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An evaluation of Clenshaw–Curtis quadrature rule for integration w.r.t. singular measures

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ABSTRACT

This work is devoted to the study of quadrature rules for integration with respect to (w.r.t.) general probability measures with known moments. Automatic calculation of the Clenshaw–Curtis rules is considered and analyzed. It is shown that it is possible to construct these rules in a stable manner for quadrature w.r.t. balanced measures. In order to make a comparison Gauss rules and their stable implementation for integration w.r.t. balanced measures are recalled. Convergence rates are tested in the case of binomial measures.

1. Introduction

In quadrature theory, much effort has been done in the analysis of the integration with respect to (w.r.t.) the Lebesgue measure or to some of its weighted variants. Among the possible generalizations of the problem, the case of singular measures naturally appears, for instance, when dealing with fractal properties of some physical phenomenon, see [3,17].

In a recent review paper [25], Trefethen compares the convergence rates of Clenshaw–Curtis rules with the Gauss ones. In this paper the author points out that the two rates of convergence are similar if the integrand function is not analytic in a suitable neighborhood of the interval of integration. In the present paper we want to compare the same two families of quadrature rules when the integration is performed w.r.t. a singular (fractal) measure.

We begin with the introduction of the convergence theory for general quadrature rules in Section 2. Then in Section 3, we introduce the Clenshaw–Curtis and Gauss families of quadrature rules and their numerical construction. On the one hand we notice that these rules converge for wide classes of functions. On the other hand, for a general measure, we observe that the automatic calculation passes through an unstable procedure which is of different origin in the two cases. In the Gauss quadrature it appears when the construction of the recurrence coefficients for orthogonal polynomials is carried out [4], while in the Clenshaw–Curtis case when the calculation of modified moments is performed [5]. In Section 4 we recall the definition of balanced measures. We show that, despite the general case, for this class of singular measures it is possible to construct in a stable manner both formulae. In the case of Gauss quadrature this has been developed in [15], while for Clenshaw–Curtis rule it is made adapting the analysis in [24]. In the same section the connection with the theory of linear refinable functionals introduced in [14] is also analyzed. As an application, in Section 5 the quadrature w.r.t. binomial measures is performed through numerical tests.

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2. Preliminary results and definitions

In this introduction we present some results valid for a general measure μ, that we will assume finite, positive and defined in a closed interval $[a, b]$. Our aim is to study how to calculate

$$I_\mu(f) \equiv \int_a^b f(x)\mathrm{d}\mu(x); \tag{1}$$

where $f \in L^1_\mu \equiv \{f : [a, b] \to \mathbb{R} : \int_a^b |f(x)|\mathrm{d}\mu(x) < \infty\}$.

In general a quadrature rule $\mathbb{I}_n$ is defined by means of $(n + 1)$ distinct points $\xi_j \in [a, b]$ called nodes and $(n + 1)$ real values w_j called weights:

$$\mathbb{I}_n(f) \equiv \sum_{j=0}^{n} w_j f(\xi_j). \tag{2}$$

In order to obtain efficient quadrature rules, we can construct $\mathbb{I}_n$ to be the exact integral of an approximating function $\tilde{f}$: $\mathbb{I}_n(f) = I_\mu(\tilde{f})$. In what follows, we assume that the moments of the measure are known:

$$\lambda_j \equiv \int_a^b x^j \mathrm{d}\mu(x) \quad \forall j = 0, 1, \ldots \tag{3}$$

and for this reason we will take as approximating function a polynomial, $\tilde{f}(x) \in \mathbb{P}^n$ where $\mathbb{P}^n$ are the polynomials of degree at most n. Such rule will be called interpolatory quadrature formula when the polynomial that we integrate exactly is the (unique) polynomial of degree n interpolating the function f at the nodes ξ_j.

We will say that a quadrature rule has degree of exactness d if

$$\sum_{j=0}^{n} w_j \xi_j^q = \lambda_q \quad \forall q \leq d, \ q \in \mathbb{N}.$$

It is well known that every quadrature rule with $n + 1$ nodes of degree of exactness at least n is interpolatory. In general the following result holds true, see [6, Section 1.3]:

Theorem 2.1. *The quadrature rule (2) has degree of exactness $d = n + k, \ k \geq 0$ if and only if both of the following conditions are satisfied:*

1. *the formula (2) is interpolatory;*
2. *the following holds true:*

$$\int_a^b \omega_n(x)p(x)\mathrm{d}\mu(x) = 0 \quad \forall p \in \mathbb{P}^{k-1} \ (\mathbb{P}^{-1} \equiv \emptyset)$$

where $\omega_n(x) = \prod_{j=0}^{n}(x - \xi_j)$ is the nodal polynomial.

