) = 0. \]
Therefore for $j \geq 2$ it holds
\[ L \left( \left( \frac{1}{z} - \alpha \right)^2 \left( z - \alpha \right) \frac{1}{z^j} \right) = \frac{1}{\alpha^j} L \left( \left( \frac{1}{z} - \alpha \right)^2 \left( z - \alpha \right) \right) \]
and this expression is also valid for $j = 0$ and $j = 1$. Next we calculate
\[ L \left( \left( \frac{1}{z} - \alpha \right)^2 \left( z - \alpha \right) \right) = L \left( \left( \frac{1}{z^2} - \frac{2z}{z^2} + \alpha^2 \right) \left( z - \alpha \right) \right) \]
\[ = \bar{c}^2 - 2\bar{c} + \alpha^2 c_1 - \alpha L \left( \left( \frac{1}{z} - \alpha \right)^2 \right). \]
Since $c_1 = 2\alpha/(1 + |\alpha|^2)$, then $\bar{c}^2 - 2\bar{c} + \alpha^2 c_1 = 0$ and
\[ L \left( \left( \frac{1}{z} - \alpha \right)^2 \left( z - \alpha \right) \right) = -\alpha L \left( \left( \frac{1}{z} - \alpha \right)^2 \right) = -\alpha(\bar{c}_2 - 2\bar{c}c_1 + \alpha^2). \]
From $\Phi_2(0) = (c_1^2 - c_2)/(1 - |c_1|^2)$ we get $\bar{c}_2 - 2\bar{c}c_1 + \alpha^2 = (\alpha^2 - \Phi_2(0))(1 - |\alpha|^2)/(1 + |\alpha|^2)$ and then
\[ L \left( \left( \frac{1}{z} - \alpha \right)^2 \left( z - \alpha \right) \right) = -\alpha(\bar{c}^2 - \Phi_2(0)) \left( \frac{1 - |\alpha|^2}{1 + |\alpha|^2} \right)^2. \]

(2) Let $j \geq 1$. We compute
\[ L \left( \left( \frac{1}{z} - \bar{\alpha} \right)^2 \frac{1}{z^j} \right) = L \left( \left( \frac{1}{z} - \bar{\alpha} \right)^2 z^2 \left( \frac{1}{z^{j+2}} - \frac{1}{\alpha^{j+2}} + \frac{1}{\alpha^{j+2}} \right) \right) \]
\[ = L \left( \left( \frac{1}{z} - \bar{\alpha} \right)^2 z^2 \left( \frac{\alpha^{j+2} - z^{j+2}}{z^{j+2}\alpha^{j+2}} \right) \right) + \frac{1}{\alpha^{j+2}} L \left( \left( \frac{1}{z} - \bar{\alpha} \right)^2 z^2 \right). \]
Since
\[ L \left( \left( \frac{1}{z} - \bar{\alpha} \right)^2 \frac{z^{j+2} - \alpha^{j+2}}{z^j} \right) = L \left( \left( \frac{1}{z} - \bar{\alpha} \right)^2 \left( z - \alpha \right) \left( \sum_{i=0}^{j+1} \alpha^i z^{j+1-i} \right) \right) \]
and we take into account that
\[ \sum_{i=0}^{j} x^i z^i = (z - \alpha) \left( \sum_{i=0}^{j} (i + 1)x^i z^i \right) + (j + 2)x^{j+1}, \]
then
\[
\mathcal{L} \left( \frac{1}{z} - \bar{\alpha} \right)^2 \frac{(z^{j+2} - \alpha^{j+2})}{z^j} \\
= \mathcal{L} \left( \frac{1}{z} - \bar{\alpha} \right)^2 \frac{(z - \alpha)^2}{z^j} \left( \sum_{i=0}^{j} (i + 1)x^i z^{j-i} \right) + \mathcal{L} \left( \frac{1}{z} - \bar{\alpha} \right)^2 \frac{(z - \alpha)}{z^j} (j + 2)x^{j+1} \\
= K(A) \mathcal{L}_0 \left( \sum_{i=0}^{j} (i + 1)x^i \frac{1}{z} \right) + (j + 2)x^{j+1} \mathcal{L} \left( \frac{1}{z} - \bar{\alpha} \right)^2 \frac{(z - \alpha)}{z^j} \\
= K(A) + (j + 2)x^{j+1} \left( -\frac{1}{x^{j+1}} \right) (\bar{x}^2 - \Phi_2(0)) \left( 1 - \frac{|x|^2}{1 + |x|^2} \right)^2,
\]
where the last equality follows from (1) above.

Since \( A = (1 - \bar{x}^2 \Phi_2(0))/(1 - |x|^4) \) then
\[
K(A) = \left( \frac{1 - |x|^2}{1 + |x|^2} \right)^2 \left( 1 + |x|^4 - 2\Re(\bar{x}^2 \Phi_2(0)) \right). \tag{4.5}
\]
Therefore, by substituting \( K(A) \),
\[
\mathcal{L} \left( \frac{1}{z} - \bar{\alpha} \right)^2 \frac{(z^{j+2} - \alpha^{j+2})}{z^j} = \left( \frac{1 - |x|^2}{1 + |x|^2} \right)^2 \left( 1 - (j + 1)|x|^4 - 2\Re(\bar{x}^2 \Phi_2(0)) \right) \\
+ (j + 2)x^2 \Phi_2(0)).
\]
On the other hand, \( \mathcal{L}((1/z - \bar{\alpha})^2 z^2) = 1 - 2\Re c_1 + \bar{x}^2 c_2 = ((1 - |x|^2)/(1 + |x|^2))^2(1 - \bar{x}^2 \Phi_2(0)) \). Therefore,
\[
\mathcal{L} \left( \frac{1}{z} - \bar{\alpha} \right)^2 \frac{1}{z^j} = \frac{1}{\alpha^j (j + 1)(\bar{x}^2 - \Phi_2(0))} \left( \frac{1 - |x|^2}{1 + |x|^2} \right)^2.
\]
It is easy to see that the relation is true for \( j = 0 \) and thus we get (2).
(3)
\[
\mathcal{L} \left( \frac{1}{z} - \bar{\alpha} \right)^2 (z - \alpha) z^{j+1} = \mathcal{L} \left( \frac{1}{z} - \bar{\alpha} \right)^2 (z - \alpha)(z^{j+1} - \alpha^{j+1} + \alpha^{j+1}) \\
= \mathcal{L} \left( \frac{1}{z} - \bar{\alpha} \right)^2 (z - \alpha)(z^{j+1} - \alpha^{j+1}) + \alpha^{j+1} \mathcal{L} \left( \frac{1}{z} - \bar{\alpha} \right)^2 (z - \alpha).
\]
For \( j \geq 1 \) it holds
\[
\mathcal{L} \left( \left( \frac{1}{z} - \overline{\alpha} \right)^2 (z - \alpha)(z - \alpha) \sum_{i=0}^{j} \alpha^i z^{j-i} \right) + \alpha^{j+1} \mathcal{L} \left( \left( \frac{1}{z} - \overline{\alpha} \right)^2 (z - \alpha) \right) = K(A) \mathcal{L}_0 \left( \sum_{i=0}^{j} \alpha^i z^{j-i} \right) + \alpha^{j+1} \mathcal{L} \left( \left( \frac{1}{z} - \overline{\alpha} \right)^2 (z - \alpha) \right).
\]

