Spectral Factorization of Wide Sense Stationary Processes on $\mathbb{Z}^2$

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The problem of prediction of wide sense stationary processes on $\mathbb{Z}^2$ with respect to the column-by-column or row-by-row lexicographic order is studied. A theorem giving an explicit realization of the corresponding canonical factorization of the spectral density is proved and conditions are given for a process to have a quarter-plane representation in terms of its innovations.


INTRODUCTION

All approaches to the prediction of wide-sense stationary processes on the plane depend on the definition of the pasts for which various authors have considered half-planes ([2], [5]), augmented half-planes ([4], [6], [7]) (cf. the definition of $H^1+$ and $H^2+$ in the text), or quarter-planes ([9], [17]).

In this paper we adopt as set of pasts the increasing sequence of augmented half-planes corresponding to the column-by-column or row-by-row lexicographic order of $\mathbb{Z}^2$ and construct our results on those of Helson and Lowdenslager in [6] and [7]. As in the case of one-parameter processes, the prediction problem for a two-parameter purely nondeterministic process can be entirely solved if the canonical factorization of its spectral density is produced. The necessary and sufficient condition for a spectral factorization to be canonical was given by Helson and Lowdenslager, but the derivation of the canonical factor was not considered. Later, a factorization algorithm was proposed by Ekstrom and Woods in [4] under a
restrictive condition on the spectral density. In a short paper [11] we presented a general method of canonical factorization with no other constraint on the spectral density than the one guaranteeing the regularity of the process. We develop in the first part of the present paper the study undertaken in this short communication.

The very aim of this paper is the study of spectral properties related to Prediction Theory introduced in Paragraph 1 by the Wold decomposition of Helson and Lowdenslager [7] to which we add the following result: the singular part of a strictly nondeterministic process has commuting horizontal and vertical projections. Processes having this property are considered in Section 2.

Taking the canonical representation of the purely nondeterministic part of a strictly nondeterministic process as a starting point, we elaborate in Paragraph 1.2 the canonical factorization theorem of the spectral density. This theorem is entirely based on the one-parameter spectral factorization techniques [3, 13] borrowed from the theory of analytic functions on the unit disc of \( \mathbb{C} \) [15]. The theory of analytic functions on the unit bidisc of \( \mathbb{C}^2 \) [14] is used in Section 2 where quarter-plane representations of a regular process in terms of a white noise are studied. Many quarter-plane representation problems are strongly connected with the commutation properties of horizontal and vertical projections. Under these properties the time domain analysis of wide sense stationary processes has been developed by Kallianpur [8] and Kallianpur and Mandrekar [9]. We give here, in terms of spectra, a necessary and sufficient condition for a process to have the commutation property with one-dimensional joint innovation spaces. An interesting situation occurs when the canonical representation of the purely nondeterministic part becomes a quarter-plane representation. We prove a criterion for the existence of such a representation and give under this criterion a spectral factorization algorithm. A quarter-plane canonical representation of a process if a kind of "causal representation" in terms of innovations, when the past in \( \mathbb{Z}^2 \) up to a point is defined as the set of all points less than or equal to that point in the usual partial order of \( \mathbb{Z}^2 \). There are however cases in which this causal representation is not causally invertible, i.e., the innovations do not have a quarter-plane representation in terms of the process. The problem of causal invertibility of a quarter-plane canonical representation reduces to that of the equality of the space \( H^2 \) on the unit bidisc with the invariant subspace generated by an outer function [14]. It is generally not an easy task to check whether or not a given outer function implies this equality. We have therefore, at the end of this work, given an elementary condition on the spectral density implying the existence of a causally invertible quarter-plane representation.
NOTATIONS

Random variables and processes considered here are complex valued and defined on a probability space \((\Omega, \mathcal{A}, P)\) fixed once and for all. Random variables are not distinguished from their equivalence classes. The norm and the scalar product on \(L^2(\Omega, \mathcal{A}, P)\) are denoted by \(\| \cdot \|\) and \((\cdot, \cdot)\), respectively. The Hilbert space generated by a family \(\{X_i: i \in I\}\) in \(L^2(\Omega, \mathcal{A}, P)\) is the smallest Hilbert subspace of \(L^2(\Omega, \mathcal{A}, P)\) containing this family and is denoted by \(H(\{X_i: i \in I\})\). If \(Y \in L^2(\Omega, \mathcal{A}, P)\) and \(H\) is a Hilbert subspace of \(L^2(\Omega, \mathcal{A}, P)\), then \((Y/H)\) denotes the orthogonal projection of \(Y\) onto \(H\). If \(H\) is the Hilbert space generated by \(\{X_i: i \in I\}\) then \((Y/H)\) is denoted by \((Y/\{X_i: i \in I\})\) as well.

The generic points of \(Z^2\) are denoted by \((k, j), (m, n), (p, q); k, m, p\) (resp. \(j, n, q\)) always represent the horizontal (resp. vertical) coordinates. In order to shorten the notations we put \(I = [-\frac{1}{2}, \frac{1}{2}]\), \(K = I \times I\), \(S = \{(m, n): m \geq 1, n \in \mathbb{Z}\} \cup \{(0, n): n \in \mathbb{N}\}\). \(D\) is the open unit disc of \(C\). \(L^p(I)\) (resp. \(L^p(K)\)) denotes the \(L^p\)-space with respect to the Lebesgue measure on \(I\) (resp. on \(K\)). \(L^p_0(I)\) (resp. \(L^p_0(K)\)) is the subspace of \(L^p(I)\) (resp. \(L^p(K)\)) generated by \(\{e^{-2\pi i su}: s \in \mathbb{N}\}\) (resp. \(\{e^{-2\pi i (mu+nu)}: (m, n) \in \mathbb{N}^2\}\)). Similarly \(L^p_+ (K)\) is the subspace generated by \(\{e^{-2\pi i (mu+nu)}: (m, n) \in S\}\).

We use the following representations for the Poisson kernel and its analytic completion:

\[
P(r, u-s) = \frac{1-r^2}{1-2r \cos 2\pi(u-s)+r^2}, \quad s, u \in I, \quad r \in [0, 1[,
\]

\[
C(z, u) = \frac{e^{-2\pi i u} + z}{e^{-2\pi i u} - z}, \quad z \in D, \quad u \in I.
\]

All along this paper, \(X = \{X_{m,n}: (m, n) \in \mathbb{Z}^2\}\) is a (nonzero) wide sense stationary centered process. \(F\) is its spectral distribution function, \(f\) its spectral density function (i.e., the Lebesgue derivative of \(F\)) and \(F^s\) the singular part of \(F\) (with respect to the Lebesgue measure). It is known that there exists a process \(Z = \{Z(u, v): (u, v) \in K\}\) with orthogonal increments, such that \(\|Z(u, v)\|^2 = F(u, v)\) and \(X_{m,n} = \int_K \exp[2\pi i (mu+nu)] dZ(u, v)\).

\(X\) is called a white noise if it is an orthonormal sequence.

The following Hilbert spaces representing various pasts of \(X\) are considered.

\[H, \quad \text{generated by} \ X = \{X_{m,n}: (m, n) \in \mathbb{Z}^2\},\]
\[H_{m,n}, \quad \text{generated by} \ \{X_{p,q}: p \leq m, q \leq n\},\]
\[H^1_{m}, \quad \text{generated by} \ \{X_{p,q}: p \leq m, q \in \mathbb{Z}\},\]
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\[ H^2_n, \] generated by \( \{ X_{p,q} : p \in \mathbb{Z}, q \leq n \} \),

\[ H_{m,n}^{1+}, \] generated by \( H^1_m \) and \( \{ X_{m+1,q} : q \leq n \} \),

\[ H_{m,n}^{2+}, \] generated by \( H^2_n \) and \( \{ X_{p,n+1} : p \leq m \} \),

and finally, \( H_{m,n} = H^1_m \cap H^2_n \), \( H_{1-\infty}^1 = \bigcap_{n=1}^{\infty} H^1_n \), \( H_{1-\infty}^2 = \bigcap_{n=1}^{\infty} H^2_n \) and \( H_{-\infty} = H_{-\infty}^1 \cap H_{-\infty}^2 \).