Given a function $f \in L^1_\mu$, we will say that a sequence of quadrature rules $\{\mathbb{I}_n\}_n$ converges in f if $\mathbb{I}_n(f) \to_n I_\mu(f)$.

Given a function $f \in C^0$, we will denote by $p_d^*(x)$ the polynomial[1] of degree at most d that gives the best approximation to f on $[a, b]$ w.r.t. the supremum norm. We will also denote by $E_d^* \equiv \|f - p_d^*\|_\infty$. With this notations, the following theorem gives the most general error estimate, see [12, Theorem 5.2.2] or [25, Theorem 4.1].

Theorem 2.2. *Let $\mathbb{I}_n$ be a quadrature rule with weights $w_j, j = 0, \ldots, n$ of degree of exactness $d \geq 0$. Then for all $f \in C^0$ we have:*

$$\left|I_\mu(f) - n(f)\right| \leq E_d^* \left[\sum_{j=0}^{n} |w_j| + \mu([a, b])\right].$$

The result is proved simply applying the definitions and the triangular inequality.

If we consider a family of rules $\{\mathbb{I}_n\}_n$ of increasing degrees of exactness d_n and such that $\sum_{j=0}^{n} |w_j| \leq K_n$ we will have that the rule converges if $K_n E_{d_n}^* \to_{n\to\infty} 0$. Notice that for interpolatory quadrature rules the constant K_n is bounded from above by the Lebesgue constant Λ_n (see [22, Eq. (8.11)]).

As corollary of the Weierstrass theorem we can state also that for every $f \in C^0$ there exists always a sequence of polynomials uniformly convergent to f, and therefore the corresponding quadrature rules will be convergent. On the other hand it is very well known that equispaced interpolatory quadrature formulae do not converge in general due to Runge phenomenon.

[1] Note that this polynomial is unique. For the theory of the best approximation see, ad example, [19, Section 3.2].

3. Quadrature rules

In this section we will introduce the properties of two families that exhibit good convergence properties: the Clenshaw–Curtis and the Gauss ones.

3.1. Clenshaw–Curtis rules

For the sake of clarity, in this section we will set $[a, b] = [-1, 1]$. We will define the Clenshaw–Curtis formula, denoted by $\mathbb{CC}_n$, as the interpolatory quadrature rule constructed on the Chebychev[2] nodes $\theta_l = \cos\left(\frac{l\pi}{n}\right)$, $l = 0, \ldots, n$ (see [12, Section 5.2.6]). The interpolating polynomial that we integrate in an exact manner, denoted by $\Pi_{f,n}(x)$, can be expressed in a compact form as (see [19, Eq. (6.27)]):

$$\Pi_{f,n}(x) = {\sum_{l=0}^{n}}' f_l^C T_l(x), \tag{4}$$

$$f_l^C = \frac{2}{n+1} {\sum_{j=0}^{n}}'' f(\theta_j) T_l(\theta_j), \tag{5}$$

where $T_j(x)$ is the Chebychev polynomial of the first type of degree j, the prime indicates that the first term is to be halved and the double prime indicates that the first and the last term are to be halved. Let us summarize some of the properties of Chebyshev polynomials of first type. The family of polynomials $\{T_j(x)\}_j$ is defined by the following:

$$T_j(\cos(\theta)) = \cos(j\theta), \quad \theta \in [0, \pi].$$

They are characterized by the recursive relation:

$$\begin{aligned} &T_0(x) = 1; \qquad T_1(x) = x \\ &T_j = 2xT_{j-1}(x) - T_{j-2}(x) \quad \forall j = 2, 3, \ldots. \end{aligned} \tag{6}$$

These polynomials are orthogonal w.r.t. the measure $\omega^C(x)\mathrm{d}x$, where $\omega^C(x) = \frac{1}{\sqrt{1-x^2}}$. In particular:

$$\langle T_i, T_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ \pi & \text{if } i = j = 0 \\ \dfrac{\pi}{2} & \text{if } i = j > 0. \end{cases}$$

With the symbol $\langle \cdot, \cdot \rangle$ we are considering the scalar product in the $\omega^C(x)\mathrm{d}x$-measure, i.e.:

$$\langle \psi, \phi \rangle = \int_{-1}^{1} \psi(x)\phi(x)\omega^C(x)\mathrm{d}x.$$

