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Fractional chromatic number of distance graphs generated by two-interval sets

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Abstract

Let D be a set of positive integers. The distance graph generated by D, denoted by $G(Z, D)$, has the set Z of all integers as the vertex set, and two vertices x and y are adjacent whenever $|x - y| \in D$. For integers $1 < a \leq b < m - 1$, define $D_{a,b,m} = \{1, 2, \ldots, a - 1\} \cup \{b + 1, b + 2, \ldots, m - 1\}$. For the special case $a = b$, the chromatic number for the family of distance graphs $G(Z, D_{a,a,m})$ was first studied by R.B. Eggleton, P. Erdős and D.K. Skilton [Colouring the real line, J. Combin. Theory (B) 39 (1985) 86–100] and was completely solved by G. Chang, D. Liu and X. Zhu [Distance graphs and T-coloring, J. Combin. Theory (B) 75 (1999) 159–169]. For the general case $a \leq b$, the fractional chromatic number for $G(Z, D_{a,b,m})$ was studied by P. Lam and W. Lin [Coloring distance graphs with intervals as distance sets, European J. Combin. 26 (2005) 25 1216–1229] and by J. Wu and W. Lin [Circular chromatic numbers and fractional chromatic numbers of distance graphs with distance sets missing an interval, Ars Combin. 70 (2004) 161–168], in which partial results for special values of a, b, m were obtained. In this article, we completely settle this problem for all possible values of a, b, m.

1. Introduction

Let D be a set of positive integers. The *distance graph* generated by D, denoted by $G(Z, D)$, has the set Z of all integers as the vertex set, and two vertices x and y are adjacent whenever $|x - y| \in D$. Initiated by Eggleton, Erdős and Skilton [5], the study of distance graphs has attracted considerable attention [2–8,11–18,20–25].

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A *fractional coloring* of a graph G is a mapping f which assigns to each independent set I of G a non-negative weight $f(I)$ such that for each vertex x, $\sum_{x \in I} f(I) \geq 1$. The *fractional chromatic number* $\chi_f(G)$ of G is the least total weight of a fractional coloring for G.

The problem of determining the fractional chromatic number for distance graphs has been studied in different research areas under different names. Firstly, it is equivalent to a sequence density problem in number theory. For a set D of positive integers, a sequence S of non-negative integers is called a *D-sequence* if $a - b \notin D$ for any $a, b \in S$. Let $S(n)$ denote $|\{0, 1, \ldots, n-1\} \cap S|$. The upper density and the lower density of S are defined, respectively, by

$$\bar{\delta}(S) = \overline{\lim}_{n\to\infty} \frac{S(n)}{n}, \qquad \underline{\delta}(S) = \underline{\lim}_{n\to\infty} \frac{S(n)}{n}.$$

We say S has density $\delta(S)$ if $\bar{\delta}(S) = \underline{\delta}(S) = \delta(S)$. The parameter of interest is the maximum density of a D-sequence, defined by

$$\mu(D) = \sup\{\delta(S) : S \text{ is a } D\text{-sequence}\}.$$

The problem of determining or estimating $\mu(D)$ was initially posed by Motzkin in an unpublished problem collection (cf. [1]), and has been studied in [1,10,19,9,18]. Note that S is a D-sequence if and only if S (as a set of integers) is an independent set of $G(Z, D)$. It was proved by Chang et al. [3] that for any finite set D,

$$\mu(D) = \frac{1}{\chi_f(G(Z, D))}.$$

Secondly, the fractional chromatic number of a distance graph is equivalent to an asymptotic problem in T-coloring. The T-coloring problem was motivated by the channel assignment problem introduced by Hale [10], in which an integer broadcast channel is assigned to each of a given set of stations or transmitters so that interference among nearby stations is avoided. Interference is modeled by a set of non-negative integers T containing 0 as the forbidden channel separations. By using a graph G to represent the broadcast network, a valid channel assignment is defined as a *T-coloring* for G, which is a mapping $f : V(G) \to Z$ such that $|f(x) - f(y)| \notin T$ whenever $xy \in E$. The *span* of a T-coloring f is the difference between the largest and the smallest numbers in $f(V)$, i.e., $\max\{|f(u) - f(v)| : u, v \in V\}$. Given T and G, the *T-span* of G, denoted by $\text{sp}_T(G)$, is the minimum span among all T-colorings of G. As for any graph G, $\text{sp}_T(G) \leq \text{sp}_T(K_{\chi(G)})$, it is useful to estimate $\text{sp}_T(K_n)$. Let σ_n denote $\text{sp}_T(K_n)$. Griggs and Liu [9] proved that for any set T the *asymptotic T-coloring ratio*

$$R(T) := \lim_{n\to\infty} \frac{\sigma_n}{n}$$

exists and is a rational number. It was proved in [3] that for any T, by letting $D = T - \{0\}$, we have $R(T) = \chi_f(G(Z, D))$.

