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Note

A Note on Generalizations of Hölder Inequalities via Convex and Concave Functions

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The Hölder inequality [2] can be presented as follows: For every positive vector $\mathbf{x} = (x_1, x_2, ..., x_n)$ and $\mathbf{y} = (y_1, y_2, ..., y_n)$ satisfying

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i = 1 \tag{1}$$

and for any real p, q with $p > 1$ the Hölder inequality is

$$\sum_{i=1}^{n} x_i^{1/p} y_i^{1/q} \leqslant 1, \qquad p^{-1} + q^{-1} = 1 \tag{2}$$

with equality iff

$$x_i = y_i, \qquad i = 1, ..., n. \tag{3}$$

The sign of the inequality in (2) is reversed if $0 < p < 1$ or $p < 0$.

To establish an upper bound c lower than 1 for $\sum_{i=1}^{n} x_i^{1/p} y_i^{1/q}$ in case $p > 1$, and a lower bound $c > 1$ in case $0 < p < 1$ or $p < 0$, we impose additional constraints on the vectors $\mathbf{x}$ and $\mathbf{y}$; we will first prove a theorem concerning a concave (convex) function.

Let us denote the increasing rearrangement of a vector $\mathbf{x}$ by $\mathbf{x}_{\uparrow} = (x_{(1)}, x_{(2)}, ..., x_{(n)})$ and the decreasing rearrangement by $\mathbf{x}_{\downarrow} = (x_{[1]}, x_{[2]}, ..., x_{[n]})$. See [3].

Our first theorem is as follows.

THEOREM 1. *Let $f(x) \in C'$ be a concave function in $0 < a \leqslant x$ and let the vectors $0 < \mathbf{x}$, $0 < \mathbf{d}$, and $0 < \mathbf{y}$ satisfy*

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} d_i = 1. \tag{4}$$

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Let

$$\frac{x_i}{y_i} \geqslant \frac{d_i}{y_i} \geqslant \frac{d_j}{y_j} \geqslant \frac{x_j}{y_j}, \qquad i = 1, ..., m, \; j = m+1, ..., n; \tag{5}$$

then

$$\sum_{i=1}^{n} y_i f\left(\frac{x_i}{y_i}\right) \leqslant \sum_{i=1}^{n} y_i f\left(\frac{d_i}{y_i}\right) \leqslant \sum_{i=1}^{n} y_{(i)} f\left(\frac{d_{(i)}}{y_{(i)}}\right). \tag{6}$$

If $f(x)$ is convex the inequalities in (6) *are reversed.*

When we choose $f(x) = x^{1/p}$ we get the following:

COROLLARY. *Let $0 < \mathbf{x}$, $0 < \mathbf{y}$, and $0 < \mathbf{d}$ satisfy conditions* (4) *and* (5). *Then*

$$\sum_{i=1}^{n} x_i^{1/p} y_i^{1/q} \leqslant \sum_{i=1}^{n} d_{(i)}^{1/p} y_{(i)}^{1/q}, \qquad p > 1 \tag{7}$$

and

$$\sum_{i=1}^{n} x_i^{1/p} y_i^{1/q} \geqslant \sum_{i=1}^{n} d_{(i)}^{1/p} y_{(i)}^{1/q}, \qquad 0 < p < 1 \text{ or } p < 0. \tag{7'}$$

To prove Theorem 1 we use the following theorems.

THEOREM B [1]. *Let $F(x)$, $G(x)$, and $M(x)$ be integrable functions over a measurable set A. Let*

$$A_1 = \{x : F(x) \leqslant G(x)\}, \qquad A_2 = \{x : F(x) > G(x)\}$$

and suppose that

$$\int_A G(x)\, dx \leqslant \int_A F(x)\, dx$$

and either of the conditions

$$0 \leqslant M(x_1) \leqslant M(x_2) \tag{8}$$

$$M(x_1) \leqslant 0 \leqslant M(x_2) \tag{9}$$

is satisfied for every pair x_1, x_2 of points such that $x_1 \in A_1$, $x_2 \in A_2$. Then

$$\int_A G(x)\, M(x)\, dx \leqslant \int_A F(x)\, M(x)\, dx. \tag{10}$$

We also need the following definition [3, p. 150]:

Let $G(x, y)$ have first partial derivatives. If $(\partial G/\partial x)(x, y)$ is increasing in y, $G(x, y)$ is called L-superadditive.

We need the following theorem too.