We can derive the convergence of the interpolation based quadrature rule from Theorem 2.2. First of all it is well known that for polynomials $\Pi_{f,n}(x)$ the Lebesgue constant[3] Λ_n is such that $\Lambda_n \leq 1 + \frac{2}{\pi}\log(n+1)$. Convergence, is thus ensured if the best approximation constant E_n^* converges more than logarithmically. This is achieved if the function $f \in C^0$ satisfies the so-called Dini–Lipschitz condition, see [23, Th. 3.4]:

$$\lim_{n\to\infty} \log(n)\omega(n^{-1}) = 0, \tag{7}$$

where the continuity modulus $\omega(\varepsilon)$ is defined by

$$\omega(\varepsilon) \equiv \sup_{\substack{|x-y|\leq\varepsilon \\ x,y\in[-1,1]}} |f(x) - f(y)|.$$

We have, thus, that if condition (7) is satisfied[4] then the sequence of Clenshaw–Curtis quadrature rules converges.

3.2. Computation of Clenshaw–Curtis rules

In this section we address the item of the computation of Clenshaw–Curtis rules. First of all we notice that from Eq. (4) we can compute the approximate integrals in this way:

$$\mathbb{CC}_n(f) = \int_{-1}^{1} \Pi_{f,n}(x)\mathrm{d}\mu = \sum_{l=0}^{n} f_l^C M_l \tag{8}$$

---

[2] For other rules on the Chebychev nodes and extremes we refer the reader to [20].

[3] For this and other estimates see [22, Section 10.3].

[4] Notice that all Lipschitz continuous functions satisfy condition (7).

where, following [5], we call M_j the following modified moments:

$$M_j \equiv \int_{-1}^{1} T_j(x)\mathrm{d}\mu. \tag{9}$$

The calculation of modified moments can be done, in principle, exactly if the moments λ_i of the measure are known, although it is well known that the direct method is unstable due to the alternating signs of the coefficients in Eq. (6). In the next section we will see how to avoid this unstability in the case of balanced measures.

The coefficients f_j^C can be calculated by means of the `fft`, see [25,26]. From Eq. (5) we can see that this is an approximation by means of a composite trapezoidal formula of the scalar product $\langle f, T_j\rangle$. Thus, using the definition of the Chebyshev polynomials, we have:

$$\begin{aligned}\langle f, T_j\rangle &= \int_{-1}^{1} f(x)T_j(x)\omega^C(x)\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} f(\cos(\theta))\cos(j\theta)\mathrm{d}\theta \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos(jx)\mathrm{d}x = \frac{1}{\pi}[a_j 2\pi] = 2a_j,\end{aligned}$$

where a_j is the real part of the Fourier coefficient of the even function $g : x \in [-\pi, \pi] \to \mathbb{R}$ defined as: $g(x) = f(\cos(x))\ \forall x \in [0, \pi]$.

For example, in Matlab notation, we can use the efficient implementation of the `fft` through the simple procedure:

```
x=cos(pi*(0:(n-1))/(n-1));
fx=eval(ftest,x)/(2*(n-1)) ;
fx=[fx(1:end) fx(end-1:-1:2)];
g= real(fft(fx));
fc=[2*g(1), g(2:(n-1))+g(2*(n-1):-1:(n+1)) , g(n)];
```

Remark 3.1. Notice that the approximate integral that we compute is, as originally considered in [2], a formula that calculates the integral of the partial sum of the Chebyshev series, where the coefficients f_l^C are approximated by means of a composite trapezoidal rule.

It is well known that the Clenshaw–Curtis rules in the case of the Lebesgue measure are of optimal degree of exactness w.r.t. the weighted measure $\omega^C(x)\mathrm{d}x$. In our case this is not true, and the weights can become negative.

Among the good properties that remain valid for these generalization of the rules, there is the possibility to consider fixed a priori known nodes (nested in the cases $n = 2^k$) and to use the `fft` procedure for the calculation of the weights.

Notice also that all the information on the measure is used in the computation of modified moments.

3.3. *Gauss formulae*

We will call Gauss quadrature $\mathbb{G}_n$ the unique quadrature rule[5] on $n + 1$ nodes of degree of exactness $2n + 1$. It is well known that these rules have positive weights, and this property implies, as corollary of Theorem 2.2, that the family $\{\mathbb{G}_n\}$ is convergent $\forall f \in C^0$.

Due to Theorem 2.1, the Gauss formulae are interpolatory quadrature rules. For this reason, we need only to define the nodes to describe the rule. The corresponding weights, called Christoffel numbers, can be calculated consequently.