Partially due to its rich connections to other problems, the fractional chromatic number for various classes of distance graphs has been studied in the literature (cf. [2,3,17,18,23,14,24,25]). If D is a singleton, trivially $\chi_f(G(Z, D)) = 2$. If $D = \{a, b\}$ and $\gcd(a, b) = 1$, it is known [1] that $\chi_f(G(Z, D)) = \frac{a+b}{\lfloor (a+b)/2 \rfloor}$. For $|D| \geq 3$, the exact values of $\chi_f(G(Z, D))$ are known only for some special sets D. For $D = \{a, b, a+b\}$, upper and lower bounds for $\chi_f(G(Z, D))$ were obtained by Rabinowitz and Proulx [19]. Let $\chi(G)$ and $\omega(G)$ denote, respectively, the chromatic number and the clique number of G. It is easy to see that $\omega(G) \leq \chi_f(G) \leq \chi(G)$ holds for any graph G, and $\chi(G(Z, D)) \leq |D| + 1$ [4,20] if D is finite. In [18], the sets D with

$\omega(G(Z, D)) \geq |D|$ were characterized and the value of $\chi_f(G(Z, D))$ for most of this class of graphs, including $D = \{a, b, a+b\}$, was determined.

For any two integers $a \leq b$, let $[a, b]$ denote the interval of consecutive integers $\{a, a+1, \ldots, b\}$. It is known that if $D = [a, b]$, then $\chi_f(G(Z, D)) = (a+b)/a$ [9,2]. For the sets D of the form $D = [1, m] - \{k, 2k, \ldots, sk\}$ for integers m, k and s, the values of $\chi_f(G(Z, D))$ were determined in [17].

For $1 < a \leq b < m-1$, let $D_{a,b,m}$ denote the two-interval set

$$D_{a,b,m} = [1, a-1] \cup [b+1, m-1].$$

Note, if $a = b$, then $D_{a,a,m} = [1, m-1] - \{a\}$. The chromatic number for $G(Z, D_{a,a,m})$ was first studied by Eggleton, Erdős and Skilton [5] and the problem was completely solved in [3]. For the general case $a \leq b$, both the fractional chromatic number and the chromatic number for $G(Z, D_{a,b,m})$ were studied by Wu and Lin [23], and by Lam and Lin [14]. Some partial results were obtained. In this article, we completely determine the fractional chromatic number of $G(Z, D_{a,b,m})$ for all $1 < a \leq b < m-1$.

2. Main result and preliminaries

For some special cases, the values of $\chi_f(G(Z, D_{a,b,m}))$ for the two-interval set $D_{a,b,m} = [1, a-1] \cup [b+1, m-1]$ were solved in [23,14]. If $b < 2a$, then $\chi_f(G(Z, D_{a,b,m}))$ is determined in [23]. Let $\Delta = m - b$. If $\Delta \leq a$ or $\Delta \geq 2a$, then $\chi_f(G(Z, D_{a,b,m}))$ is determined in [14]. Some other special cases (which cannot be easily described) are discussed in [14].

The main result of this article is the following which completely determines the value of $\chi_f(G(Z, D_{a,b,m}))$ for all $1 < a \leq b < m-1$.

Theorem 1. *For integers* $1 < a \leq b < m-1$. *Suppose* $G = G(Z, D_{a,b,m})$ *where* $D_{a,b,m} = [1, a-1] \cup [b+1, m-1]$. *Let* $\Delta = m - b$, $s = \lfloor b/a \rfloor$, *and* $q = \lfloor m/\Delta \rfloor$.

- *If* $\Delta \geq 2a$, *then* $\chi_f(G) = (sa+m)/(s+1)$.
- *If* $\Delta \leq a$, *then* $\chi_f(G) = \max\{a, m/(s+1)\}$.
- *If* $a < \Delta < 2a$, *then*

$$\chi_f(G) = \begin{cases} \dfrac{sa+m}{s+1}, & \text{if } 2qa \leq m < a + q\Delta \text{ or if } m \geq (2q+1)a; \\ \dfrac{m}{q}, & \text{if } m < \min\{q\Delta + a, 2qa\}; \\ \dfrac{(2q-1)m + a}{2q^2}, & \text{if } q\Delta + a \leq m < (2q+1)a. \end{cases}$$

The cases for $\Delta \geq 2a$ and $\Delta \leq a$ were solved in [14]. However, for completeness, we include these cases in the statement and give a short proof for them.

Recall the result in [3] mentioned in Section 1, the fractional chromatic number of G is equal to the reciprocal of $\mu(D_{a,b,m})$, which is the maximum density of a $D_{a,b,m}$-sequence. Let $I = \{x_1, x_2, \ldots\}$ be a $D_{a,b,m}$-sequence where $x_i < x_{i+1}$. Let $\delta_i = x_{i+1} - x_i$. The sequence $\Omega = (\delta_1, \delta_2, \ldots)$ is called the *gap sequence* of I. In the following, we call a sequence $(\delta_1, \delta_2, \ldots)$ a *D-gap sequence* if it is the gap sequence of a D-sequence. Observe that a sequence $(\delta_1, \delta_2, \ldots)$ is a D-gap sequence if and only if for any $j \leq j'$, $\sum_{i=j}^{j'} \delta_i \notin D$. In particular, the following observation is frequently used, usually implicitly, in our proofs.