By substituting \( K(A) \) by expression (4.5) and using (1) for \( j = 0 \) we finally obtain (3). Besides it is easy to check the equality for \( j = 0 \):
\[
\mathcal{L} \left( \left( \frac{1}{z} - \overline{\alpha} \right)^2 (z - \alpha) \right) = \mathcal{L} \left( \left( \frac{1}{z} - \overline{\alpha} \right)^2 (z - \alpha)(z - \alpha + \alpha) \right) = K(A) \mathcal{L}_0(1) + \alpha \mathcal{L} \left( \left( \frac{1}{z} - \overline{\alpha} \right)^2 (z - \alpha) \right) = (1 - \alpha^2) \Phi_1(0) \left( \frac{1 - |\alpha|^2}{1 + |\alpha|^2} \right)^2.
\]

(4)
\[
\mathcal{L} \left( \left( \frac{1}{z} - \overline{\alpha} \right)^2 z^{j+2} \right) = \mathcal{L} \left( \left( \frac{1}{z} - \overline{\alpha} \right)^2 (z^{j+2} - \alpha^{j+2}) \right) + \alpha^{j+2} \mathcal{L} \left( \left( \frac{1}{z} - \overline{\alpha} \right)^2 \right)
\]
\[
= \mathcal{L} \left( \left( \frac{1}{z} - \overline{\alpha} \right)^2 (z - \alpha) \sum_{i=0}^{j} \alpha^{i+1} \left( z^{j-i} + (j+2) \alpha^{i+1} \right) \right) + \alpha^{j+2} \mathcal{L} \left( \left( \frac{1}{z} - \overline{\alpha} \right)^2 \right)
\]
\[
= K(A) \mathcal{L}_0 \left( \sum_{i=0}^{j} \alpha^{i+1} z^{j-i} \right) + \alpha^{j+2} \mathcal{L} \left( \left( \frac{1}{z} - \overline{\alpha} \right)^2 \right).
\]

By using (1) and (2) for \( j = 0 \) we get
\[
\mathcal{L} \left( \left( \frac{1}{z} - \overline{\alpha} \right)^2 z^{j+2} \right) = (j+1) \alpha^j K(A) + (-j+2) \alpha^{j+2} \alpha^{j+2} (\overline{\alpha}^2 - \overline{\Phi}_2(0)) \left( \frac{1 - |\alpha|^2}{1 + |\alpha|^2} \right)^2,
\]
from which, by using (4.5), the result follows. \( \square \)

In the next lemma two representations of the orthogonal polynomials \( \{ \Phi_n(z) \} \) are given. These representations are necessary for obtaining the values \( \Phi_n(\alpha) \) and \( \Phi_n'(\alpha) \) in Theorem 6.

**Lemma 3.** If \( \mathcal{L} \) is a regular and hermitian solution of \( (z - \alpha)^2 (1/z - \overline{\alpha})^2 \mathcal{L} = \mathcal{L}_A \) with \( |\alpha| \neq 1 \) and initial conditions \( \mathcal{L}(1) = 1, \ \mathcal{L}(z) = 2\alpha/(1 + |\alpha|^2) \) and \( \mathcal{L}(z^2) = (-A(|\alpha|^2 - 1)^3 + 3|\alpha|^2 - 1)/(1 + |\alpha|^2) \overline{\alpha}^2 \), then \( E_n \neq 0 \), with
\[
E_n = \left( \frac{1 - |\alpha|^2}{1 + |\alpha|^2} \right)^2 \left( \frac{(\overline{\alpha}^2 - \overline{\Phi}_2(0))}{\alpha^{n-2}} \right) [(n-1)\Phi_n(\alpha) - \alpha \Phi_n'(\alpha)]
\]
\[
+ 1 + |\alpha|^4 - 2R(\overline{\alpha}^2 \Phi_2(0)) \quad \text{for} \ n \geq 3.
\]
Besides the orthogonal polynomial sequence \( \{ \Phi_n(z) \} \) related to \( \mathcal{L} \) is given by the following representations:

