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A CHARACTERIZATION OF *F*-COMPLETE TYPE ASSIGNMENTS*

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Abstract. The aim of this paper is to investigate the soundness and completeness of the intersection type discipline (for terms of the (untyped) λ-calculus) with respect to the F-semantics (F-soundness and F-completeness).

As pointed out by Scott, if D is the domain of a λ-model, there is a subset F of D whose elements are the 'canonical' representatives of functions. The F-semantics of types takes into account that the intuitive meaning of "$\sigma \to \tau$" is 'the type of functions with domain σ and range τ' and interprets $\sigma \to \tau$ as a subset of F.

The type theories which induce F-complete type assignments are characterized. It follows that a type assignment is F-complete iff equal terms get equal types and, whenever M has a type $\varphi \wedge \omega^n \to \omega$, where φ is a type variable and ω is the 'universal' type, the term $\lambda z_1 \ldots z_n . M z_1 \ldots z_n$ has type φ. Here we assume that $z_1, \ldots, z_n$ do not occur free in M.

Contents

Introduction

A rigorous polymorphic type discipline for terms of the (untyped) λ-calculus was first introduced by Curry [15, Chapter 8; 16, Chapter 17; 20]. In Curry's approach,

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types are built inductively from a set At of type variables by means of the exponentiation operator "$\rightarrow$". Types are assigned to terms by a natural deduction system; in general, a term may have more than one type.

In [7, 8, 11, 12, 2, 9], an extension of the set of types has been proposed by adding the constant type "ω", which plays the role of universal type, and the intersection operator "$\wedge$" of type formation (*intersection type discipline*). By this we obtain a set of type assignment systems, one for each preorder relation on types which satisfy some conditions (*type theory*) (cf. Definitions 1.2 and 1.4). The features of the system presented in [2] essentially are that all solvable terms have types other than ω while a term has a normal form iff it has a type without ω occurrences.

In [30], Milner gives a polymorphic type discipline for a (nonimperative) fragment of the language ML. This system can be viewed as an extension of that of Curry for a λ-calculus augmented with operators such as **if** ... **then** ... **else** ..., **fix** ... and **let** ... **in** .... In [4], the intersection type discipline is modified to handle this fragment of ML, obtaining a type for many functions which have no type in Milner's discipline.

A different extension of Curry's types (*quantification type discipline*) has been described in [32, 31]. This type discipline is based on the F-system of Girard [19] (called second-order lambda-calculus in [33, 18]).

Leivant [27] has recently compared the above polymorphic type disciplines and proved that the type system of [2] is the most powerful in the sense that the set of terms that are typable in it strictly contains the sets of terms typable in all other disciplines.

Given a system of type assignment it is natural to ask for a semantics of types. In literature there are essentially four different ways of interpreting Curry's types in a model of the untyped λ-calculus which can be naturally extended to intersection types. We will mostly follow the nomenclature of [21].

Given a λ-model $\langle D, \cdot, [\![\ ]\!]\rangle$ (for the definition of λ-model see [1, Chapter 5; 25]) the *simple semantics* of types associates to each valuation of type variables $\mathcal{V}: \mathrm{At} \rightarrow \mathcal{P}(D)$ a valuation of types inductively defined as follows:

(1) $\mathcal{V}(\omega) = D$,

(2) $\mathcal{V}(\sigma \rightarrow \tau) = \{d \in D \mid \forall c \in \mathcal{V}(\sigma)\colon d.c \in \mathcal{V}(\tau)\}$,

(3) $\mathcal{V}(\sigma \wedge \tau) = \mathcal{V}(\sigma) \cap \mathcal{V}(\tau)$.

This semantics has been proposed in [35].

Following Scott [37], the *quotient set semantics* takes into account that we want to consider two functions as being equivalent iff they give equivalent results when applied to equivalent arguments. Types are interpreted as equivalence relations on subsets of D rather than simply as subsets of D. In this case a valuation $\mathcal{V}$ of type variables associates to each type variable φ a transitive and symmetric relation $\sim_\varphi^{\mathcal{V}}$ on D.

$\mathcal{V}$ can be extended to all types by defining inductively

(1*) $d \sim_\omega^{\mathcal{V}} d'$ for all $d, d' \in D$,

(2*) $d \sim_{\sigma\rightarrow\tau}^{\mathcal{V}} d'$ iff $\forall c, c'$ such that $c \sim_\sigma^{\mathcal{V}} c'$, $d.c \sim_\tau^{\mathcal{V}} d'.c'$, (3*) $d \sim^{\mathscr{V}}_{\sigma\wedge\tau} d'$ iff $d \sim^{\mathscr{V}}_{\sigma} d'$ and $d \sim^{\mathscr{V}}_{\tau} d'$.

If we define $\mathscr{V}(\tau)=\{d \mid d \sim^{\mathscr{V}}_{\tau} d\}$, we have that $\sim^{\mathscr{V}}_{\tau}$ is an equivalence relation on $\mathscr{V}(\tau)$.

As Scott has pointed out [38], the key of a λ-model is the set $F \subseteq D$ of the elements which represent functions. In fact, using F we can obtain a first-order axiomatization of the notion of λ-model [38]. Each element $d \in D$ represents a function (since "." is always defined), but the interpretation $[\![\ ]\!]$ of terms chooses 'canonical representatives' of functions, i.e., elements which are meanings of terms starting with an initial abstraction (in a suitable environment). More precisely, in [21] F is defined by

$$F=\{d \in D \mid \exists y, M, \xi \text{ such that } d=[\![\lambda y.M]\!]_{\xi}\}.$$

Notice that F may also be defined as the range of the retraction $\varepsilon=[\![\lambda xy.xy]\!]_{\xi}$ (ξ is arbitrary). We can show that each representable function from D to D has a *unique* canonical representative in F.

The *F-semantics* of types (as defined in [21]) takes into account that the intuitive meaning of "$\sigma \to \tau$" is 'the type of functions with domain σ and range τ' and interprets $\sigma \to \tau$ as a subset of F. Therefore, the F-semantics is obtained from the simple semantics by replacing clause (2) with

(2′) $\mathscr{V}(\sigma \to \tau)=\{d \in F \mid \forall c \in \mathscr{V}(\sigma){:}\ d.c \in \mathscr{V}(\tau)\}$.

It is easy to prove (cf. the discussion after Definition 1.3) that in this semantics $\mathscr{V}(\omega \to \omega)=F$. Notice that other semantics could be defined by choosing a subset of D different from F.

Lastly, the semantics of types proposed by Scott in [36] is obtained from the quotient set semantics taking into account the relations between F and $\sigma \to \tau$ for all types σ, τ (*F-quotient set semantics*). More precisely, the elements which are $\sim^{\mathscr{V}}_{\sigma\to\tau}$-equivalent must belong to F, i.e., clause (2*) is replaced by:

(2″) $d \sim^{\mathscr{V}}_{\sigma\to\tau} d'$ iff $d, d' \in F$ and $\forall c, c'$ such that $c \sim^{\mathscr{V}}_{\sigma} c'$; $d.c \sim^{\mathscr{V}}_{\tau} d'.c'$.

In [30], the semantic domain D is a cpo satisfying a suitable domain equation, and types are interpreted as ideals, i.e., downward closed and direct complete subsets of D (*Milner's semantics*).

The semantics of the quantification type disciplines is given in [32, 31].

Once one has introduced formal systems of type assignment and type valuations, it is natural to ask for soundness and completeness results. Coppo has proved [21] that for Curry's type discipline, completeness for the simple semantics implies completeness for the quotient set semantics. This is because the simple semantics is a particular case of the quotient set semantics. Coppo's argument naturally extends to the intersection type discipline, giving completeness for the (F-)quotient set semantics from the completeness for the (F-)simple semantics.

For Curry's types, soundness for the simple semantics has been proved in [3] and for the other semantics in [21]. In order to prove the completeness result, the most natural way is to prove that a type system is complete with respect to a fixed λ-model. Different completeness proofs for the four semantics have been done using terms models [21, 22] and the graph model P_ω [6]. In [2] Curry's system has been proved complete for the simple semantics using a filter λ-model (also defined in [2]).

The type assignment of [2] is proved to be sound and complete for the simple semantics in [2] using a filter λ-model and in [23] using a term model. On the other hand, it is easy to see that this type system is neither sound nor complete with respect to F-semantics (cf. the remarks after Theorems 2.9 and 4.6). In [9] (using filter λ-models), and in [13] (using the term model) the type theories which yield complete type assignments for the simple semantics are characterized. It turns out that a type assignment is complete iff $\omega \leq \omega \to \omega$ belongs to the associated type theory and equal terms get equal types.

In the case of ML, both Milner's type discipline and the extension of [4] have been proved to be sound, but there are very simple examples that they are not complete with respect to Milner's semantics. In [5, 14], a nontrivial subset of ML is given for which Milner's type assignment is complete. Moreover, a semantics characterization of typed terms is exhibited.

In [31], Mitchell proves soundness and completeness results for the quantification type discipline using the term model of β-equality.

The aim of the present paper is to investigate the soundness and completeness for the F-semantics (*F-soundness* and *F-completeness*) of the intersection type discipline. As noted by Hindley [21], this type discipline seems to be strong enough to express the differences between the simple semantics and the F-semantics of types by the following arguments.

(1) As mentioned before, the system of [2] is sound and complete for the simple semantics but neither F-sound nor F-complete.

(2) P_ω and the filter λ-models used to prove completeness for the simple semantics are sensible, while we must look at non-sensible λ-models to prove F-completeness (recall that a λ-model is sensible iff its theory equates all unsolvable terms, cf. [1, Chapter 16]). Let $\Delta \equiv \lambda z.zz$. We cannot deduce $\omega \to \omega\Delta\Delta$ in the systems discussed in Section 4 while $[\![\Delta\Delta]\!]_\xi \in F$ (since $[\![\lambda y.\Delta\Delta]\!]_\xi \in F$) for all sensible models and all environments ξ. This is contradictory since (as mentioned before) $\mathcal{V}(\omega \to \omega) = F$ in the F-semantics.

(3) (This argument is due to Coppo.) The term model of β-equality $\mathfrak{M}_\beta$ does not help in proving the F-completeness for the system $\vdash^{s}$ as defined in Definition 4.1. First we notice that if $\xi(z) = [Z]$ and there are y, M such that $Z\mathbf{I} \twoheadrightarrow_\beta \lambda y.M$, then, a fortiori, $Z\mathbf{1} \twoheadrightarrow_\beta \lambda y.M'$ for some M' (where, as usual, $\mathbf{I} \equiv \lambda u.u$ and $\mathbf{1} \equiv \lambda uv.uv$). So $[\![Z\mathbf{I}]\!]_\xi \in F$ implies $[\![Z\mathbf{1}]\!]_\xi \in F$ for all environment ξ. Therefore, we have:

$$(\varphi \to \varphi) \to \omega \to \omega z \overset{s}{\vdash} \omega \to \omega z\mathbf{I}$$

$$\Rightarrow \mathfrak{M}_\beta, (\varphi \to \varphi) \to \omega \to \omega z \overset{s}{\models} \omega \to \omega\mathbf{I}$$

by $\mathcal{T}$-F-soundness (proved in Theorem 4.6)

$$\Rightarrow \mathfrak{M}_\beta, (\varphi \to \varphi) \to \omega \to \omega z \overset{s}{\models} \omega \to \omega z\mathbf{1}$$

from above since $F = \mathcal{V}(\omega \to \omega)$. It is, however, easy to check that

$$(\varphi \to \varphi) \to \omega \to \omega z \overset{s}{\nvdash} \omega \to \omega z \mathbf{1}.$$

Notice that $\vdash^{s}$ is proved to be F-complete (cf. Theorem 4.8).

(4) The following rule scheme (proposed by Hindley) is sound for the F-semantics (cf. Theorem 2.9)

$$\text{(HR)}\quad \frac{\varphi \wedge \omega^{n} \to \omega M}{\varphi \lambda y_1 \ldots y_n . M y_1 \ldots y_n} \quad \text{if } y_1, \ldots, y_n \notin \mathrm{FV}(M).$$

Notice that this is not a derived rule for the system of [2].

The present paper is a systematic exposition and a development of some results and ideas which have been discussed at length by Coppo and Hindley with the present authors. The main result is the characterization of the type theories which induce F-complete type assignments (Theorem 2.9).

In Section 1 we will define the notions of type theory, of type assignment and we will characterize the type theories yielding (F-filter) λ-models (Theorem 1.12). In Section 2 we will prove that a type assignment induced by a type theory is F-complete iff (Eq_β) and (HR) are derived rules. In Section 3 we will prove an Approximation Theorem for F-filter λ-models satisfying suitable conditions. In Section 4 we will discuss four particular type theories. We will prove that all these theories give rise to F-filter λ-models but only two of the induced type assignments are F-complete. In Section 5 we will look at a new syntax of types by limiting the application of the operator "$\wedge$" of intersection. We will prove that this does not change the set of typable terms, but it is the only system (presented in this paper) in which the Normal Form and the Head Normal Form Theorems hold (Theorem 5.6) and which is also F-complete (Theorem 5.11).

Reference [17] is a preliminary and incomplete version of this paper which has been presented at the 'International Symposium on the Semantics of Data Types' (Sophia Antipolis, 1984).

1. Type theories and F-filter λ-models

We introduce the notions of type scheme and of type theory mostly following [2, 9].

1.1. Definition. (1) The set T of *type schemes* is inductively defined by

(i) $\varphi_0, \varphi_1, \ldots \in T$ type variables, $\omega \in T$ type constant,

(ii) $\sigma, \tau \in T \Rightarrow (\sigma \to \tau), (\sigma \wedge \tau) \in T$.

(2) The preorder relation $\leq_f$ on T is the smallest relation satisfying:

(i) $\tau \leq_f \omega$;

(ii) $\tau \to \omega \leq_f \omega \to \omega$;

(iii) $\tau \leq_f \tau \wedge \tau$;

(iv) $\sigma \wedge \tau \leq_f \sigma$, $\sigma \wedge \tau \leq_f \tau$;
(v) $(\sigma \to \tau) \wedge (\sigma \to \tau') \leq_f \sigma \to (\tau \wedge \tau')$;
(vi) $\sigma \leq_f \sigma'$, $\tau \leq_f \tau' \Rightarrow \sigma \wedge \tau \leq_f \sigma' \wedge \tau'$;
(vii) $\sigma' \leq_f \sigma$, $\tau \leq_f \tau' \Rightarrow \sigma \to \tau \leq_f \sigma' \to \tau'$
plus transitivity and reflexivity.
(3) $\sigma \sim_f \tau$ iff $\sigma \leq_f \tau \leq_f \sigma$.

Note that, e.g., $\omega \to \omega \to \omega \leq_f \sigma \to \omega \to \omega$ for all σ, but the converse does not hold.
In what follows, we will simply say 'types' instead of 'type schemes'. We will write equality "=" between types, with the convention that $\sigma \wedge \tau = \tau \wedge \sigma$, $\sigma = \sigma \wedge \omega$, and $(\sigma \to \tau) \wedge (\sigma \to \tau') = \sigma \to (\tau \wedge \tau')$.

1.2. Definition. (1) If $\sigma, \tau \in T$, then $\sigma \leq \tau$ is a *formula*.
(2) A *type theory* $\mathcal{T}$ is any set of formulas closed under (i)-(vii) of Definition 1.1(2) plus reflexivity and transitivity. $\sigma \leq_{\mathcal{T}} \tau$ stands for $\sigma \leq \tau \in \mathcal{T}$. We write $\sigma \sim_{\mathcal{T}} \tau$ iff $\sigma \leq_{\mathcal{T}} \tau \leq_{\mathcal{T}} \sigma$.
(3) If Σ is any set of formulas, then $\mathcal{T}(\Sigma)$ is the minimal type theory which includes Σ. We will write $\leq_\Sigma$ for $\leq_{\mathcal{T}(\Sigma)}$.

$\mathcal{T}_f$ denotes the least type theory, i.e., $\mathcal{T}_f = \mathcal{T}(\Phi)$ (where Φ is the empty set). Obviously, $\mathcal{T}_f \subseteq \mathcal{T}$ for all type theories $\mathcal{T}$.

The difference between the notion of type theory introduced here and that of [9] is that each type theory of [9] contains $\omega \leq \omega \to \omega$. We had to exclude this formula for considering F-semantics, as will become clear later on (cf. the discussions after Definition 1.3 and before 4.1).

Following [21] we want to interpret the types, taking into account that we can distinguish between the elements of the domain D of a λ-calculus model those elements which are interpretations of terms of the shape $\lambda y.M$ and those that are not (M need not be closed). More precisely, if $\mathfrak{M} = \langle D, \cdot, [\![\ ]\!]^{\mathfrak{M}} \rangle$ is a λ-model and ξ is a valuation of term variables in D, we define $F = \{d \in D \mid \exists y, M, \xi\colon d = [\![\lambda y.M]\!]^{\mathfrak{M}}_{\xi}\}$.

1.3. Definition (*F-semantics*). Let $\mathcal{V}\colon \{\varphi_j \mid j \in \mathbb{N}\} \to \mathcal{P}D = \{X \mid X \subseteq D\}$. Then, $\mathcal{V}$ extends to all $\tau \in T$ as follows ($\mathcal{V}$ is a *type interpretation*):
(1) $\mathcal{V}(\omega) = D$,
(2) $\mathcal{V}(\sigma \to \tau) = \{d \in F \mid \forall c \in \mathcal{V}(\sigma)\colon d.c \in \mathcal{V}(\tau)\}$,
(3) $\mathcal{V}(\sigma \wedge \tau) = \mathcal{V}(\sigma) \cap \mathcal{V}(\tau)$.