- *A sequence* $(\delta_1, \delta_2, \ldots)$ *is a* $D_{a,b,m}$*-gap sequence if and only if*
  (1) $\delta_i \geq a$ *for each* i; *and*
  (2) *for any* $j \leq j'$, *either* $\sum_{i=j}^{j'} \delta_i \leq b$ *or* $\sum_{i=j}^{j'} \delta_i \geq m$.

  By definition,

$$\mu(D_{a,b,m}) = \max \lim_{n\to\infty} \frac{|I \cap [0, n-1]|}{n},$$

where the maximum is taken over all $D_{a,b,m}$-sequences I. Hence

$$\chi_f(G) = \frac{1}{\mu(D_{a,b,m})} = \min \lim_{n\to\infty} \frac{n}{|I \cap [0, n-1]|} = \min \lim_{k\to\infty} \sum_{i=1}^{k} \frac{\delta_i}{k}.$$

Again, the minimum is taken over all $D_{a,b,m}$-sequences I with gap sequence $(\delta_1, \delta_2, \ldots)$.

For an interval of integers $[a, b]$, we call its cardinality $|[a, b]|$ the *length* of $[a, b]$. Given a $D_{a,b,m}$-gap sequence $Y = (\delta_1, \delta_2, \delta_3, \ldots)$, the *average gap length* of Y is $\lim_{k\to\infty} \sum_{i=1}^{k} \frac{\delta_i}{k}$ (if exists). Thus to determine the fractional chromatic number of $G(Z, D_{a,b,m})$, it amounts to determine the minimum average gap length of a $D_{a,b,m}$-gap sequence. Usually, the gap sequences we concern are periodic. For a periodic gap sequence, it suffices to present one period of the sequence. We shall denote by $\langle y_1, y_2, \ldots, y_k\rangle$ the infinite periodic sequence with period k. That is, $\langle y_1, y_2, \ldots, y_k\rangle = (y_1, y_2, \ldots, y_j, \ldots)$ where for $j > k$, $y_j = y_{j-k}$. For convenience, we denote by $p \otimes t$, for any integers p and t, the *p repetitions* of t. For example, $\langle 3 \otimes 5, 2 \otimes 7\rangle$ is the periodic sequence $\langle 5, 5, 5, 7, 7\rangle = (5, 5, 5, 7, 7, 5, 5, 5, 7, 7, \ldots)$.

We now give a short proof for the cases $\Delta \leq a$ and $\Delta \geq 2a$. As each gap of a $D_{a,b,m}$-gap sequence is at least a, we have $\chi_f(G) \geq a$. If $m \leq (s+1)a$, then $\langle a\rangle$ is a $D_{a,b,m}$-gap sequence with average gap length a. Hence $\chi_f(G) = a$. Assume $m > (s+1)a$ and $\Delta \leq a$. Then the sequence $\langle s \otimes a, m - sa\rangle$ is a $D_{a,b,m}$-gap sequence of average gap length $m/(s+1)$. So $\chi_f(G) \leq m/(s+1)$. On the other hand, for any $D_{a,b,m}$-gap sequence $(\delta_1, \delta_2, \ldots)$, since $\sum_{i=1}^{s+1} \delta_i \geq (s+1)a \geq b+1$, we must have $\sum_{i=1}^{s+1} \delta_i \geq m$. Hence the average gap length is at least $m/(s+1)$. So $\chi_f(G) = m/(s+1)$.

Assume $\Delta \geq 2a$. It is easy to verify that the sequence $\langle s \otimes a, m\rangle$ is a $D_{a,b,m}$-gap sequence with average gap length $(m+sa)/(s+1)$. Hence $\chi_f(G) \leq (m+sa)/(s+1)$. On the other hand, if $\chi_f(G) = 1/\mu(D_{a,b,m}) < (m+sa)/(s+1)$, then there is a $D_{a,b,m}$-sequence I with $|[0, sa+m-1] \cap I| \geq s+2$. Without loss of generality, we may assume $0 \in I$. Let $I' = \{i : i \in I, i \leq b\} \cup \{i - m + a : i \in I, i \geq m - a\}$. It is easy to verify that $|I| = |I'|$, $I' \subseteq [0, (s+1)a - 1]$ and for any $x, y \in I'$, $|x - y| \geq a$. This is in contrary to the assumption that $|I| \geq s+2$. Therefore we have $\chi_f(G) = (m+sa)/(s+1)$.

3. Proof of the upper bound

In the rest of the paper, we assume that $a < \Delta < 2a$, and let

$$\tau(D_{a,b,m}) = \begin{cases} \dfrac{sa+m}{s+1}, & \text{if } 2qa \leq m < a + q\Delta \text{ or if } m \geq (2q+1)a; \\ \dfrac{m}{q}, & \text{if } m < \min\{q\Delta + a, 2qa\}; \\ \dfrac{(2q-1)m + a}{2q^2}, & \text{if } q\Delta + a \leq m < (2q+1)a. \end{cases}$$

In this section, we prove that $\chi_f(G) \leq \tau(D_{a,b,m})$. This amounts to present a $D_{a,b,m}$-gap sequence whose average gap length is at most $\tau(D_{a,b,m})$.