If we need to specify that these spaces \( H \) and functions \( F, f, F', \) \( Z \) correspond to \( X \), we shall denote them as \( H(X) \) and \( F_X, f_X, F'_X, Z_X \).

We denote by \( X^m \) (resp. \( X^n \)) the one-parameter process \( \{ X^m = X_{m,n} : n \in \mathbb{Z} \} \) (resp. \( \{ X^n = X_{m,n} : m \in \mathbb{Z} \} \)) and by \( H_n(X^m) \) (resp. \( H_m(X^n) \)) the Hilbert space generated by \( \{ X^m = X_{m,q} : q \leq n \} \) (resp. \( \{ X^n = X_{p,n} : p \leq m \} \)).

The prediction problem studied in the next section is related to the pasts \( H_1^1 \) and \( H_2^1 \). We mainly express the results related to \( H_1^1 \), those related to \( H_2^1 \) are deduced from them by symmetry.

1. LINEAR PREDICTION AND SPECTRAL FACTORIZATION

To fully express the spectral properties studied in this paper we present here some important features of the time domain analysis related to the prediction problem.

1.1. The Wold Decomposition

The Wold decomposition expressed here by Propositions 1.1.2. and 1.1.6. is contained in Helson and Lowdenslager's work [7]. It corresponds to the column-by-column lexicographic order, i.e., to the pasts \( H_1^1 \). A comparison with the symmetric results corresponding to the row-by-row lexicographic order provides a symmetric representation of the singular component of a strictly nondeterministic process from which we deduce a commutation property studied in Section 2.

**Definition 1.1.1.** The prediction error

\[ I_{m,n}^{1+} = X_{m,n} - (X_{m,n}/H_{m-1,n-1}^{1+}) \]

is called the innovation of \( X \) at \( (m, n) \). \( X \) is said to be singular if \( I_{m,n}^{1+} \) vanishes and strictly nondeterministic (SND) otherwise. If more precision is needed \( I_{m,n}^{1+} \) will be called \( H_1^1 \)-innovation.

According to [6],

\[ \| I_{m,n}^{1+} \| = \exp \frac{1}{2} \int \log f(u, v) \, du \, dv, \]

\( I_{m,n} \)
where the right-hand side should be considered as vanishing if \( \int_{\mathbb{R}^{2}} \log f(u,v) \, du \, dv = -\infty \). Therefore, \( X \) is SND iff \( \log f \) is Lebesgue-integrable.

Let us suppose now that \( X \) is SND and put

\[
\begin{align*}
    c_{0,0} &= \|I_{m,n}^{+}\| \quad \text{and} \quad v_{m,n}^{+} = c_{0,0}^{-1} I_{m,n}^{+}. 
\end{align*}
\]  

(1.1.3)

The normalized innovation process \( v_{m,n}^{+} \) defined by the last equation is a white noise. The \((H^{2+})\)-innovations \( P_{m,n}^{+} \) and \( v_{m,n}^{+} \) can be defined in a similar way.

**Theorem 1.1.2.** (The Wold Decomposition). Let \( X \) be SND. Then it has uniquely the following orthogonal decomposition,

\[
X_{m,n} = U_{m,n} + V_{m,n},
\]  

(1.1.4)

where \( U \) and \( V \) are mutually orthogonal, stationary processes and stationarily correlated with \( X \),

\[
U_{m,n} = \sum_{(p,q) \in \mathcal{S}} c_{p,q} v_{m-p,n-q}^{+},
\]  

(1.1.5)

with

\[
\sum_{(p,q) \in \mathcal{S}} |c_{p,q}|^2 < \infty,
\]  

(1.1.6)

\( U \) is a SND process such that

\[
P_{m,n}^{+} = U_{m,n} - (U_{m,n}/H_{m-1,n-1}^{1+}(U))
\]  

(1.1.7)

and \( V \) is a singular process such that

\[
H(v_{m,n}^{+}) \perp H(V).
\]  

(1.1.8)

Let \( L \) be a Borel set in \( K \) of Lebesgue measure zero supporting the measure defined by \( F^* \). Then we have

\[
U_{m,n} = \int_{K \setminus L} \exp[2\pi i (mu + nv)] \, dZ_X(u,v), \]  

(1.1.9)

\[
V_{m,n} = \int_{L} \exp[2\pi i (mu + nv)] \, dZ_X(u,v). \]  

(1.1.10)

Therefore,

\[
f(u,v) = \left| \sum_{(p,q) \in \mathcal{S}} c_{p,q} \exp[2\pi i (pu + qv)] \right|^2,
\]  

(1.1.11)

\[
F^*(u,v) = F_{V^*}(u,v). \]  

(1.1.12)
**DEFINITION 1.1.3.** We say that \( X \) is purely nondeterministic (PND) if it is SND and if its singular part \( V \) in (1.1.4) vanishes, i.e., if

\[
X_{m,n} = \sum_{(p,q) \in S} c_{p,q} v_{m-p,n-q}^{1+}.
\]  

(1.1.13)

Because of the equality \( H_{m,n}^{1+}(X) = H_{m,n}^{1+}(v^{1+}) \) we call this representation canonical.

We note that representations (1.1.9) and (1.1.10) show that the PND component \( U \) and the singular component \( V \) related to the prediction problem with respect to \( H^{1+} \) are also the PND and singular components related to the prediction problem with respect to \( H^{2+} \).

The criterion for the pure nondeterminism of \( X \) is given in the following

**THEOREM 1.1.4.** \( X \) is PND iff \( F \) is absolutely continuous and \( \log f \in L^1(K) \).

We would like to give more details on the structures of the PND and singular components.

**DEFINITION 1.1.5.** Let

\[
Y_{m,n} = X_{m,n} - (X_{m,n} / H_{m-1}^1).
\]  

(1.1.14)

Then for any fixed \( m \), the process \( Y_{m,0,n} = \{ Y_{m,n} : n \in \mathbb{Z} \} \) generates \( H_{m} \otimes H_{m-1}^1 \). We call \( Y_{m,n} \) the horizontal innovation at the \( m \)th column and the process \( Y^1 = \{ Y_{m,n} : m \in \mathbb{Z} \} \) (with values in \( \mathbb{C}^\mathbb{Z} \) and also denoted by \( \{ Y_{m,n} : (m, n) \in \mathbb{Z}^2 \} \) is called the horizontal innovation process of \( X \). (The vertical innovation at the \( n \)th row \( Y_{2,n} \) and the vertical innovation process are defined in the same way).

We have \( I_{m,n}^{1+} = Y_{m,n} - (Y_{m,n} / H_{m}(Y_{1,m})) \), i.e., \( I_{m,n}^{1+} \) is the innovation of \( Y_{1,m} \) at the point \( n \). Consequently, \( X \) is SND iff \( Y_{1,m} \) is nondeterministic. In this case \( Y_{1,m} \) has the following representation.

\[
Y_{m,n} = Y_{n,m}^{1,m} = \sum_{q=0}^{\infty} c_{0,q} v_{m,n-q}^{1+} + T_{m,n}^{1,m}
\]  

(1.1.15)

where \( T_{m,n}^{1,m} \) is the deterministic part of \( Y_{1,m} \). Similar properties can be announced for the vertical innovation \( Y_{2,n} \) whose deterministic part is denoted by \( T_{2,n} \).