Clearly, if $\mathfrak{M}$ is an extensional λ-model, then $F = D$ and the F-semantics coincides with the simple semantics of types as Hindley proved in [21, Section 4].
From Definition 1.3 it follows that, for all $\mathcal{V}$ and τ,

$$\mathcal{V}(\tau \to \omega) = \{d \in F \mid \forall c \in \mathcal{V}(\tau)\colon d.c \in \mathcal{V}(\omega)\}$$
$$= \{d \in F \mid \forall c \in \mathcal{V}(\tau)\colon d.c \in D\} = F.$$

The motivation for the definition of $\leqslant_f$ is that $\sigma \sim_f \rho$ ($\sigma \leqslant_f \rho$) implies for all $\mathfrak{M}$, $\mathcal{V}$: $\mathcal{V}(\sigma) = \mathcal{V}(\rho)$ ($\mathcal{V}(\sigma) \subseteq \mathcal{V}(\rho)$) (this will be proved in Theorem 2.4(3)). Therefore, we may assume $\tau \to \omega \sim_f \omega \to \omega$ for all τ. On the contrary, we cannot assume $\omega \leqslant_f \omega \to \omega$ (as in [2]) since we would obtain, for all $\mathfrak{M}$, $\mathcal{V}$: $\mathcal{V}(\omega) \subseteq \mathcal{V}(\omega \to \omega)$, i.e., $D \subseteq F$ and this means that we would restrict our attention to extensional λ-models.

Each type theory $\mathcal{T}$ induces a system of type assignment, in the sense of [2], for the set Λ of terms.

1.4. Definition. (1) A *statement* is of the form τM with $\tau \in T$ and $M \in \Lambda$. M is the *subject* and τ the *predicate* of τM.

(2) A *basis* B is a set of statements with only variables as subjects.

(3) The *type assignment* induced by the type theory $\mathcal{T}$ is defined by the following natural deduction system:

$$(\to I): \quad \frac{\begin{array}{c}[\sigma y]\\ \vdots \\ \tau M\end{array}}{\sigma \to \tau \lambda y.M} \; (*) \qquad (\to E): \quad \frac{\sigma \to \tau M \quad \sigma N}{\tau MN}$$

$$(\wedge I): \quad \frac{\sigma M \quad \tau M}{\sigma \wedge \tau M} \qquad (\wedge E): \quad \frac{\sigma \wedge \tau M}{\sigma M} \quad \frac{\sigma \wedge \tau M}{\tau M}$$

$$(\leqslant_{\mathcal{T}}): \quad \frac{\sigma M \quad \sigma \leqslant_{\mathcal{T}} \tau}{\tau M} \qquad (\omega): \quad \frac{}{\omega M}$$

($*$): if y is not free in assumptions on which τM depends other than σy.

(4) $B \vdash^{\mathcal{T}} \tau M$ if τM is derivable from the basis B in the system induced by $\mathcal{T}$. If $\mathcal{D}$ is a derivation showing this, we write $\mathcal{D} : B \vdash^{\mathcal{T}} \tau M$.

Rule ($\wedge$ E) is superfluous, since it is directly derivable from rule ($\leqslant_{\mathcal{T}}$).

Notice that typing is preserved under substitution in the type assignment induced by $\mathcal{T}_f$, but this is not true for an arbitrary $\mathcal{T}$ (cf. the examples after Definition 4.1).

We are interested in building the complete algebraic lattices whose elements are (abstract) filters of types. In Theorem 1.12 we will give a characterization of the type theories which give rise to lattices which are domains of λ-calculus models. Similar results with a slightly different definition of type theory have been shown in [9, 13].

1.5. Definition. Let $\mathcal{T}$ be a type theory.

(1) An *abstract filter* of $\mathcal{T}$ is a subset $d \subseteq T$ such that

(i) $\omega \in d$,

(ii) $\sigma, \tau \in d \Rightarrow \sigma \wedge \tau \in d$,

(iii) $\sigma \;{}_{\mathcal{T}}\!\geqslant \tau \in d \Rightarrow \sigma \in d$.

(2) If $S \subseteq T$, ${}^{\mathcal{T}}\!\uparrow S$ is the minimal abstract filter of $\mathcal{T}$ which includes S. We use the abbreviation ${}^{\mathcal{T}}\!\uparrow \tau$ for ${}^{\mathcal{T}}\!\uparrow \{\tau\}$.

(3) $|\mathcal{T}|$ is the set of abstract filters of $\mathcal{T}$.

Notice that $(T, \omega, \mathrm{Con}, \vdash_{\mathcal{T}})$, where Con consists of all finite subsets of T and $\{\sigma_1, \ldots, \sigma_n\} \vdash_{\mathcal{T}} \tau$ iff $\sigma_1 \wedge \cdots \wedge \sigma_n \leq_{\mathcal{T}} \tau$ (for some type theory $\mathcal{T}$), is an information system in the sense of Scott [39]. Moreover, $|\mathcal{T}|$ is the domain determined by this information system.

1.6. Lemma. *$\langle |\mathcal{T}|, \subseteq \rangle$ is a complete algebraic lattice, where ${}^{\mathcal{T}}\!\uparrow\omega$ and T are the least and the largest elements (respectively). Moreover, if d, $c \in |\mathcal{T}|$, then*

(i) $d \sqcup c = {}^{\mathcal{T}}\!\uparrow(d \cup c)$;

(ii) $d \sqcap c = (d \cap c)$;

(iii) *if $X \subseteq |\mathcal{T}|$ is a directed set, then $\bigsqcup X = \bigcup X$;*

(iv) *the finite elements are exactly the principal filters, i.e., $d = \bigcup \{ {}^{\mathcal{T}}\!\uparrow\tau \mid {}^{\mathcal{T}}\!\uparrow\tau \subseteq d\}$.*

The proof is a simple routine (cf. [39]).

1.7. Lemma. (1) *$\{\tau \mid B \vdash^{\mathcal{T}} \tau M\}$ is an abstract filter.*

(2) *$B \vdash^{\mathcal{T}} \sigma y \Leftrightarrow \sigma \in {}^{\mathcal{T}}\!\uparrow\{\tau \mid \tau y \in B$ or $\tau \equiv \omega\}$.*

(3) *If τM is derived from $\sigma_1 M, \ldots, \sigma_n M$ by means of rules $(\wedge I)$, $(\wedge E)$, and $(\leq_{\mathcal{T}})$ only, then $\tau \,{}_{\mathcal{T}}\!\geq \sigma_1 \wedge \cdots \wedge \sigma_n$.*

Proof. (1): By rules (ω), $(\wedge I)$ and $(\leq_{\mathcal{T}})$.

(2): By induction on derivations.

(3): From (2) since, in the rules in question, M behaves as a variable. □

B/z denotes the basis obtained from B by deleting all statements whose subject is the variable z:

$$B/z = \{\tau y \mid \tau y \in B \text{ and } y \not\equiv z\}.$$

$B \restriction M$ denotes the basis obtained by considering only those statements of B whose subjects are variables occurring free in the term M:

$$B \restriction M = \{\tau y \mid \tau y \in B \text{ and } y \in \mathrm{FV}(M)\}.$$

1.8. Lemma. (1) *$B \vdash^{\mathcal{T}} \tau MN$, $\tau \not\sim_{\mathcal{T}} \omega \Rightarrow \exists \sigma \in T$:*

$$[B \overset{\mathcal{T}}{\vdash} \sigma \to \tau M \quad \textit{and} \quad B \overset{\mathcal{T}}{\vdash} \sigma N].$$

(2) $$\forall \sigma, \tau\colon [B/y \cup \{\sigma y\} \overset{\mathcal{T}}{\vdash} \tau M \;\Rightarrow\; B/y \cup \{\sigma y\} \overset{\mathcal{T}}{\vdash} \tau N]$$

$$\Rightarrow \forall \rho\colon [B \overset{\mathcal{T}}{\vdash} \rho \lambda y.M \;\Rightarrow\; B \overset{\mathcal{T}}{\vdash} \rho \lambda y.N].$$

(3) *$B \vdash^{\mathcal{T}} \tau M$ iff $B \restriction M \vdash^{\mathcal{T}} \tau M$.*

(4) *$B/y \cup \{\sigma y\} \vdash^{\mathcal{T}} \tau M$ and $z \notin \mathrm{FV}(M) \Rightarrow B/z \cup \{\sigma z\} \vdash^{\mathcal{T}} \tau M[y/z]$.*

The proof of (1)–(4) is done by induction on derivations.

We now introduce a notion of application "." between abstract filters and an interpretation $[\![\ ]\!]^{\mathcal{T}}$ of terms in $|\mathcal{T}|$.

1.9. Definition. Let $\mathcal{T}$ be a type theory.

(1) For $d, c \in |\mathcal{T}|$ define $d.c = {}^{\mathcal{T}}\!\uparrow\omega \cup \{\tau \mid \exists\sigma \in c\colon \sigma \to \tau \in d\}$.

(2) Let V be the set of term variables and $\xi\colon V \to |\mathcal{T}|$. Then, $B_\xi = \{\sigma y \mid \sigma \in \xi(y)\}$.

(3) For $M \in \Lambda$, define $[\![M]\!]^{\mathcal{T}}_\xi = \{\tau \mid B_\xi \vdash^{\mathcal{T}} \tau M\}$ ($\in |\mathcal{T}|$ by Lemma 1.7(1)).

(4) $\xi^{\mathcal{T}}_B(y) = {}^{\mathcal{T}}\!\uparrow\{\sigma \mid \sigma \equiv \omega \text{ or } \sigma y \in B\}$.

1.10. Lemma. (1) $d, c \in |\mathcal{T}| \Rightarrow d.c \in |\mathcal{T}|$.

(2) $B \vdash^{\mathcal{T}} \tau M \Leftrightarrow B_{\xi^{\mathcal{T}}_B} \vdash^{\mathcal{T}} \tau M$.

Proof. (1): Let $S = \{\tau \mid \exists\sigma \in c\colon \sigma \to \tau \in d\}$.

$$\sigma \to \tau \in d \;\Rightarrow\; \sigma \to \omega \in d \quad (\text{by} \leq_{\mathcal{T}})$$

and therefore, $S \neq \Phi \Leftrightarrow {}^{\mathcal{T}}\!\uparrow\omega \subseteq S$. So it is sufficient to verify that $S \neq \Phi \Rightarrow S \in |\mathcal{T}|$.

(i) $\omega \in S$;

(ii) $\tau_1, \tau_2 \in S \;\Rightarrow\; \exists\sigma_1, \sigma_2 \in c\colon \sigma_1 \to \tau_1, \sigma_2 \to \tau_2 \in d$
$\;\Rightarrow\; (\sigma_1 \wedge \sigma_2) \to (\tau_1 \wedge \tau_2) \in d$
$\;\Rightarrow\; \tau_1 \wedge \tau_2 \in S$;

(iii) $\tau \in S,\ \tau \leq_{\mathcal{T}} \rho \;\Rightarrow\; \exists\sigma \in c\colon \sigma \to \tau \in d$
$\;\Rightarrow\; \sigma \to \rho \in d \;\Rightarrow\; \rho \in S$.

(2): Routine. □

Now we are able to characterize the type theories such that $\mathfrak{M}_{\mathcal{T}} = \langle |\mathcal{T}|, ., [\![\ ]\!]^{\mathcal{T}} \rangle$ are λ-models (*F-filter λ-models*). We point out that we are using the definition of λ-model given in [25], which is essentially equivalent to other accepted definitions (cf. [1, Chapter 5]). Our result is that $\mathfrak{M}_{\mathcal{T}}$ is an F-filter λ-model iff types are invariant under β-conversion of terms, i.e., iff the following rule:

$$(\mathrm{Eq}_\beta)\colon \quad \frac{\tau M \quad M =_\beta N}{\tau N}$$

is derivable in the system $\vdash^{\mathcal{T}}$.

1.11. Lemma. *Let $\mathcal{T}$ be a type theory such that* (Eq_β) *is a derived rule for the type assignment induced by $\mathcal{T}$; then*

$$B \overset{\mathcal{T}}{\vdash} \sigma \to \tau\lambda y.M \;\Leftrightarrow\; B/y \cup \{\sigma y\} \overset{\mathcal{T}}{\vdash} \tau M.$$

Proof. ($\Leftarrow$): This is immediate by rule ($\to I$) and Lemma 1.8(3).

($\Rightarrow$):

$$\begin{aligned} B \overset{\mathcal{T}}{\vdash} \sigma \to \tau\lambda y.M &\;\Rightarrow\; B/z \cup \{\sigma z\} \overset{\mathcal{T}}{\vdash} \tau(\lambda y.M)z && \text{where } z \notin \mathrm{FV}(M) \\ &\;\Rightarrow\; B/z \cup \{\sigma z\} \overset{\mathcal{T}}{\vdash} \tau M[y/z] && \text{by hypothesis} \\ &\;\Rightarrow\; B/y \cup \{\sigma y\} \overset{\mathcal{T}}{\vdash} \tau M && \text{by Lemma 1.8(4).} \end{aligned}$$

□

1.12. Theorem. *$\mathfrak{M}_{\mathcal{T}}$ is an F-filter λ-model iff rule* (Eq_β) *is a derived rule for the type assignment induced by $\mathcal{T}$.*

Proof. (⇐): Just mimic the proof of [2, Theorem 3.5] using Lemmas 1.7, 1.8, 1.11, and the definition of λ-model given in [25].

(⇒): Immediate from the definition of λ-model. □

If $\mathcal{T}$ yields an F-filter λ-model, we call $F_{\mathcal{T}}$ the subset of $|\mathcal{T}|$ whose elements represent functions, i.e.,

$$F_{\mathcal{T}} = \{d \in |\mathcal{T}| \mid \exists y, M, \xi \text{ such that } d = [\![\lambda y.M]\!]_\xi^{\mathcal{T}}\}$$

$$= \{d \in |\mathcal{T}| \mid \exists y, M, B \text{ such that } \tau \in d \Leftrightarrow B \overset{\mathcal{T}}{\vdash} \tau\lambda y.M\}.$$

2. Hindley's rule and *F*-completeness results

The following rule scheme (HR) of type assignment has been suggested by Hindley during many discussions we had about F-semantics for the intersection type discipline. He has proved that (HR) is sound for the F-semantics, so each F-complete type assignment must satisfy (HR).

The idea under this rule is that if the meaning of a term M is the 'canonical' representative of an n-ary function, then the meaning of M coincides with the meaning of $\lambda z_1 \ldots z_n . M z_1 \ldots z_n$ where $z_1, \ldots, z_n \notin \text{FV}(M)$. This will be formalized in Lemma 2.6.

Let $\omega^n \to \omega$ abbreviate

$$\underbrace{\omega \to \cdots \to \omega}_{n} \to \omega.$$

2.1. Definition (*Hindley's rule scheme*).

$$(\text{HR}) \quad \frac{\varphi \wedge \omega^n \to \omega M}{\varphi \lambda z_1 \ldots z_n . M z_1 \ldots z_n} \quad \text{if } z_1, \ldots, z_n \notin \text{FV}(M)$$

for all type variables φ.

We define, as usual, the notion of semantics satisfiability (⊨).

2.2. Definition. Let $\mathfrak{M} = \langle D, \cdot, [\![\ ]\!]^{\mathfrak{M}} \rangle$ be a λ-model.

(1) $\mathfrak{M}, \xi, \mathcal{V} \models \tau M \Leftrightarrow [\![M]\!]_\xi^{\mathfrak{M}} \in \mathcal{V}(\tau)$;

$\mathfrak{M}, \xi, \mathcal{V} \models B \Leftrightarrow \mathfrak{M}, \xi, \mathcal{V} \models \sigma x$ for all $\sigma x \in B$.

$B \models \tau M \Leftrightarrow \forall \mathfrak{M}, \xi, \mathcal{V}: \mathfrak{M}, \xi, \mathcal{V} \models B \Rightarrow \mathfrak{M}, \xi, \mathcal{V} \models \tau M$.

(2) $\Sigma_{\mathcal{V}} = \{\sigma \leq \tau \mid \mathcal{V}(\sigma) \subseteq \mathcal{V}(\tau)\}$.

(3) $\mathcal{V}$ *agrees* with $\mathcal{T}$ iff $\mathcal{T} \subseteq \mathcal{T}(\Sigma_{\mathcal{V}})$.

(4) $\mathfrak{M}, B \models^{\mathcal{T}} \tau M \Leftrightarrow \forall \xi, \mathcal{V}$ which agree with $\mathcal{T}$:

$$\mathfrak{M}, \xi, \mathcal{V} \models B \Rightarrow \mathfrak{M}, \xi, \mathcal{V} \models \tau M.$$

$$B \overset{\mathcal{T}}{\models} \tau M \Leftrightarrow \forall \mathfrak{M}: \mathfrak{M}, B \overset{\mathcal{T}}{\models} \tau M.$$

(5) $\mathcal{V}_{\mathcal{T}}(\varphi) = \{d \in |\mathcal{T}| \mid \varphi \in d\}$ for all type variables φ.

2.3. Definition. Let $\mathcal{T}$ be a type theory.

(1) The type assignment $\vdash^{\mathcal{T}}$ is *F-sound* iff $B \vdash^{\mathcal{T}} \tau M \Rightarrow B \models \tau M$.
(2) The type assignment $\vdash^{\mathcal{T}}$ is *$\mathcal{T}$-F-sound* iff $B \vdash^{\mathcal{T}} \tau M \Rightarrow B \models^{\mathcal{T}} \tau M$.
(3) The type assignment $\vdash^{\mathcal{T}}$ is *F-complete* iff $B \models \tau M \Rightarrow B \vdash^{\mathcal{T}} \tau M$.

The $\mathcal{T}$-F-soundness of all type assignments induced by type theories is easily proved.

2.4. Theorem ($\mathcal{T}$-F-soundness). (1) $\sigma \leqslant_{\mathcal{T}} \tau \Rightarrow \forall \mathfrak{M}, \mathcal{V}$ *which agree with* $\mathcal{T}$: $\mathcal{V}(\sigma) \subseteq \mathcal{V}(\tau)$.

(2) $B \vdash^{\mathcal{T}} \tau M \Rightarrow B \models^{\mathcal{T}} \tau M$.
(3) $\sigma \leqslant_{\mathrm{f}} \tau \Rightarrow \forall \mathfrak{M}, \mathcal{V}$: $\mathcal{V}(\sigma) \subseteq \mathcal{V}(\tau)$.
(4) $B \vdash^{\mathcal{T}_{\mathrm{f}}} \tau M \Rightarrow B \models \tau M$.

Proof. (1): Immediate from Definition 2.2(3).