Lemma 2. *Suppose $G = G(Z, D_{a,b,m})$. Then $\chi_f(G) \leq \tau(D_{a,b,m})$.*

Proof. First note that the following are two $D_{a,b,m}$-gap sequences:

$$\langle s \otimes a, m\rangle \quad \text{and} \quad \langle (q-1) \otimes \Delta,\ m - ((q-1)\Delta)\rangle,$$

where the average gap lengths, respectively, are $(sa+m)/(s+1)$ and m/q. This proves the result for all the cases, except the very last one.

For the last case, $q\Delta + a \leq m < (2q+1)a$, the gap sequence is more complicated. We shall define some special sequences, then combine them to form the required periodic sequence.

For $i = 1, 2, \ldots, q-1$, let Y_i, Y_i' and W be finite sequences of integers defined as follows:

$$\begin{aligned} Y_i &= (i \otimes \Delta, a, (q-1-i) \otimes \Delta, m - (a + (q-1)\Delta)) \\ Y_i' &= ((i-1) \otimes \Delta, \Delta + a, (q-1-i) \otimes \Delta, m - (a + (q-1)\Delta)) \\ W &= (a). \end{aligned}$$

Let

$$Y_q' = ((q-1) \otimes \Delta, m - (q-1)\Delta).$$

For finite sequences $A = (a_1, a_2, \ldots, a_s)$ and $B = (b_1, b_2, \ldots, b_t)$, the *concatenation* of A and B, denoted by AB, is the sequence

$$AB = (a_1, a_2, \ldots, a_s, b_1, b_2, \ldots, b_t).$$

The concatenation of sequences is associative. Thus for finite sequences $A_1, A_2, \ldots, A_t$, the sequence $A_1 A_2 \ldots A_t$ is well-defined. Define the periodic gap sequence as

$$\langle Y_q' Y_{q-1} Y_{q-1}' Y_{q-2} Y_{q-2}' \cdots Y_1 Y_1' W\rangle.$$

Now we show that this sequence is indeed a $D_{a,b,m}$-gap sequence. Since

$$m - (a + (q-1)\Delta) = m - q\Delta - a + \Delta \geq \Delta > a,$$

each entry of the sequence is at least a. It remains to show that the sum of any number of consecutive entries of the sequence is either at most b or at least m. Observe that the sum of the entries in each Y_i or Y_i' is equal to m. Consider the sum of any t consecutive entries in the sequence. Straightforward calculation shows that if $t \geq q+1$, then the sum is at least m; if $t \leq q-1$, then the sum is at most b; if $t = q$, then the sum is either equal to m or at most b. (Here we use the condition that $(q-1)\Delta + a \leq (q-1)\Delta + m - q\Delta = b$.) Thus the sequence defined above is a $D_{a,b,m}$-gap sequence.

Straightforward calculation shows that this gap sequence has average gap length $\frac{(2q-1)m+a}{2q^2}$. ■

4. Proof of the lower bound

To complete the proof of Theorem 1, it remains to show that $\chi_f(G) \geq \tau(D_{a,b,m})$. To this end, we need some more definitions.

In the following, we assume that $I = \{x_1, x_2, \ldots\}$ is a $D_{a,b,m}$-sequence, i.e., an independent set in $G = G(Z, D_{a,b,m})$. We shall prove that the gap sequence of I has average gap length at least $\tau(D_{a,b,m})$.

Let

$$L = \{i : x_{i+1} - x_i \geq \Delta\}.$$

For each $x_i \in I$, we associate it with a set X_i of integers as follows.

$$X_i = \begin{cases} [x_i, x_i + \Delta - 1], & \text{if } i \in L; \\ [x_i, x_i + a - 1] \cup [x_i + m, x_i + m + a - 1], & \text{if } i \notin L. \end{cases}$$

Lemma 3. *If $i \neq j$, then $X_i \cap X_j = \emptyset$.*

Proof. Assume $i < j$. If $i \in L$, then $X_i = [x_i, x_i + \Delta - 1]$ and by definition, $x_j \geq x_i + \Delta$. As $t \in X_j$ implies that $t \geq x_j$, we have $X_i \cap X_j = \emptyset$. Assume $i \notin L$. Then $X_i = [x_i, x_i + a - 1] \cup [x_i + m, x_i + m + a - 1]$. As $x_j \geq x_i + a$, we know that $X_j \cap [x_i, x_i + a - 1] = \emptyset$. Assume $X_j \cap [x_i + m, x_i + m + a - 1] \neq \emptyset$. Then by the definition of X_j, we have either $x_j \in [x_i + m - \Delta + 1, x_i + m - 1]$ or $x_j \in [x_i + m, x_i + m + a - 1]$. The former case implies $b + 1 \leq x_j - x_i \leq m - 1$; and the latter case implies $b + 1 \leq x_j - x_{i+1} \leq m - 1$ (since $i \notin L$, we have $a \leq x_{i+1} - x_i < \Delta$). For both cases, it contradicts the assumption that I is a $D_{a,b,m}$-sequence. ■

We call intervals of the form $[x_i + m, x_i + m + a - 1]$ for $i \notin L$ *Type-B I-intervals*. Intervals of the form $[x_i, x_i + \Delta - 1]$ for $i \in L$, and intervals of the form $[x_i, x_i + a - 1]$ for $i \notin L$ are called *Type-A I-intervals*. Both Type-A and Type-B I-intervals are referred as I-intervals. The length of an I-interval is either Δ or a, and they are called, respectively, *long* or *short* I-intervals.