**THEOREM 1.1.6.** (The Wold Decomposition Continued). The following orthogonal decomposition holds.

\[
H_{m,n}^{1+}(X) = H_{m}^1(X) \oplus H_{m+1}(T^1) \oplus H_{m,n}^{1+}(v^{1+}).
\]  

(1.1.16)
Proof. By using the Wold decomposition (1.1.15) of $Y_{1,m}$ we obtain:

$$H_{m,n}^1(X) = H_{-\infty}^1(X) \oplus \left[ \bigoplus_{k=0}^{\infty} H(Y_{1,m-k}) \right] \oplus H_n(Y_{1,m+1}) \quad (1.1.17)$$

$$= H_{-\infty}^1(X) \oplus \left[ \bigoplus_{k=0}^{\infty} H(Y_{1,m+1-k}) \right] \oplus H_{m,n}^1(Y_{1,m+1}),$$

where $H_{-\infty}^1(Y_{1,p})$ is the remote past of $Y_{1,p}$. Decomposition (1.1.16) is then immediate.

**Definition 1.1.7.** A process $X$ is said to be **horizontally deterministic** if $H_{m}^1(X) = H_{-\infty}^1(X)$, it is said to be **horizontally evanescent** if $H_{m}^1(X) = H_{m}^1(X_T)$. Vertically deterministic and vertically evanescent processes are defined in a similar way.

It is shown in [7] that processes $\{(X_{m,n}/H_{-\infty}^1(X)); (m,n) \in \mathbb{Z}^2\}$ and $\{(X_{m,n}/H_{m}^1(X_T)); (m,n) \in \mathbb{Z}^2\}$ are horizontally deterministic and horizontally evanescent processes, respectively.

A nonzero horizontally evanescent process $X$ considered as an infinite dimensional process $\{X^m; m \in \mathbb{Z}\}$ is purely nondeterministic, in the sense that its remote past $H_{-\infty}^1(X)$ reduces to $\{0\}$, whereas its $H_{-\infty}^1+$-innovations vanish. A horizontally deterministic process, considered again as an infinite dimensional process, is also deterministic in the sense that its space $H(X)$ coincides with its remote past $H_{-\infty}^1(X)$. This justifies our definition of a SND process as the one having nonvanishing innovations $I_{m,n}^1$, $I_{m,n}^2$, $Y_{1,m}$, $Y_{2,n}$. Our definition of a PND process as the one expressed in terms of its $H_{-\infty}^1+$-innovations (or equivalently its $H_{-\infty}^2+$-innovations) extends the usual definition of one-parameter PND processes.

Before ending this introductory part of the paper we prove some useful properties of the singular component of a SND process.

**Proposition 1.1.8.** For a SND process, the following relations hold.

$$H(T^1) \perp H(T^2), \quad (1.1.18)$$

$$H_m^1(V) = H(T^2) \oplus \tilde{H}_{-\infty}^1 \oplus H_m^1(T^1), \quad (1.1.19)$$

$$H_n^2(V) = H(T^1) \oplus \tilde{H}_{-\infty}^2 \oplus H_n^2(T^2), \quad (1.1.20)$$

$$\tilde{H}_{m,n}(V) = H_m^1(T^1) \oplus H_n^2(T^2) \oplus \tilde{H}_{-\infty}, \quad (1.1.21)$$

$$H_m^1(V) \ominus \tilde{H}_{m,n}(V) \perp H_n^2(V) \ominus \tilde{H}_{m,n}(V). \quad (1.1.22)$$

**Proof.** From (1.1.16) we deduce

$$H(V) = H_{-\infty}^1(X) \oplus H(T^1) = H_{-\infty}^2(X) \oplus H(T^2). \quad (1.1.23)$$
On the other hand we can write

\[ dF_\gamma(u, v) = g^1(u) \, du \, dv + du \, dG^1(u), \]

\[ dF_\zeta(u, v) = g^2(u) \, du \, dv + dG^2(u) \, dv, \] (1.1.24)

where \( g^1 \) (resp. \( g^2 \)) is the spectral density of \( Y_1 \) (resp. \( Y_2 \)) and \( G^1 \) (resp. \( G^2 \)) is the spectral distribution function of \( T_1 \) (resp. \( T_2 \)) (cf. (1.1.15)). As \( G^1 \) and \( G^2 \) are singular with respect to the Lebesgue measure, \( du \, dG^1(u) \) and \( dG^2(u) \, dv \) generate mutually singular measures. This implies that \( T_1 \) and \( T_2 \) are mutually uncorrelated. Hence (1.1.18) holds. According to (1.1.23),

\[ H(T_1) \subset H^1_{-\infty} \quad \text{and} \quad H(T_2) \subset H^1_{-\infty}. \] (1.1.25)

From (1.1.18), (1.1.23), and (1.1.25) we deduce

\[ (H(T_1) \oplus H(T_2))^\perp = H(T_1)^\perp \cap H(T_2)^\perp = \tilde{H}_{-\infty} \] (1.1.26)

with orthogonal complements, denoted by \( ^\perp \), taken in \( H(V) \). Therefore, \( H^1_{-\infty} = H(T_2) \oplus \tilde{H}_{-\infty} \quad \text{and} \quad H^2_{-\infty} = H(T_1) \oplus \tilde{H}_{-\infty}. \) These two decompositions imply (1.1.19) and (1.1.20) from which (1.1.21) and (1.1.22) are deduced.

A comparison of (1.1.16) with (1.1.19) gives a four-fold orthogonal decomposition of \( H^1_{m,n} \) produced in

**Corollary 1.1.9.** For a SND \( X \), we have

\[ H^1_{m,n} (X) = H(T_2) \oplus \tilde{H}_{-\infty} \oplus H^1_{m+1} (T_1) \oplus H^1_{m,n} (v_1^1) \] (1.1.27)

and similarly

\[ H^2_{m,n} (X) = H(T_1) \oplus \tilde{H}_{-\infty} \oplus H^2_{n+1} (T_2) \oplus H^2_{m,n} (v_2^2). \] (1.1.28)

To write down the representation of \( X_{m,n} \) in terms of its projections onto the components of the right-hand sides of (1.1.27) and (1.1.28) it would be interesting to have algorithms of extraction for \( T_1, T_2, v_1^1, \) and \( v_2^2 \). To the best of our knowledge the problem of extraction of \( T_1 \) and \( T_2 \) is an open problem. We develop in the following paragraph a method of computation for \( v_1^1 \) (or equivalently for \( v_2^2 \)). This would allow the computation of the PND and singular parts of a SND process.

### 1.2. Spectral Factorization

In this paragraph we deal with the canonical factorization of the spectral density \( f \). It is known that [6] \( X \) is PND iff its spectral distribution
function is absolutely continuous and there exists a function \( \varphi \in L^2(K) \) with the Fourier series,

\[
\varphi(u, v) = \sum_{(p,q) \in S} a_{p,q} e^{-2\pi i(pu + qv)}
\]

such that \( a_{0,0} \neq 0 \) and \( f(u, v) = |\varphi(u, v)|^2 \) a.e. (1.2.1)

By an application of Karhunen's theorem [10], it can be shown that this is also equivalent to the existence of a white noise \( v \) such that \( X_{m,n} \) has the following representation

\[
X_{m,n} = \sum_{(p,q) \in S} a_{p,q} v_{m-p-n-q}, \quad a_{0,0} \neq 0
\]

**DEFINITION 1.2.1.** The function \( \varphi \) in (1.2.1) is said to factorize \( f \). The factorization is said to be **canonical** if \( f^1 = a_{0,0} v_{m,n} \).

The canonicity of the spectral factorization (1.2.1) is equivalent [6] to the equality:

\[
|a_{0,0}|^2 = \exp \int_K \log f(u, v) \, du \, dv.
\]

If a function \( \varphi \in L^2(K) \) realizes the canonical factorization of \( f \) then it is unique, in the sense that any other element of \( L^2(K) \) realizing the canonical factorization of \( f \) is deduced from \( \varphi \) by multiplying it by a constant of modulus 1 [7]. Consequently, with an abuse of language we can say that the element \( \varphi \) realizes the canonical factorization of \( f \).