THEOREM L [3, p. 156]. *For all vectors* $\mathbf{x}$ *and* $\mathbf{y}$ *in* R^n *and all* n

$$\sum_{i=1}^{n} G(x_{(i)}, y_{[i]}) \leqslant \sum_{i=1}^{n} G(x_i, y_i) \leqslant \sum_{i=1}^{n} G(x_{(i)}, y_{(i)}) \qquad \text{iff } G(x, y) \text{ is } L\text{-superadditive.}$$

Proof of Theorem 1. Let us consider the function

$$H(t) = \sum_{i=1}^{n} y_i f\left(\frac{(1-t)x_i + td_i}{y_i}\right), \qquad 0 \leqslant t \leqslant 1$$

when x_i, d_i, y_i for $i = 1, ..., n$ are as defined in the theorem. Hence,

$$\frac{dH(t)}{dt} = \sum_{i=1}^{n} (d_i - x_i) f'\left(\frac{(1-t)x_i + td_i}{y_i}\right) = \sum_{i=1}^{m} (d_i - x_i) f'\left(\frac{(1-t)x_i + td_i}{y_i}\right) + \sum_{j=m+1}^{n} (d_j - x_j) f'\left(\frac{(1-t)x_j + td_j}{y_j}\right).$$

It follows from (5) that

$$\frac{(1-t)x_i + td_i}{y_i} \geqslant \frac{(1-t)x_j + td_j}{y_j}, \qquad 0 \leqslant t \leqslant 1 \quad \text{for} \quad i = 1, ..., m, j = m+1, ..., n.$$

f' is decreasing therefore

$$f'\left(\frac{(1-t)x_i + td_i}{y_i}\right) \leqslant f'\left(\frac{(1-t)x_j + td_j}{y_j}\right) \qquad \text{for} \quad i = 1, ..., m, j = m+1, ..., n.$$

Let us assume that $f'(x) > 0$ because otherwise we will replace $f(x)$ by $g(x) = f(x) + kx$, where k is chosen to satisfy $g'(x) > 0$ in the interval we are interested in, and the inequalities (6) for $g(x)$ generate immediately (6) for $f(x)$.

We will use Theorem B as follows.

Define

$$F(x) = d_i, \qquad i-1 \leqslant x < i$$

$$G(x) = x_i, \qquad i-1 \leqslant x < i$$

$$M(x) = f'\left(\frac{(1-t)x_i + td_i}{y_i}\right), \qquad i-1 \leqslant x < i \qquad \text{for} \quad i = 1, ..., n.$$

$$A_1 = \{x : 0 \leqslant x \leqslant m\}, \qquad A_2 = \{x : m < x \leqslant n\}.$$

Applying Theorem B to these functions and domains we get that $dH(t)/dt \geqslant 0$ and in particular $H(0) \leqslant H(1)$ which means that

$$\sum_{i=1}^{n} y_i f\left(\frac{x_i}{y_i}\right) \leqslant \sum_{i=1}^{n} y_i f\left(\frac{d_i}{y_i}\right). \tag{11}$$

Let us now note that when $f(x)$ is concave, the function $G(x, y) = yf(x/y)$ is L-superadditive. Hence applying Theorem L we get

$$\sum_{i=1}^{n} y_i f\left(\frac{d_i}{y_i}\right) \leqslant \sum_{i=1}^{n} y_{(i)} f\left(\frac{d_{(i)}}{y_{(i)}}\right). \tag{12}$$

Equations (11) and (12) give (6) and Theorem 1 is proven. ∎

The following Theorems 2 and 3 will show us how Theorem 1 can be used to derive lower upper bound $c < 1$ for $\sum_{i=1}^{n} x_i^{1/p} y_i^{1/q}$ when $p > 1$ and higher lower bound $c > 1$ when $0 < p < 1$ or $p < 0$ when we impose some constraints on $\mathbf{x}$ and $\mathbf{y}$. In proving these therems we use the following theorems:

THEOREM III [4, p. 165]. *Let $p_1, p_2, ..., p_n$ be arbitrary real numbers. If $\mathbf{x} = \mathbf{x}_\uparrow$ and $\mathbf{y} = \mathbf{y}_\uparrow$ and if*

$$\sum_{i=1}^{k} p_i x_i \leqslant \sum_{i=1}^{k} p_i y_i, \qquad k = 1, 2, ..., n-1$$

$$\sum_{i=1}^{n} p_i x_i = \sum_{i=1}^{n} p_i y_i$$

then inequality

$$\sum_{i=1}^{n} p_i f(x_i) \leqslant \sum_{i=1}^{n} p_i f(y_i)$$

holds for every concave function f.