Nodes of the Gauss quadrature formulae $\mathbb{G}_n$ can be characterized as the zeros of the monic polynomial π_{n+1} of degree $n + 1$ $\mathrm{d}\mu$-orthogonal to the ones of lower degree. These polynomials can be constructed from a three-term recurrence relation:

$$\begin{aligned}&\pi_{k+1}(x) = (x - \alpha_k)\pi_k(x) - \beta_k\pi_{k-1}(x), \quad k = 0, 1, 2, \ldots \\ &\pi_{-1}(x) = 0, \qquad \pi_0(x) = 1.\end{aligned} \tag{10}$$

The coefficients α_k and β_k are defined by the following:

$$\begin{aligned}\alpha_k &= \frac{(x\pi_k(x), \pi_k(x))_{\mathrm{d}\mu}}{(\pi_k(x), \pi_k(x))_{\mathrm{d}\mu}}, \quad k = 0, 1, 2, \ldots \\ \beta_k &= \frac{(\pi_k(x), \pi_k(x))_{\mathrm{d}\mu}}{(\pi_{k-1}(x), \pi_{k-1}(x))_{\mathrm{d}\mu}}, \quad k = 1, 2, 3, \ldots\end{aligned} \tag{11}$$

where with the symbol $(\cdot, \cdot)_{\mathrm{d}\mu}$ we refer to the scalar product in the $\mathrm{d}\mu$ measure, i.e.:

$$(\phi(x), \psi(x))_{\mathrm{d}\mu} = \int_a^b \phi(x)\psi(x)\,\mathrm{d}\mu(x).$$

[5] For a complete survey on the theory concerning Gauss quadrature and various extensions we refer to [6]. In particular in all this section we will follow, without explicit reference, Sections 1.4 and 5.

The numerical calculation of the coefficients α_k and β_k is in general unstable. In the next section we will address this feature for a wide class of measures.

From the first n of these coefficients we can construct the following symmetric tridiagonal matrix (usually refereed to as Jacobi matrix):

$$J_n \equiv \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & & 0 & \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & & \\ & \sqrt{\beta_1} & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \sqrt{\beta_{n-1}} \\ & 0 & & & \sqrt{\beta_{n-1}} & \alpha_{n-1} \end{bmatrix}.$$

The nodes of the Gauss formulae can be calculated as eigenvalues of the matrix J_n and the Christoffel numbers are constructed from the corresponding eigenvectors (see [9] or [7, th. 3.1]).

Numerically the calculation of the eigenvalues can be done in a stable manner, and the calculation costs $O(n^2)$ flops, see [9].

4. Balanced measures and linear refinable functionals

In this section we will see how for a wide class of measures it is possible to compute in a stable manner the coefficients in Eq. (11) and the modified moments defined in (9). In general we will be able to do this for balanced measures, that are particular invariant measures, see [10, Section 4]. Relevance of these measures for applications has been highlighted in [1,18]. In this section we will also see the connection with the recently introduced linear refinable functionals [14]. This connection is very important to notice, as it allows taking ideas from the wide literature on quadrature w.r.t. refinable weight functions, see for example [24,8,11,13].

For the sake of clarity, and without loss of generality all the results are given in the interval $[-1, 1]$ and in the probability case, i.e. $\mu([-1, 1]) = 1$.

Following [15–17], we will call δ-homogeneous linear iterated function system (δ-HLIFS) balanced measure the unique measure μ such that:

$$\int_{-1}^{1} f\,\mathrm{d}\mu = \sum_{i=0}^{M} p_i \int_{-1}^{1} f(\phi_i(x))\mathrm{d}\mu$$
$$\sum p_i = 1, \qquad \phi_i(x) = \delta x + \beta_i; \quad 0 < \delta < 1,\ \beta_i \in [-1, 1].$$

In this framework, δ is called contraction ratio and p_i and β_i are, respectively, the probabilities and the fixed points of the LIFS.

Now we present the definition of refinable linear functional as given in [14]. A linear functional $L : P \to \mathbb{R}$ where P are all the polynomials with real coefficients is called refinable if there exists an $(N + 1)$-uple $[\gamma_j]_{j=0,\ldots,N}$ called mask such that:

$$L[f] = \sum_{j=0}^{N} \gamma_j L\left[f\left(\frac{x+j}{2}\right)\right] \quad L[1] = 1.$$

If we assume that the functional is positive, by Riesz theorem, the functional L acts as integration w.r.t. a positive Borel measure μ. Rescaling in $[-1, 1]$ we can write:

$$L[f] = \int_{-1}^{1} f\,\mathrm{d}\mu = 1/2 \sum_{j=0}^{N} \gamma_j \int_{-1}^{1} f(E_j^N(x))\mathrm{d}\mu$$
$$\sum \gamma_j = 2, \qquad E_j^N(x) = \frac{x-1}{2} + \frac{j}{N}.$$

For this reason we can notice that a positive refinable linear functional is given by integration w.r.t. a δ-HLIFS balanced measure with contraction ratio $\delta = 1/2$, probabilities $p_i = \gamma_i/2$ and fixed points $\beta_i = \frac{2i-N}{2N}$ $i = 0 \ldots N$.