\[
\Phi_n(z) = z^n + z^{n-1} \left( -2\alpha - [(n - 2)\Phi_n(z) - z\Phi'_n(z)] \frac{(\bar{\alpha}^2 - \Phi_2(0))}{x^{n-2}(1 + |x|^4 - 2\Re(\bar{\alpha}^2 \Phi_2(0)))} \right) \\
+ z^{n-2} \left( \alpha^2 + [(n - 1)\Phi_n(z) - z\Phi'_n(z)] \frac{(\bar{\alpha}^2 - \Phi_2(0))}{x^{n-4}(1 + |x|^4 - 2\Re(\bar{\alpha}^2 \Phi_2(0)))} \right) \\
+ z\Phi'_n(z) \frac{1 - \bar{\alpha}^2 \Phi_2(0)}{1 + |x|^4 - 2\Re(\bar{\alpha}^2 \Phi_2(0))} + \left[ \Phi_n(z) - z\Phi'_n(z) \right] \\
\times \frac{1 - \bar{\alpha}^2 \Phi_2(0)}{1 + |x|^4 - 2\Re(\bar{\alpha}^2 \Phi_2(0))}, \quad \forall n \geq 3; \quad (4.7)
\]

\[
\Phi_2(z) = z^2 - 2\left( \alpha + \frac{\Phi_2(0)\bar{\alpha}}{1 + |x|^2} \right)z + \Phi_2(0),
\]

\[
\Phi_1(z) = z - \frac{2\alpha}{1 + |x|^2},
\]

\[
\Phi_n(z) = z^n + z^{n-1} \left( -2\alpha - [(n - 2)\Phi_n(z) - z\Phi'_n(z)] \frac{(\bar{\alpha}^2 - \Phi_2(0))}{x^{n-2}(1 + |x|^4 - 2\Re(\bar{\alpha}^2 \Phi_2(0)))} \right) \\
+ z^{n-2} \left( \alpha^2 + [(n - 1)\Phi_n(z) - z\Phi'_n(z)] \frac{(\bar{\alpha}^2 - \Phi_2(0))}{x^{n-4}(1 + |x|^4 - 2\Re(\bar{\alpha}^2 \Phi_2(0)))} \right) \\
+ z\Phi'_n(z) \frac{1 - \bar{\alpha}^2 \Phi_2(0)}{1 + |x|^4 - 2\Re(\bar{\alpha}^2 \Phi_2(0))} + \left[ \Phi_n(z) - z\Phi'_n(z) \right] \\
\times \frac{1 - \bar{\alpha}^2 \Phi_2(0)}{1 + |x|^4 - 2\Re(\bar{\alpha}^2 \Phi_2(0))}, \quad \forall n \geq 3. \quad (4.8)
\]

**Proof.** We distinguish two cases:

(i) We suppose that \( K(A) \neq 0 \) (or equivalently, \( 2\Re A \neq 1 \)). We can write for \( n \geq 3 \):

\[
\frac{\Phi_n(z) - \Phi_n(\alpha) - \Phi'_n(\alpha)(z - \alpha)}{(z - \alpha)^2} = z^{n-2} + \sum_{k=0}^{n-3} \lambda_{nk} z^k.
\]

Applying \((z - \alpha)^2(\frac{1}{z} - \bar{\alpha})^2\mathcal{L}\) to both members we get

\[
K(A)\mathcal{L}_0 \left( \frac{\Phi_n(z) - \Phi_n(\alpha) - \Phi'_n(\alpha)(z - \alpha)}{(z - \alpha)^2} z^{-j} \right) = \begin{cases} 
K(A) & \text{if } j = n - 2, \\
K(A)\lambda_{nj} & \text{if } j \in \{0, \ldots, n - 3\},
\end{cases}
\]

i.e.,
\[
L \left( \left( \frac{1}{z - \bar{\alpha}} \right)^2 \Phi_n(z) z^{-j} \right) - \Phi_n(\alpha) L \left( \left( \frac{1}{z - \bar{\alpha}} \right)^2 z^{-j} \right)
\]
\[
- \Phi_n'(\alpha) L \left( \left( \frac{1}{z - \bar{\alpha}} \right)^2 (z - \alpha) z^{-j} \right)
\]
\[
= \begin{cases} 
K(A) & \text{for } j = n - 2, \\
K(A) \lambda_{nj} & \text{for } j \in \{0, \ldots, n - 3\}.
\end{cases}
\] (4.9)

Since
\[
L \left( \left( \frac{1}{z - \bar{\alpha}} \right)^2 \Phi_n(z) z^{-j} \right) = L(z^{-(j+2)} \Phi_n(z))
\]
\[
= \begin{cases} 
E_n & \text{for } j = n - 2, \\
0 & \text{for } j \in \{0, \ldots, n - 3\},
\end{cases}
\] (4.10)

if we take into account the previous lemma we obtain for \( j \in \{0, \ldots, n - 3\} \)
\[
\lambda_{nj} = -\frac{\alpha^{-j} \left( \overline{\alpha}^2 - \Phi_2(0) \right)}{1 + |\alpha|^4 - 2 \Re(\overline{\alpha}^2 \Phi_2(0))} \left[(j + 1) \Phi_n(\alpha) - \lambda \Phi_n'(\alpha)\right]
\]

and
\[
\frac{\Phi_n(z) - \Phi_n(\alpha) - \Phi_n'(\alpha) (z - \alpha)}{(z - \alpha)^2} = z^{n-2} - \lambda \Phi_n(\alpha) \sum_{j=0}^{n-3} (j + 1) \left( \frac{z}{\alpha} \right)^j + \lambda \alpha \Phi_n'(\alpha) \sum_{j=0}^{n-3} \left( \frac{z}{\alpha} \right)^j,
\]

with
\[
\lambda = \frac{\left( \overline{\alpha}^2 - \Phi_2(0) \right)}{1 + |\alpha|^4 - 2 \Re(\overline{\alpha}^2 \Phi_2(0))}.
\]