(2): By induction on derivations. For rule ($\leqslant_{\mathcal{T}}$) use (1).
(3): By induction on $\leqslant_{\mathrm{f}}$.
(4): As (2). □

Theorem 2.4(3) means that $\mathcal{T}_{\mathrm{f}}$ is F-sound, i.e., that if $\sigma \leqslant_{\mathrm{f}} \tau \in \mathcal{T}_{\mathrm{f}}$, then this containment between types is valid in all models.

We now show that (HR) characterizes the F-completeness of type assignment systems induced by type theories (provided that they yield F-filter λ-models). To this aim, following [38] we introduce a further classification of the elements of the domain of a λ-model. If we distinguish, inside the domain D of a λ-model, the subset F of elements which represent functions, a further (natural) step is then to distinguish inside F the elements which represent one-place functions, two-place functions, etc. In this way we obtain a chain of subsets of D, which can be used to define the notion of λ-model, as suggested by Scott [38].

2.5. Definition. Let $\mathfrak{M} = \langle D, \cdot, [\![\ ]\!]^{\mathfrak{M}} \rangle$ be a λ-model, then $F^{(n)}$ is inductively defined as follows:

$$F^{(0)} = D, \qquad F^{(n+1)} = \{d \in F \mid \forall c \in D: d.c \in F^{(n)}\}.$$

It is easy to verify that $F^{(1)} = F$ and, for all $n > 0$, $F^{(n)} \subseteq F^{(n-1)}$.

2.6. Lemma. *For all λ-models* $\mathfrak{M}$,

(1) $\mathcal{V}(\omega^n \to \omega) = F^{(n)}$;

(2) $d \in F^{(n)} \Leftrightarrow d = [\![\lambda z_1 \ldots z_n.M]\!]_\xi^{\mathfrak{M}}$ *for some* $z_1, \ldots, z_n$, M, ξ;

(3) $d \in F^{(n)} \Leftrightarrow d = [\![\lambda z_1 \ldots z_n.yz_1 \ldots z_n]\!]_{\xi[y/d]}^{\mathfrak{M}}$ *for* $n \geq 0$.

Proof. (1) and (2) are proved by induction on n.

(3) ($\Leftarrow$): Trivial.

($\Rightarrow$): $d \in F^{(n)} \Rightarrow d = [\![\lambda z_1 \ldots z_n.M]\!]_\xi^{\mathfrak{M}}$ for some $z_1, \ldots, z_n$, M, ξ by (2). Then,

$$\begin{aligned} d &= [\![\lambda z_1 \ldots z_n.(\lambda z_1 \ldots z_n.M)z_1 \ldots z_n]\!]_\xi^{\mathfrak{M}} && \text{by } \beta\text{-conversion} \\ &= [\![\lambda z_1 \ldots z_n.yz_1 \ldots z_n]\!]_{\xi[y/[\![\lambda z_1 \ldots z_n.M]\!]_\xi^{\mathfrak{M}}]}^{\mathfrak{M}} && \text{by [25, Lemma 2.8]} \\ &= [\![\lambda z_1 \ldots z_n.yz_1 \ldots z_n]\!]_{\xi[y/d]}^{\mathfrak{M}}. \quad \Box \end{aligned}$$

Theorem 2.6(3) for $n = 1$ is proved in [22]. The proof in the general case has also been given in Hindley [24].

2.7. Lemma. *Let* $\mathcal{T}$ *be a type theory such that* (Eq_β) *and* (HR) *are derived rules for the induced type assignment.*

(1) $\forall d \in |\mathcal{T}|$: $[\omega \to \omega \in d \Leftrightarrow d \in F_{\mathcal{T}}]$.

(2) $\mathcal{V}_{\mathcal{T}}(\tau) = \{d \in |\mathcal{T}| \mid \tau \in d\}$ *for all* $\tau \in T$.

(3) $\mathfrak{M}_\tau$, $\xi_B^{\mathcal{T}}$, $\mathcal{V}_{\mathcal{T}} \models B$.

(4) $\sigma \leq_{\mathcal{T}} \tau \Leftrightarrow \forall \mathfrak{M}$, $\mathcal{V}$ *which agree with* $\mathcal{T}$: $\mathcal{V}(\sigma) \subseteq \mathcal{V}(\tau)$.

Proof. (1): By Lemma 2.6(3), it is sufficient to prove that

$$\omega \to \omega \in d \;\Leftrightarrow\; d = [\![\lambda z.yz]\!]_{\xi[y/d]}^{\mathcal{T}}.$$

($\Leftarrow$): Trivial.

($\Rightarrow$): We prove that if $\omega \to \omega \in d$, and (Eq_β), (HR) are derived rules for the type assignment induced by $\mathcal{T}$, then $\sigma \in d \Leftrightarrow \sigma \in [\![\lambda z.yz]\!]_{\xi[y/d]}^{\mathcal{T}}$. Use induction on σ for "$\Rightarrow$", and induction on the derivation $\mathcal{D}$: $B_{\xi[y/d]} \vdash^{\mathcal{T}} \sigma\lambda z.yz$ for "$\Leftarrow$". For "$\Rightarrow$" the only interesting case is $\sigma \equiv \varphi$:

$$\begin{aligned} \varphi \in d &\Rightarrow B_{\xi[y/d]} \vdash^{\mathcal{T}} \varphi \wedge \omega \to \omega y && \text{by Definition 1.9(2)} \\ &\Rightarrow B_{\xi[y/d]} \vdash^{\mathcal{T}} \varphi\lambda z.yz && \text{by (HR)} \\ &\Rightarrow \varphi \in [\![\lambda z.yz]\!]_{\xi[y/d]}^{\mathcal{T}}. \end{aligned}$$

For "$\Leftarrow$", if the last applied rule is ($\to I$), we have

$$\frac{\begin{matrix}[\mu z] \\ \vdots \\ \nu yz\end{matrix}}{\mu \to \nu\lambda z.yz}\;(\to I).$$

By Lemmas 1.8(1) and 1.7(2), if $\mu \to \nu \not\leqslant_{\mathcal{T}} \omega \to \omega$, there is ρ such that $\mu \leqslant_{\mathcal{T}} \rho$ and $B_{\xi[y/d]} \vdash^{\mathcal{T}} \rho \to \nu y$. Therefore, by $(\leqslant_{\mathcal{T}})$, $B_{\xi[y/d]} \vdash^{\mathcal{T}} \mu \to \nu y$ which implies $\mu \to \nu \in d$ since $[\![y]\!]^{\mathcal{T}}_{\xi[y/d]} = d$.

(2): By induction on τ. The only interesting case is $\tau \equiv \sigma \to \rho$.

$$\begin{aligned}\mathcal{V}_{\mathcal{T}}(\tau) &= \{d \in F_{\mathcal{T}} \mid \forall c \in \mathcal{V}_{\mathcal{T}}(\sigma)\colon d.c \in \mathcal{V}_{\mathcal{T}}(\rho)\}\\ &= \{d \in F_{\mathcal{T}} \mid \forall c \ni \sigma\colon d.c \ni \rho\} \quad \text{by the induction hypothesis}\\ &= \{d \in F_{\mathcal{T}} \mid \sigma \to \rho \in d\} \quad \text{by definition of "."}.\end{aligned}$$

Notice that $\sigma \to \rho \in d \Rightarrow \omega \to \omega \in d$ (by $\leqslant_{\mathcal{T}}$)$\Rightarrow d \in F_{\mathcal{T}}$ (by (1)). Therefore, $\mathcal{V}_{\mathcal{T}}(\tau) = \{d \in |\mathcal{T}| \mid \sigma \to \rho \in d\}$.

(3): $\tau y \in B \Rightarrow \tau \in [\![y]\!]^{\mathcal{T}}_{\xi_B}$ (by Definition 1.9(3) and (4))$\Rightarrow [\![y]\!]^{\mathcal{T}}_{\xi_B} \in \mathcal{V}(\tau)$ (by (2)).

(4): ($\Rightarrow$): Immediate from Definition 2.2(3).

($\Leftarrow$): Take $\mathfrak{M} = \mathfrak{M}_{\mathcal{T}}$, $\mathcal{V} = \mathcal{V}_{\mathcal{T}}$ ($\mathcal{V}_{\mathcal{T}}$ agrees with $\mathcal{T}$ by (2)).

$$\mathcal{V}_{\mathcal{T}}(\sigma) \subseteq \mathcal{V}_{\mathcal{T}}(\tau) \Rightarrow \{d \in |\mathcal{T}| \mid \sigma \in d\} \subseteq \{d \in |\mathcal{T}| \mid \tau \in d\} \Rightarrow \sigma \leqslant_{\mathcal{T}} \tau. \qquad \square$$

The meaning of Lemma 2.7(4)($\Leftarrow$) is that the type theory $\mathcal{T}$ is semantically complete, i.e., every containment between types that is valid in all models is a formula of $\mathcal{T}$.

From Lemma 2.7(1) we easily obtain a property of the elements of $|\mathcal{T}|$ when (Eq_β) and (HR) are provable in $\vdash^{\mathcal{T}}$.

2.8. Corollary. *Let $\mathcal{T}$ be a type theory such that* (Eq_β) *and* (HR) *are derived rules for the induced type assignment. Then $\forall d \in |\mathcal{T}|$: either $d \in F_{\mathcal{T}}$, or $\forall e \in |\mathcal{T}|$: $d.e = {}^{\mathcal{T}}\!\uparrow \omega$.*

Proof

$$\begin{aligned} d.e \neq {}^{\mathcal{T}}\!\uparrow\omega &\Rightarrow \exists \sigma \to \tau \in d \quad \text{by definition of "."}\\ &\Rightarrow \omega \to \omega \in d \quad \text{since } \sigma \to \tau \leqslant_{\mathcal{T}} \omega \to \omega\\ &\Rightarrow d \in F_{\mathcal{T}} \quad \text{by Lemma 2.7(1).} \qquad \square\end{aligned}$$

In other words, if $f^{\mathcal{T}}_\omega : |\mathcal{T}| \to |\mathcal{T}|$ is the function always equal to ${}^{\mathcal{T}}\!\uparrow\omega$, then each $d \in |\mathcal{T}|$ which represents a function different from $f^{\mathcal{T}}_\omega$ belongs to $F_{\mathcal{T}}$. That is, only $f^{\mathcal{T}}_\omega$ is represented by more than one filter of $|\mathcal{T}|$ (for example, ${}^{\mathcal{T}}\!\uparrow\omega$ and ${}^{\mathcal{T}}\!\uparrow\omega \to \omega$ both represent $f^{\mathcal{T}}_\omega$).

2.9. Theorem. *Let $\mathcal{T}$ be a type theory. The induced type assignment system is F-complete iff* (Eq_β) *and* (HR) *are derived rules.*

Proof. ($\Rightarrow$): To have F-completeness we must obviously require invariance of types under β-conversion of subjects. We show that (HR) is sound for all $\mathfrak{M}$, ξ, $\mathcal{V}$ (this proof is due to Hindley).

$$\mathfrak{M}, \xi, \mathcal{V} \models \varphi \wedge \omega^n \to \omega M \Rightarrow [\![M]\!]_\xi^{\mathfrak{M}} \in \mathcal{V}(\varphi) \cap F^{(n)}$$

by Lemma 2.6(1). Let $d = [\![M]\!]_\xi^{\mathfrak{M}}$; then,

$$\begin{aligned} d \in F^{(n)} &\Rightarrow d = [\![\lambda z_1 \ldots z_n . y z_1 \ldots z_n]\!]_{\xi[y/d]}^{\mathfrak{M}} && \text{by Lemma 2.6(3)} \\ &\Rightarrow d = [\![\lambda z_1 \ldots z_n . M z_1 \ldots z_n]\!]_\xi^{\mathfrak{M}} && \text{by [25, Theorem 2.8]} \\ &\Rightarrow [\![\lambda z_1 \ldots z_n . M z_1 \ldots z_n]\!]_\xi^{\mathfrak{M}} \in \mathcal{V}(\varphi) \\ &\Rightarrow \mathfrak{M}, \xi, \mathcal{V} \models \varphi \lambda z_1 \ldots z_n . M z_1 \ldots z_n. \end{aligned}$$

($\Leftarrow$): Notice that $\mathfrak{M}_{\mathcal{T}}$ is a λ-model by Theorem 1.12.

$$\begin{aligned} B \models \tau M &\Rightarrow \mathfrak{M}_{\mathcal{T}}, \xi_B^{\mathcal{T}}, \mathcal{V}_{\mathcal{T}} \models \tau M && \text{by Lemma 2.7(3)} \\ &\Rightarrow [\![M]\!]_{\xi_B^{\mathcal{T}}}^{\mathcal{T}} \in \mathcal{V}_{\mathcal{T}}(\tau) && \text{by Definition 2.2(1)} \\ &\Rightarrow \tau \in [\![M]\!]_{\xi_B^{\mathcal{T}}}^{\mathcal{T}} && \text{by Lemma 2.7(2)} \\ &\Rightarrow B_{\xi_B^{\mathcal{T}}} \overset{\mathcal{T}}{\vdash} \tau M && \text{by Definition 1.9(3)} \\ &\Rightarrow B \overset{\mathcal{T}}{\vdash} \tau M && \text{by Lemma 1.10(2).} \quad \square \end{aligned}$$

Notice that $\mathfrak{M}_{\mathcal{T}}$ is the λ-model used in Theorem 2.9 to prove the completeness of $\vdash^{\mathcal{T}}$.

As an immediate consequence of Theorem 2.9, we have that the type assignment of [2] is not F-complete.

3. Approximation Theorem

In this section we prove, under suitable conditions on $\mathcal{T}$, an Approximation Theorem for the F-filter λ-models $\mathfrak{M}_{\mathcal{T}}$. This result, which is similar to the Approximation Theorem proved in [26, 42] for D_∞ and P_ω, is interesting in itself and useful in subsequent sections.

We use a variant of λ-Ω-calculus (called λ-Ω^*-calculus here and λ-β-Ω_1-calculus in [29], cf. also [28]) obtained from λ-calculus by adding the constant Ω to the formation rules of terms and the reduction rule (Ω^*): $\Omega M \to \Omega$, only (besides rules α and β). The congruence relations $=_{\Omega^*}$ and $=_{\beta\Omega^*}$ are defined as usual. A λ-Ω^*-term A is *β-Ω^*-normal form* (β-Ω^*-n.f.) iff A cannot be further reduced. A β-Ω^*-n.f. A is the β-Ω^*-n.f. of a λ-Ω^*-term M iff M reduces to A using rules α, β and Ω^*.

Let M be a λ-Ω^*-term and A a β-Ω^*-n.f., A is an *approximate normal form* (a.n.f.) *of* M ($A \sqsubseteq^* M$) iff $\exists M' =_\beta M$ such that A matches M' except at occurrences of Ω in A. Lastly, define $\mathscr{A}^*(M)=\{A \mid A \sqsubseteq^* M\}$. As usual, we say that a λ-Ω^*-term M is of order 0 if there are no y, N such that $M =_\beta \lambda y.N$.

The type assignment given in Definition 1.4(3) can be extended to λ-Ω^*-terms without modifications.

We need some properties of approximants.

3.1. Lemma. (1) $\forall M$: *$\mathscr{A}^*(M)$ is a directed set with respect to $\sqsubseteq^*$.*

(2) *If $A \sqsubseteq^* Mz$ and $z \notin \mathrm{FV}(M)$, then there is $A' \sqsubseteq^* M$ such that $A =_{\beta\Omega^*} A'z$.*

(3) *If $A \sqsubseteq^* M$ and A' is the β-Ω^*-n.f. of $A\Omega$, then $A' \sqsubseteq^* MN$ for all N.*

(4) *$B \vdash^{\mathscr{T}} \sigma\Omega$ implies $\omega \sim_{\mathscr{T}} \sigma$.*

Proof. (1): Confer [28, Proposition 3.2].

(2): If M is not of order 0, i.e., $\lambda z.Mz =_\beta M$, we have $A' \equiv \lambda z.A$. If M is of order 0 and $A \equiv \Omega$, then $A' \equiv \Omega$. If M is of order 0 and $A \equiv xA_1 \ldots A_n z$, then $A' \equiv xA_1 \ldots A_n$.

(3): Immediate from $A \sqsubseteq^* M$ and $\forall N$: $\Omega \sqsubseteq^* N$.

(4): By induction on deductions (notice that we can use only rules (ω), $(\wedge I)$, $(\wedge E)$, and $(\leqslant_{\mathscr{T}})$). $\square$

It is easy to check that if (Eq_β) is a derived rule for $\vdash^{\mathscr{T}}$, then also

$$(\mathrm{Eq}_{\beta\Omega^*}):\quad \frac{\tau M \quad M =_{\beta\Omega^*} N}{\tau N}$$

is a derived rule for $\vdash^{\mathscr{T}}$.

3.2. Lemma. *If (Eq_β) is a derived rule for the type assignment induced by $\mathscr{T}$, then $(\mathrm{Eq}_{\beta\Omega^*})$ is a derived rule, too.*

Proof. First we show that $\sigma \to \tau \sim_{\mathscr{T}} \omega$ implies $\tau \sim_{\mathscr{T}} \omega$. It is easy to check that if $\sigma \to \tau \sim_{\mathscr{T}} \omega$, then $\{\sigma z\} \vdash^{\mathscr{T}} \tau y$ for all variables z, y:

$$\cfrac{\cfrac{\cfrac{}{\omega\lambda x.y}\,(\omega)}{\sigma \to \tau\lambda x.y}\,(\leqslant_{\mathscr{T}}) \qquad \sigma z}{\tau(\lambda x.y)z}\,(\to E)$$

and $\{\sigma z\} \vdash^{\mathscr{T}} \tau(\lambda x.y)z$ implies $\{\sigma z\} \vdash^{\mathscr{T}} zy$ by (Eq_β). But $\{\sigma z\} \vdash^{\mathscr{T}} \tau y$ implies $\tau \in {}^{\mathscr{T}}\!\uparrow\omega$ by Lemma 1.7(2), i.e., $\tau \sim_{\mathscr{T}} \omega$. Let

$$(\mathrm{Eq}_{\Omega^*}):\quad \frac{\tau M \quad M =_{\Omega^*} N}{\tau N}.$$

Clearly, $(\mathrm{Eq}_{\beta\Omega^*})$ is derivable iff both (Eq_β) and (Eq_{Ω^*}) are derivable. So it is sufficient to prove that $B \vdash^{\mathscr{T}} \tau\Omega M \Rightarrow B \vdash^{\mathscr{T}} \tau\Omega$. This proof is by induction on the deduction $\mathscr{D}$: $B \vdash^{\mathscr{T}} \tau\Omega M$. If the last applied rule is (ω), $(\wedge E)$, $(\wedge I)$, or $(\leqslant_{\mathscr{T}})$, it is trivial. If the last applied rule is $(\to E)$,

$$\frac{\sigma \to \tau\Omega \quad \sigma M}{\tau\Omega M} \quad (\to E),$$

by Lemma 3.1(4), $\sigma \to \tau \sim_{\mathscr{T}} \omega$ which implies $\tau \sim_{\mathscr{T}} \omega$ by the remarks above. □

The technique used to state the Approximation Theorem is a variant of Tait's 'computability' [41] proposed in [13] (a similar technique is used also by Stenlund [40]).