Stable algorithms for the calculation of the scalar products in (11) for these measures have been introduced in [15] using the balance equation and successively in [14] using the property that the functional is refinable. These algorithms rely on the idea of projecting the composite functions that define the balance equation on the same set of functions: $\pi_k(\phi(x)) = \sum_{i=0}^{k} c_i \pi_i(x)$. This can be done if the $\pi_k(x)$ are a basis of polynomials of increasing degree k, because from the balance equation we can easily compute the leading coefficient c_k and then the other coefficients by induction. Clearly, for Gauss quadrature we take as polynomials $\pi_k(x)$ the μ-orthogonal polynomials described in Eq. (10).

Table 1
Some quadrature rules for integration w.r.t. binomial measures μ_α.

Rule	Nodes	Weights
$\mathbb{G}_1$	$\frac{4}{7}(2\alpha-1) \mp \frac{\sqrt{3}}{21}\sqrt{-88\alpha^2+88\alpha+27}$	$\frac{1}{2} \pm \frac{3\sqrt{3}(2\alpha-1)}{2}\frac{\sqrt{-88\alpha^2+88\alpha+27}}{88\alpha^2-88\alpha-27}$
$\mathbb{CC}_1$	$[-1, 1]$	$[(1-\alpha), \alpha]$
$\mathbb{CC}_2$	$[-1, 0, 1]$	$\left[\frac{(1-\alpha)(3-4\alpha)}{3}, \frac{8\alpha(1-\alpha)}{3}, \frac{\alpha(4\alpha-1)}{3}\right]$

The same procedure via projections can be adopted for the calculation of the modified moments (9), by using as polynomials $\pi_k(x)$ the Chebyshev ones described in Eq. (6). This has been done in the case of integration w.r.t. refinable weight functions, see [24], and we have simply used this idea applying it for integration w.r.t. balanced measures.

Details of these procedures can be found in the cited References [15,14,24]. Notice that the computational cost of these algorithm for the scalar products of Eq. (11) and the integrals in (9) is of $O(n^2)$ flops.

5. Case study: Integration with respect to binomial measure

We will compare the two families of quadrature rules on a special class of balanced measures, the binomial ones. Quadrature w.r.t. these measures has been already addressed in [1], where the convergence of the composite rules and automatic quadrature with local error estimates has been explored. The binomial measure μ_α, where $0 < \alpha < 1$ is a parameter, is a probability measure (i.e. $\mu_\alpha([a, b]) = 1$) that is characterized by the following (self-similar) property [3]: let J a dyadic subinterval of $[a, b]$ and bisect J in the left and right parts $J = J_L \cup J_R$; then

$$\mu_\alpha(J_R) = \alpha\mu_\alpha(J). \tag{12}$$

When $\alpha = 1/2$ we trivially obtain the probability measure proportional to the Lebesgue measure on $[a, b]$. Some other important properties of μ_α are:

- $\{\mu_\alpha\}_\alpha$ is a family of pairwise mutually singular Borel measures: $\mu_{\alpha_1} \perp \mu_{\alpha_2}$ if $\alpha_1 \neq \alpha_2$;
- Each μ_α is a continuous measure, i.e. $\mu_\alpha(\{x\}) = 0\ \forall x \in [a, b]$;
- μ_α and $\mu_{1-\alpha}$ are connected by the following property of symmetry: $\mu_\alpha([c, d]) = \mu_{1-\alpha}([a+b-d, a+b-c])\ \forall [c, d] \subset [a, b]$.