By computing the sums
\[
\sum_{j=0}^{n-3} \left( \frac{z}{\alpha} \right)^j = \frac{z^{n-2} - \alpha^{-n-2}}{\alpha^{-n-3}(z - \alpha)} \quad \text{and} \quad \sum_{j=0}^{n-3} (j + 1) \left( \frac{z}{\alpha} \right)^j = \frac{(n - 2)z^{n-1} - \alpha(n - 1)z^{n-2} + \alpha^{-n-1}}{\alpha^{-n-3}(z - \alpha)^2}
\]

we have for \( n \geq 3 \)
\[
\Phi_n(z) = z^n + z^{n-1} \left(-2\alpha - \lambda(n - 2) \frac{\Phi_n(\alpha)}{\alpha^{n-3}} + \lambda \frac{\Phi_n'(\alpha)}{\alpha^{n-4}}\right)
\]
\[
+ z^{n-2} \left(\alpha^2 + \lambda(n - 1) \frac{\Phi_n(\alpha)}{\alpha^{n-4}} - \lambda \frac{\Phi_n'(\alpha)}{\alpha^{n-5}}\right) + z(1 - \lambda \alpha^2) \Phi_n'(\alpha)
\]
\[
+ (1 - \lambda \alpha^2)[\Phi_n(\alpha) - \alpha \Phi_n'(\alpha)].
\] (4.11)
Taking into account that
\[ 1 - \lambda x^2 = \frac{(1 - \overline{x}^2 \Phi_2(0))}{1 + |x|^4 - 2\Re(x^2 \Phi_2(0))} \]
we obtain (4.7). Besides, from (4.9) and (4.10) for \(j = n-2\) and the previous lemma we get (4.6):
\[
E_n = \left(1 - \frac{|x|^2}{1 + |x|^2}\right)^2 \left(1 + |x|^4 - 2\Re(x^2 \Phi_2(0))\right) \left(n - 1\right) \frac{\Phi_n(x)}{\alpha^{n-2}} - \lambda \frac{\Phi_n'(x)}{\alpha^{n-3}} + 1.
\] (4.12)
Therefore, since \(\mathcal{L}\) is regular \(E_1 = ((1 - |x|^2)/(1 + |x|^2))^2 \neq 0\), \(E_2 = (1 - \Phi_2(0)^2)((1 - |x|^2)/(1 + |x|^2))^2 \neq 0\), and for \(n \geq 3\)
\[
\left(1 - \frac{\Phi_n(x)}{\alpha^{n-2}} - \frac{\Phi_n'(x)}{\alpha^{n-3}}\right) \frac{(\alpha^2 - \Phi_2(0))}{1 + |x|^4 - 2\Re(x^2 \Phi_2(0))} + 1 \neq 0.
\]
In order to obtain \(\Phi_n(z)\) we need to know \(\Phi_n(x)\) and \(\Phi_n'(x)\). To solve this problem we proceed in a similar way as before: Since for \(n \geq 3\)
\[
\frac{\Phi_n(z) - \Phi_n(z^{1/2}) - \Phi_n'(z^{1/2})(z - \frac{1}{2})}{(z - \frac{1}{2})^2} = z^{n-2} + \sum_{k=0}^{n-3} \mu_{nk} z^k
\]
then
\[
z^2 \frac{x^2 \overline{\Phi_n(z^{1/2})}}{(z - \overline{x})^2} = z^{n-2} + \sum_{k=0}^{n-3} \mu_{nk} \frac{1}{z^k}.
\]
Applying \((z - \alpha)^2(1/z - \overline{\alpha})^2\mathcal{L}\) to both members we get
\[
\mathcal{L} \left(\left(\frac{1}{z - \overline{x}}\right)^2 x^2 \overline{\Phi_n(z^{1/2})} \left(\frac{1}{z}\right) z^{j+2} \right) - \Phi_n(1/\overline{x}) \mathcal{L} \left(\left(\frac{1}{z - \overline{x}}\right)^2 x^2 z^{j+2}\right)
\]
\[
+ \Phi_n'(1/\overline{x}) \mathcal{L} \left(\left(\frac{1}{z - \overline{x}}\right)^2 (z - \alpha)xz^{j+1}\right)
\]
\[
= \begin{cases} K(A) & \text{for } j = n-2, \\
K(A)\mu_{nj} & \text{for } j \in \{0, \ldots, n-3\}. \end{cases}
\]
For \(j \in \{0, \ldots, n-3\}\), taking into account Lemma 2 and applying that
\[
\mathcal{L} \left(\left(\frac{1}{z - \overline{x}}\right)^2 \Phi_n(z^{1/2}) \left(\frac{1}{z}\right) z^{j+2}\right) = \begin{cases} \alpha^2 E_n & \text{for } j = n-2, \\
0 & \text{for } j \in \{0, \ldots, n-3\}, \end{cases}
\]
we deduce
\[
\mu_{nj} = - \left[\Phi_n(1/\overline{x}) \overline{\alpha}(j + 1) - \Phi_n'(1/\overline{x})\right] \overline{x}^{j+1} \frac{(1 - \alpha^2 \overline{\Phi_2(0)})}{1 + |x|^4 - 2\Re(x^2 \Phi_2(0))}.
\]
Therefore, if we denote by
\[ \mu = \frac{(1 - x^2\Phi_2(0))}{1 + |x|^2 - 2\Re(x^2\Phi_2(0))} \]
it holds that
\[ \frac{\Phi_n(z) - \Phi_n\left(\frac{1}{\alpha}\right) - \Phi_n'\left(\frac{1}{\alpha}\right) (z - \frac{1}{\alpha})}{(z - \frac{1}{\alpha})^2} = z^{n-2} + \mu \sum_{j=0}^{n-3} \left( -\Phi_n\left(\frac{1}{\alpha}\right) \overline{\alpha}^{j+2}(j+1) + \Phi_n'\left(\frac{1}{\alpha}\right) \overline{\alpha}^{j+1} \right) z^j. \]

If we compute the sums and make some calculations we get
\[ \Phi_n(z) - \Phi_n\left(\frac{1}{\alpha}\right) - \Phi_n'\left(\frac{1}{\alpha}\right) (z - \frac{1}{\alpha}) = z^{n-2} \left( z - \frac{1}{\alpha} \right)^2 - \mu \Phi_n\left(\frac{1}{\alpha}\right) \left[ (n-2)(\overline{\alpha}z)^{n-1} - (n-1)(\overline{\alpha}z)^{n-2} + 1 \right] + \mu \Phi_n'\left(\frac{1}{\alpha}\right) \left( (\overline{\alpha}z)^{n-2} - 1 \right) \left( z - \frac{1}{\alpha} \right). \]

Finally, if we substitute \( \mu \) in (4.13) we obtain (4.8).