Lemma 4. *If $T = [x_i, x_i + a - 1]$ is a short Type-A I-interval, then the first I-interval $T' = [u, v]$ with $u \geq x_i + a$ is Type-A.*

Proof. Assume to the contrary that $T' = [u, v] = [x_j + m, x_j + m + a - 1]$ for some j. As $x_j + m \geq x_i + a$, which implies $x_i - x_j \leq m - a$, we have $x_i - x_j \leq b$. So $x_j + m \geq x_i + \Delta$. In addition, since T is a short Type-A I-interval, $x_{i+1} < x_i + \Delta$. Hence, $x_{i+1} < x_j + m$, contradicting the choice of T'. ■

Lemma 5. *There are at most s short consecutive I-intervals that are of the same type.*

Proof. First we show that there are at most s short consecutive Type-A I-intervals. Assume $T_1 = [u_1, v_1], T_2 = [u_2, v_2], \ldots, T_j = [u_j, v_j]$ are consecutive I-intervals and $T_1, T_2, \ldots, T_{j-1}$ are short and Type-A. By Lemma 4, T_j is also Type-A. So $u_1, u_2, \ldots, u_j \in I$. We prove by induction on i that $u_i \leq u_1 + b$ for $i = 1, 2, \ldots, j$. It is trivial for $i = 1$. Assume $i < j$ and $u_i \leq u_1 + b$. By definition of I-intervals, $u_{i+1} - u_i < \Delta$. Hence $u_{i+1} < u_i + \Delta \leq u_1 + m$. As $u_1, u_{i+1} \in I$, it follows that $u_{i+1} \leq u_1 + b$.

Because $s = \lfloor b/a \rfloor$ and $|T_i| \geq a$, we conclude that there are at most s consecutive short Type-A I-intervals. By definition, consecutive Type-B I-intervals correspond to consecutive short Type-A I-intervals. So the result follows. ■

Suppose T is an I-interval. Define the *weight* of T by

$$w(T) = \begin{cases} 1, & \text{if } T \text{ is long;} \\ 1/2, & \text{if } T \text{ is short.} \end{cases}$$

For any interval of integers $[u, v]$, let

$$w([u, v]) = \sum_{T \text{is an } I\text{-interval and } T \subseteq [u,v]} w(T).$$

By definition, every integer in I creates either a long interval of weight 1 or two short intervals of weight $1/2$ each. By Lemma 3, all these intervals are disjoint, and by definition the two short intervals induced by an integer in I are of distance $m - a$ apart. Hence, by Lemma 5, for any n,

$$w([0, n-1]) - s/2 \le |I \cap [0, n-1]| \le w([0, n-1]) + s/2.$$

Thus to prove that $\lim_{n\to\infty} \frac{n}{|I\cap[0,n-1]|} \ge \tau(D_{a,b,m})$, it suffices to show that $\lim_{n\to\infty} \frac{n}{w([0,n-1])} \ge \tau(D_{a,b,m})$.

An interval $W = [x, y]$ of integers is called *neat* if every I-interval is either contained in W or disjoint from W. Suppose W is a neat interval. We define the *X-ratio* of W to be

$$r(W) = \frac{|W|}{w(W)}.$$

To prove that $\lim_{n\to\infty} \frac{n}{|I\cap[0,n-1]|} \ge \tau(D_{a,b,m})$, it suffices to find integers $a_1 < a_2 < \cdots$ such that for any i, $R_i = [a_i, a_{i+1} - 1]$ is a neat interval and $r(R_i) \ge \tau(D_{a,b,m})$.

We say an integer p has property (*) if

(*) *for the first Type-B I-interval $[u, u + a - 1]$ with $u \ge p$, we have $u \ge p + \Delta$.*

Lemma 6. *Each $x_i \in I$ has property $(*)$. Moreover, if $i \in L$, then $x_i + m$ also has property $(*)$ and $[x_i, x_i + m - 1]$ is neat.*

Proof. If $i \notin L$, by Lemma 4, x_i has property (*). Assume $i \in L$. By definition, x_i has property (*). Suppose $x_i + m$ does not have property (*). Then, there exists some u with $x_i + m \le u < x_i + m + \Delta$ such that $[u, u+a-1]$ is a Type-B I-interval. By definition, $u - m \in I$ and $[u - m, u - m + a - 1]$ is Type-A. This is impossible as $x_i \le u - m < x_i + \Delta \le x_{i+1}$ but $i \in L$. Hence, $x_i + m$ has property (*).

Now, assume to the contrary that $[x_i, x_i + m - 1]$ is not neat. Let $T = [u, v]$ be an I-interval that $T \cap [x_i, x_i + m - 1] \neq \emptyset$ and $T \not\subseteq [x_i, x_i + m - 1]$. By definition and as $i \in L$, T must be Type-A. Hence, $u \in I$. Let $u = x_t$ for some t. Then $x_i + m - \Delta + 1 \le x_t \le x_i + m - 1$. This implies $b + 1 \le x_t - x_i \le m - 1$, a contradiction. ■

To complete the proof of Theorem 1, it suffices to find an infinite sequence of integers $a_1 < a_2 < \cdots$ such that the following hold for all i:

(1) a_i has property (*),
(2) $R_i = [a_i, a_{i+1} - 1]$ is neat, and
(3) $r(R_i) \ge \tau(D_{a,b,m})$.