We will consider $I \equiv [-1, 1]$ and refer to the measure on this interval as μ^*_α, while with μ_α we will refer to the case with support in [0, 1], the most usual in the framework of fractals. Trivially, there exists the following relation:

$$\int_{-1}^{1} f(x)\mathrm{d}\mu^*_\alpha(x) = \int_0^1 f(2y-1)\mathrm{d}\mu_\alpha(y). \tag{13}$$

The general balance equation that defines the measure is the following:

$$\int_{-1}^{1} f(x)\mathrm{d}\mu^*_\alpha(x) = (1-\alpha)\int_{-1}^{1} f\left(\frac{x-1}{2}\right)\mathrm{d}\mu^*_\alpha(x) + \alpha\int_{-1}^{1} f\left(\frac{x+1}{2}\right)\mathrm{d}\mu^*_\alpha(x). \tag{14}$$

We have reported in Table 1 the rules $\mathbb{G}_1$, $\mathbb{CC}_1$ and $\mathbb{CC}_2$. From a comparison between the two-point rules it can be seen how it is difficult to write down the Gauss rule while the Clenshaw–Curtis one is easily described. In the rule $\mathbb{CC}_2$ it can be seen that the weights corresponding to the endpoints become negative[6] for some values of α, as noticed in Remark 3.1.

In order to construct our reference solution, we have applied an adaptive quadrature algorithm based on a six-point local quadrature modulus and an error estimate with Null-Rules. The description can be found in [1] (`modheralpha.m`) where it has been also proved the efficiency and the reliability of this procedure.

Test functions are taken as in Reference [25]:

$$\text{Test functions:} \quad \begin{bmatrix} f_1 = x^{20} & f_2 = \mathrm{e}^x & f_3 = \mathrm{e}^{-x^2} \\ f_4 = \dfrac{1}{1+16x^2} & f_5 = \mathrm{e}^{-1/x^2} & f_6 = |x^3|. \end{bmatrix}$$

For the construction of the Gauss quadrature rules from the coefficients α_k, β_k, we have used the Matlab route *eig*, as in the package *OPQ: a Matlab Suite of Programs for Generating Orthogonal Polynomials and Related Quadrature Rules*, described in [7].

We have tested the convergence of the two families of rules up to 60 nodes in the cases $\mu_{\alpha_i} = \mu_{0.i}$, $i = 1, \ldots, 5$ for the six test functions. Notice that the last case is the Lebesgue one that has been considered in Reference [25] and has been included for a quick comparison. Relative errors are plotted in Figs. 1–5. As we can see, these tests give convergence of the

[6] However a numerical check, performed for n up to 20 and every value of α gave (for the sum of absolute values of the weights) values very close to 1 (maximum value around 1.1).

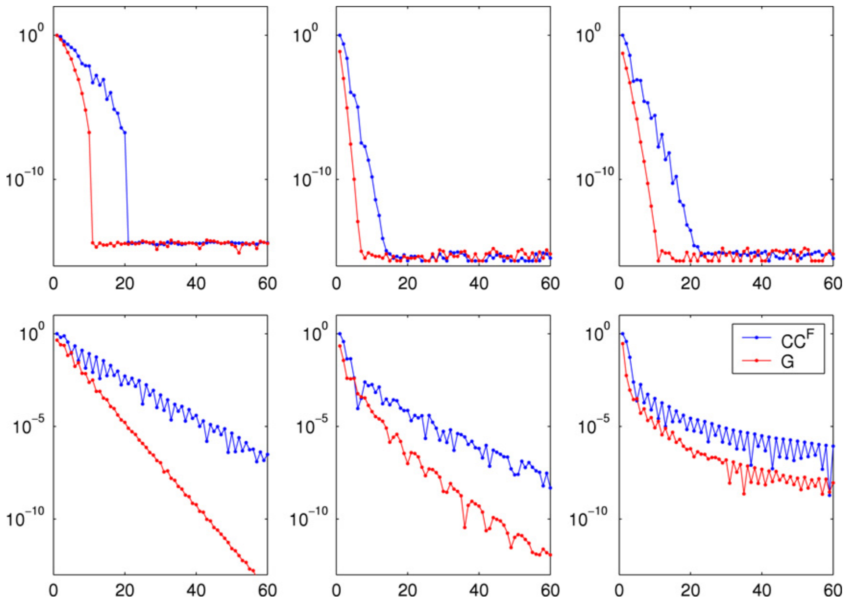

Fig. 1. Calculated errors in semilog scale, case $\mu_\alpha = \mu_{0.1}$.