(ii) Let us now assume that \( K(A) = 0 \) (2\Re A = 1). Proceeding in the same way as in the previous case we obtain from (4.9) for \( j \in \{0, \ldots, n-3\} \):
\[ \left( z_2 - \overline{\Phi}_2(0) \right) \left( 1 - |x|^2 \right)^2 (j + 1) \Phi_n(x) - x \Phi_n'(x) = 0. \]

Since \( A \neq 1 \) then \( \Phi_2(0) \neq x^2 \) and the last equation implies
\[ (j + 1) \Phi_n(x) = x \Phi_n'(x), \quad \forall j \in \{0, \ldots, n-3\} \text{ and } n \geq 3. \]

Then
\[ \Phi_n(x) = \Phi_n'(x) = 0, \quad \forall n \geq 3. \]

For \( j = n - 2 \) from (4.9) and (4.10) we obtain that \( E_n = 0, n \geq 3 \) in contradiction with the regularity of \( L \).

Theorem 6. If \( L \) is a hermitian solution of \((z - x)^2(1/z - \overline{x})^2 L = L_A \), with \( |x| \neq 1 \) and initial conditions \( L(1) = 1, L(z) = 2x/(1 + |x|^2), L(z^2) = (-A(|x|^2 - 1)^3 + 3|x|^2 - 1)/(1 + |x|^2\overline{x}^2) \) and \( |\Phi_2(0)| \neq 1 \), then \( L \) is regular if and only if
\[ \forall n \geq 3 \quad \alpha^{2n-7} D_n \neq 0, \quad (4.14) \]

with
\[ \alpha^{2n-7} D_n = |x^2 - \Phi_2(0)|^4 |x|^2 + 1 - x^2 \overline{\Phi}_2(0)|^4 |x|^{2(2n-7)} - |x^2 - \Phi_2(0)|^2 (1 - x^2 \overline{\Phi}_2(0))^2 ((n-2)^2 |x|^4 - 2(n-1)(n-3)|x|^2 + (n-2)^2) |x|^{2(n-4)}. \]

(4.15)
Besides the orthogonal polynomial sequence \( \{ \Phi_n(z) \} \) related to \( \mathcal{L} \) is given by (4.7) with

\[
\Phi_n(x) = \frac{(1 - |x|^2)^2}{x^D_n} [1 + |x|^4 - 2\Re(\bar{x}^2 \Phi_2(0))][x^2 - \Phi_2(0)]
\]

\[
\times \left( \frac{|x^2 - \Phi_2(0)|^2}{\alpha^n - 4} + |1 - x^2 \Phi_2(0)|^2 x^{n-4} \left[ 1 - n + (n - 2)|x|^2 \right] \right),
\]

(4.16)

\[
\Phi'_n(x) = \frac{(1 - |x|^2)^2}{x^D_n} [1 + |x|^4 - 2\Re(\bar{x}^2 \Phi_2(0))][x^2 - \Phi_2(0)]
\]

\[
\times \left( \frac{|x^2 - \Phi_2(0)|^2}{\alpha^n - 3} \left[ (n - 2) - n|x|^2 \right] - |1 - x^2 \Phi_2(0)|^2 x^{n-3} \left[ -n + (n - 2)|x|^2 \right] \right).
\]

(4.17)

**Proof.** \( \Rightarrow \): Taking into account Lemma 3, first we compare coefficients between (4.7) and (4.8). If we identify the coefficients of \( z \) we get

\[
\forall n \geq 3 \quad \bar{x}^2(x^2 - \Phi_2(0)) \Phi'_n \left( \frac{1}{x} \right) = (1 - \bar{x}^2 \Phi_2(0)) \Phi'_n(x).
\]

Therefore, if \( \Phi_2(0) = x^2 \) then \( \Phi'_n(x) = 0 \quad \forall n \geq 3 \), and if \( \Phi_2(0) \neq x^2 \) then

\[
\Phi'_n \left( \frac{1}{x} \right) = \frac{(1 - \bar{x}^2 \Phi_2(0))}{\bar{x}^2(x^2 - \Phi_2(0))} \Phi'_n(x).
\]

(4.18)

In the same way, identifying the independent terms

\[
\bar{x}^2(x^2 - \Phi_2(0)) \left[ \Phi_n \left( \frac{1}{x} \right) - \frac{1}{\alpha} \Phi'_n \left( \frac{1}{x} \right) \right] = (1 - \bar{x}^2 \Phi_2(0))[\Phi_n(x) - x \Phi'_n(x)],
\]

thus if \( \Phi_2(0) = x^2 \) then \( \Phi_n(x) = 0 \quad \forall n \geq 3 \), and \( \Phi_n(z) = x^{n-2}(z - x)^2 \). If \( \Phi_2(0) \neq x^2 \) and we take into account (4.18) then

\[
\Phi_n \left( \frac{1}{x} \right) = \frac{(1 - \bar{x}^2 \Phi_2(0))}{\bar{x}^2(x^2 - \Phi_2(0))} [\bar{x} \Phi_n(x) + (1 - |x|^2) \Phi'_n(x)].
\]

(4.19)

From the coefficients of \( z^{n-1} \) we deduce

\[
(\bar{x}^2 - \Phi_2(0)) \left( \frac{n - 2}{\alpha^n - 3} \Phi_n(x) + \frac{1}{\alpha^n - 4} \Phi'_n(x) \right) + (1 - \bar{x}^2 \Phi_2(0))
\]

\[
\times \left( (n - 2)\bar{x}^{n-1} \Phi_n \left( \frac{1}{x} \right) - \bar{x}^{n-2} \Phi'_n \left( \frac{1}{x} \right) \right)
\]

\[
= 2 \frac{|x|^2 - 1}{\alpha} [1 + |x|^4 - 2\Re(\bar{x}^2 \Phi_2(0))].
\]