We define sets of 'approximable' and 'computable' terms. The computable terms are defined by induction on types, and every computable term is shown to be approximable. Using induction on typings, we then show that every term is computable.

3.3. Definition. Let $\mathscr{T}$ be a type theory:

(1) $\text{App}_{\mathscr{T}}(B, \tau, M) \Leftrightarrow \exists A \in \mathscr{A}^*(M): B \overset{\mathscr{T}}{\vdash} \tau A.$

(2) $\text{Comp}_{\mathscr{T}}(B, \omega, M)$ is true,

$\text{Comp}_{\mathscr{T}}(B, \varphi, M) = \text{App}_{\mathscr{T}}(B, \varphi, M)$,

$\text{Comp}_{\mathscr{T}}(B, \sigma \to \tau, M)$ and $\tau \sim_{\mathscr{T}} \omega \Leftrightarrow \text{App}_{\mathscr{T}}(B, \sigma \to \tau, M)$,

$\text{Comp}_{\mathscr{T}}(B, \sigma \to \tau, M)$ and $\tau \not\sim_{\mathscr{T}} \omega$

$\Leftrightarrow [\text{Comp}_{\mathscr{T}}(B', \sigma, N) \Rightarrow \text{Comp}_{\mathscr{T}}(B \cup B', \tau, MN)]$,

$\text{Comp}_{\mathscr{T}}(B, \sigma \wedge \tau, M) \Leftrightarrow \text{Comp}_{\mathscr{T}}(B, \sigma, M)$ and $\text{Comp}_{\mathscr{T}}(B, \tau, M)$.

It is easy to verify, by induction on types, that $\text{Comp}_{\mathscr{T}}$ is invariant under β-Ω^*-conversion of terms. That is, if $M =_{\beta\Omega^*} N$, then $\text{Comp}_{\mathscr{T}}(B, \tau, M)$ iff $\text{Comp}_{\mathscr{T}}(B, \tau, N)$.

We can show that in the systems $\vdash^{\mathscr{T}}$ for which (Eq_β) is derivable, $B \vdash^{\mathscr{T}} \tau A$ and $A \sqsubseteq^* M$ imply $B \vdash^{\mathscr{T}} \tau M$ as follows. By definition, there is $M' =_\beta M$ such that A matches M' except at occurrences of Ω in A. Thanks to Lemma 3.1(4) and $(\leqslant_{\mathscr{T}})$ we may simply obtain a deduction of $B \vdash^{\mathscr{T}} \tau M'$ by using rule (ω) to assign type ω to the terms which are replaced by Ω in A. Lastly we have $B \vdash^{\mathscr{T}} \tau M$ by (Eq_β).

We characterize the type theories for which the converse holds (i.e., $B \vdash^{\mathscr{T}} \tau M$ implies that there is an $A \sqsubseteq^* M$ such that $B \vdash^{\mathscr{T}} \tau A$). They are all and the only theories $\mathscr{T}$ such that

(i) rule (Eq_β) is derivable in the system $\vdash^{\mathscr{T}}$;

(ii) $\sigma \leqslant_{\mathscr{T}} \tau$ implies $\text{Comp}_{\mathscr{T}}(B, \sigma, M) \Rightarrow \text{Comp}_{\mathscr{T}}(B, \tau, M)$.

The proof of the sufficiency of these conditions is done by showing (by induction on types using condition (i)) that $\text{Comp}_{\mathscr{T}}(B, \tau, M)$ implies $\text{App}_{\mathscr{T}}(B, \tau, M)$ and by showing (by induction on deductions using condition (ii)) that $B \vdash^{\mathscr{T}} \tau M$ implies $\text{Comp}_{\mathscr{T}}(B, \tau, M)$. The necessity of conditions (i) and (ii) are shown in Theorem

3.10. Notice that, given a term M and a basis B, we can assume that there are infinitely many variables which are all distinct and do not occur in B and in M. This is proved in [21, 23].

Let $\boldsymbol{M}$ denote a sequence $M_1, \ldots, M_n$ $(n \geq 0)$ of terms and let $x\boldsymbol{M}$ stand for $xM_1 \ldots M_n$. FV(B) is the set of variables which are subjects of statements in B.

3.4. Lemma. *Let $\mathcal{T}$ be a type theory such that* (Eq_β) *is a derived rule for the induced type assignment.*

(1) $\mathrm{App}_{\mathcal{T}}(B, \tau, x\boldsymbol{M}) \Rightarrow \mathrm{Comp}_{\mathcal{T}}(B, \tau, x\boldsymbol{M})$.

(2) $\mathrm{Comp}_{\mathcal{T}}(B, \tau, M) \Rightarrow \mathrm{App}_{\mathcal{T}}(B, \tau, M)$.

Proof. We prove (1) and (2) simultaneously by induction on τ. $\tau \equiv \varphi$, $\tau \equiv \omega$, $\tau \equiv \sigma \to \rho$ with $\rho \sim_{\mathcal{T}} \omega$ follows from the definition. $\tau \equiv \sigma \wedge \rho$ is easily proved.

$\tau \equiv \sigma \to \rho$, $\rho \not\sim_{\mathcal{T}} \omega$. We prove (1) first. Note that $\mathrm{Comp}_{\mathcal{T}}(B', \sigma, N)$ implies $\mathrm{App}_{\mathcal{T}}(B', \sigma, N)$ by the induction hypothesis. Therefore, from $\mathrm{App}_{\mathcal{T}}(B, \sigma \to \rho, x\boldsymbol{M})$ and $\mathrm{App}_{\mathcal{T}}(B', \sigma, N)$ we have $\mathrm{App}_{\mathcal{T}}(B \cup B', \rho, x\boldsymbol{M}N)$ which implies $\mathrm{Comp}_{\mathcal{T}}(B \cup B', \rho, x\boldsymbol{M}N)$ again by the induction hypothesis. We conclude $\mathrm{Comp}_{\mathcal{T}}(B, \sigma \to \rho, x\boldsymbol{M})$.

(2): Take $z \notin \mathrm{FV}(M) \cup \mathrm{FV}(B)$. Notice that, by (1), $\mathrm{App}_{\mathcal{T}}(\{\sigma z\}, \sigma, z)$ implies $\mathrm{Comp}_{\mathcal{T}}(\{\sigma z\}, \sigma, z)$. Hence, $\mathrm{Comp}_{\mathcal{T}}(B, \sigma \to \rho, M)$ and

$$\begin{aligned}\mathrm{Comp}_{\mathcal{T}}(\{\sigma z\}, \sigma, z) &\Rightarrow \mathrm{Comp}_{\mathcal{T}}(B \cup \{\sigma z\}, \rho, Mz)\\ &\Rightarrow \mathrm{App}_{\mathcal{T}}(B \cup \{\sigma z\}, \rho, Mz) \quad \text{by the induction hypothesis}\\ &\Rightarrow \exists A \sqsubseteq^* Mz\end{aligned}$$

such that $B \cup \{\sigma z\} \vdash^{\mathcal{T}} \rho A$.

Notice that $A \sqsubseteq^* Mz$ implies that there is an $A' \sqsubseteq^* M$ such that $A =_{\beta\Omega^*} A'z$ by Lemma 3.1(2).

$$B \cup \{\sigma z\} \overset{\mathcal{T}}{\vdash} \rho A \Rightarrow B \cup \{\sigma z\} \overset{\mathcal{T}}{\vdash} \rho A'z$$

by $(\mathrm{Eq}_{\beta\Omega^*})$ (cf. Lemma 3.2)

$$\Rightarrow \exists \mu: B \cup \{\sigma z\} \overset{\mathcal{T}}{\vdash} \mu \to \rho A' \quad \text{and} \quad B \cup \{\sigma z\} \vdash^{\mathcal{T}} \mu z$$

by Lemma 1.8(1)

$$\Rightarrow \exists \mu: B \cup \{\sigma z\} \overset{\mathcal{T}}{\vdash} \mu \to \rho A' \quad \text{and} \quad \sigma \leq_{\mathcal{T}} \mu$$

by Lemma 1.7(2) since $z \notin \mathrm{FV}(B)$

$$\Rightarrow B \cup \{\sigma z\} \overset{\mathcal{T}}{\vdash} \sigma \to \rho A' \quad \text{by } (\leq_{\mathcal{T}})$$

$$\Rightarrow B \overset{\mathcal{T}}{\vdash} \sigma \to \rho A'$$

by Lemma 1.8(3) since $z \notin \mathrm{FV}(M)$

$$\Rightarrow \mathrm{App}_{\mathcal{T}}(B, \sigma \to \rho, M). \qquad \square$$

3.5. Definition. A type theory $\mathcal{T}$ is *approximable* if $\sigma \leqslant_{\mathcal{T}} \tau$ implies $\text{Comp}_{\mathcal{T}}(B, \sigma, M) \Rightarrow \text{Comp}_{\mathcal{T}}(B, \tau, M)$.

3.6. Lemma. *Let Σ be a set of formulas such that $\sigma \leqslant_{\mathcal{T}} \tau \in \Sigma$ implies $\text{Comp}_{\mathcal{T}(\Sigma)}(B, \sigma, M) \Rightarrow \text{Comp}_{\mathcal{T}(\Sigma)}(B, \tau, M)$. Then, $\mathcal{T}(\Sigma)$ is an approximable type theory.*

The proof is done by induction on $\leqslant_{\mathcal{T}(\Sigma)}$.

3.7. Lemma. *Let $\mathcal{T}$ be an approximable theory, $B = \{\sigma_1 x_1, \ldots, \sigma_n x_n\}$ and $\text{Comp}_{\mathcal{T}}(B_i, \sigma_i, N_i)$ for $1 \leqslant i \leqslant n$. Then, $B \vdash^{\mathcal{T}} \tau M \Rightarrow \text{Comp}_{\mathcal{T}}(B_1 \cup B_2 \cup \cdots \cup B_n, \tau, M[x_1/N_1, x_2/N_2, \ldots, x_n/N_n])$.*

Proof. By induction on the derivation $\mathcal{D}$: $B \vdash^{\mathcal{T}} \tau\mathrm{M}$. If the last applied rule is $(\leqslant_{\mathcal{T}})$, use Definition 3.5. If the last applied rule is $(\rightarrow E)$, i.e., $M \equiv PQ$ and

$$\frac{\sigma \rightarrow \tau P \quad \sigma Q}{\tau PQ} \ (\rightarrow E),$$

we have

$$\text{Comp}_{\mathcal{T}}(B_1 \cup \cdots \cup B_n, \sigma \rightarrow \tau, P[x_1/N_1, \ldots, x_n/N_n])$$

and

$$\text{Comp}_{\mathcal{T}}(B_1 \cup \cdots \cup B_n, \sigma, Q[x_1/N_1, \ldots, x_n/N_n])$$

by the induction hypothesis which implies

$$\text{Comp}_{\mathcal{T}}(B_1 \cup \cdots \cup B_n, \tau, PQ[x_1/N_1, \ldots, x_n/N_n])$$

by definition. If the last applied rule is $(\rightarrow I)$, let $M \equiv \lambda x.P$. We distinguish two cases:

Case 1: $\tau \equiv \sigma \rightarrow \rho$ with $\rho \sim_{\mathcal{T}} \omega$:

$$\frac{\begin{array}{c}[\sigma x]\\ \vdots \\ \rho P\end{array}}{\sigma \rightarrow \rho \lambda x.P} \ (\rightarrow I).$$

In this case, we have $\text{App}_{\mathcal{T}}(B, \sigma \rightarrow \rho, \lambda x.P)$ since $\lambda x.\Omega \sqsubseteq^* \lambda x.P$ and

$$\cfrac{\cfrac{\cfrac{[\sigma x]}{\omega\Omega}\ (\omega)}{\rho\Omega}\ (\leqslant_{\mathcal{T}})}{\sigma \rightarrow \rho\lambda x.\Omega}\ (\rightarrow I),$$

so $\text{Comp}_{\mathcal{T}}(B, \sigma \rightarrow \rho, M)$ by definition.

Case 2: $\tau \equiv \sigma \rightarrow \rho$ with $\rho \not\sim_{\mathcal{T}} \omega$:

$$\frac{\begin{array}{c}[\sigma x]\\ \vdots \\ \rho P\end{array}}{\sigma \rightarrow \rho \lambda x.P} \ (\rightarrow I).$$

$\text{Comp}_{\mathcal{T}}(B', \sigma, N)$ implies

$$\text{Comp}_{\mathcal{T}}(B' \cup B_1 \cup \cdots \cup B_n, \rho, P[x/N, x_1/N_1, \ldots, x_n/N_n])$$

by the induction hypothesis. Then we have

$$\text{Comp}_{\mathcal{T}}(B' \cup B_1 \cup \cdots \cup B_n, \rho, (\lambda x.P[x_1/N_1, \ldots, x_n/N_n])N)$$

since $\text{Comp}_{\mathcal{T}}$ is invariant under β-Ω^*-conversion (note that $x \notin \text{FV}(N)$). Hence, by definition,

$$\text{Comp}_{\mathcal{T}}(B_1 \cup \cdots \cup B_n, \sigma \to \rho, \lambda x.P[x_1/N_1, \ldots, x_n/N_n]).$$

The other cases are trivial. □

3.8. Theorem. *Let $\mathcal{T}$ be an approximable theory such that* (Eq_β) *is a derived rule for the type assignment induced by $\mathcal{T}$. $B \vdash^{\mathcal{T}} \tau M \Leftrightarrow \exists A \sqsubseteq^* M$ such that $B \vdash^{\mathcal{T}} \tau A$.*

Proof. ($\Leftarrow$): Obvious (cf. the discussion after Definition 3.3).

($\Rightarrow$): Notice that $\sigma x \in B \Rightarrow \text{Comp}_{\mathcal{T}}(B, \sigma, x)$ by Lemma 3.4(1).

$$\begin{aligned} B \overset{\mathcal{T}}{\vdash} \tau M &\Rightarrow \text{Comp}_{\mathcal{T}}(B, \tau, M) \quad \text{by Lemma 3.7} \\ &\Rightarrow \text{App}_{\mathcal{T}}(B, \tau, M) \quad \text{by Lemma 3.4(2).} \quad \square \end{aligned}$$

Let us extend $[\![\ ]\!]^{\mathcal{T}}_{\xi}$ to λ-Ω^*-terms by assuming $[\![\Omega]\!]^{\mathcal{T}}_{\xi} = {}^{\mathcal{T}}\!\uparrow\omega$. Notice that, by Lemma 3.1(4), ${}^{\mathcal{T}}\!\uparrow\omega = \{\tau \mid B_\xi \vdash^{\mathcal{T}} \tau\Omega\}$ for all ξ.

3.9. Theorem (Approximation Theorem for $\mathfrak{M}_{\mathcal{T}}$). *Let $\mathcal{T}$ be an approximable theory such that* (Eq_β) *is a derived rule for the type assignment induced by $\mathcal{T}$. Then $[\![M]\!]^{\mathcal{T}}_{\xi} = \bigsqcup\{[\![A]\!]^{\mathcal{T}}_{\xi} \mid A \in \mathcal{A}^*(M)\}$.*

The proof is immediate from Theorem 3.8.

When $\mathcal{T}$ satisfies the conditions of Theorem 3.9, the local structure of the λ-model $\mathfrak{M}_{\mathcal{T}}$ has some interesting properties. Firstly, $[\![M]\!]^{\mathcal{T}}_{\xi} = {}^{\mathcal{T}}\!\uparrow\omega$ for all unsolvable terms of order 0 and all environments ξ. Moreover, defining the tree $T(M)$ of a term M and the partial order relation $\subseteq$ between trees as in [29], we have that $T(M) \subseteq T(N)$ implies that the value of M is less than or equal to the value of N in $\mathfrak{M}_{\mathcal{T}}$. In fact, it is easy to verify that $A \sqsubseteq^* M$ iff $T(A) \subseteq T(M)$. An immediate consequence of this is that all fixed point combinators of the λ-calculus coincide (since they have the same tree) and represent the fixed point operator in $\mathfrak{M}_{\mathcal{T}}$.

In [10], the class of λ-models $\mathfrak{M}_{\mathcal{T}}$ for all $\mathcal{T}$ is characterized, proving that it properly includes (up to isomorphism) all D_∞-λ-models [34]. So we can argue that different $\mathcal{T}$'s induce λ-models with different local structures.

3.10. Theorem. *Let $\mathcal{T}$ be a type theory which does not satisfy one of the conditions of Theorem 3.9. Then the Approximation Theorem fails for $\mathfrak{M}_{\mathcal{T}}$.*

Proof. If (Eq_β) is not a derived rule for $\vdash^{\mathcal{T}}$, $\mathfrak{M}_{\mathcal{T}}$ is not a λ-model.

If $\mathcal{T}$ is not an approximable theory, let $\sigma \leqslant_{\mathcal{T}} \tau$, $\mathrm{Comp}_{\mathcal{T}}(B, \sigma, M) = \mathrm{true}$ and $\mathrm{Comp}_{\mathcal{T}}(B, \tau, M) = \mathrm{false}$. Assume, in order to derive a contradiction, that the Approximation Theorem holds for $\mathfrak{M}_{\mathcal{T}}$.