THEOREM 2. *Let $d>0$ and $p_i \geqslant 1$, $i=1, ..., n-1$. Let $\mathbf{y}>0$ be a given vector satisfying*

$$y_{(1)} < d, \qquad y_{(i+1)} \geqslant p_i y_{(i)} \qquad \text{for} \quad i=1, 2, ..., n-1$$
$$\sum_{i=1}^{n} y_i = 1. \tag{13}$$

Suppose that the set A of vectors $\mathbf{x}>0$ satisfying

$$x_{(1)} \geqslant d, \qquad x_{(i+1)} \geqslant p_i x_{(i)} \qquad \text{for} \quad i=1, 2, ..., n-1$$
$$\sum_{i=1}^{n} x_i = 1 \tag{14}$$

is non-empty. Then there is a vector $0<\mathbf{d}=(d_1, ..., d_n)$ in A such that

$$d_1 = d$$
$$d_i = p_{i-1} d_{i-1} \qquad \text{for} \quad i=2, ..., m$$
$$d_i = k y_{(i)} \qquad \text{for} \quad i=m+1, ..., n$$

when m and k satisfy

$$\frac{d_m}{y_{(m)}} > \frac{1-\sum_{i=1}^{m} d_i}{1-\sum_{i=1}^{m} y_{(i)}} \equiv k \geqslant \frac{d_m p_m}{y_{(m+1)}}.$$

The vector $0<\mathbf{d}^*$ defined by

$$d_1^* = d$$
$$d_i^* = p_{i-1} d_{i-1}^* \qquad \text{for} \quad i=2, ..., n-1$$
$$d_n^* = 1 - \sum_{i=1}^{n-1} d_i^*$$

belongs to A. For these $\mathbf{d}$ and $\mathbf{d}^*$ the following inequalities hold:

$$\sum_{i=1}^{n} y_{[i]} f\left(\frac{d_i^*}{y_{[i]}}\right) \leqslant \sum_{i=1}^{n} y_i f\left(\frac{x_i}{y_i}\right) \leqslant \sum_{i=1}^{n} y_{(i)} f\left(\frac{d_i}{y_{(i)}}\right) \tag{15}$$

for every $\mathbf{x} \in A$ and every concave function f. In particular

$$\sum_{i=1}^{n} d_i^{*1/p} y_{[i]}^{1/q} \leqslant \sum_{i=1}^{n} x_i^{1/p} y_i^{1/q}$$
$$\leqslant \sum_{i=1}^{n} d_i^{1/p} y_{(i)}^{1/q} < 1 \qquad \text{when} \quad p>1. \tag{16}$$

These inequalities are reversed when $0<p<1$ or $p<0$.

Proof. Let us prove first the existence of $\mathbf{d}$.

Define $\mathbf{d}' = (d_1', \ldots, d_n')$ by

$$d_1' = d, \qquad d_{i+1}' = p_i d_i' \qquad \text{for} \quad i = 1, 2, \ldots, n-1.$$

Then by (14)

$$\sum_{i=1}^{n} d_i' \leqslant \sum_{i=1}^{n} x_i = 1,$$

hence

$$d_n' \leqslant 1 - \sum_{i=1}^{n-1} d_i'.$$

Since $y_{(1)} < d = d_1'$, we have

$$\frac{1 - d_1'}{1 - y_{(1)}} < \frac{d_1'}{y_{(1)}}.$$

Therefore we can determine m $(<n)$ as the largest natural number m for which

$$\frac{1 - \sum_{i=1}^{m} d_i'}{1 - \sum_{i=1}^{m} y_{(i)}} < \frac{d_m'}{y_{(m)}}.$$

We claim

$$\frac{1 - \sum_{i=1}^{m} d_i'}{1 - \sum_{i=1}^{m} y_{(i)}} \equiv k \geqslant \frac{d_{m+1}'}{y_{(m+1)}}.$$

In fact, otherwise we have

$$\frac{1 - \sum_{i=1}^{m+1} d_1'}{1 - \sum_{i=1}^{m+1} y_{(i)}} < \frac{d_{m+1}'}{y_{(m+1)}},$$

contradicting the maximum property of m. These m and k meet the requirement for the definition of $\mathbf{d}$ in A.

It is immediately seen that

$$\frac{x_{(i)}}{y_{(i)}} \geqslant \frac{d_i}{y_{(i)}} > k = \frac{d_j}{y_{(j)}} \geqslant \frac{x_{(j)}}{y_{(j)}} \qquad \text{for} \quad i = 1, \ldots, m \text{ and } j = m+1, \ldots, n.$$

With a suitable permutation π of $\{1, 2, \ldots, n\}$ such that $\pi_1 = i$ for $i = 1, 2, \ldots, m$, we can see that for some m' $(\geqslant m)$

$$\frac{x_{(\pi_i)}}{y_{(\pi_i)}} \geqslant \frac{d_{(\pi_i)}}{y_{(\pi_i)}} \geqslant \frac{d_{(\pi_j)}}{y_{(\pi_j)}} \geqslant \frac{x_{(\pi_j)}}{y_{(\pi_j)}} \qquad \text{for} \quad i = 1, \ldots, m' \text{ and } j = m'+1, \ldots, n.$$