(4.20)
From the coefficients of \( z^n \) we get

\[
(n-1) \left( \frac{1}{\bar{x}} \right) - \bar{x}^{n-3} \Phi_n \left( \frac{1}{x} \right) = \frac{1 - |x|^2}{\bar{x}^2} [1 + |x|^4 - 2\Re(\bar{x}^2 \Phi_2(0))].
\]

(4.21)

We eliminate \( \Phi_n'(1/\bar{x}) \) between (4.20) and (4.21) and we substitute \( \Phi_n(1/\bar{x}) \) by using (4.19) obtaining

\[
\left( \frac{|x|^2 - \Phi_2(0)}{x^2 - \Phi_2(0)} \right) \left[ (n-1)|x|^2 + (n-2) - |1 - x^2 \Phi_2(0)|^2 \bar{x}^{n-3} \right] \Phi_n(x)
\]

\[
+ \left( \frac{|x|^2 - \Phi_2(0)}{x^2 - \Phi_2(0)} \right) \left[ (n-1)|x|^2 + (n-2) - |1 - x^2 \Phi_2(0)|^2 \bar{x}^{n-4} \right] \Phi_n'(x)
\]

\[
= \frac{1 - |x|^2}{\bar{x}} [1 + |x|^4 - 2\Re(\bar{x}^2 \Phi_2(0))].
\]

(4.22)

From (4.20) and (4.21) we eliminate \( \Phi_n(1/\bar{x}) \) and taking into account (4.18) we have

\[
(n-1)(n-2)(\bar{x}^2 - \Phi_2(0)) \left( \frac{|x|^2 - 1}{\bar{x}^2 - \Phi_2(0)} \right) \Phi_n(x)
\]

\[
+ \left( \frac{|x|^2 - \Phi_2(0)}{x^2 - \Phi_2(0)} \right) \left[ (n-1)|x|^2 + (n-2) - |1 - x^2 \Phi_2(0)|^2 \bar{x}^{n-4} \right] \Phi_n'(x)
\]

\[
= \frac{(1 - |x|^2)}{\bar{x}} \left( 1 + |x|^4 - 2\Re(\bar{x}^2 \Phi_2(0)) \right) (n - (n-2)|x|^2).
\]

(4.23)

We can consider the two last equations as a linear system in the unknowns \( \Phi_n(x) \) and \( \Phi_n'(x) \). If we denote the determinant of the matrix coefficients by \( D_n \) we know, since \( \mathcal{L} \) is regular, that \( D_n \) must be different from 0 for all \( n \geq 3 \). Taking into account that

\[
D_n = \frac{|x|^2 - \Phi_2(0)}{x^{n-7}} |x|^2 - |x|^2 - \Phi_2(0) |^2 \bar{x}^{n-4} \frac{x^2 - \Phi_2(0)}{x^{n-3}}
\]

\[
\times (n-2)|x|^4 - 2(n-1)(n-3)|x|^2 + (n-2)^2 + |1 - x^2 \Phi_2(0)|^4 \bar{x}^{2n-7},
\]

(4.24)

we deduce that \( x^{2n-7} \in \mathbb{R} \) and we obtain (4.15). By solving the system we get the expressions of the unknowns \( \Phi_n(x) \) and \( \Phi_n'(x) \) given by (4.16) and (4.17). Finally, from (4.13) we obtain

\[
\bar{x}D_n E_n \left( \frac{1 + |x|^2}{1 - |x|^2} \right) \frac{1}{1 + |x|^4 - 2\Re(\bar{x}^2 \Phi_2(0))}
\]

\[
= \bar{x}D_n + \frac{\bar{x}^2 - \Phi_2(0)}{1 + |x|^4 - 2\Re(\bar{x}^2 \Phi_2(0))] \bar{x}^{n-2} ((n-1) \bar{x}D_n \Phi_n(x) - \alpha \Phi_n'(x) \bar{x}D_n).\]
If we substitute (4.16) and (4.17) and operate we get

$$
\frac{\bar{D}_n E_n}{1 + |x|^2} \left( \frac{1 + |x|^2}{1 - |x|^2} \right)^2 = \left[ 1 + |x|^4 - 2\Re(\bar{x}^2 \Phi_2(0)) \right] \frac{D_{n+1}}{\bar{x}},
$$

from which we conclude

$$
\forall n \geq 3 \quad E_n = \frac{D_{n+1}}{|x|^4 \alpha^{2n-5} D_n} \left( \frac{1 - |x|^2}{1 + |x|^2} \right)^2 \left[ 1 + |x|^4 - 2\Re(\bar{x}^2 \Phi_2(0)) \right].
$$

Thus, if $\mathcal{L}$ is regular the result holds.

$\Leftarrow$: We suppose that $D_n \neq 0 \quad \forall n \geq 3$ and we prove that the sequence of polynomials given by (4.7) is the MOPS related to $\mathcal{L}$. We shall calculate $\mathcal{L}(\Phi_n(z)z^{-k})$ distinguishing several cases since the result depends on whether the moments have positive or negative subindex. From (4.11)

$$
\Phi_n(z) = z^{n-2}(z - \chi)^2 - \frac{\lambda \Phi_n(\chi)}{\alpha^{n-3}} [(n-2)z - (n-1)\chi]z^{n-2}
$$

$$
+ \frac{\lambda \Phi_n(\chi)}{\alpha^{n-4}} z^{n-2}(z - \chi) + (1 - \lambda x^2)\Phi_n(\chi)(z - \chi) + (1 - \lambda x^2)\Phi_n(\chi),
$$

$$
\mathcal{L}(\Phi_n(z)z^{-k}) = \mathcal{L}(z^{n-k-2}(z - \alpha)^2) - \frac{\lambda \Phi_n(\chi)}{\alpha^{n-3}} \mathcal{L}(z^{n-k-2}[(n-2)z - (n-1)\chi])
$$