We shall construct such a sequence of integers $a_1 < a_2 < \cdots$ inductively. Initially, set $a_1 = x_1$. By Lemma 6, a_1 has property (*). Assume we have determined $a_1, a_2, \ldots, a_i$, where (1–3) in the above are satisfied. We shall determine a_{i+1} so that (1–3) still hold.

Let $[u, v]$ be the first I-interval with $u \ge a_i$. If $[u, v]$ is Type-B, then as a_i has property (*), $u \ge a_i + \Delta$. Let $a_{i+1} = x_t$, where x_t is the smallest element of I for which $x_t > a_i$. Then all the I-intervals contained in $R_i = [a_i, a_{i+1} - 1]$ are Type-B, and R_i is neat. Assume R_i contains j Type-B I-intervals. By Lemma 5, $j \le s$. Since $w(R_i) = j/2$ and $|R_i| \ge \Delta + ja$, it follows that

$$r(R_i) \ge \frac{2(\Delta + ja)}{j} \ge 2a + \frac{2\Delta}{s} \ge \tau(D_{a,b,m}).$$

(Observe that $\frac{sa+m}{s+1} < a + \frac{b}{s+1} + \frac{\Delta}{s+1} < 2a + \frac{\Delta}{s+1}$. If $m < 2qa$, then $\frac{m}{q} < 2a$. If $m < (2q+1)a$, then $\frac{(2q-1)m+a}{2q^2} < 2a$.) Moreover, by Lemma 6, $a_{i+1} = x_t$ has property (*). Thus (1–3) in the above are satisfied.

In the following, assume $[u, v]$ is Type-A. Then $u \in I$. Let x_h be the first element of I such that $x_h \geq u$ and $h \in L$. Let $a_{i+1} = x_h + m$. By Lemma 6, $R_i = [a_i, a_{i+1} - 1]$ is neat and a_{i+1} has property (*).

It remains to show (3). Assume the interval $[u, x_h - 1]$ contains j I-intervals for some $j \geq 0$. By Lemma 4, all the I-intervals contained in $[u, x_h - 1]$ are Type-A and short.

Since an I-interval of weight 1 has length Δ and an I-interval of weight 1/2 has length $a > \Delta/2$, so for any interval T of length m, we have

$$w(T) \leq \begin{cases} q, & \text{if } m < q\Delta + a; \\ q + \dfrac{1}{2}, & \text{if } m \geq q\Delta + a. \end{cases}$$

Because $R_i = [a_i, x_h - 1] \cup [x_h, x_h + m - 1]$, it follows that

$$w(R_i) \leq \begin{cases} q + \dfrac{j}{2}, & \text{if } m < q\Delta + a; \\ q + \dfrac{j+1}{2}, & \text{if } m \geq q\Delta + a. \end{cases}$$

Now we consider three cases.

Case 1. $m < q\Delta + a$. As $|R_i| \geq ja + m$, by the above discussion, $r(R_i) \geq \frac{ja+m}{q+j/2}$. Observe that $\frac{ja+m}{q+j/2}$ is a function of j which is increasing if $m \leq 2qa$ and decreasing if $m \geq 2qa$. Hence, as $j \leq s$, we have

- if $m \geq 2qa$, then $r(R_i) \geq \frac{sa+m}{q+\frac{s}{2}} \geq \frac{sa+m}{s+1}$;
- if $m < 2qa$, then $r(R_i) \geq \frac{0a+m}{q+0} \geq \frac{m}{q}$.

Hence, (3) holds.

Case 2. $m \geq (2q+1)a$. Similar to Case 1, we have $r(R_i) \geq \frac{ja+m}{q+(j+1)/2}$. Because $m \geq (2q+1)a$, which implies that $\frac{ja+m}{q+(j+1)/2}$ is a decreasing function of j, we conclude that $r(R_i) \geq \frac{sa+m}{q+(s+1)/2}$. As $\frac{b}{a} = \frac{m}{a} - \frac{\Delta}{a} \geq 2q + 1 - 2$, we have $s = \lfloor b/a \rfloor \geq 2q - 1$, i.e., $q \leq (s+1)/2$. Hence $r(R_i) \geq (sa+m)/(s+1)$, so (3) holds.

Case 3. $a + q\Delta \leq m < (2q+1)a$. Then $r(R_i) \geq \frac{ja+m}{q+(j+1)/2}$. Because $m < (2q+1)a$, $\frac{ja+m}{q+(j+1)/2}$ is an increasing function of j. If $j \geq 1$, then $r(R_i) \geq \frac{a+m}{q+1} > \frac{(2q-1)m+a}{2q^2}$. If $j = 0$ and $w(R_i) \leq q$, then $r(R_i) \geq \frac{m}{q} > \frac{(2q-1)m+a}{2q^2}$, and we are done.