Therefore (4) and (5) satisfied for $(x_{(\pi_i)}, ..., x_{(\pi_n)})$, $(y_{(\pi_i)}, ..., y_{(\pi_n)})$, and $(d_{\pi_i}, ..., d_{\pi_n})$. Since $yf(x/y)$ is L-superadditive, we have by Theorem 1

$$\sum_{i=1}^{n} y_i f\left(\frac{x_i}{y_i}\right) \leqslant \sum_{i=1}^{n} y_{(i)} f\left(\frac{x_{(i)}}{y_{(i)}}\right) = \sum_{i=1}^{n} y_{(\pi_i)} f\left(\frac{x_{(\pi_i)}}{y_{(\pi_i)}}\right)$$
$$\leqslant \sum_{i=1}^{n} y_{(\pi_i)} f\left(\frac{d_{\pi_i}}{y_{(\pi_i)}}\right) = \sum_{i=1}^{n} y_{(i)} f\left(\frac{d_i}{y_{(i)}}\right).$$

This proves the right inequalities of (15) and (16).

The vector $\mathbf{d}^*$ obviously belongs to A. Since $yf(x/y)$ is L-superadditive,

$$\sum_{i=1}^{n} y_{[i]} f\left(\frac{x_{(i)}}{y_{[i]}}\right) \leqslant \sum_{i=1}^{n} y_i f\left(\frac{x_i}{y_i}\right).$$

Since

$$\frac{x_{(i)}}{y_{[i]}} \leqslant \frac{x_{(i+1)}}{y_{[i+1]}}, \qquad \frac{d_i^*}{y_{[i]}} \leqslant \frac{d_{i+1}^*}{y_{[i+1]}}, \qquad i = 1, ..., n-1$$

and

$$\sum_{i=1}^{k} y_{[i]} \frac{d_i^*}{y_{[i]}} = \sum_{i=1}^{k} d_i^*$$
$$\leqslant \sum_{i=1}^{k} x_{(i)} = \sum_{i=1}^{k} y_{[i]} \frac{x_{(i)}}{y_{[i]}}$$

(with equality for $k = n$) for $k = 1, 2, ..., n$, inequality

$$\sum_{i=1}^{n} y_{[i]} f\left(\frac{d_i^*}{y_{[i]}}\right) \leqslant \sum_{i=1}^{n} y_{[i]} f\left(\frac{x_{(i)}}{y_{[i]}}\right)$$

follows immediately from Theorem III. This proves the left inequalities of (15) and (16). ▌

In a similar way one can prove another example of the use of Theorem 1.

THEOREM 3. *Let $\eta > 0$ and let $\mathbf{y} > 0$ be a given vector satisfying*

$$y_{(1)} < d, \qquad y_{(1)} \leqslant y_{(2)} - \eta, \qquad \sum_{i=1}^{n} y_i = 1.$$

Suppose that the set A of vectors $\mathbf{x} > 0$ satisfying

$$x_{(1)} \geqslant d, \qquad x_{(1)} \leqslant x_{(2)} - \eta, \qquad \sum_{i=1}^{n} x_i = 1$$

is non-empty. Then there is a vector $0 < \mathbf{d} = (d_1, ..., d_n)$ *in* A *such that*

$$d_1 = d$$
$$d_i = d_1 + \eta \qquad \text{for} \quad i = 2, ..., m$$
$$d_i = k y_{(i)} \qquad \text{for} \quad i = m+1, ..., n$$

when m *and* k *satisfy*

$$\frac{d+\eta}{y_{(m)}} > \frac{1 - md_i}{1 - \sum_{i=1}^{m} y_{(i)}} \equiv k \geqslant \frac{d+\eta}{y_{(m+1)}}.$$

The vector $0 < \mathbf{d}^*$ *defined by*

$$d_1^* = d$$
$$d_i^* = d_1^* + \eta \qquad \text{for} \quad i = 2, ..., n-1$$
$$d_n^* = 1 - \sum_{i=1}^{n-1} d_i^*$$

belongs to A*. For these* $\mathbf{d}$ *and* $\mathbf{d}^*$ *the following inequalities hold:*

$$\sum_{i=1}^{n} y_{[i]} f\left(\frac{d_i^*}{y_{[i]}}\right) \leqslant \sum_{i=1}^{n} y_i f\left(\frac{x_i}{y_i}\right) \leqslant \sum_{i=1}^{n} y_{(i)} f\left(\frac{d_i}{y_{(i)}}\right)$$

for every $\mathbf{x} \in A$ *and every concave function* f*. In particular*

$$\sum_{i=1}^{n} d_i^{*\,1/p} y_{[i]}^{1/q} \leqslant \sum_{i=1}^{n} x_i^{1/p} y_i^{1/q} \leqslant \sum_{i=1}^{n} d_i^{1/p} y_{(i)}^{1/q}$$

when $p > 1$*. These inequalities are reversed when* $0 < p < 1$ *or* $p < 0$*.*

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