$$
+ \frac{\lambda \Phi_n(\chi)}{\alpha^{n-4}} \mathcal{L}(z^{n-k-2}(z - \alpha)) + (1 - \lambda x^2)\Phi_n(\chi)\mathcal{L}(z^{-k}(z - \alpha))
$$

$$
+ (1 - \lambda x^2)\Phi_n(\chi)\mathcal{L}(z^{-k}) \quad (4.25)
$$

(i) If $1 \leq k \leq n - 2$, from (4.2) and the hermitian character of $\mathcal{L}$, we obtain

$$
\mathcal{L}(z^{n-k-2}(z - \alpha)^2) = \mathcal{L} \left( z^{-n+k+2} \left( \frac{1}{z - \alpha} \right)^2 \right)
$$

$$
= \frac{1}{\alpha^{n-2-k}}(n - 1 - k)(\alpha^2 - \Phi_2(0)) \left( \frac{1 - |x|^2}{1 + |x|^2} \right)^2. \quad (4.26)
$$

The coefficient of $\Phi_n(\chi)$ in (4.25) is

$$
\frac{\bar{x}^k}{(1 - |x|^2)(1 + |x|^2)[1 + |x|^4 - 2\Re(\bar{x}^2 \Phi_2(0))]} \times 
\left[ \frac{|x|^2 - \Phi_2(0)}{|x|^{2(n-3)}} \right]^{2} \left[-(n-2)(n-k-2)

+ ((n-2)(n-k) + (n-1)(n-k-3))|x|^2 - (n-1)(n-k-1)|x|^4\right] \times

+ \left[ 1 - \bar{x}^2 \Phi_2(0) \right]^{2} [(k+1) - (k-1)|x|^2] \right], \quad (4.27)
$$
and the coefficient of $\Phi'_n(z)$ is

$$
\frac{(1 - |z|^2)z^{n-1}}{(1 - |z|^4)(1 + |z|^2)[1 + |z|^4 - 2\Re(\bar{z}^2 \Phi_2(0))]} \\
\times \left[ \frac{|z| - \Phi_2(0)}{|z|^{2(n-1)}} \right] \\
- \frac{1}{|z|^{2(n-2)}} \right] (n - k - 2) - (n - k - 1)|z|^2 + |1 - \bar{z}^2 \Phi_2(0)|^2 [k - (k - 1)|z|^2].
$$

(4.28)

Taking into account the previous expressions and the values of $D_n$, $\Phi_n(z)$ and $\Phi'_n(z)$ given by (4.15) - (4.17) we deduce that (4.25) is 0 for $1 \leq k \leq n - 2$ and $\forall n \geq 3$.

(ii) If $k = n - 1$, one can proceed as in case (i). The coefficients obtained for $\Phi_n(z)$ and for $\Phi'_n(z)$ are the same as in the previous case.

(iii) If $k = 0$, same as (ii).

(iv) If $k = n$ we compute the first addend in (4.25) from (4.4) for $j = 0$. Besides, the coefficient of $\Phi_n(z)$ in (4.25) is

$$
\bar{z}^n \\
(1 - |z|^4)(1 + |z|^2)(1 + |z|^4 - 2\Re(\bar{z}^2 \Phi_2(0))) \left[ \frac{|z| - \Phi_2(0)}{|z|^{2(n-2)}} \right] [(n - 1) - (n + 1)|z|^2] \\
+ |1 - \bar{z}^2 \Phi_2(0)|^2 [(n + 1) - (n - 1)|z|^2] \\
- \frac{(n - 1)(1 - \bar{z}^2 \Phi_2(0))(|z|^2 - 1)}{|z|^{2(n-2)}|z|^2}. 
$$

(4.29)

and the coefficient of $\Phi'_n(z)$ is

$$
\frac{(1 - |z|^2)z^{n-1}}{(1 - |z|^4)(1 + |z|^2)|1 + |z|^4 - 2\Re(\bar{z}^2 \Phi_2(0))|} \\
\times \left[ \frac{|z| - \Phi_2(0)}{|z|^{2(n-1)}} \right] \\
- \frac{(1 - |z|^2)^2}{|z|^2} \left[ \frac{|z|^2 - \Phi_2(0)}{|z|^{2(n-3)}} (1 - \bar{z}^2 \Phi_2(0))(|z|^2 - \Phi_2(0)) \\
+ |1 - \bar{z}^2 \Phi_2(0)|^2 [n - (n - 1)|z|^2] \right].
$$

(4.30)

Next one can proceed as in case (i) obtaining that $\mathcal{L}^*(\Phi_n(z)z^{-n}) = E_n$, where $E_n$ is given as follows:

$$
E_n = \left( \frac{1 - |z|^2}{1 + |z|^2} \right)^2 [1 + |z|^4 - 2\Re(\bar{z}^2 \Phi_2(0))] D_{n+1} z^2 D_n. 
$$

Theorem 7. The only functionals $\mathcal{L}$ which solve our problem in the positive-definite case are the following:
(i) If $|x| < 1$ then $\mathcal{L}(z^n) = u_n$ ($n \geq 0$), i.e., $A = 1$. The MOPS related to $\mathcal{L}$ is given by $\Phi_n(z) = z^{n-2} (z - x)^2 \forall n \geq 2$ and $\Phi_1(z) = z - 2x/(1 + |x|^2)$.

(ii) If $|x| > 1$ then $\mathcal{L}(z^n) = v_n$ ($n \geq 0$), i.e., $A = 0$. The MOPS related to $\mathcal{L}$ is given by $\Psi_n(z) = z^{n-2} (z - 1/\bar{x})^2 \forall n \geq 2$ and $\Psi_1(z) = z - 2x/(1 + |x|^2)$.