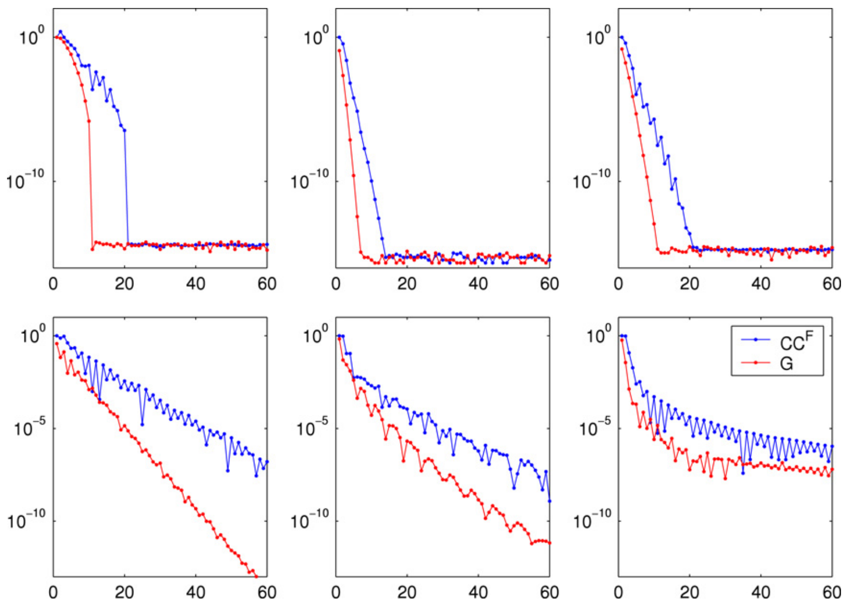

Fig. 2. Calculated errors in semilog scale, case $\mu_\alpha = \mu_{0.2}$.

two families and thus stability of the considered algorithms. The Gauss formulae converge more rapidly, as expected; we can notice that the convergence velocity becomes similar for the two formulae only when the function has less regularity and for values of α closer to $1/2$. For the Lebesgue measure this has been pointed out in Reference [25], due also to the similar distributions of the nodes of the two quadrature rules.

Thinking at the question given in the title of Reference [25] we can note the following:

- Clenshaw–Curtis rules are very simple to compute in the general case and converge almost in the same hypotheses.
- The proposed algorithms for the numerical construction of the formulae in the case of balanced measures are numerically stable both for Gauss and Clenshaw–Curtis rules.
- In the case of binomial measures Gauss quadrature leads to better results, especially in the case of measures that are very different from the Lebesgue one, while Clenshaw–Curtis rules perform almost in the same manner when the function is less regular.
- Clenshaw–Curtis rules have a priori fixed nodes that are nested in the case of 2^k nodes, see [21]. This implies that an automatic quadrature routine that doubles the number of nodes can reuse the calculated values of the function, while

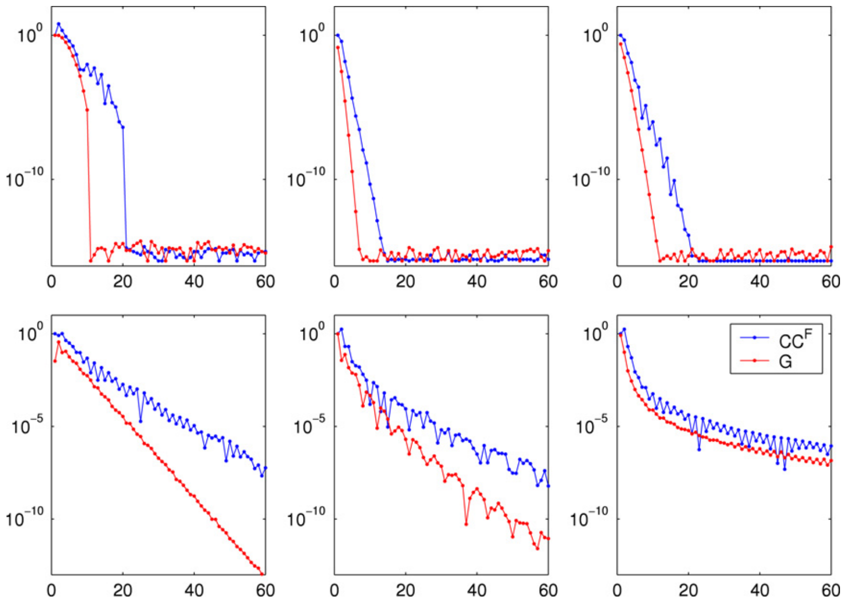

Fig. 3. Calculated errors in semilog scale, case $\mu_\alpha = \mu_{0.3}$.