Recalling the conventions about "=" between types, it is easy to verify that, for each type ν, we can find types $\mu_1^{(i)}, \ldots, \mu_{n_i}^{(i)}, \psi^{(i)}$ (where $n_i \geqslant 0$ and $1 \leqslant i \leqslant m$) such that

$$\nu = \bigwedge_{1 \leqslant i \leqslant m} \mu_1^{(i)} \to \cdots \to \mu_{n_i}^{(i)} \to \psi^{(i)}$$

and each $\psi^{(i)}$ is ω or a type variable.

Therefore, $\mathrm{Comp}(B, \tau, M) = \mathrm{false}$ implies that we can assume, without loss of generality, $\tau = \mu_1 \to \cdots \to \mu_n \to \psi \wedge \rho$ $(n \geqslant 0)$, where ψ is ω or a type variable and $\mathrm{Comp}_{\mathcal{T}}(B, \mu_1 \to \cdots \to \mu_n \to \psi, M) = \mathrm{false}$.

$$\mathrm{Comp}_{\mathcal{T}}(B.\sigma, M) \Rightarrow \mathrm{App}_{\mathcal{T}}(B, \sigma, M) \quad \text{by Lemma 3.4(2)}$$

$$\Rightarrow \mathrm{App}_{\mathcal{T}}(B, \mu_1 \to \cdots \to \mu_n \to \psi, M)$$

since $\sigma \leqslant_{\mathcal{T}} \mu_1 \to \cdots \to \mu_n \to \psi$

$$\Rightarrow B \overset{\mathcal{T}}{\vdash} \mu_1 \to \cdots \to \mu_n \to \psi M.$$

Moreover, for $1 \leqslant i \leqslant n$,

$$\mathrm{Comp}_{\mathcal{T}}(B_i, \mu_i, N_i) \Rightarrow \mathrm{App}_{\mathcal{T}}(B_i, \mu_i, N_i) \quad \text{by Lemma 3.4(2)}$$

$$\Rightarrow B_i \overset{\mathcal{T}}{\vdash} \mu_i N_i.$$

Therefore, we conclude $B \cup B_1 \cup \cdots \cup B_n \vdash^{\mathcal{T}} \psi M N_1 \ldots N_n$.

$$B \cup B_1 \cup \cdots \cup B_n \overset{\mathcal{T}}{\vdash} \psi M N_1 \ldots N_n$$

$$\Rightarrow \mathrm{App}_{\mathcal{T}}(B \cup B_1 \cup \cdots \cup B_n, \psi, M N_1 \ldots N_n) \quad \text{by hypothesis}$$

$$\Rightarrow \mathrm{Comp}_{\mathcal{T}}(B \cup B_1 \cup \cdots \cup B_n, \psi, M N_1 \ldots N_n) \quad \text{by Definition 3.3(2)}$$

$$\Rightarrow \mathrm{Comp}_{\mathcal{T}}(B, \mu_1 \to \cdots \to \mu_n \to \psi, M) \quad \text{by Definition 3.3(2).}$$

□

4. Some type assignments

In this section we study four type assignments induced by type theories. The choice of these theories has been suggested by the following considerations.

Type theories $\mathcal{T}$ which give rise to always empty interpretations of types are pathological since, if $\mathcal{T}$ implies $\mathcal{V}(\tau) = \Phi$, the type assignment τx can never be satisfied. In this case, $\{\tau x\} \models^{\mathcal{T}} \sigma M$ will hold for every σ and M.

Therefore, in addition to $\mathcal{T}_f$, we consider type theories in which some relations between type variables and $\omega \to \omega$ are axiomatized. More precisely, we study:

(1) the type theory $\mathcal{T}_e$ which forces the interpretation of each type variable to be a subset of F ($\forall \varphi$: $\mathcal{V}(\varphi) \subseteq F$);

(2) the type theory $\mathcal{T}_i$ which forces the interpretation of each type variable to contain F ($\forall \varphi$: $F \subseteq \mathcal{V}(\varphi)$);

(3) the type theory $\mathcal{T}_s$ which forces the intersection between F and the interpretation of the type variable φ to be the interpretation of the type $\omega \to \varphi$ ($\forall \varphi$: $F \cap \mathcal{V}(\varphi) = \mathcal{V}(\omega \to \varphi)$).

4.1. Definition. (1) $\Sigma_f = \Phi$, $\Sigma_e = \{\varphi_j \leqslant \omega \to \omega \mid j \in \mathbb{N}\}$, $\Sigma_i = \{\omega \to \omega \leqslant \varphi_j \mid j \in \mathbb{N}\}$, $\Sigma_s = \{\omega \to \omega \wedge \varphi_j \sim \omega \to \varphi_j \mid j \in \mathbb{N}\}$.

(2) $\mathcal{T}_x = \mathcal{T}(\Sigma_x)$ for $x =$ f, e, i, s.

(3) $\leqslant_x, \sim_x, \ldots$ are short for $\leqslant_{\mathcal{T}_x}, \sim_{\mathcal{T}_x}, \ldots$ where $x =$ f, e, i, s.

Clearly, $\leqslant_f$ is as defined in Definition 1.1(2). In the type assignments $\vdash^i$ and $\vdash^s$ typing is not preserved by substitution since, for example, $\{\omega \to \omega y\} \vdash^i \varphi y$, $\{\omega \to \omega y\} \nvdash^i \varphi \to \varphi y$ and $\{\omega \to \omega \wedge \varphi y\} \vdash^s \omega \to \varphi y$, $\{\omega \to \omega \wedge \varphi \to \varphi y\} \nvdash^s \omega \to \varphi \to \varphi y$.

In order to prove that $\mathfrak{M}_x$ (for $x =$ f, e, i, s) are F-filter λ-models we need two technical lemmas.

4.2. Definition. $\#(\tau)$ is inductively defined by

(i) $\#(\varphi_j) = \#(\omega) = 0$ for all $j \in \mathbb{N}$,

(ii) $\#(\sigma \to \tau) = 1 + \#(\tau)$,

(iii) $\#(\sigma \wedge \tau) = \max(\#(\sigma), \#(\tau))$.

4.3. Lemma. *For $x =$ f, e, i, s, if $(\mu_1 \to \nu_1) \wedge \cdots \wedge (\mu_n \to \nu_n) \leqslant_x \sigma \to \tau$ and if $\tau \not\sim_x \omega$, then there are $p_1, \ldots, p_q \in \{1, \ldots, n\}$ such that $\mu_{p_1} \wedge \cdots \wedge \mu_{p_q}\, {}_x\!\geqslant \sigma$ and $\nu_{p_1} \wedge \cdots \wedge \nu_{p_q} \leqslant_x \tau$.*

Proof. Let

$$\gamma = (\mu_1 \to \nu_1) \wedge \cdots \wedge (\mu_n \to \nu_n) \wedge \varphi_{j_1} \wedge \cdots \wedge \varphi_{j_m},$$

$$\delta = (\sigma_1 \to \tau_1) \wedge \cdots \wedge (\sigma_{n'} \to \tau_{n'}) \wedge \varphi_{j'_1} \wedge \cdots \wedge \varphi_{j'_{m'}}.$$

Define the properties (P1), ..., (P5) as follows:

(P1) $\forall l (1 \leqslant l \leqslant n')\, \exists h (1 \leqslant h \leqslant n)$: $\#(\sigma_l \to \tau_l) \leqslant \#(\mu_h \to \nu_h)$;

(P2) $\{j_1, \ldots, j_m\} \supseteq \{j'_1, \ldots, j'_{m'}\}$;

(P3x) $\forall l (1 \leqslant l \leqslant n')$: $\tau_l \not\sim_x \omega$ implies $\exists \{h_1, \ldots, h_k\} \subseteq \{1, \ldots, n\}$ such that $\nu_{h_1}, \ldots, \nu_{h_k}$ are $\not\sim_x \omega$ and

$$\mu_{h_1} \wedge \cdots \wedge \mu_{h_k}\, {}_x\!\geqslant \sigma_l, \qquad \nu_{h_1} \wedge \cdots \wedge \nu_{h_k} \leqslant_x \tau_l$$

for $x = \mathrm{f, e, i, s}$;

(P4) $\forall l(1 \leq l \leq n')$: $\tau_l \not\leq_{\mathrm{s}} \omega$ implies $\exists\{h_1, \ldots, h_k\} \subseteq \{1, \ldots, n\}$ $\exists\{r_1, \ldots, r_p\} \subseteq \{j_1, \ldots, j_m\}$ such that $\nu_{h_1}, \ldots, \nu_{h_k}$ are $\not\leq_{\mathrm{s}} \omega$,

$$\omega \wedge \mu_{h_1} \wedge \cdots \wedge \mu_{h_k}\, {}_{\mathrm{s}}\!\geq \sigma_l \quad \text{and} \quad \nu_{h_1} \wedge \cdots \wedge \nu_{h_k} \wedge \varphi_{r_1} \wedge \cdots \wedge \varphi_{r_p} \leq_{\mathrm{s}} \tau_l.$$

(P5) $\forall q(1 \leq q \leq m')\, \exists\{h_1, \ldots, h_k\} \subseteq \{1, \ldots, n\}$, $\exists\{r_1, \ldots, r_p\} \subseteq \{j_1, \ldots, j_m\}$ such that $\nu_{h_1}, \ldots, \nu_{h_k}$ are $\not\leq_{\mathrm{s}} \omega$,

$$\omega \wedge \mu_{h_1} \wedge \cdots \wedge \mu_{h_k}\, {}_{\mathrm{s}}\!\geq \omega \quad \text{and} \quad \nu_{h_1} \wedge \cdots \wedge \nu_{h_k} \wedge \varphi_{r_1} \wedge \cdots \wedge \varphi_{r_p} \leq_{\mathrm{s}} \varphi_{j'_q}.$$

By straightforward induction on the definitions of $\leq_{\mathrm{f}}$, $\leq_{\mathrm{e}}$, $\leq_{\mathrm{i}}$, and $\leq_{\mathrm{s}}$ we can prove that

(i) $\gamma \leq_{\mathrm{f}} \delta \Rightarrow$(P1), (P2), (P3f).
(ii) $\gamma \leq_{\mathrm{e}} \delta \Rightarrow$(P2), (P3e).
(iii) $\gamma \leq_{\mathrm{i}} \delta \Rightarrow$(P1), (P3i).
(iv) $\gamma \leq_{\mathrm{s}} \delta \Rightarrow$(P1), (P4), (P5). □

Remarks. (1) In properties (P4) and (P5), one of the two sets can be empty. This is true, for example, for (P4) in the case $\omega \to \omega \wedge \varphi \leq_{\mathrm{s}} \sigma \to \varphi$ (where $\sigma \not\leq_{\mathrm{s}} \omega$), and for (P5) in the case $\omega \to \varphi \leq_{\mathrm{s}} \varphi$.

(2) $\leq_{\mathrm{e}}$ does not satisfy (P1) (take $\varphi \leq_{\mathrm{e}} \omega \to \omega$).

(3) $\leq_{\mathrm{i}}$ does not satisfy (P2). Take, for example, $\omega \to \omega \leq_{\mathrm{i}} \varphi$.

(4) $\leq_{\mathrm{s}}$ does not satisfy (P2) (take $\omega \to \varphi \leq_{\mathrm{s}} \varphi$), and
(P3s) (take $\omega \to \omega \wedge \varphi \leq_{\mathrm{s}} \omega \to \varphi$).

4.4. Lemma. *For* $x = \mathrm{f, e, i, s}$,

(1) $B \vdash^x \sigma \to \tau\lambda y.M \Rightarrow B/y \cup \{\sigma y\} \vdash^x \tau M$;

(2) *Rule* (Eq_β) *is a derived rule for the systems* $\vdash^x$.

Proof. (1): We may suppose $\tau \not\leq_x \omega$. Let $\mu_j \to \nu_j\lambda y.M$ $(1 \leq j \leq n)$ be all the statements in $\mathscr{D}$: $B \vdash^x \sigma \to \tau\lambda y.M$ on which $\sigma \to \tau\lambda y.M$ depends and which are conclusions of $(\to I)$:

$$\frac{\begin{matrix}[\mu_j y]\\ \vdots \\ \nu_j M\end{matrix}}{\mu_j \to \nu_j\lambda y.M}\ (\to I).$$

By Lemma 1.7(3),

$$(\mu_1 \to \nu_1) \wedge \cdots \wedge (\mu_n \to \nu_n) \leq_x \sigma \to \tau.$$

By Lemma 4.3, there are $p_1, \ldots, p_q \in \{1, \ldots, n\}$ such that $\mu_{p_1} \wedge \cdots \wedge \mu_{p_q}\, {}_x\!\geq \sigma$ and

$\nu_{p_1} \wedge \cdots \wedge \nu_{p_q} \leqslant_x \tau$. Therefore,

$$\cfrac{\cfrac{\sigma y}{\mu_{p_k} y}\ (\leqslant_x)}{\begin{array}{c}\vdots \quad 1 \leqslant k \leqslant q \\ \cfrac{\cfrac{\nu_{p_k} M}{\nu_{p_1} \wedge \cdots \wedge \nu_{p_q} M}\ (\wedge I)}{\tau M}\ (\leqslant_x).\end{array}}$$

(2): Clearly, it is sufficient to prove

$$B \overset{x}{\vdash} \tau(\lambda y.M)N \Leftrightarrow B \overset{x}{\vdash} \tau M[y/N] \quad \text{for } \tau \not\sim_x \omega.$$

($\Rightarrow$): $B \vdash^x \tau(\lambda y.M)N$ and $\tau \not\sim_x \omega$ imply $\exists \sigma$: $B \vdash^x \sigma \to \tau\lambda y.M$ and $B \vdash^x \sigma N$ by Lemma 1.8(1). Hence, $\exists \sigma$: $B/y \cup \{\sigma y\} \vdash^x \tau M$ and $B \vdash^x \sigma N$ by (1).

Therefore, we obtain a deduction of $B \vdash^x \tau M[y/N]$ by replacing each premise σy by a deduction of σN and y by N in $\mathscr{D}$: $B/y \cup \{\sigma y\} \vdash^x \tau M$.

($\Leftarrow$): If y does not occur in M, this is trivial. Otherwise, let $\mathscr{D}$: $B \vdash^x \tau M[y/N]$ and $\sigma_1 N, \ldots, \sigma_n N$ be all the statements in $\mathscr{D}$ whose subject is N. Then we can obtain a deduction $\mathscr{D}'$: $B/y \cup \{\sigma_1 y, \ldots, \sigma_n y\} \vdash^x \tau M$ by simply replacing the deduction of $\sigma_j N$ by the premise $\sigma_j y$ for $1 \leqslant j \leqslant n$ and N by y in $\mathscr{D}$. Lastly, by applying rules ($\wedge E$), ($\to I$), ($\wedge I$), and ($\to E$), we conclude $B \vdash^x \tau(\lambda y.M)N$. $\square$

4.5. Theorem. *$\mathfrak{M}_x = \langle |\mathscr{T}_x|, \cdot, [\![\ ]\!]^x \rangle$ are F-filter λ-models, for x* = f, e, i, s.

The proof is immediate from Theorem 1.12 and Lemma 4.4(2).
From Theorem 2.4 we have the following theorem.

4.6. Theorem ($\mathscr{T}$-F-soundness). *For x* = f, e, i, s, $B \vdash^x \tau M \Rightarrow B \vDash^x \tau M$.

Notice that the system $\vdash^*$ of [2] is not F-sound. In fact, if $\Delta \equiv \lambda x.xx$, we have $\vdash^* \omega \to \omega\Delta\Delta$ (using $\omega \leqslant \omega \to \omega$). However, in the F-filter λ-model $\mathfrak{M}_f$, say, we have, since the Approximation Theorem holds (cf. Theorem 4.13):

$$[\![\Delta\Delta]\!]^f_\xi = {}^f\!\uparrow\omega \notin F_f = \mathscr{V}(\omega \to \omega) \quad \text{for all } \xi, \mathscr{V}.$$

In fact, ${}^f\!\uparrow\omega$ and ${}^f\!\uparrow\omega \to \omega$ have the same functional behavior (i.e., $\forall d \in |\mathscr{T}_f|$: ${}^f\!\uparrow\omega.d = {}^f\!\uparrow\omega \to \omega.d = {}^f\!\uparrow\omega$) and ${}^f\!\uparrow\omega \to \omega \in F_f$ by Lemma 2.6(3) since

$$[\![\lambda y.zy]\!]^f_{\xi[z/{}^f\uparrow\omega\to\omega]} = {}^f\!\uparrow\omega \to \omega \quad \text{for all } \xi.$$

Moreover, it is easy to verify that $\vdash^e$, $\vdash^i$, and $\vdash^s$ are not F-sound.

Thanks to Theorem 2.9, if we want to establish whether the type assignments $\vdash^x$ (x = f, e, i, s) are F-complete, we need only know if (HR) is provable in them.

4.7. Lemma. (1) (HR) *is a derived rule in the type assignments* $\vdash^x$ *for x* = i, s.
(2) (HR) *is not a derived rule in the type assignments* $\vdash^x$ *for x* = f, e.