**Proof.** If $\mathcal{L}$ is positive definite the series $S(z)$ defined in Theorem 3 is

$$S(z) = \frac{1}{1 + |x|^2} \sum_{n=0}^{+\infty} \left[ \frac{A[(n + 1) + (1 - n)|x|^2](\bar{x}z)^n + (1 - A)[(1 - n) + (n + 1)|x|^2]z^n}{(x)^n} \right].$$

(4.31)

Since $|x| \neq 1$ we distinguish two possibilities:

(i) If $|x| < 1$ there exists $z$ such that $|x| < |z| < 1$. In this case $|\bar{x}z| < 1$ and the series $\sum_{n=0}^{+\infty} A[(n + 1) + (1 - n)|x|^2](\bar{x}z)^n$ converges $\forall A \in \mathbb{C}$. On the other hand, since $|z/\bar{x}| > 1$ the series $\sum_{n=0}^{+\infty} (1 - A)[(1 - n) + (n + 1)|x|^2](z/\bar{x})^n$ diverges if $A \neq 1$. Then in order that $S(z)$ converges for all $z$ in the unit disk, $A$ must be equal to 1.

(ii) If $|x| > 1$ there exists $z$ such that $1/|x| < |z| < 1$. In this case $|z/\bar{x}| < 1$. Therefore the series $\sum_{n=0}^{+\infty} (1 - A)[(1 - n) + (n + 1)|x|^2](z/\bar{x})^n$ converges $\forall A \in \mathbb{C}$ and the series $\sum_{n=0}^{+\infty} A[(n + 1) + (1 - n)|x|^2](\bar{x}z)^n$ diverges except for $A = 0$. Then in order that $S(z)$ converges for all $z$ in the unit disk, $1 - A$ must be equal to 1. □

The orthogonal polynomials appearing in this theorem coincide with those related to rational modifications of the Lebesgue measure, i.e., orthogonal polynomials with respect to the measure $d\omega(\theta) = \frac{d\theta}{2\pi} / |x - \alpha|^2$, $z = e^{i\theta}$, $|x| < 1$, which is a semiclassical measure and the corresponding induced functional belongs to the class $(2, 2)$ (see [3]).

It seems natural to ask for the existence of families of semiclassical orthogonal polynomials belonging to the class $(2, 2)$ which are different from the Bernstein-Szegő polynomials obtained in the positive definite case. The next example shows that for every complex number $\alpha$, if we choose $\Phi_2(0) = \alpha^2 + 1/\alpha^2$ then the functional is regular and therefore there exists the corresponding sequence of orthogonal polynomials.

**Example (A Regular case).** Let us consider $\Phi_2(0) = \alpha^2 + 1/\alpha^2$ then $1 - |\Phi_2(0)|^2 = -(1 + |x|^4 + |x|^4)/|x|^4 < 0$.

From (4.15) $\forall n \geq 3$,

$$\alpha^{2n-7}D_n = \frac{1}{|x|^6} [1 - |x|^{2(n+2)} - |x|^{n+1}(n - 2)(|x|^2 - 1)]$$

$$\times [1 - |x|^{2(n+2)} + |x|^{n+1}(n - 2)(|x|^2 - 1)].$$

Putting $z = |x|$ we can consider $|x|^6 \alpha^{2n-7}D_n$ as a product of two polynomials in $z$ as follows:

$$[z^{2(n+2)} + (n - 2)z^{n+3} - (n - 2)z^{n+1} - 1][z^{2(n+2)} - (n - 2)z^{n+3} + (n - 2)z^{n+1} - 1].$$
We look for the positive roots of the following polynomials:

\[ P_{2n+4}(z) = z^{2(n+2)} + (n - 2)z^{n+3} - (n - 2)z^{n+1} - 1, \]

\[ Q_{2n+4}(z) = z^{2(n+2)} - (n - 2)z^{n+3} + (n - 2)z^{n+1} - 1. \]

By applying Descartes's rules of signs [6] we deduce that the polynomial \( P_{2n+4}(z) \) has exactly one positive root \( z = 1 \).

Next we analyse the polynomial \( Q_{2n+4}(z) \):

\[ Q_{2n+4} \left( \frac{1}{z} \right) = - \frac{Q_{2n+4}(z)}{z^{2n+4}}, \]

hence if \( z \) is root of \( Q_{2n+4}(z) \) then \( 1/z \) is also a root.

If \( n \geq 4 \) is even

\[ R_{2n+2}(z) = \frac{Q_{2n+4}(z)}{z^2 - 1} = z^{2n+2} + z^{2n} + \cdots + z^{n+2} - (n - 2)z^{n+1} + z^n + \cdots + z^2 + 1. \]

If \( n \geq 4 \) is odd

\[ R_{2n+2}(z) = \frac{Q_{2n+4}(z)}{z^2 - 1} = z^{2n+2} + z^{2n} + \cdots + z^{n+3} - (n - 3)z^{n+1} + z^{n-1} + \cdots + z^2 + 1. \]

Let us assume that \( n \geq 4 \) and \( n \) is even

\[ zR_{2n+2}(z) - R_{2(n+1)+2}(z) = - \frac{(z^{n+3} - 1)(z^{n+2} - 1)}{z + 1} < 0, \quad \forall z > 1 \]

then

\[ zR_{2n+2}(z) < R_{2n+4}(z), \quad \forall z > 1. \]

Taking into account that

\[ R_{10}(z) = z^{10} + z^8 + z^6 - 2z^5 + z^4 + z^2 + 1 > 0, \quad \forall z > 1 \]

we deduce that \( R_{10} > zR_{10} > 0, \quad \forall z > 1. \)

On the other hand,

\[ R_{2(n+1)+2}(z) - R_{2(n+2)+2}(z) = z^{n+2}(1 - z)z^{n+3} + z^{n+2} + \cdots + z + 1 - n < 0 \quad \text{if} \quad z > 1. \]

From the last expression for \( n = 4 \) we obtain that \( R_{14} > R_{12} > 0, \quad \forall z > 1. \) Proceeding inductively from the recursion relation one easily sees that \( R_{2n+2}(z) > 0, \quad \forall z > 1 \) and therefore the polynomial \( Q_{2n+4}(z) \neq 0, \quad \forall z > 0, \text{and} \quad z \neq 1 \) because if it had one real root lesser than 1 then it would had to have one other larger than 1.

References