**PROPOSITION 1.2.2.** Suppose \( X \) is SND. Then the spectral density of \( Y^1 \) as a two-parameter process is given by

\[
f_{Y^1}(u, v) = g^1(v) \quad \text{a.e.,}
\]

where \( g^1 \) is the spectral density function of \( Y^{1,m} \) (for all \( m \)) and is given by

\[
g^1(v) = \exp \int f(u, v) \, du \quad \text{a.e.}
\]

**Proof.** Representation (1.1.15) implies that \( f_{Y^1} \) is of the form (1.2.4) where \( g^1 \) is the spectral density of \( Y^{1,0} = \{ Y_{0,n} : n \in \mathbb{Z} \} \). We thus have

\[
g^1(v) = \left| \sum_{q=0}^{\infty} c_{0,q} \exp(-2\pi iqv) \right|^2
\]
with the series converging in \( L^2(I) \). Let functions \( c_p, p \geq 0 \), in \( L^2(I) \) be defined by

\[
c_0(v) = \sum_{q=0}^{\infty} c_{0,q} \exp(-2\pi i q v),
\]

\[
c_p(v) = \sum_{q=-\infty}^{\infty} c_{p,q} \exp(-2\pi i q v), \quad \text{for } p \geq 1,
\]

where coefficients \( c_{p,q} \) are those given in Theorem 1.1.2. We have

\[
\int I \sum_{p=0}^{\infty} |c_p(v)|^2 dv = \sum_{p=0}^{\infty} \int I |c_p(v)|^2 dv = \sum_{(p,q) \in \mathcal{S}} |c_{p,q}|^2 < \infty.
\]

Therefore, there is a set \( A_1 \) of Lebesgue measure 1 such that, for all \( v \in A_1 \), \( \sum_{p=0}^{\infty} |c_p(v)|^2 < \infty \). Let us fix \( v \in A_1 \) and put

\[
G(z, v) = \sum_{p=0}^{\infty} c_p(v) z^p, \quad z \in D,
\]

\[
G_r(u, v) = G(re^{-2\pi i u}, v).
\]

Then \( G(\cdot, v) \) belongs to the Hardy space \( H^2(D) \) and there is a Borel set \( B_v \) of measure 1 such that, for all \( u \in B_v \),

\[
\lim_{r \to 1} G_r(u, v) = \sum_{p=0}^{\infty} c_p(v) e^{-2\pi i pu},
\]

where the right-hand side is the Fourier series of an element of \( L^2(I) \) that we denote by \( G'(\cdot, v) \), [15]. Consequently, there is an analytic function \( M(z, v) \) on \( D \) such that \( |M(z, v)| \leq 1 \) and

\[
G(z, v) = M(z, v) \exp \frac{1}{2} \int I C(z, u) \log |G'(u, v)|^2 \, du.
\]

Let us denote by \( G^* \) the element of \( L^2_+ (K) \) defined by the Fourier series \( \sum_{(p,q) \in \mathcal{S}} c_{p,q} \exp[-2\pi i (pu + qv)] \). We have

\[
\int K |G^*(u, v) - G_r(u, v)|^2 du dv \leq \int I \left[ \sum_{p=0}^{\infty} |c_p(v)|^2 (1 - r^p)^2 \right] dv. \tag{1.2.10}
\]

An application of the Dominated Convergence Theorem shows that \( G_r \) converges to \( G^* \) in \( L^2(K) \) as \( r \to 1 \). Therefore, there is an increasing sequence \( \{r_n : n \in \mathbb{N}\} \) in \( [0, 1] \), converging to 1, such that \( \{G_{r_n} : n \in \mathbb{N}\} \) converges to \( G^* \) a.e. on \( K \). We deduce that there is a set \( A_2 \) of Lebesgue
measure 1 such that for each \( v \in A_2 \), \( \{G_n(u, v) : n \in \mathbb{N}\} \) converges to \( G^*(u, v) \) for almost all \( u \). Hence, for \( v \in A_1 \cap A_2 \), \( G'(u, v) = G^*(u, v) \) for almost all \( u \) and we may replace in (1.2.9) \( G' \) by \( G^* \). Finally, taking into account the fact that \( |G^*|^2 = f \) a.e. we can write

\[
G(z, v) = M(z, v) \exp \frac{1}{2} \int_I C(z, u) \log f(u, v) \, du
\]  

(1.2.11)

for almost all \( v \).

According to (1.2.6), (1.2.7), and (1.2.8) we have \( g'(v) = |G(0, v)|^2 \) a.e. Therefore,

\[
g'(v) \leq \exp \int_I \log f(u, v) \, du \quad \text{a.e.} \quad (1.2.12)
\]

But (1.1.2) and (1.1.5) imply that

\[
\exp \int_I \log g'(v) \, dv = \exp \int_I \log f(u, v) \, du \, dv.
\]

This in turn implies that in (1.2.12) only equality holds.

Formulas (1.2.5) and (1.2.11) show that \( |M(0, v)| = 1 \). Therefore, by the Maximum Modulus Theorem [15], \( M(z, v) \) does not depend on \( z \) and, hence, formula (1.2.11) becomes

\[
G(z, v) = \frac{c_0(v)}{|c_0(v)|} \exp \left[ \frac{1}{2} \int_I C(z, u) \log f(u, v) \, du \right].
\]  

(1.2.13)

This formula and the preceding proof suggest the following factorization theorem.

**THEOREM 1.2.3.** We suppose \( X \) is SND and define

\[
\Gamma(z, v) = \exp \frac{1}{2} \int_I C(z, u) \log f(u, v) \, du, \quad z \in D, \text{ a.e. } v \in I, \quad (1.2.14)
\]

\[
\gamma(w) = \exp \frac{1}{2} \int_K C(w, v) \log f(u, v) \, du \, dv, \quad w \in D. \quad (1.2.15)
\]

Let \( \Gamma_r \) and \( \gamma_r \) be defined by

\[
\Gamma_r(u, v) = \Gamma(re^{-2\pi i u}, v), \quad r \in [0, 1[, \ u \in I, \text{ a.e. } v \in I, \quad (1.2.16)
\]

\[
\gamma_r(v) = \gamma(re^{-2\pi i u}), \quad r \in [0, 1[, \ v \in I. \quad (1.2.17)
\]
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Then $\Gamma_r \in L^2(K)$ (resp. $\gamma_r \in L^2(I)$) and it converges in $L^2(K)$ (resp. in $L^2(I)$) to a function $\Gamma^*$ (resp. $\gamma^*$), when $r \to 1$, and the element $G^*$ of $L^2_+(K)$ defined by

$$G^*(u, v) = \gamma^*(v)|\gamma^*(v)|^{-1} \Gamma^*(u, v) \quad a.e. \ (u, v) \in K \quad (1.2.18)$$

realizes the canonical factorization of $f$.

Proof. Many of the relations between functions of $v$ that we shall reproduce below, as well as the ones considered in the statement of the theorem, are only valid for almost all $v \in I$. To shorten the expressions we shall write these relations without always putting the mention "for almost all $v \in I$.”

By putting

$$b_r(v) = \int I e^{2\pi i pu} \log f(u, v) \ du \quad (1.2.19)$$

we can write

$$\frac{1}{2} \int I C(z, u) \log f(u, v) \ du = \frac{1}{2} b_0(v) + \sum_{\rho = 1}^{\infty} b_\rho(v) z^\rho \quad (1.2.20)$$

and also

$$\Gamma(z, v) = \sum_{\rho = 0}^{\infty} a_\rho(v) z^\rho. \quad (1.2.21)$$

We note that $|\Gamma(0, v)|^2 = |a_0(v)|^2 = g^1(v)$, the spectral density of the horizontal innovation at the $m$th column, considered in Proposition 1.2.2.