Assume $j = 0$ and $w(R_i) = q + 1/2$. Then $u = x_h$ and $r(R_i) \geq m/(q + 1/2)$. As $\frac{m}{q+1/2} < \frac{(2q-1)m+a}{2q^2} = \tau(D_{a,b,m})$, this "$a_{i+1}$" does not satisfy our requirement. We need to find a different a_{i+1} so that (1–3) are satisfied. In the following, we re-name the interval $[u, u+m-1]$ just obtained by R_i^1. (The correct R_i is not found yet.)

Since $w(R_i^1) = q + 1/2$, R_i^1 contains a short I-interval. Let $p_1 \leq q$ be the total weight of I-intervals preceding the last short I-interval in R_i^1. As $w(R_i^1) = q + 1/2$ and the first I-interval of R_i^1 is long, we know that $p_1 \geq 1$ is an integer.

Before reaching the correct interval R_i, we may need a (finite) sequence of intervals R_i^j, where R_i^1 is just the first one of them. In the following, we describe the inductive step of finding R_i^j.

Suppose z is an integer, $1 \leq z \leq 2q-1$, and for $j = 1, 2, \ldots, z$, we have obtained $R_i^j = [x_{i_j}, x_{i_j} + m - 1]$ with the following properties:

- $x_{i_j} \in I$ and $i_j \in L$, and for $j \geq 2$, $x_{i_{j-1}} + m \leq x_{i_j} < x_{i_{j-1}} + m + a$.
- $w(R_i^j) = q + 1/2$.

Observe that if $w(R_i^j) = q + 1/2$, the I-intervals in R_i^j must be "tightly packed." Namely, if a neat sub-interval H of R_i^j has length $\geq \alpha\Delta + \beta a$, where α, β are non-negative integers, then $w(H) \geq \alpha + \beta/2$. For otherwise, $w(R_i^j)$ will be less than $q + 1/2$.

Let p_j be the total weight of I-intervals preceding the last short I-interval in R_i^j. Since $w(R_i^j) = q + 1/2$, R_i^j does contain a short I-interval. Since the first interval of R_i^j is a long interval, we have $p_j \geq 1$.

Let $[x_{s'}, x_{s'} + \Delta - 1]$ be the first long I-interval with $x_{s'} \geq x_{i_z} + m$. If $x_{s'} \geq x_{i_z} + m + a$, let $a_{i+1} = x_{s'}$. Then $R_i = [a_i, a_{i+1} - 1]$ is neat, $|R_i| \geq zm + ja$ for some $j \geq 1$, and $w(R_i) \leq z(q + 1/2) + j/2$. Hence

$$r(R_i) \geq \frac{zm + ja}{z(q + \frac{1}{2})} + \frac{j}{2} \geq \frac{zm + a}{z(q + \frac{1}{2})} + \frac{1}{2} \geq \frac{(2q-1)m + a}{2q^2}.$$

Note, all the I-intervals contained in $[x_{i_z} + m, x_{s'} - 1]$, if any, are short. The last inequality in the above holds since $z \leq 2q - 1$ and $\frac{zm+a}{z(q+\frac{1}{2})}$ is a decreasing function of z. As $a_{i+1} \in I$ has property (*), we are done for this case.

Assume $x_{s'} \leq x_{i_z} + m + a - 1$. Let $R_i^{z+1} = [x_{s'}, x_{s'} + m - 1]$. If $w(R_i^{z+1}) \leq q$, then let $a_{i+1} = x_{s'} + m$. By Lemma 6, $R_i = [a_i, a_{i+1} - 1]$ is neat and a_{i+1} has property (*). To verify (3), we note that $w(R_i) \leq (z+1)q + z/2$ and

$$r(R_i) \geq \frac{(z+1)m}{(z+1)q + \frac{z}{2}} \geq \frac{2qm}{2q^2 + q - \frac{1}{2}} \geq \frac{(2q-1)m + a}{2q^2}.$$

The second inequality in the above holds because $\frac{(z+1)m}{(z+1)q+z/2}$ is a decreasing function of z. The third inequality holds because $m \geq a(q+1)$. Thus we assume that $w(R_i^{z+1}) = q + 1/2$.

Claim. $p_{z+1} \leq p_z$. Moreover, if $p_{z+1} = p_z$, then the last short I-interval contained in R_i^z is Type-A, and the last short I-interval in R_i^{z+1} is Type-B.

Proof of Claim. Let $T = [u, u + a - 1]$ and $T' = [u', u' + a - 1]$ be the last short I-interval in R_i^z and R_i^{z+1}, respectively. If T' is Type-B, then $T'' = [u' - m, u' - m + a - 1]$ is a short Type-A I-interval contained in R_i^z. Note, as $|[u' - m, x_{i_{s'}} - 1]| = |[u', x_{i_{s'}} + m - 1]|$ and $x_{i_{s'}} \geq x_{i_z} + m$, we have $|[x_{i_z}, u' - m - 1]| \geq |[x_{i_{s'}}, u' - 1]|$. Hence, $[x_{i_z}, u' - m - 1]$ is capable of containing I-intervals of total weight at least p_{z+1}. As the I-intervals in R_i^z are tightly packed, the I-intervals contained in $[x_{i_z}, u' - m - 1]$ has total weight at least p_{z+1}. Therefore $p_z \geq p_{z+1}$, and if the equality holds then the last short I-interval in R_i^z is Type-A.