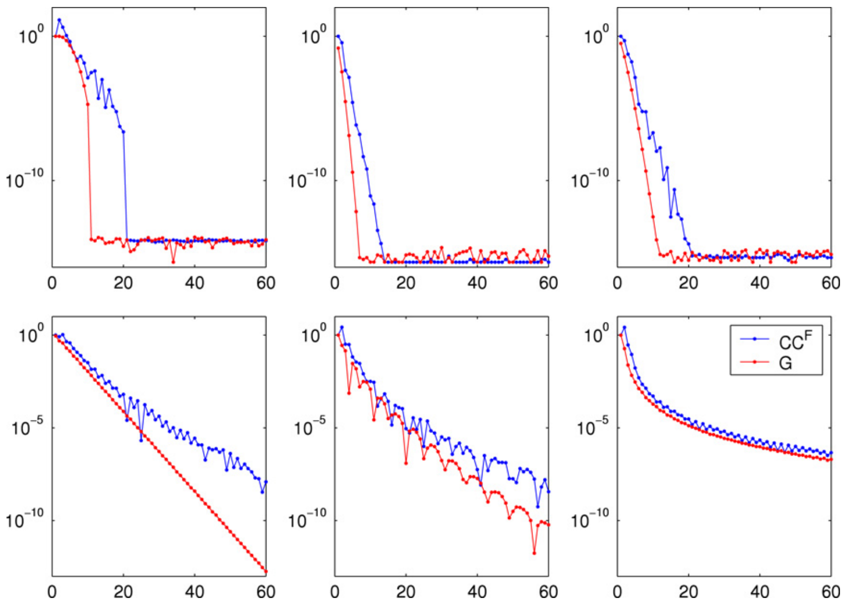

Fig. 4. Calculated errors in semilog scale, case $\mu_\alpha = \mu_{0.4}$.

for Gauss quadrature each time the order is changed the nodes – and consequently the function evaluations – are to be recomputed.

- Fixed to n the number of function evaluations, the corresponding Gauss quadrature rule maintain as in the Lebesgue case a cost of $O(n^2)$ flops. The Clenshaw–Curtis rule takes advantage of the `fft` procedure but for the calculation of the modified moments has a cost of $O(n^2)$ flops, thus the two have the same leading computational cost.

Concluding and summarizing, we have explored the convergence properties of the Gauss and Clenshaw–Curtis quadrature families and the numerical calculation of these formulae for general probability measures with known moments. In the case of Gauss rules this was done and we have only described and utilized available software, in the case of Clenshaw–Curtis we propose a construction in such a way that we preserve the possibility to use the `fft` procedure. In this way we have constructed in an efficient fashion a family that converges for a very general class of functions. The tests of these two families in the case of binomial measure seem to indicate that Gauss formulae are preferable both in the case of functions with high regularity, as pointed out in [25], and in the case of measures more irregular. The latter is probably

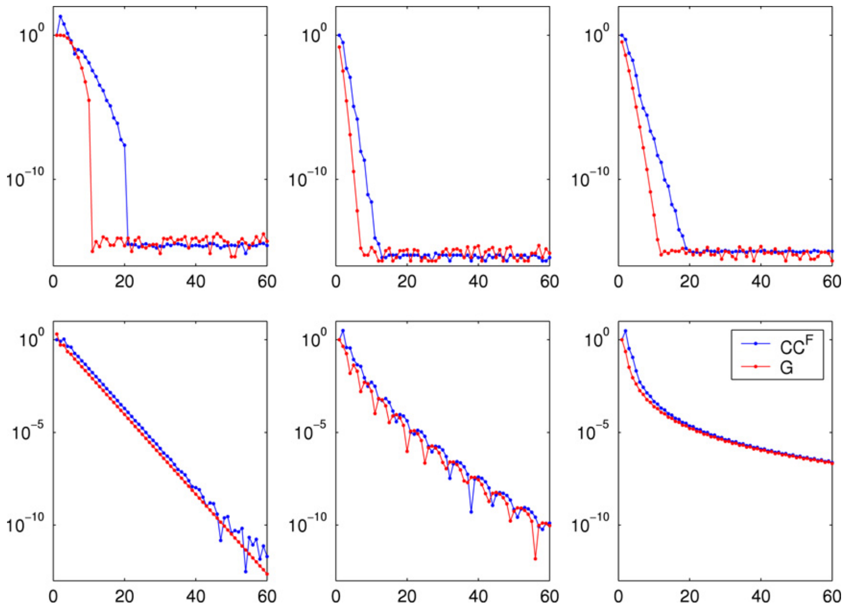

Fig. 5. Calculated errors in semilog scale, case $\mu_\alpha = \mu_{0.5}$.

due to the symmetry of the distributions of the nodes of Clenshaw–Curtis quadrature family while the measures are not symmetric.

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