Obviously $\Gamma_r$ is a measurable function of $(u, v)$. We have, by an application of Jensen’s inequality,

$$|\Gamma_r(u, v)|^2 = \exp \int I P(r, s - u) \log f(s, v) \ ds \quad (1.2.22)$$

$$\leq \int I P(r, s - u) f(s, v) \ ds$$

and we get by the Fubini Theorem

$$\int_K |\Gamma_r(u, v)|^2 \ du \ dv \leq \int_K f(u, v) \ du \ dv < \infty. \quad (1.2.23)$$

Therefore, $\Gamma_r \in L^2(K)$. 

Coefficients $a_p$ in (1.2.21) are measurable, for $a_p(v)$ is the value of $(1/p!)(\partial^p f(z, v)/\partial z^p)$ at $z = 0$. On the other hand, by (1.2.22), we have

$$\int_I |\Gamma_r(u, v)|^2 \, du = \sum_{p=0}^{\infty} |a_p(v)|^2 \, r^{2p} \leq \int_I f(u, v) \, du$$

and

$$\lim_{r \to 1} \sum_{p=0}^{\infty} |a_p(v)|^2 \, r^{2p} = \sum_{p=0}^{\infty} |a_p(v)|^2 \leq \int_I f(u, v) \, du.$$

We thus have

$$\int_I \sum_{p=0}^{\infty} |a_p(v)|^2 \, dv \leq \int_K f(u, v) \, du \, dv < \infty.$$

This implies that $a_p \in L^2(I)$ for all $p \in \mathbb{N}$ and $\sum_{p=0}^{n} a_p(v) e^{-2\pi i p u}$ converges in $L^2(K)$, as $n \to \infty$, to an element $\Gamma^*$ that we may represent by

$$\Gamma^*(u, v) = \sum_{p=0}^{\infty} a_p(v) e^{-2\pi i p u}. \quad (1.2.24)$$

Now, by using an inequality of the type (1.2.10), we can see that $\Gamma_r$ converges to $\Gamma^*$ in $L^2(K)$ as $r \to 1$.

Let us put

$$G_r(v) = \int_I \left| \int_I P(r, s-u) \log f(s, v) \, ds - \log f(u, v) \right| \, du.$$

It is known that $[15]$ for almost all $v$, $\lim_{r \to 1} G_r(v) = 0$. On the other hand, $G_r(v) \leq 2 \int_I |\log f(u, v)| \, du$. Therefore, by the Dominated Convergence Theorem, we have $\lim_{r \to 1} \int_K G_r(v) \, dv = 0$, i.e.,

$$\lim_{r \to 1} \int_K \left| \int_I P(r, s-u) \log f(s, u) \, ds - \log f(u, v) \right| \, du \, dv = 0.$$

We conclude from this that there is an increasing sequence $\{r_n, n \in \mathbb{N}\}$ in $[0, 1]$, converging to 1, such that the sequence $\{\exp \int_I P(r_n, s-u) \log f(s, v) \, ds, n \in \mathbb{N}\}$ converges to $f(u, v)$, almost everywhere on $K$. But, as a consequence of the $L^2$-convergence of $\Gamma_r$ to $\Gamma^*$,

$$|\Gamma_{r_n}(u, v)|^2 - \exp \int_I P(r_n, s-u) \log f(s, v) \, ds$$

converges in $L^1(K)$ to $|\Gamma^*|^2$. Therefore, $f = |\Gamma^*|^2$ a.e. on $K$. 

The statement concerning $y$ is the classical one-parameter factorization of the density $g^1$ given by (1.2.5), (cf. [3, 13]). It remains to prove that $G^*$ realizes the canonical factorization of $f$. In fact, it is easily seen that $G^*$ has the following representation in $L^2(K)$,

$$G^*(u, v) = \sum_{(p, q) \in \mathbb{S}} c_{p, q} \exp[-2\pi i(pu + qv)],$$

where

$$c_{p, q} = \int f e^{2\pi iuv} \gamma^*(v) |\gamma^*(v)|^{-1} a_p(v) \, dv.$$  

We have $|G^*|^2 = |\Gamma^*|^2 = f$ a.e. and by (1.2.15)

$$c_{0,0}^2 = \exp \int f(u, v) \, du \, dv.$$  

Therefore, $G^*$ realizes the canonical factorization of $f$. 

**Remark 1.2.4.** Let $X$ be regular and let $b_{p,q}$ be defined by

$$b_{p,q} = \int e^{2\pi i(pu + qv)} \log f(u, v) \, du \, dv.$$  

(1.2.25)

In case,

$$\sum_{(p, q) \in \mathbb{Z}^2} |b_{p,q}| < \infty,$$  

(1.2.26)

the formulation of the canonical factor $G^*$ given by (1.2.18) is equivalent to the one proposed in [4].

Let us put

$$b_p(v) = \sum_{q \in \mathbb{Z}} b_{p, q} e^{-2\pi iqv}, \quad v \in I,$$

$$H(z, v) = \frac{i}{2} b_0(v) + \sum_{p = 1}^{\infty} b_p(v) z^p, \quad (z, v) \in \bar{D} \times I,$$

$$h(w) = \frac{i}{2} b_{0,0} + \sum_{q = 1}^{\infty} b_{0, q} w^q, \quad w \in \bar{D},$$

where $\bar{D}$ is the closure of $D$. Under condition (1.2.26), these series converge uniformly and define continuous functions. Then following the notations of the preceding theorem, we have

$$\Gamma^*(u, v) = \exp H(e^{-2\pi iu}, v),$$

$$\gamma^*(v) = \exp h(e^{-2\pi iv}).$$
Finally, by putting
\[ L(u, v) = H(e^{-2\pi i u}, v) - \frac{1}{2} b_0(v) + h(e^{-2\pi i v}) \]
\[ = \frac{1}{2} b_{0,0} + \sum_{q=1}^{\infty} b_{0,q} e^{-2\pi i q u} + \sum_{p=1}^{\infty} \sum_{q=-\infty}^{\infty} b_{p,q} e^{-2\pi i (pu + qv)} \]
we obtain, as in [4], \( G^*(u, v) = \exp L(u, v) \).

2. QUARTER-PLANE REPRESENTATIONS

In this section we study various types of quarter-plane representations of PND processes. These representations are connected with commutation properties of horizontal and vertical projections or equivalently with conditional independence properties (in the wide sense) that we consider in the following paragraph.

2.1. Time Domain Considerations

**Definition 2.1.1.** We say that \( X \) has **Property \( F_4^\circ \)** if
\[ \forall (m, n), \quad H^1_m \ominus H_{m,n} \perp H^2_n \ominus H_{m,n} \] (2.1.1)
and that it has **Property \( F_4^\sim \)** if
\[ \forall (m, n), \quad H^1_m \ominus \tilde{H}_{m,n} \perp H^2_n \ominus \tilde{H}_{m,n} . \] (2.1.2)

In the theory of two-parameter martingales, properties \( F_4^\circ \) concern \( \sigma \)-algebras rather than Hilbert spaces [1, 12]. In case \( X \) is a Gaussian process \( F_4^\circ \) (resp. \( F_4^\sim \)) expresses the fact that the \( \sigma \)-algebras generated by \( H^1_m \) and \( H^2_n \) are conditionally independent given the \( \sigma \)-algebra generated by \( H_{m,n} \) (resp. \( \tilde{H}_{m,n} \)).

In what follows \( P_{m,n}, \tilde{P}_{m,n}, P^1_{m,n}, P^2_{m,n}, P^1_{m,n}^+, P^2_{m,n}^+ \) will denote the orthogonal projections onto \( H_{m,n}, \tilde{H}_{m,n}, H^1_m, H^2_n, H^1_{m,n}, H^2_{m,n} \), respectively.