Proof. (1): For $x = \mathrm{i}$,

$$\dfrac{\dfrac{\dfrac{[\omega z_1]}{\omega\lambda z_2 \ldots z_n.Nz_1 \ldots z_n}\,(\omega)}{\omega \to \omega\lambda z_1 \ldots z_n.Nz_1 \ldots z_n}\,(\to I)}{\varphi\lambda z_1 \ldots z_n.Nz_1 \ldots z_n}\,(\leqslant_{\mathrm{i}}).$$

For $x = \mathrm{s}$,

$$\dfrac{\dfrac{\dfrac{\varphi \wedge \omega^n \to \omega N}{\omega \to \varphi N}\,(\leqslant_s) \quad [\omega z_1]}{\varphi N z_1}\,(\to E) \qquad \dfrac{\dfrac{\varphi \wedge \omega^n \to \omega N}{\omega^n \to \omega N}\,(\wedge E) \quad [\omega z_1]}{\omega^{n-1} \to \omega N z_1}\,(\to E)}{\varphi \wedge \omega^{n-1} \to \omega N z_1}\,(\wedge I)$$

$$\vdots$$

$$\dfrac{\varphi N z_1 \ldots z_n}{\vdots}\,(\to I)$$

$$\dfrac{\omega^n \to \varphi\lambda z_1 \ldots z_n.Nz_1 \ldots z_n}{\varphi\lambda z_1 \ldots z_n.Nz_1 \ldots z_n}\,(\leqslant_{\mathrm{s}}).$$

(2): $\{\omega \to \omega \wedge \varphi y\} \nvdash^x \varphi\lambda z.yz$ for $x = \mathrm{f}, \mathrm{e}$. □

4.8. Theorem (F-completeness). *The type assignments $\vdash^{\mathrm{i}}$ and $\vdash^{\mathrm{s}}$ are F-complete while $\vdash^{\mathrm{f}}$ and $\vdash^{\mathrm{e}}$ are not F-complete.*

4.9. Remark. Notice that whereas, for all $\mathcal{V}$, $\mathcal{V}$ agrees with $\mathcal{T}_{\mathrm{f}}$ by Theorem 2.4(3) and $\mathcal{V}_x$ agrees with $\mathcal{T}_x$ for $x = \mathrm{i}, \mathrm{s}$ (by Lemma 2.7(2) and Lemma 4.7(1)), $\mathcal{V}_{\mathrm{e}}$ does not agree with $\mathcal{T}_{\mathrm{e}}$. In fact, let φ be any type variable; we have, by definition, ${}^{\mathrm{e}}\!\uparrow\varphi \in \mathcal{V}_{\mathrm{e}}(\varphi)$ and from $\leqslant_{\mathrm{e}}$, $\mathcal{V}_{\mathrm{e}}(\varphi) \subseteq \mathcal{V}_{\mathrm{e}}(\omega \to \omega)$ which would imply ${}^{\mathrm{e}}\!\uparrow\varphi \in F_{\mathrm{e}}$. But ${}^{\mathrm{e}}\!\uparrow\varphi$ for all type variables φ and ${}^{\mathrm{e}}\!\uparrow\omega \to \omega$ have the same functional behavior (i.e., $\forall d \in |\mathcal{T}_{\mathrm{e}}|$: ${}^{\mathrm{e}}\!\uparrow\varphi.d = {}^{\mathrm{e}}\!\uparrow\omega \to \omega.d = {}^{\mathrm{e}}\!\uparrow\omega$) and therefore, they cannot be all elements of F_{e}. ${}^{\mathrm{e}}\!\uparrow\omega \to \omega \in F_{\mathrm{e}}$ by Lemma 2.6(3) since $[\![\lambda y.zy]\!]^{\mathrm{e}}_{\xi[z/{}^{\mathrm{e}}\uparrow\omega\to\omega]} = {}^{\mathrm{e}}\!\uparrow\omega \to \omega$, for all ξ.

It is natural to consider the type assignments obtained by adding (HR) to $\vdash^{\mathrm{e}}$, $\vdash^{\mathrm{f}}$. We call these systems $\vdash^{\mathrm{He}}$, $\vdash^{\mathrm{Hf}}$, respectively. The following lemma compares derivability in the systems with and without (HR).

4.10. Lemma. *For $x = \mathrm{f}, \mathrm{e}$,*

(1) $B \vdash^x \tau M \Rightarrow B \vdash^{\mathrm{H}x} \tau M$,

(2) $B \vdash^{\mathrm{H}x} \tau M \Rightarrow \exists M'$ *such that* $M \twoheadrightarrow_\eta M'$ *and* $B \vdash^x \tau M'$.

Proof. (1): Trivial.

(2): Simply replace each application of (HR)

$$\dfrac{\varphi \wedge \omega^n \to \omega N}{\varphi\lambda z_1 \ldots z_n.Nz_1 \ldots z_n}\quad \text{(HR) if } z_1, \ldots, z_n \notin \mathrm{FV}(N)$$

by an application of ($\wedge$ E)

$$\dfrac{\varphi \wedge \omega^n \to \omega N}{\varphi N}\quad (\wedge E). \qquad \square$$

It is easy to see that $\vdash^{\mathrm{He}}$ and $\vdash^{\mathrm{Hf}}$ are not F-complete since types are not invariant under subject expansion. In fact, $\{\varphi \wedge \omega \to \omega y\} \vdash^{\mathrm{Hx}} \varphi\lambda z.yz$ while

$$\{\varphi \wedge \omega \to \omega y\} \overset{\mathrm{Hx}}{\nvdash} \varphi\lambda z.(\lambda u.yz)v \quad \text{for } x = \mathrm{f, e}.$$

This also proves that $\langle|\mathcal{T}_x|, \cdot, [\![\ ]\!]^{\mathrm{Hx}}\rangle$ is not an F-filter λ-model (if we define $[\![M]\!]_\xi^{\mathrm{Hx}} = \{\tau \mid B_\xi \vdash^{\mathrm{Hx}} \tau M\}$) for $x = \mathrm{f, e}$. Instead, $\vdash^{\mathrm{Hf}}$ and $\vdash^{\mathrm{He}}$ satisfy the subject reduction property.

4.11. Lemma (Subject reduction for $\vdash^{\mathrm{Hx}}$). *For* $x = \mathrm{f, e}$,

(1) $B \vdash^{\mathrm{Hx}} \tau MN,\ \tau \not\sim_x \omega \Leftrightarrow \exists\sigma\colon B \vdash^{\mathrm{Hx}} \sigma \to \tau M,\ B \vdash^{\mathrm{Hx}} \sigma N$;

(2) $B \vdash^{\mathrm{Hx}} \sigma \to \tau\lambda y.M \Leftrightarrow B/y \cup \{\sigma y\} \vdash^{\mathrm{Hx}} \tau M$;

(3) $B \vdash^{\mathrm{Hx}} \tau M,\ M \twoheadrightarrow_\beta M' \Rightarrow B \vdash^{\mathrm{Hx}} \tau M'$.

Proof. (1): Immediate from the proof of Lemma 1.8(1) since the last applied rule cannot be (HR).

(2): The proof of Lemma 4.4(1) remains valid since (HR) assigns only type variables.

(3): Just mimic the proof of Lemma 4.4(2) ($\Rightarrow$). □

Notice that the systems $\vdash^{\mathrm{Hf}\beta}$ and $\vdash^{\mathrm{He}\beta}$ obtained by adding (Eq_β) to $\vdash^{\mathrm{Hf}}$ and $\vdash^{\mathrm{He}}$ do not induce F-filter λ-models since property (2) of Lemma 1.8 fails. In fact, if B is any basis, from $B/y \cup \{\varphi \wedge \omega \to \omega y\}$, we assign only type ω to yz and tz, but

$$B/y \cup \{\varphi \wedge \omega \to \omega y\} \overset{\mathrm{Hx}\beta}{\vdash} \varphi\lambda z.yz \quad \text{and}$$

$$B/y \cup \{\varphi \wedge \omega \to \omega y\} \overset{\mathrm{Hx}\beta}{\nvdash} \varphi\lambda z.tz \qquad (x = \mathrm{f, e}).$$

However, the question whether these systems are F-complete remains open.

To prove the Approximation Theorem for $\mathfrak{M}_x$, we show that $\mathcal{T}_x$ are approximable ($x = \mathrm{f, e, i, s}$).

4.12. Lemma. *The type theories* $\mathcal{T}_x$ *for* $x = \mathrm{f, e, i, s}$ *are approximable.*

Proof. By Lemma 3.6 it is sufficient to show that $\sigma \leqslant_x \tau \in \Sigma_x$ implies $\mathrm{Comp}_{\mathcal{T}_x}(B, \sigma, M) \Rightarrow \mathrm{Comp}_{\mathcal{T}_x}(B, \tau, M)$. For Σ_{f}, the proof is trivial. For Σ_{e} and Σ_{i}, the proof is easy using Definition 3.3.

For Σ_{s}, we have to prove:

$$\mathrm{Comp}_{\mathrm{s}}(B, \omega \to \omega \wedge \varphi, M) \Leftrightarrow \mathrm{Comp}_{\mathrm{s}}(B, \omega \to \varphi, M),$$

which, by Definition 3.3, is equivalent to

$$\mathrm{App}_{\mathrm{s}}(B, \omega \to \omega, M) \quad \text{and} \quad \mathrm{App}_{\mathrm{s}}(B, \varphi, M) \Leftrightarrow \forall N\colon \mathrm{App}_{\mathrm{s}}(B, \varphi, MN).$$

Notice that ($Eq_{\beta\Omega^*}$) is a derived rule for $\vdash^s$ by Lemmas 4.4(2) and 3.2.

(⇒):

$$\mathrm{App_s}(B, \omega\to\omega, M) \text{ and } \mathrm{App_s}(B, \varphi, M)$$

$$\Rightarrow \exists A_1, A_2 \in \mathscr{A}^*(M)\colon B \overset{s}{\vdash} \omega\to\omega A_1 \text{ and } B \overset{s}{\vdash} \varphi A_2$$

$$\Rightarrow \exists A' \in \mathscr{A}^*(M)\colon A_1 \sqsubseteq^* A', A_2 \sqsubseteq^* A'$$

since $\mathscr{A}^*(M)$ is directed

$$B \overset{s}{\vdash} \omega\to\omega A' \quad\text{and}\quad B \overset{s}{\vdash} \varphi A'$$

by the discussion after Definition 3.3

$$\Rightarrow B \overset{s}{\vdash} \omega\to\varphi A' \quad \text{by } (\wedge I) \text{ and } (\leqslant_s)$$

$$\Rightarrow B \overset{s}{\vdash} \varphi A'\Omega \quad \text{by } (\omega) \text{ and } (\to E)$$

$$\Rightarrow B \overset{s}{\vdash} \varphi A''$$

where A'' is the β-Ω^*-n.f. of $A'\Omega$ (by ($Eq_{\beta\Omega^*}$))

$$\Rightarrow \forall N \; \mathrm{App_s}(B, \varphi, MN)$$

since $A'' \sqsubseteq^* MN$ by Lemma 3.1(3).

(⇐): $\forall N$: $\mathrm{App_s}(B, \varphi, MN) \Rightarrow \mathrm{App_s}(B, \varphi, Mz)$, where $z \notin FV(M) \cup FV(B)$

$$\Rightarrow \exists A \sqsubseteq^* Mz \text{ such that } B \overset{s}{\vdash} \varphi A$$

$$\Rightarrow \exists A' \sqsubseteq^* M \text{ such that } B \overset{s}{\vdash} \varphi A' z$$

by Lemma 3.1(2) and ($Eq_{\beta\Omega^*}$)

$$\Rightarrow \exists A' \sqsubseteq^* M \text{ such that } B \overset{s}{\vdash} \omega\to\varphi A' \text{ and } B \overset{s}{\vdash} \omega z$$

by Lemmas 1.8(1) and 1.7(2) since $z \notin FV(B)$

$$\Rightarrow \exists A' \sqsubseteq^* M \text{ such that } B \overset{s}{\vdash} \omega\to\omega\wedge\varphi A' \quad \text{by } (\leqslant_s)$$

$$\Rightarrow \mathrm{App_s}(B, \omega\to\omega\wedge\varphi, M). \qquad \square$$

4.13. Theorem. *For* $x = f, e, i, s$,

(1) $B \vdash^x \tau M \Leftrightarrow \exists A \sqsubseteq^* M$ *such that* $B \vdash^x \tau A$;

(2) *The Approximation Theorem holds for* $\mathfrak{M}_x$;

(3) $\mathfrak{M}_x$ *are not sensible.*

Proof. (1) and (2) are immediate from Theorems 3.8 and 3.9, and Lemmas 4.4(2) and 4.12.

(3): Simply notice that $\mathscr{A}^*(\Delta\Delta)=\{\Omega\}$, while $\mathscr{A}^*(\lambda y.\Delta\Delta)=\{\Omega, \lambda y.\Omega\}$, so $[\![\Delta\Delta]\!]^x_\xi = {}^x{\uparrow}\omega$, whereas $[\![\lambda y.\Delta\Delta]\!]^x_\xi = {}^x{\uparrow}\omega \to \omega$, for all ξ. □

In the remainder of the present section we will connect the types that can be assigned to the terms with the normalization properties of the terms themselves. More precisely, we will prove that

(1) in the systems $\vdash^{\mathrm{f}}$, $\vdash^{\mathrm{e}}$, $\vdash^{\mathrm{s}}$, $\vdash^{\mathrm{Hf}}$, and $\vdash^{\mathrm{He}}$, all and only the terms with head normal form (h.n.f.) have tailproper types (see Definition 4.14 below);

(2) in the systems $\vdash^{\mathrm{f}}$, $\vdash^{\mathrm{e}}$, $\vdash^{\mathrm{Hf}}$, and $\vdash^{\mathrm{He}}$, all and only the terms with normal form (n.f.) have types without ω-occurrences.

4.14. Definition. The set TT of *tailproper types schemes* is defined by:

$$\varphi_0, \varphi_1, \ldots \in \mathrm{TT},$$

$$\tau \in \mathrm{TT} \Rightarrow \sigma \to \tau, \sigma \wedge \tau, \tau \wedge \sigma \in \mathrm{TT} \quad \text{for all } \sigma \in T.$$

4.15. Lemma. *For* $x=\mathrm{f}$, e, s,

(1) $\sigma \leqslant_x \tau$ *and* $\sigma \notin \mathrm{TT} \Rightarrow \tau \notin \mathrm{TT}$;

(2) $B \vdash^x \tau A$, *where A is an unsolvable* β-Ω^*-n.f.$\Rightarrow \tau \notin \mathrm{TT}$.

Proof. (1): By induction on $\leqslant_x$.

(2): By induction on derivations using (1). □

Similar properties do not hold for $\vdash^{\mathrm{i}}$ since, for example, $\vdash^{\mathrm{i}} \varphi\lambda y.\Omega$ and $\vdash^{\mathrm{i}} \varphi \to \varphi\lambda yz.\Omega$. Notice that $\omega \to \omega \leqslant_{\mathrm{i}} \varphi$, $\omega \to \omega \notin \mathrm{TT}$, and $\varphi \in \mathrm{TT}$.

4.16. Theorem. *For* $x=\mathrm{f}$, e, s, $\exists B$, $\tau \in \mathrm{TT}$: $[B \vdash^x \tau M] \Leftrightarrow M$ *has a h.n.f.*

Proof. ($\Rightarrow$): $\exists B$, $\tau \in \mathrm{TT}$: $B \vdash^x \tau M \Rightarrow \exists A \sqsubseteq^* M$ such that $B \vdash^x \tau A$ by Theorem 4.13(1) implying that A is solvable by Lemma 4.15(2), whence M is solvable.

($\Leftarrow$): Let $\lambda z_1 \ldots z_n.yM_1 \ldots M_m$ be the h.n.f. of M and $y \in \mathrm{FV}(M)$. Clearly, $\{\omega^m \to \varphi y\} \vdash^x \varphi y M_1 \ldots M_m$ using (ω) and ($\to E$) and therefore, by applying ($\to I$), we obtain

$$\{\omega^m \to \varphi y\} \overset{x}{\vdash} \omega^n \to \varphi\lambda z_1 \ldots z_n.yM_1 \ldots M_m.$$

The case $y \notin \mathrm{FV}(M)$ is similar. □

4.17. Theorem. *For* $x=\mathrm{f}$, e, $\exists B$, τ: $[B \vdash^x \tau M$ *and* ω *not in* B, $\tau] \Leftrightarrow M$ *has a n.f.*

Proof. ($\Rightarrow$): $B \vdash^x \tau M \Rightarrow \exists A \sqsubseteq^* M$: $B \vdash^x \tau A$ by Theorem 4.13(1). We prove that Ω does not occur in A by induction on A. The only interesting case is $A \equiv zA_1 \ldots A_n$.

$B \overset{x}{\vdash} \tau z A_1 \ldots A_n \Rightarrow \exists \sigma_1, \ldots, \sigma_n: B \overset{x}{\vdash} \sigma_1 \to \cdots \to \sigma_n \to \tau z$ and

$$B \overset{x}{\vdash} \sigma_l A_l \quad 1 \leq l \leq n \text{ by Lemma 1.8(1)}$$

$$\Rightarrow \sigma_1 \to \sigma_2 \to \cdots \to \sigma_n \to \tau \,{}_x\!\geq (\mu_1 \to \nu_1) \wedge \cdots \wedge (\mu_m \to \nu_m) \wedge \varphi_{j_1} \wedge \cdots \wedge \varphi_{j_p},$$

where $\{\mu_1 \to \nu_1 z, \ldots, \mu_m \to \nu_m z, \varphi_{j_1} z, \ldots, \varphi_{j_p} z\} \subseteq B$ by Lemma 1.7(2),

$$\Rightarrow \exists \{h_1, \ldots, h_k\} \subseteq \{1, \ldots, m\} \text{ such that } \sigma_1 \leq_x \mu_{h_1} \wedge \cdots \wedge \mu_{h_k}$$

$$\text{and } \sigma_2 \to \cdots \to \sigma_n \to \tau \,{}_x\!\geq \nu_{h_1} \wedge \cdots \wedge \nu_{h_k}$$

since for $x = \mathrm{f}, \mathrm{e}$, P3$x$ holds (cf. the proof of Lemma 4.3)

$$\Rightarrow B \overset{x}{\vdash} \mu_{h_1} \wedge \cdots \wedge \mu_{h_k} A_1 \quad \text{by } (\leq_x).$$

Reasoning in a similar way from

$$\sigma_2 \to \cdots \to \sigma_n \to \tau_x \geq \nu_{h_1} \wedge \cdots \wedge \nu_{h_k},$$

we can prove that, for each A_l, $2 \leq l \leq n$, $\exists \mu_1^{(l)}, \ldots, \mu_{p_l}^{(l)}$ which occur in B such that $B \vdash^x \mu_1^{(l)} \wedge \cdots \wedge \mu_{p_l}^{(l)} A_l$. Now, the inductive hypothesis can be applied.

($\Leftarrow$): By structural induction on the n.f. of M. □

A counterexample to the Normal Form Theorem for the system $\vdash^{\mathrm{s}}$ (due to the fact that property P3s of Lemma 4.3 is not true for $\vdash^{\mathrm{s}}$) is

$$\{\varphi \wedge \psi \to \psi y\} \overset{\mathrm{s}}{\vdash} \varphi y(\Delta\Delta).$$

Moreover, notice that although property P3i of Lemma 4.3 holds, the Normal Form Theorem fails for the system $\vdash^{\mathrm{i}}$ since it does not satisfy Theorem 4.16.