Assume T' is Type-A. Thus $u' = x_{i^*} \in I$ for some i^*. Since T' is short, $x_{i^*+1} \leq x_{i^*} + \Delta - 1$. Note, as $s' \in L$ and $x_{i^*}, x_{i^*+1} \in I$, we have $x_{i^*+1} \leq x_{s'} + b$ and $[x_{i^*} - m + 1, x_{i^*+1} - b - 1] \cap I = \emptyset$.

Consider the interval $[x_{i_z}, x_{i^*+1} - b - 1]$. If this is a sub-interval of R_i^z, then since

$$|[x_{i_z}, x_{i^*+1} - b - 1]| \geq |[x_{i_{s'}}, x_{i^*+1} - 1]| + \Delta,$$

and the interval $[x_{i_z}, x_{i^*+1} - b - 1]$ is tightly packed, we conclude that the total weight of the I-intervals that intersect with $[x_{i_z}, x_{i^*+1} - b - 1]$ is at least $p_{z+1} + 1 + 1/2$. Moreover, since $|[x_{i^*} - m + 1, x_{i^*+1} - b - 1]| \geq \Delta + a - 1$ and $[x_{i^*} - m + 1, x_{i^*+1} - b - 1] \cap I = \emptyset$, we conclude that the last I-interval intersecting with $[x_{i_z}, x_{i^*+1} - b - 1]$ is Type-B. The total weight of the I-intervals of R_i^z preceding this Type-B I-interval is at least $p_{z+1} + 1$. Therefore, $p_{z+1} < p_z$.

Assume $[x_{i_z}, x_{i^*+1} - b - 1]$ is not a sub-interval of R_i^z. Then $x_{i^*+1} - b - 1 \geq x_{i_z} + m$. Since $x_{i_z} \geq x_{s'} - m - a + 1$, we have $x_{i^*+1} \geq x_{s'} + b - a + 3$. This implies that $[x_{i^*+1} + 1, x_{s'} + m] \cap I = \emptyset$, and $[x_{i^*+1}, x_{i^*+1} + \Delta - 1]$ is the last I-interval contained in R_i^{z+1}. Hence $p_{z+1} = q - 1$. If $p_z \leq q - 2$, then the conclusion follows.

Assume $p_z = q - 1$. Then the last I-interval in R_i^z is a long interval. Denote this long I-interval by $[w, w']$. Since $w \in I$ but $[x_{i^*} - m + 1, x_{i_z} + m - 1] \cap I = \emptyset$, we have $w \leq x_{i^*} - m$. As all I-intervals are disjoint, the last short I-interval of R_i^z must be within $[x_{i_z}, x_{i^*} - m - 1]$. Therefore, the interval $[x_{i_z}, x_{i^*} - m - 1]$ has length at least $(q - 1)\Delta + a$. Moreover, since

$$|[x_{i^*} - m, x_{s'} - 1]| \geq |[x_{i^*} - m, x_{i^*+1} - b - 1]| \geq \Delta + a,$$

we conclude, $[x_{i_z}, x_{s'} - 1]$ has length at least $q\Delta + 2a$. Let $a_{i+1} = x_{s'}$. Then

$$r(R_i) \geq \frac{(z-1)m + q\Delta + 2a}{z(q + \frac{1}{2})} \geq \frac{(2q-2)m + q\Delta + 2a}{(2q-1)(q + \frac{1}{2})}.$$

The second inequality holds because the formula is a decreasing function on z and $z \geq 2q - 1$. To complete the proof of the Claim, it suffices to show

$$\frac{(2q-2)m + q\Delta + 2a}{(2q-1)(q + \frac{1}{2})} \geq \frac{(2q-1)m + a}{2q^2}.$$

Write $m = q\Delta + 2a - \lambda$, where $0 < \lambda \leq a$. The above inequality is equivalent to

$$2q^2\lambda - (2q^2 - 1/2)a - m(1/2 - q) \geq 0.$$

By definition, we have:

(1) $\lambda \geq 2a - \Delta + 1$ (since $q = \lfloor m/\Delta \rfloor$)

(2) $\Delta \leq 2a - 1$ (since $2a > \Delta$).

Therefore,

$$\begin{aligned}
&2q^2\lambda - (2q^2 - 1/2)a - m(1/2 - q) \\
&\quad = (2q^2 - q + 1/2)\lambda - a(2q^2 - 2q + 1/2) - \Delta(q/2 - q^2) \\
&\quad \geq a(2q^2 + 1/2) - \Delta(q^2 - q/2 + 1/2) + (2q^2 - q + 1/2) \quad \text{(by (1))} \\
&\quad \geq a(q - 1/2) + 3q^2 - (3q)/2 + 1 \quad \text{(by (2))} \\
&\quad \geq 0 \quad \text{(since } q \geq 1\text{)}
\end{aligned}$$

This completes the proof of the Claim. □

Since $p_i \geq 1$, so p_{2q} does not exist. Thus the procedure above terminates at the k-th step for some $k \leq 2q$, when the valid a_{i+1} is obtained. This completes the proof of Theorem 1. ■

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