**Proposition 2.1.2.** (i) \( F_4^\sim \) is equivalent to the following commutation property
\[ \forall (m, n), \quad P^1_m P^2_n = P^2_n P^1_m . \] (2.1.3)
The projection defined by this equality necessarily coincides with \( \tilde{P}_{m,n} \).

(ii) \( F_4^\circ \) holds iff
\[ \forall (m, n), \quad P^1_m P^2_n = P_{m,n} \] (or equivalently \( P^2_n P^1_m = P_{m,n} \)). (2.1.4)
Therefore, \( F_4^\circ \) implies \( F_4^\sim \).
Proof. Let $H^x_{m,n}$ represent either $H_{m,n}$ or $\tilde{H}_{m,n}$ and $P^x_{m,n}$ represent either $P_{m,n}$ or $\tilde{P}_{m,n}$. Then the proposition is an immediate consequence of the following equivalence: $H^1_n \perp H^2_n \ominus H^x_{m,n} \iff P^1_n (P^2_n - P^x_{m,n}) = 0$ (or equivalently) $H^1_n \perp H^2_n \ominus P^x_{m,n} \iff P^2_n (P^1_n - P^x_{m,n}) = 0$.

Here is an intermediate property between $F^2$ and $F^{\sim}_\alpha$.

Definition 1.2.3. A SND process is said to have Property $F_\alpha$ if $I^1_{m,n} = I^2_{m,n}$.

Proposition 2.1.4. For a SND process $F^2 \Rightarrow F_\alpha \Rightarrow F^{\sim}_\alpha$.

Proof. Let $X$ be a SND process having Property $F_\alpha$. We proved in Proposition 1.1.8 that its singular part has Property $F^{\sim}_\alpha$. Denoting by $v_{m,n}$ the common innovation $v^1_{m,n} = v^2_{m,n}$ and by $U$ the PND part of $X$ we can write $H^1_m (U) = H^1_m (v)$ and $H^2_m (U) = H^2_m (v)$. As a white noise, $v$ has Property $F_\alpha$, hence Property $F^{\sim}_\alpha$. Therefore $U$ and $X$ have Property $F^{\sim}_\alpha$. Now suppose that $X$ has Property $F^{\sim}_\alpha$. Then $P^1_n X_{m,n-k} = P^2_n X_{m,n-k}$. From this we deduce that $P^1_n X_{m,n-k} = P^2_n X_{m,n-k}$ for all $k \geq 0$. We then see that $H_n (Y^{1,m}) = H_{m,n} \ominus H_{m-1,n}$ where $Y^{1,m}$ is the horizontal innovation at the $m$th column. Consequently,

$$P^1_{m-1,n-1} X_{m,n} = P^1_{m-1} X_{m,n} + (X_{m,n} / H_{n-1} (Y^{1,m}))$$

$$= (P_{m-1,n} + P_{m,n-1} - P_{n-1,n-1}) X_{m,n}.$$  

But by a symmetric argument we see that this is also equal $P^2_{m-1,n-1} X_{m,n}$. Therefore, $I^1_{m,n} = I^2_{m,n}$.

Definition 2.1.5. A PND process is said to have a quarter-plane representation in terms of a white noise $v$ if

$$X_{m,n} = \sum_{(p,q) \in \mathbb{N}^2} c_{p,q} v_{m-p,n-q}$$

with $c_{0,0} \neq 0$ and $\sum_{(p,q) \in \mathbb{N}^2} |c_{p,q}|^2 < \infty$.  

We say that the quarter-plane representation is canonical if $c_{0,0} v_{m,n} = I^1_{m,n} = I^2_{m,n}$ and that it is invertible if $H_{m,n} (X) = H_{m,n} (v)$.

Proposition 2.1.6. Let $X$ be a PND process. Then

(i) $X$ has a canonical quarter-plane representation if it has Property $F_\alpha$.  


(ii) \( X \) has an invertible quarter-plane representation iff it has Property \( F_4 \).

Proof. (i) is obvious. Suppose that \( X \) has an invertible quarter-plane representation. Since \( v \) has Property \( F_4 \) then so does \( X \). Conversely, if \( X \) has Property \( F_4 \) then it has Property \( F_4 \) and we see that its canonical representation (1.1.13) reduces to (2.1.5) with \( v = v^{1+} = v^{2+} \). This implies that \( H_{m,n}(X) \subset H_{m,n}(v) \). On the other hand, according to the proof of Proposition 2.1.4 we have, under Property \( F_4 \), \( P^1_{m-1,n-1} X_{m,n} \in H_{m,n}(X) \) and \( P^2_{m-1,n-1} X_{m,n} \in H_{m,n}(X) \). This implies that \( I^1_{m,n} \) and \( I^2_{m,n} \) are in \( H_{m,n}(X) \) and hence \( H_{m,n}(v) \subset H_{m,n}(X) \). Therefore \( H_{m,n}(X) = H_{m,n}(v) \).

Now, let \( X \) have the commutation property \( F_4 \). It is easy to see that its PND and singular parts both have this property. On the other hand, we have \( P^1_{m-1} X_{m,n} = \tilde{P}^1_{m-1,n} X_{m,n} \). This implies that \( Y^1_{m,n} \in \tilde{R}_{m,n} \subset H^2_n \). Consequently, the remote past \( H(T^1 \cdot n) = H_{-\infty}(Y^1 \cdot m) \) of the horizontal innovation \( Y^1 \cdot m \) is a subspace of \( H^2_n \). Taking into account (1.1.23) we see that \( H(T^1) \perp H(T^2) \). We recall that this orthogonality relation was proved in Proposition 1.1.8 under the hypothesis that \( X \) should be SND.

**Definition 2.1.7.** The following space

\[
A_{m,n} = H(Y^{1,m}) \cap H(Y^{2,n})
\]

is called the joint innovation space at \((m, n)\).

Joint innovation spaces \( A_{m,n} \) (called two-dimensional innovation spaces in [9]) played an important role in the time domain analysis of wide sense stationary processes on \( \mathbb{Z}^2 \), developed in [8] and [9]. The moving average representation proved in [9] and also derived from [16] under Property \( F_4 \) can be widened under Property \( F_4 \) as shown in

**Proposition 2.1.8.** Let \( X \) have Property \( F_4 \) and let \( U \) and \( V \) be its PND and singular components, respectively. Then

\[
U_{m,n} = \sum_{(p,q) \in \mathbb{N}^2} (X_{m,n}/A_{m-p,n-q}),
\]

\[
V_{m,n} = \sum_{p=0}^{\infty} (X_{m,n}/T^{1,m-p}) + \sum_{q=0}^{\infty} (X_{m,n}/T^{2,n-q}) + (X_{m,n}/\tilde{R}_{-\infty}).
\]

**Proof.** Representation (2.1.8) is an immediate consequence of (1.1.21) which is valid under \( F_4 \) even if \( X \) is singular. To prove (2.1.7) it is enough to show that, under \( F_4 \), we have \( H(U) = \bigoplus_{(m,n) \in \mathbb{Z}^2} H(A_{m,n}) \). In fact, if \( F_4 \) holds, then
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\[ \bigoplus_{n = -\infty}^{\infty} H(A_{m,n}) = \bigoplus_{n = -\infty}^{\infty} \left[ H(Y^{1,m}) \cap H(Y^{2,n}) \right] = H(Y^{1,m}) \cap (H \ominus H_{-\infty}^{2}) = H(Y^{1,m}) \ominus H_{-\infty}^{2} (Y^{1,m}). \]

Therefore,

\[ \bigoplus_{(m,n) \in \mathbb{Z}^2} H(A_{m,n}) = \bigoplus_{m = -\infty}^{\infty} \left[ H(Y^{1,m}) \ominus H_{-\infty}^{2} (Y^{1,m}) \right] = H(u^{1,1}) = H(U). \]

Representation (2.1.7) would be a quarter-plane representation in the sense of Definition 2.1.5 if the joint innovation spaces \( A_{m,n} \) were one-dimensional. It would reduce to a canonical quarter-plane representation if \( X \) has Property \( F_2 \) and to an invertible quarter-plane representation if \( X \) (or rather \( U \)) had Property \( F_2 \) (cf. [9] for the last case). In the next section spectral criteria will be given for \( X \) to have these properties \( F_4 \).