4.18. Theorem. *For* $x = \mathrm{f}, \mathrm{e}$,

(1) $\exists B, \tau \in \mathrm{TT}$: $[B \vdash^{\mathrm{H}x} \tau M] \Leftrightarrow M$ *has an h.n.f.*;

(2) $\exists B, \tau$: $[B \vdash^{\mathrm{H}x} \tau M$ *and* ω *not in* $B, \tau] \Leftrightarrow M$ *has an n.f.*

Proof. (1) ($\Rightarrow$):

$$B \overset{\mathrm{H}x}{\vdash} \tau M \Rightarrow \exists M': M \twoheadrightarrow_\eta M' \quad B \overset{x}{\vdash} \tau M' \quad \text{by Lemma 4.10(2)}$$

$$\Rightarrow M' \text{ has h.n.f.} \qquad \text{by Theorem 4.16}$$

$$\Rightarrow M \text{ has h.n.f.}$$

($\Leftarrow$): M has h.n.f. implies $\exists B, \tau \in \mathrm{TT}$: $B \vdash^x \tau M$ by Theorem 4.16, whence $B \vdash^{\mathrm{H}x} \tau M$ by Lemma 4.10(1).

(2) is proved in a similar way. □

5. Restricted types

In the preceding sections we have seen that many of the problems of F-completeness arise from the necessity of giving a type interpretation $\mathcal{V}$ such that $\mathcal{V}(\varphi) \cap F$ is nonempty.

These type interpretations are necessary because interpretation of types like $\varphi \wedge \omega \to \omega$ should not always be empty. An alternative approach could be to avoid intersections of this kind by allowing "$\wedge$" to be a partial function satisfying some conditions. Informally, $\sigma \wedge \tau$ is legal only if σ and τ have the same number of "$\to$"s or $\sigma \notin \mathrm{TT}$ (as defined in Definition 4.14) has less "$\to$"s than τ.

The type assignment $\vdash^{\mathrm{R}}$, constructed on this subset $\mathrm{RT} \subseteq T$ of restricted types, is the only system (to the author's knowledge) which is F-sound and F-complete. Another interesting feature of this system is that the Head Normal Form and Normal Form Theorems hold. Moreover, we want to mark the connection between $\vdash^{\mathrm{R}}$ and $\vdash^{\mathrm{f}}$ established in Theorem 5.5. This result is relevant and rather unexpected since the definitions and motivations for $\mathcal{T}_{\mathrm{f}}$ and $\mathcal{T}_{\mathrm{R}}$ look at first sight, totally unrelated.

Note that (HR) is now irrelevant since $\varphi \wedge \omega^n \to \omega \notin \mathrm{RT}$.

5.1. Definition. (1) The set $\mathrm{RT} \subseteq T$ of *restricted types* is inductively defined by

(i) $\varphi_0, \varphi_1, \ldots \in \mathrm{RT}$;

(ii) $\omega \in \mathrm{RT}$;

(iii) $\sigma, \tau \in \mathrm{RT} \Rightarrow \sigma \to \tau \in \mathrm{RT}$;

(iv) $\sigma, \tau \in \mathrm{RT}, \#(\sigma) = \#(\tau) \Rightarrow \sigma \wedge \tau \in \mathrm{RT}$;

(v) $\sigma, \tau \in \mathrm{RT}, \sigma \notin \mathrm{TT}$, and $\#(\sigma) \leqslant \#(\tau) \Rightarrow \sigma \wedge \tau, \tau \wedge \sigma \in \mathrm{RT}$.

(2) $\sigma \leqslant_{\mathrm{R}} \tau$ iff $\sigma, \tau \in \mathrm{RT}$ and $\sigma \leqslant_{\mathrm{f}} \tau$. $\sigma \sim_{\mathrm{R}} \tau$ iff $\sigma \leqslant_{\mathrm{R}} \tau \leqslant_{\mathrm{R}} \sigma$.

(3) $\mathcal{T}_{\mathrm{R}} = \{\sigma \leqslant \tau \mid \sigma \leqslant_{\mathrm{R}} \tau\}$.

Notice that $\sigma \to \tau \wedge \mu \to \nu \in \mathrm{RT}$ implies $\tau \wedge \nu \in \mathrm{RT}$. Let us remark that $(\mathrm{RT}, \omega, \mathrm{Con}_{\mathrm{R}}, \vdash_{\mathrm{R}})$, where $\{\sigma_1, \ldots, \sigma_n\} \in \mathrm{Con}_{\mathrm{R}}$ iff $\sigma_1 \wedge \cdots \wedge \sigma_n \in \mathrm{RT}$ and $\{\sigma_1, \ldots, \sigma_n\} \vdash_{\mathrm{R}} \tau$ iff $\sigma_1 \wedge \cdots \wedge \sigma_n \leqslant_{\mathrm{R}} \tau$, is an information system in the sense of Scott [39].

In order to build the formal system of type assignment $\vdash^{\mathrm{R}}$ we need to modify the definitions of Section 1 slightly.

5.2. Definition. (1) A set $S \subseteq \mathrm{RT}$ is *consistent* iff σ and $\tau \in S$ imply $\sigma \wedge \tau \in \mathrm{RT}$.

(2) A *restricted basis* B is defined by adding to Definition 1.4(2) the condition that, for each variable y, $B \restriction y$ is consistent.

(3) $B \vdash^{\mathrm{R}'} \tau M$ iff τM is derivable from the restricted basis B in the type assignment induced by $\mathcal{T}_{\mathrm{R}}$, where $(\wedge I)$ has been restricted as follows:

$$(\wedge I') \quad \frac{\sigma M \quad \tau M \quad \sigma \wedge \tau \in \mathrm{RT}}{\sigma \wedge \tau M}.$$

(4) $\vdash^{\mathrm{R}}$ is the type assignment obtained by adding rule (Eq_β) to $\vdash^{\mathrm{R}'}$.

It is straightforward to verify (by induction on deductions) that if $\mathcal{D}$: $B \vdash^{\mathrm{R}} \tau M$, then each predicate of statement which occurs in $\mathcal{D}$ belongs to RT. In particular, $\tau \in \mathrm{RT}$.

It is easy to show that the subject reduction property holds for $\vdash^{\mathrm{R}'}$.

5.3. Lemma. (1) *If* $(\mu_1 \to \nu_1) \wedge \cdots \wedge (\mu_n \to \nu_n) \leqslant_{\mathrm{R}} \sigma \to \tau$ *and* $\tau \not\leqslant_{\mathrm{R}} \omega$, *then there are* $p_1, \ldots, p_q \in \{1, \ldots, n\}$ *such that, for* $1 \leqslant j \leqslant q$, $\mu_{p_j\,\mathrm{R}}\!\geqslant \sigma$ *and* $\nu_{p_1} \wedge \cdots \wedge \nu_{p_q} \leqslant_{\mathrm{R}} \tau$.

(2) *If* $\sigma \to \tau M$ *is derived from* $(\mu_1 \to \nu_1)M, \ldots, (\mu_n \to \nu_n)M$ *only by means of rules* $(\wedge I')$, $(\wedge E)$, *and* $(\leqslant_{\mathrm{R}})$, $\mu_l \to \nu_l \in \mathrm{RT}$ *for* $1 \leqslant l \leqslant n$ *and* $\tau \not\leqslant_{\mathrm{R}} \omega$, *then there are* $p_1, \ldots, p_q \in \{1, \ldots, n\}$ *such that, for* $1 \leqslant j \leqslant q$, $\mu_{p_j\,\mathrm{R}}\!\geqslant \sigma$ *and* $\nu_{p_1} \wedge \cdots \wedge \nu_{p_q} \leqslant_{\mathrm{R}} \tau$.

(3) $B \vdash^{\mathrm{R}'} \tau MN$, $\tau \not\leqslant_{\mathrm{R}} \omega \Rightarrow \exists \sigma \in \mathrm{RT}$:

$$[B \overset{\mathrm{R}'}{\vdash} \sigma \to \tau M \quad \textit{and} \quad B \overset{\mathrm{R}'}{\vdash} \sigma N].$$

(4) $B \vdash^{\mathrm{R}'} \sigma \to \tau \lambda y.M \Rightarrow B/y \cup \{\sigma y\} \vdash^{\mathrm{R}'} \tau M$.

(5) $B \vdash^{\mathrm{R}'} \tau M$, $M \twoheadrightarrow_\beta M' \Rightarrow B \vdash^{\mathrm{R}'} \tau M'$.

Proof. (1): By Lemma 4.3, there are $p_1, \ldots, p_q \in \{1, \ldots, n\}$ such that

$$\mu_{p_1} \wedge \cdots \wedge \mu_{p_q\,\mathrm{f}}\!\geqslant \sigma \quad \text{and} \quad \nu_{p_1} \wedge \cdots \wedge \nu_{p_q} \leqslant_{\mathrm{f}} \tau.$$

Notice that $\sigma \in \mathrm{RT}$ since $\sigma \to \tau \in \mathrm{RT}$ and $\nu_{p_1} \wedge \cdots \wedge \nu_{p_q} \in \mathrm{RT}$ since $(\mu_1 \to \nu_1) \wedge \cdots \wedge (\mu_n \to \nu_n) \in \mathrm{RT}$. However, $\mu_{p_1} \wedge \cdots \wedge \mu_{p_q} \in \mathrm{RT}$ can be false.

(2): By inductions on derivations one can show that if $(\sigma_1 \to \tau_1) \wedge \cdots \wedge (\sigma_m \to \tau_m)M$ is derived from $(\mu_1 \to \nu_1)M, \ldots, (\mu_n \to \nu_n)M$ only by means of rules $(\wedge I')$, $(\wedge E)$, and $(\leqslant_{\mathrm{R}})$, then, $\forall l$ $(1 \leqslant l \leqslant m)$ such that $\tau_l \not\leqslant_{\mathrm{R}} \omega$, there are $p_1, \ldots, p_q \in \{1, \ldots, n\}$ such that, for $1 \leqslant j \leqslant q$, $\mu_{p_j\,\mathrm{R}}\!\geqslant \sigma_l$ and $\nu_{p_1} \wedge \cdots \wedge \nu_{p_q} \leqslant_{\mathrm{R}} \tau_l$. If the last applied rule is $(\leqslant_{\mathrm{R}})$ use (1) and the induction hypothesis.

(3): By induction on derivations.

(4 and 5): Just mimic the proof of Lemma 4.4(1) and (2) $(\Rightarrow)$ using (2) and (3). □

Contrary to the subject reduction property, subject expansion fails for $\vdash^{\mathrm{R}'}$. For example,

$$\{((\varphi \to \varphi) \to \sigma \to \tau z, ((\varphi \to \psi) \to \varphi \to \psi)\} \to \sigma y \overset{\mathrm{R}'}{\vdash} \tau z\mathbf{I}(y\mathbf{I}),$$

while

$$\{((\varphi \to \varphi) \to \sigma \to \tau z, ((\varphi \to \psi) \to \varphi \to \psi)\} \to \sigma y \overset{\mathrm{R}'}{\nvdash} (\lambda t.zt(yt))\mathbf{I}.$$

So $\vdash^{\mathrm{R}'}$ is not F-complete. We will prove instead the F-completeness of $\vdash^{\mathrm{R}}$.

We will prove that when B is restricted and $\tau \in \mathrm{RT}$, the inference systems $\vdash^{\mathrm{f}}$ and $\vdash^{\mathrm{R}}$ have the same expressive power.

5.4. Lemma. *$\rho \leqslant_f \sigma_1 \to \cdots \to \sigma_m \to \tau$ and $\tau \not\leqslant_f \omega$ implies*

$$\rho = (\mu_1^{(1)} \to \cdots \to \mu_m^{(1)} \to \nu^{(1)}) \wedge \cdots \wedge (\mu_1^{(s)} \to \cdots \to \mu_m^{(s)} \to \nu^{(s)}) \wedge \rho'$$

for some $\mu_1^{(j)}, \ldots, \mu_m^{(j)}$, $\nu^{(j)}$ such that $\mu_i^{(j)} {}_f\!\geqslant \sigma_i$, with $(1 \leqslant i \leqslant m)$, $(1 \leqslant j \leqslant s)$, and $\nu^{(1)} \wedge \cdots \wedge \nu^{(s)} \leqslant_f \tau$.

Proof. By induction on m: Let $\tau \equiv \sigma_{m+1} \to \tau'$. By the induction hypothesis,

$$\rho = (\mu_1^{(1)} \to \cdots \to \mu_m^{(1)} \to \nu^{(1)}) \wedge \cdots \wedge (\mu_1^{(s)} \to \cdots \to \mu_m^{(s)} \to \nu^{(s)}) \wedge \rho'$$

for some $\mu_1^{(j)}, \ldots, \mu_m^{(j)}$, $\nu^{(j)}$ such that $\mu_i^{(j)} {}_f\!\geqslant \sigma_i$, with $(1 \leqslant i \leqslant m)$, $(1 \leqslant j \leqslant s)$, and $\nu^{(1)} \wedge \cdots \wedge \nu^{(s)} \leqslant_f \tau$. Without loss of generality, we can assume

$$\nu^{(1)} \wedge \cdots \wedge \nu^{(s)} = (\alpha_1 \to \beta_1) \wedge \cdots \wedge (\alpha_t \to \beta_t) \wedge \varphi_{k_1} \wedge \cdots \wedge \varphi_{k_r}.$$

By (P3f) as defined in the proof of Lemma 4.3, there exist $\{p_1, \ldots, p_q\} \subseteq \{1, \ldots, t\}$ such that

$$\alpha_{p_1} \wedge \cdots \wedge \alpha_{p_q} {}_f\!\geqslant \sigma_{m+1} \quad \text{and} \quad \beta_{p_1} \wedge \cdots \wedge \beta_{p_q} \leqslant_f \tau'.$$

Therefore, we can conclude (recall that $(\gamma \to \delta) \wedge (\gamma \to \delta') = \gamma \to \delta \wedge \delta'$):

$$\rho = (\mu_1^{(h(p_1))} \to \cdots \to \mu_m^{(h(p_1))} \to \alpha_{p_1} \to \beta_{p_1}) \wedge \cdots \wedge$$
$$(\mu_1^{(h(p_q))} \to \cdots \to \mu_m^{(h(p_q))} \to \alpha_{p_q} \to \beta_{p_q}) \wedge \rho'',$$

where $h : \{p_1, \ldots, p_q\} \to \{1, \ldots, s\}$ is defined by $h(p_l) = j$ iff $\nu^{(j)} = (\alpha_{p_l} \to \beta_{p_l}) \wedge \bar{\nu}^{(j)}$ for some $\bar{\nu}^{(j)}$. □

5.5. Theorem. *Let B be restricted and $\tau \in$ RT, then $B \vdash^f \tau M \Leftrightarrow B \vdash^R \tau M$.*

Proof. ($\Rightarrow$): Notice that $B \vdash^f \tau M$ implies $\exists A \in \mathscr{A}^*(M)$ such that $B \vdash^f \tau A$ by Theorem 4.13(1). We prove by induction on A that $B \vdash^R \tau A$. $A \equiv x A_1 \ldots A_m$ $(m \geqslant 0)$ and $\tau \not\leqslant_f \omega$. $B \vdash^f \tau x A_1 \ldots A_m$ implies $\exists \sigma_1, \ldots, \sigma_m$ such that $B \vdash^f \sigma_i A_i$ $(1 \leqslant i \leqslant m)$ and $B \vdash^f \sigma_1 \to \cdots \to \sigma_m \to \tau x$ by Lemma 1.8(1).

$$B \overset{f}{\vdash} \sigma_1 \to \cdots \to \sigma_m \to \tau x$$
$$\Rightarrow \exists \rho_1 x, \ldots, \rho_n x \in B : \rho_1 \wedge \cdots \wedge \rho_n \leqslant_f \sigma_1 \to \cdots \to \sigma_m \to \tau \quad \text{Lemma 1.7(2)}$$
$$\Rightarrow \rho_1 \wedge \cdots \wedge \rho_n = (\mu_1^{(1)} \to \cdots \to \mu_m^{(1)} \to \nu^{(1)}) \wedge \cdots \wedge (\mu_1^{(s)} \to \cdots \to \mu_m^{(s)} \to \nu^{(s)}) \wedge \rho'$$

for some $\mu_1^{(j)}, \ldots, \mu_m^{(j)}$, $\nu^{(j)}$ such that $\mu_i^{(j)} {}_f\!\geqslant \sigma_i$ $(1 \leqslant i \leqslant m)$ $(1 \leqslant j \leqslant s)$ and $\nu^{(1)} \wedge \cdots \wedge \nu^{(s)} \leqslant_f \tau$ by Lemma 5.4.

$$B \overset{f}{\vdash} \sigma_i A_i \Rightarrow B \overset{f}{\vdash} \mu_i^{(j)} A_i \quad \text{for } 1 \leqslant j \leqslant s \text{ by } (\leqslant_f)$$
$$\Rightarrow B \overset{R}{\vdash} \mu_i^{(j)} A_i \quad \text{for } 1 \leqslant j \leqslant s$$

by the induction hypothesis since $\mu_i^{(j)} \in$ RT. Therefore, we obtain a deduction of

$B \vdash^{\mathrm{R}} \tau A$ as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\rho_1 x \ldots \rho_n x}{\rho_1 \wedge \cdots \wedge \rho_n x}(\wedge I')}{\mu_1^{(j)} \to \cdots \to \mu_m^{(j)} \to \nu^{(j)} x}(\leq_{\mathrm{R}}) \quad \begin{matrix}\vdots \\ \mu_1^{(j)} A_1\end{matrix}}{\mu_2^{(j)} \to \cdots \to \mu_m^{(j)} \to \nu^{(j)} x A_1}(\to E)}{\vdots}}{\mu_m^{(j)} \to \nu^{(j)} x A_1 \ldots A_{m-1}} \quad \begin{matrix}\vdots \\ \mu_m^{(j)} A_m\end{matrix}}{\nu^{(j)} x A_1 \ldots A_m}(\to E)}{\nu^{(1)} \wedge \cdots \wedge \nu^{(s)} x A_1 \ldots A_m}(\wedge I')}{\tau x A_1 \ldots A_m}(\leq_{\mathrm{R}}).$$

Notice that $\rho_1 \wedge \cdots \wedge \rho_n$, $\nu^{(1)} \wedge \cdots \wedge \nu^{(s)} \in \mathrm{RT}$ since B is a restricted basis.

$A \equiv \lambda y.A'$. Notice that $B \vdash^{\mathrm{f}} \mu \wedge \nu A \Rightarrow B \vdash^{\mathrm{f}} \mu A$ and $B \vdash^{\mathrm{R}} \mu A$, $B \vdash^{\mathrm{R}} \nu A \Rightarrow B \vdash^{\mathrm{R}} \mu \wedge \nu A$ using $(\wedge I')$ since $\mu \wedge \nu \in \mathrm{RT}$. So we can restrict our attention to the case $B \vdash^{\mathrm{f}} \sigma \to \rho \lambda y.A'$.

$$B \overset{\mathrm{f}}{\vdash} \sigma \to \rho \lambda y.A' \Rightarrow B/y \cup \{\sigma y\} \overset{\mathrm{f}}{\vdash} \rho A' \quad \text{by Lemma 4.4(1)}$$

$$\Rightarrow B \overset{\mathrm{R}}{\vdash} \sigma \to \rho \lambda y.A' \quad \text{by the induction hypothesis}$$

(notice that $B/y \cup \{\sigma y\}$ is a restricted basis and $\rho \in \mathrm{RT}$)

$$\Rightarrow B \overset{\mathrm{R}}{\vdash} \sigma \to \rho \lambda y.A' \quad \text{by } (\to I).$$

Now, from $B \vdash^{\mathrm{R}} \tau A$ with $A \in \mathscr{A}^*(M)$, we have $B \vdash^{\mathrm{R}} \tau M$. Just mimic the argument given after Definition 3.3 (notice that (Eq_β) is a rule of $\vdash^{\mathrm{R}}$).

($\Leftarrow$): Immediate since (Eq_β) is a derived rule for $\vdash^{\mathrm{f}}$. □

As a consequence of Theorem 5.5 we have the Head Normal Form and the Normal Form Theorems for the system $\vdash^{\mathrm{R}}$.

5.6. Theorem. (1) $\exists B, \tau \in \mathrm{TT}$: $[B \vdash^{\mathrm{R}} \tau M] \Leftrightarrow M$ *has an h.n.f.*

(2) $\exists B, \tau$: $[B \vdash^{\mathrm{R}} \tau M$ *and* ω *not in* $B, \tau] \Leftrightarrow M$ *has an n.f.*

Proof. (1) ($\Rightarrow$): By Theorems 5.5 and 4.16.

($\Leftarrow$): The proof of Theorem 4.16 ($\Leftarrow$) remains valid.

(2) ($\Rightarrow$): By Theorems 5.5 and 4.17.

($\Leftarrow$): Let $\| \ \|: \Lambda \to \mathbb{N}$ be defined by

(i) $\|z\| = 1$,

(ii) $\|MN\| = \max(\|M\| + 1, \|N\|)$,

(iii) $\|\lambda z.M\| = \|M\|$.

Clearly, if M is an n.f., $\|M\|$ is the maximum number of components of the subterms of M. We will prove by structural induction on the n.f. M that $\forall n \geq \|M\|: \exists B, \tau$ such that

(i) $B \vdash^{R} \tau M$;
(ii) ω not in B, τ;
(iii) $\rho y \in B$ implies $\#(\rho) = n$.

$M \equiv \lambda z.M'$ is trivially proved.

$M \equiv zM_1 \ldots M_m$ $(m \geq 0)$. By the induction hypothesis there are B_i, σ_i such that $B_i \vdash^{R} \sigma_i M_i$, ω not in B_i, σ_i and $\rho y \in B_i$ implies $\#(\rho) = n$. Notice that if $n \geq \|M\|$, then $n \geq m$. It is easy to verify that a correct choice is $\tau \equiv \varphi_{m+1} \to \cdots \to \varphi_n \to \varphi$ and

$$B = B_1 \cup \cdots \cup B_m \cup \{\sigma_1 \to \cdots \to \sigma_m \to \tau z\}. \qquad \square$$

The F-soundness of $\vdash^{R}$ follows from Theorem 5.5 and Lemma 2.4(4).

5.7. Theorem (F-soundness).

$$B \vdash^{R} \tau M \Rightarrow B \models \tau M.$$

To prove the F-completeness of $\vdash^{R}$ we use $\langle |\mathcal{T}_f|, \cdot, [\![\ ]\!]^{f} \rangle$ which we know from Lemma 4.5 to be an F-filter λ-model. Therefore, we interpret the types belonging to RT as subsets of $|\mathcal{T}_f|$. Notice that $\langle |\mathcal{T}_R|, \cdot, [\![\ ]\!]^{R} \rangle$ is not an F-filter λ-model since $\varphi \to \varphi$, $(\varphi \to \psi) \to \varphi \to \psi \in [\![\lambda x.x]\!]^{R}_{\xi}$, but $(\varphi \to \varphi) \wedge ((\varphi \to \psi) \to \varphi \to \psi) \notin \mathrm{RT}$.

5.8. Definition. (1) $R(\tau) = \{d \in |\mathcal{T}_f| \mid \tau \in d\}$, where $\tau \in \mathrm{RT}$.

(2) $\mathcal{V}_R(\varphi) = R(\varphi)$.

(3) The set $\mathrm{FT} \subseteq T$ of *functional types* is defined by
  (i) $\omega \in \mathrm{FT}$,
  (ii) $\sigma, \tau \in T \Rightarrow \sigma \to \tau \in \mathrm{FT}$,
  (iii) $\sigma, \tau \in \mathrm{FT} \Rightarrow \sigma \wedge \tau \in \mathrm{FT}$.

(4) The set $\mathrm{AT} \subseteq T$ of *argument types* is defined by
  (i) $\varphi_0, \varphi_1, \ldots \in \mathrm{AT}$,
  (ii) $\omega \in \mathrm{AT}$,
  (iii) $\sigma, \tau \in \mathrm{AT} \Rightarrow \sigma \wedge \tau \in \mathrm{AT}$.

Notice that $\mathrm{RT} \subsetneq \mathrm{FT} \cup \mathrm{AT} \subsetneq T$ since, for example, $(\varphi \wedge \varphi \to \psi) \to \psi \in \mathrm{FT}$, but $\notin \mathrm{RT}$ and $\varphi \wedge \varphi \to \psi \in T$, but $\notin \mathrm{FT} \cup \mathrm{AT}$.

5.9. Lemma. (1) $\sigma \in \mathrm{FT}$, $\sigma \leq_f \tau \Rightarrow \tau \in \mathrm{FT}$.

(2) *Let* $S = B \upharpoonright y$. *If* ${}^{f}\!\uparrow S \neq {}^{f}\!\uparrow \omega$ *and* $S \subseteq \mathrm{FT}$ *then, for all* τ, $B \vdash^{f} \tau y \Leftrightarrow B \vdash^{f} \tau \lambda z.yz$.

(3) ${}^{f}\!\uparrow S \neq {}^{f}\!\uparrow \omega$ *and* $S \subseteq \mathrm{FT}$ *imply* ${}^{f}\!\uparrow S \in F_f$.

(4) *S being consistent implies* $\forall c \in |\mathcal{T}_f|: \exists U$ *consistent such that* ${}^{f}\!\uparrow S.c = {}^{f}\!\uparrow U$.

Proof. (1) is proved by induction on $\leqslant_{\mathrm{f}}$.

(2) ($\Rightarrow$): $B \vdash^{\mathrm{f}} \tau y \Rightarrow \exists \sigma_1 x, \ldots, \sigma_m x \in B$ such that $\sigma_1 \wedge \cdots \wedge \sigma_m \leqslant_{\mathrm{f}} \tau$ by Lemma 1.7(2) implying that $\tau \in \mathrm{FT}$ by (1) since $\sigma_1 \wedge \cdots \wedge \sigma_m \in \mathrm{FT}$ by hypothesis.

The proof of $B \vdash^{\mathrm{f}} \tau \lambda z.yz$ by induction on τ is straightforward.

($\Leftarrow$): We will prove $B \vdash^{\mathrm{f}} \tau\lambda z.yz \Rightarrow B \vdash^{\mathrm{f}} \tau y$ by induction on τ.

Case 1: $\tau \equiv \sigma \to \rho$, $\rho \sim_{\mathrm{f}} \omega$. By hypothesis, there is a $\mu y \in B$ such that $\mu \not\sim_{\mathrm{f}} \omega$ and $\mu \in \mathrm{FT}$ which imply that $\mu \leqslant_{\mathrm{f}} \sigma \to \rho$ since $\rho \sim_{\mathrm{f}} \omega$.

Case 2: $\tau \equiv \sigma \to \rho$, $\rho \not\sim_{\mathrm{f}} \omega$. $B \vdash^{\mathrm{f}} \sigma \to \rho\lambda z.yz$ implies $\exists \nu$: $B/z \cup \{\sigma z\} \vdash^{\mathrm{f}} \nu z$ and $B/z \cup \{\sigma z\} \vdash^{\mathrm{f}} \nu \to \rho y$ by Lemmas 4.4(1) and 1.8(1). Hence, $\exists \nu$: $\sigma \leqslant_{\mathrm{f}} \nu$ and $B/z \cup \{\sigma z\} \vdash^{\mathrm{f}} \nu \to \rho y$ by Lemma 1.7(2). And thus, $B \vdash^{\mathrm{f}} \tau y$ by ($\leqslant_{\mathrm{f}}$) and Lemma 1.8(3).

(3): $S \subseteq \mathrm{FT} \Rightarrow {}^{\mathrm{f}}\!\uparrow S \subseteq \mathrm{FT}$ by (1). Let $d = {}^{\mathrm{f}}\!\uparrow S$. Then

$$\begin{aligned} d = [\![y]\!]^{\mathrm{f}}_{\xi[y/d]} &= \{\sigma \mid B_{\xi[y/d]} \overset{\mathrm{f}}{\vdash} \sigma y\} && \text{by Definition 1.9(3)} \\ &= \{\sigma \mid B_{\xi[y/d]} \overset{\mathrm{f}}{\vdash} \sigma\lambda z.yz\} && \text{by (2)} \\ &= [\![\lambda z.yz]\!]^{\mathrm{f}}_{\xi[y/d]} && \text{by Definition 1.9(3).} \end{aligned}$$

So we conclude $d \in F_{\mathrm{f}}$ by Lemma 2.6(3).

(4): It is easy to verify using Lemma 4.3 that a correct choice is

$$U = \{\tau \mid \tau \equiv \omega \text{ or } \exists \sigma \in c\colon \sigma \to \tau \in S\}. \qquad \square$$

5.10. Lemma. (1) *$\mathcal{V}_{\mathrm{R}}(\tau) = R(\tau)$ for all $\tau \in \mathrm{AT}$.*

(2) *$\mathcal{V}_{\mathrm{R}}(\tau) \subseteq R(\tau)$ for all $\tau \in \mathrm{RT}$.*

(3) *S consistent and $\tau \in S$ imply ${}^{\mathrm{f}}\!\uparrow S \in \mathcal{V}_{\mathrm{R}}(\tau)$.*

(4) *$\mathfrak{M}_{\mathrm{f}}$, ξ^{f}_B, $\mathcal{V}_{\mathrm{R}} \models B$ for every restricted basis B.*

Proof. (1): By cases on τ. The only interesting case is $\tau \equiv \varphi_1 \wedge \cdots \wedge \varphi_n$.

$$\begin{aligned} \mathcal{V}_{\mathrm{R}}(\tau) &= \mathcal{V}_{\mathrm{R}}(\varphi_1) \cap \cdots \cap \mathcal{V}_{\mathrm{R}}(\varphi_n) \\ &= R(\varphi_1) \cap \cdots \cap R(\varphi_n) && \text{by Definition 5.8(2)} \\ &= \{d \in |\mathcal{T}_{\mathrm{f}}| \mid \varphi_1, \ldots, \varphi_n \in d\} && \text{by Definition 5.8(1)} \\ &= R(\tau). \end{aligned}$$

(2 and 3): Simultaneously by induction on τ. The only interesting case is $\tau \equiv \sigma \to \rho$. We first prove (2)

$$\begin{aligned} d \in \mathcal{V}_{\mathrm{R}}(\tau) &\Rightarrow \forall c \in \mathcal{V}_{\mathrm{R}}(\sigma)\colon d.c \in \mathcal{V}_{\mathrm{R}}(\rho) \\ &\Rightarrow d.{}^{\mathrm{f}}\!\uparrow\sigma \in \mathcal{V}_{\mathrm{R}}(\rho) \end{aligned}$$

since ${}^{\mathrm{f}}\!\uparrow\sigma \in \mathcal{V}_{\mathrm{R}}(\sigma)$ by the induction hypothesis

$$\Rightarrow \rho \in d.{}^{\mathrm{f}}\!\uparrow\sigma$$

since $\mathcal{V}_{\mathrm{R}}(\rho) \subseteq R(\rho)$ by the induction hypothesis. Hence, $\tau \in d$ by definition of ".".

For (3), recall that $\mathcal{V}_{\mathrm{R}}(\sigma \to \rho) = \{d \in F_{\mathrm{f}} \mid \forall c \in \mathcal{V}_{\mathrm{R}}(\sigma){:}\ d.c \in \mathcal{V}_{\mathrm{R}}(\rho)\}$. By Lemma 5.9(3), ${}^{\mathrm{f}}\!\uparrow S \in F_{\mathrm{f}}$ since $\tau \in S$ and S consistent imply ${}^{\mathrm{f}}\!\uparrow S \neq {}^{\mathrm{f}}\!\uparrow\omega$ and $S \subseteq \mathrm{FT}$.

If $c \in \mathcal{V}_{\mathrm{R}}(\sigma)$, then $\sigma \in c$ (by the induction hypothesis) which implies $\rho \in {}^{\mathrm{f}}\!\uparrow S.c$ since $\sigma \to \rho \in S$. By Lemma 5.9(4), $\exists U$ consistent such that ${}^{\mathrm{f}}\!\uparrow S.c = {}^{\mathrm{f}}\!\uparrow U$. Therefore, by the induction hypothesis, ${}^{\mathrm{f}}\!\uparrow S.c \in \mathcal{V}_{\mathrm{R}}(\rho)$, so we conclude ${}^{\mathrm{f}}\!\uparrow S \in \mathcal{V}_{\mathrm{R}}(\tau)$.

(4): Let $S = \{\tau \mid \tau y \in B \text{ or } \tau \equiv \omega\}$. By Lemma 1.7(2) and Definition 1.9(3), $[\![y]\!]^{\mathrm{f}}_{\xi^{\mathrm{f}}_B} = {}^{\mathrm{f}}\!\uparrow S$. S is consistent since B is a restricted basis. Therefore, $[\![y]\!]^{\mathrm{f}}_{\xi^{\mathrm{f}}_B} \in \mathcal{V}_{\mathrm{R}}(\tau)$ for all $\tau \in S$ by (3). □

5.11. Theorem (*F*-completeness). *Let B be restricted and* $\tau \in \mathrm{RT}$. $B \models \tau M \Rightarrow B \vdash^{\mathrm{R}} \tau M$.

Proof

$$\begin{aligned}
B \models \tau M &\Rightarrow \mathfrak{M}_{\mathrm{f}}, \xi^{\mathrm{f}}_B, \mathcal{V}_{\mathrm{R}} \models \tau M && \text{by Lemma 5.10(4)}\\
&\Rightarrow [\![M]\!]^{\mathrm{f}}_{\xi^{\mathrm{f}}_B} \in \mathcal{V}_{\mathrm{R}}(\tau) && \text{by Definition 2.2(1)}\\
&\Rightarrow \tau \in [\![M]\!]^{\mathrm{f}}_{\xi^{\mathrm{f}}_B} && \text{by Lemma 5.10(2)}\\
&\Rightarrow B_{\xi^{\mathrm{f}}_B} \overset{\mathrm{f}}{\vdash} \tau M && \text{by Definition 1.9(3)}\\
&\Rightarrow B \overset{\mathrm{f}}{\vdash} \tau M && \text{by Lemma 1.10(2)}\\
&\Rightarrow B \overset{\mathrm{R}}{\vdash} \tau M && \text{by Theorem 5.5.} \quad \square
\end{aligned}$$

Conclusion

The present paper has not been intended to be a final answer to the problem of finding a type system of the intersection type discipline which is complete for the *F*-semantics. We simply propose three natural answers to this problem, i.e., the inference systems $\vdash^{\mathrm{i}}$, $\vdash^{\mathrm{s}}$, and $\vdash^{\mathrm{R}}$.

Lastly, we mention that Hindley [24] has proposed another semantics of types, which takes into account the meaning of $F^{(j)} \subseteq D$ for $j \geq 0$ as defined in Definition 2.5. More precisely, each $F^{(j)}$ is the set of elements which represents j-place functions, and therefore, Hindley defines the valuation of $\sigma_1 \to \cdots \to \sigma_j \to \varphi$ for all types $\sigma_1, \ldots, \sigma_j$ and type variable φ as a subset of $F^{(j)}$:

$$\begin{aligned}
&\mathcal{V}(\sigma_1 \to \cdots \to \sigma_j \to \varphi)\\
&\quad = \{d \in F^{(j)} \mid \forall c_1 \in \mathcal{V}(\sigma_1), \ldots, \forall c_j \in \mathcal{V}(\sigma_j){:}\ d.c_1.\ldots.c_j \in \mathcal{V}(\varphi)\}.
\end{aligned}$$

As noted by Coppo, the problem with this semantics is that also Curry's system becomes not sound since, for example, we have $\vdash (\varphi \to \psi) \to \varphi \to \psi \lambda y.y$, but clearly, $[\![\lambda y.y]\!]_\xi^{\mathfrak{M}} \notin F^{(2)}$ for all $\mathfrak{M}$, ξ. The same example shows that typings are not preserved by substitution since $\models \varphi \to \varphi \lambda y.y$, but $\not\models (\varphi \to \psi) \to \varphi \to \psi \lambda y.y$.

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