2.2. Spectral Considerations

We shall consider the functions \( \Gamma, \Gamma^*, \gamma^*, G^*, \) etc. used in the canonical spectral factorization of Theorem 1.2.3. To emphasize the fact that these functions concern the prediction problem with respect to \( H^1 \) we shall indicate them by \( \Gamma^1, \Gamma^*^1, \gamma^*^1, G^*^1, \) etc., and similarly for those corresponding to \( H^2 \) we shall write \( \Gamma^2, \Gamma^*^2, \gamma^*^2, G^*^2, \) etc.

**Theorem 2.2.1.** A SND process \( X \) with spectral density \( f \) has the commutation property \( F_4 \) with one-dimensional joint innovation spaces \( A_{m,n} \) iff there exist Borel functions \( M^1 \) and \( M^2 \) with moduli equal to 1 a.e. such that

\[ M^1(v) \Gamma^1(u, v) = M^2(u) \Gamma^*^2(u, v) \quad \text{a.e.} \quad (2.2.1) \]

In particular, \( X \) has Property \( F_4 \) iff

\[ \frac{\gamma^*^1(v)}{a_0^1(v)} \Gamma^1(u, v) = \frac{\gamma^*^2(u)}{a_0^2(u)} \Gamma^*^2(u, v) \quad \text{a.e.} \quad (2.2.2) \]

or equivalently

\[ G^*^1(u, v) = G^*^2(u, v) \quad \text{a.e.} \quad (2.2.2') \]

**Proof.** Since the singular part of a SND process always has Property \( F_2 \) as well as Property \( F_4 \) we may suppose, without any loss of generality, that \( X \) is PND. Then according to Theorem 1.2.3, \( X \) has the following representation.
We recall that \((a_0^1)^2\) (resp. \((a_0^2)^2\)) is the spectral density of \(Y^{1,m}\) (resp. \(Y^{2,n}\)). Suppose that \(X\) has the commutation property \(F_4\) with one-dimensional innovation spaces \(A_{m,n}\) and let \(\eta_{0,0}\) be a unit vector generating \(A_{0,0}\). We define a white noise \(\eta\) by \(\eta_{m,n} = S_{m,n}\eta_{0,0}\), where \(S_{m,n}\) is the shift operator on \(H\) defined by \(S_{m,n}X_{0,0} = X_{m,n}\). Then \(X\) and \(\eta\) are stationarily correlated and therefore there are functions \(\delta^1, \delta^2, \varepsilon = 1, 2\), in \(L_2^0(I)\) such that \(|\delta^1|^2 = (a_0^1)^2\) a.e. and that

\[
\begin{align*}
Y_{m,n}^{1} & = \int \frac{e^{2\pi i (mu + nv)}}{a_0^{1}(v)} \delta^1(v) \, dZ_{\eta}(u, v), \\
Y_{m,n}^{2} & = \int \frac{e^{2\pi i (mu + nv)}}{a_0^{2}(u)} \delta^2(u) \, dZ_{\eta}(u, v).
\end{align*}
\]

Therefore equalities (2.2.3) can be written as

\[
X_{m,n} = \int \frac{e^{2\pi i (mu + nv)}}{a_0^{1}(v)} \delta^1(v) \, \Gamma^{*1}(u, v) \, dZ_{\eta}(u, v)
\]

\[
- \int \frac{e^{2\pi i (mu + nv)}}{a_0^{2}(u)} \delta^2(u) \, \Gamma^{*2}(u, v) \, dZ_{\eta}(u, v).
\]

By putting \(M^* = \delta^\prime(v)/a_0^{1}(v)\), \(\varepsilon = 1, 2\), we obtain (2.2.1) where \(|M^*| = 1\) a.e.

If \(X\) has Property \(F_4\) then \(A_{m,n}\) is generated by the normalized innovation \(v_{m,n} = v_{m,n}^{1+} = v_{m,n}^{2+}\) and we have

\[
X_{m,n} = \int \frac{e^{2\pi i (mu + nv)}}{a_0^{1}(v)} \, G^{*\varepsilon}(u, v) \, dZ_{\eta}(u, v), \quad \varepsilon = 1, 2
\]

From this, equality (2.2.2) (or (2.2.2')) is deduced.

Conversely, suppose that there are functions \(M^1\) and \(M^2\) satisfying the condition of the theorem. Since \(f(u, v) = |\Gamma^{*\varepsilon}(u, v)|^2\) a.e. \(\varepsilon = 1, 2\), then by Karhunen’s theorem there is a white noise \(\eta\) such that

\[
X_{m,n} = \int \frac{e^{2\pi i (mu + nv)}}{a_0^{1}(v)} \, M^1(v) \, \Gamma^{*1}(u, v) \, dZ_{\eta}(u, v)
\]

\[
= \int \frac{e^{2\pi i (mu + nv)}}{a_0^{2}(u)} \, M^2(u) \, \Gamma^{*2}(u, v) \, dZ_{\eta}(u, v).
\]
On the other hand $r_{1}$ and $r_{2}$ have the following expansions

$$f(y, u) = f(u, z, v) e^{-2\pi i u v},$$

$$g(y, u) = g(u, z, v) e^{-2\pi i u v},$$

where $a_{1}, a_{2} \in L^{2}(I)$ and the series converge in $L^{2}(K)$. But comparing representation (2.2.4) with (2.2.3) we see that $dZ_{1}(u, v) = a_{1}(v) M^{1}(v) dZ_{v}(u, v)$ and $dZ_{2}(u, v) = a_{2}(u) M^{2}(u) dZ_{v}(u, v)$. The a.e. positivity and the a.e. finiteness of $a_{1}$ and $a_{2}$ imply that $H(Y_{1,m}) = H(\eta_{m})$ and $H(Y_{2,n}) = H(\eta_{n})$. Consequently $H_{m}(X) = H_{m}(\eta)$, $H_{n}(X) = H_{n}(\eta)$ and $H(Y_{1,m}) \cap H(Y_{2,n}) = H(\eta_{m}) \cap H(\eta_{n}) = \text{space generated by } \eta_{m,n}$. We see that $X$ has the commutation property $F_{m,n}$ and that $\Delta_{m,n}$ is generated by $\eta_{m,n}$.

Now suppose that (2.2.2) holds. Then the canonical representations of $X$ in terms of $v_{1}^{+}$ and $v_{2}^{+}$ are quarter-plane representations. Since $G^{*1} = G^{*2}$ a.e. and $|G^{*1}|^{2} = |G^{*2}|^{2} = \delta$ a.e. there is a white noise $v$ such that

$$X_{m,n} = \int_{K} e^{2\pi i (mu + nv)} G^{*1}(u, v) dZ_{v}(u, v), \quad e = 1, 2.$$ 

Since canonical representations are always unique we have $v = v_{1}^{+} = v_{2}^{+}$. Therefore, $X$ has Property $F_{m,n}$.

The following theorems gives a more direct criterion than (2.2.2) for $X$ to have Property $F_{m,n}$ and offers a shorter way of computing the canonical factor $G^{*1} = G^{*2}$ that we denote again by $G^{*}$.

To shorten the notations we shall write

$$h(r, u) = h(re^{-2\pi i u}, re^{-2\pi i u}, r \in [0, 1[$$

for an element $h$ of the Hardy space $H^{p}(D^{2})$, $p \geq 1$. It is known [14] that the radial limit $h^{*}(u, v) = \lim h_{r}(u, v)$ exists a.e. and the convergence also holds in $L^{p}(K)$.

We shall also put

$$P[h](z, w) = \int_{K} P(r, s - u) P(p, t - v) h(s, t) ds dt,$$

where $z = re^{-2\pi i u}, w = pe^{-2\pi i w}$.

The $b_{p,q}$'s denote the Fourier coefficients of $\log f$ as defined by (1.2.25).

**Theorem 2.2.2.** (i) $X$ has Property $F_{m}$ iff $b_{p,q} = 0$ for $pq < 0$.

(ii) Suppose that this property holds and put

$$H(z, w) = \frac{1}{2} b_{0,0} + \sum_{(p,q) \in \mathbb{N}^{2}} b_{p,q} z^{p} w^{q}, \quad (z, w) \in D^{2},$$

(2.2.5)
where \( N_0^2 = \mathbb{N}^2 \setminus \{(0,0)\} \), and
\[
G(z, w) = \exp H(z, w). \quad (2.2.6)
\]
Then \( H \) is analytic on \( D^2 \) and \( G \) belonging to \( H^2(D^2) \) has the following expansion:
\[
G(z, w) = \sum_{(p,q) \in N^2} c_{p,q} z^p w^q \text{ with } c_{0,0} > 0 \text{ and } \sum_{(p,q) \in N^2} |c_{p,q}|^2 < \infty. \quad (2.2.7)
\]
Finally the radial limit \( G^* \in L^2(K) \) which is defined by
\[
G^*(u, v) = \sum_{(p,q) \in N^2} c_{p,q} \exp[-2\pi i (pu + qv)] \quad (2.2.8)
\]
realizes the canonical factorization of \( f \) in the sense of Definition 1.2.1, with respect to both \( H^1^+ \) and \( H^2^+ \).

**Proof.** First, suppose that \( b_{p,q} = 0 \) for \( pq < 0 \). Since, for \( r, \rho \in [0, 1[ \), \[
|H(re^{-2\pi i u}, \rho e^{-2\pi i u})| = (1 - r)^{-1} (1 - \rho)^{-1} \int_K |\log f(u, v)| \, du \, dv < \infty, \quad H \text{ is analytic on } D^2 \text{ and so is } G \text{ which has moreover, no zeros in } D^2. \text{ On the other hand, according to Jensen's inequality, we have}
\]
\[
|G_r(u, v)|^2 = \exp P[\log f], (u, v) \leq P[f], (u, v).
\]
Therefore, by the Fubini theorem
\[
\int_K |G_r(u, v)|^2 \, du \, dv \leq \int_K f(u, v) \, du \, dv.
\]
This shows that \( G \in H^2(D^2) \). The fact that \( G^* \) is the radial limit of \( G \) and it satisfies the equality \( |G^*|^2 = f \text{ a.e.} \) is an elementary consequence of the well-known convergence theorems [14]. Since \( c_{0,0}^2 = \exp \int_K \log f(u, v) \, du \, dv \), \( G^* \) realizes the canonical factorization of \( f \) corresponding to the prediction problem with respect to both pasts \( H^1^+ \) and \( H^2^+ \). Consequently, \( X \) has Property \( F_4 \).

Conversely, let us suppose that \( X \) has this property, and let (2.1.5) be its canonical representation of the PND part of \( X \) in terms of the joint innovation \( v = v^1 = v^2 \). Then the function \( G \) defined by
\[
G(z, w) = \sum_{(p,q) \in N^2} c_{p,q} z^p w^q \quad (2.2.9)
\]
is an element of \( H^2(D^2) \) and satisfies the equality: \( \log |G(0,0)| = \int_K \log |G^*(u, v)| \, du \, dv. \) Therefore, \( G \) is an outer function (cf. Paragraph 4.4.3 in [14]). This implies that \( P[\log |G^*|] \), that is then equal to \( \log |G| \), is the
real part of a homomorphic function on $D^2$. According to Theorem 2.4.1 of [14], the Fourier coefficients $b_{p,q}$ of $\log |G*|^2 = \log f$ vanish for $pq < 0$.

We would like to mention the fact that Property $F_4$ does not necessarily imply $F_4$. This is proved by using the notion of invariant subspaces in $H^2(D^2)$, [14]. We recall that an invariant subspace in $H^2(D^2)$ generated by an element $h \in H^2(D^2)$ is the closed linear subspace of $H^2(D^2)$ generated by \{ $z^pw^qh(z,w) : (p, q) \in \mathbb{N}^2$ \} and denoted by $S(h)$.

**Theorem 2.2.3.** (i) Let a PND process $X$ have Property $F_4$. Then it has Property $F_4^*$ iff $S(G) = H^2(D^2)$ where $G$ is defined by (2.2.6). (ii) There is a PND process $X$ having Property $F_4^*$ without having Property $F_4$.  

**Proof.** (i) It is clear that Property $F_4^*$ implies Property $F_4$ iff $H_{m,n}(X) = H_{m,n}(v)$, i.e., iff the set \{ $X_{m-p,n-q}$ : $(p, q) \in \mathbb{N}^2$ \} generates $H_{m,n}(v)$. But this holds iff the set \{ $e^{-2\pi i(pu+qv)} G*(u,v) : (p, q) \in \mathbb{N}^2$ \}, where $G^*$ is the radial limit (2.2.8), is total in $L_0^2(K)$. This last subspace is isometric to $H^2(D^2)$ under the correspondence $e^{-2\pi i(pu+qv)} \leftrightarrow z^p w^q$. So the assertion holds.

(ii) Let $G$ be defined on $D^2$ by $G(z, w) = \exp(z + w + 2)/(z + w - 2)$. Obviously $G \in H^2(D^2)$. Let $G^*$ be the radial limit of $G$. Then $f = |G*|^2$ is a continuous bounded positive function defined on $D^2 \\setminus \{ (1, 1) \}$. There is a centered gaussian PND process whose spectral density is $f$. The function $G$ given above coincides with the one defined by (2.2.6), because of the uniqueness of the canonical factorization. On the other hand, it is proved in [14] that $S(G) \neq H^2(D^2)$. Therefore, $X$ has Property $F_4^*$ without having Property $F_4$.

Finally, to end this paragraph, we would like to express a sufficient condition on $f$ for $X$ to have Property $F_4^*$.

**Proposition 2.2.4.** If a PND process has Property $F_4$, $f \in L^\infty(K)$ and $f^{-1} \in L^1(K)$, then $X$ has Property $F_4^*$.

**Proof.** We can apply to $f^{-1}$ what we developed here for $f$ and obtain that $\varphi = G^{-1}$ is an element of $H^2(D^2)$ such that $\varphi* = G^{-1}$ realizes the quarter-plane factorization of $f^{-1}$, where $G$ is defined by (2.2.6). Let us put then

$$\varphi^*(u,v) = \sum_{(p,q) \in \mathbb{N}^2} a_{p,q} e^{-2\pi i(pu+qv)}.$$  

The series converges in $L^2(K)$. Since $f \in L^\infty(K)$ it also converges in $L^2(K, \mu)$, where $\mu$ is the measure generated by the density $f$. Now, by putting again $v = v^1 + v^2$, we can write...
$$v_{m,n} = \int_K e^{2\pi i (mu + nv)} dZ_v(u, v)$$

$$= \int_K e^{2\pi i (mu + nv)} \varphi^*(u, v) G^*(u, v) dZ_v(u, v)$$

$$= \int_K e^{2\pi i (mu + nv)} \varphi^*(u, v) dZ_{\chi}(u, v).$$

Therefore, $v_{m,n} \in H_{m,n}(X)$. This implies that $H_{m,n}(v) \subset H_{m,n}(X)$ and hence $H_{m,n}(v) = H_{m,n}(X)$.

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REFERENCES


