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ORIGINAL ARTICLE

On the convergence of Homotopy perturbation method

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Abstract In many papers, Homotopy perturbation method has been presented as a method for solving non-linear equations of various kinds. Using Homotopy perturbation method, it is possible to find the exact solution or a closed approximate to the solution of the problem. But, only a few works have been considered the problem of convergence of the method. In this paper, convergence of Homotopy perturbation method has been elaborated briefly.

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1. Introduction

Mathematical modeling of many physical systems leads to functional equations, in various fields of physics and engineering. There are some methods to obtain approximate solutions of this kind of equations. One of them is Homotopy perturbation method. The method introduced by He in 1998 [1] has been developed and improved by himself. He applied it to boundary value problems, non-linear wave equations, and many other subjects [2–6]. Homotopy perturbation method can be considered as a universal method capable of solving various kinds of non-linear functional equations. For example, it was also applied to non-linear Schrodinger equations [7], to systems of Volterra integral equations of the second kind [8], to generalized Hirota–Satsuma coupled KdV equation [11], and to many other equations [2–12].

In this method the solution is considered as the summation of an infinite series which usually converge rapidly to the exact solution. This method continuously deforms, the difficult problems under study into a simple problem, easy to solve. Almost all perturbation methods are based on the assumption of the existence of a small parameter in the equation. But most non-linear problems have not such a small parameter. This method has been proposed to eliminate the small parameters. In this paper, a proof of convergence of the HPM is presented.

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2. Basic idea of homotopy perturbation method

To illustrate the basic concept of homotopy perturbation method, consider the following non-linear functional equation

$$A(u) - f(r) = 0 \ , r \in \Omega, \tag{1}$$

with boundary conditions;

$$B(u, \partial u/\partial n) = 0 \ , r \in \Gamma, \tag{2}$$

where A is a general functional operator, B is a boundary operator, $f(r)$ is a known analytic function, and Γ is the boundary of the domain Ω. Generally speaking, the operator A can be divided into two parts L and N, where L is a linear, while N is a non-linear operator. Eq. (1), therefore, can be rewritten as follows;

$$L(u) + N(u) - f(r) = 0. \tag{3}$$

We construct a homotopy $v(r,p):\Omega \times [0,1] \to R$ which satisfies

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \tag{4}$$

or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \tag{5}$$

where $p \in [0,1]$ is an embedding parameter, and u_0 is an initial approximation for the solution of Eq. (1), which satisfies the boundary conditions. According to HPM, we can first use the embedding parameter p as a small parameter, and assume that the solution of Eq. (5) can be written as a power series in p:

$$v = v_0 + v_1 p + v_2 p^2 + \cdots = \sum_{i=0}^{\infty} v_i p^i. \tag{6}$$

Considering $p = 1$, the approximate solution of Eq. (2) will be obtained as follows;

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots \tag{7}$$

3. Convergence of the method

Let's rewrite the Eq. (5) as the following;

$$L(v) - L(u_0) = p[f(r) - L(u_0) - N(v)]. \tag{8}$$

Substituting (7) into (8) leads to;

$$L\left(\sum_{i=0}^{\infty} v_i p^i\right) - L(u_0) = p\left[f(r) - L(u_0) - N\left(\sum_{i=0}^{\infty} v_i p^i\right)\right]. \tag{9}$$

So

$$\sum_{i=0}^{\infty} L(v_i) - L(u_0) = p\left[f(r) - L(u_0) - N\left(\sum_{i=0}^{\infty} v_i p^i\right)\right]. \tag{10}$$

According to Maclaurin expansion of $N(\sum_{i=0}^{\infty} v_i p^i)$ with respect to p, we have;

$$N\left(\sum_{i=0}^{\infty} v_i p^i\right) = \sum_{n=0}^{\infty}\left(\frac{1}{n!}\frac{\partial^n}{\partial p^n} N\left(\sum_{i=0}^{\infty} v_i p^i\right)\right)_{p=0} P^i. \tag{11}$$

From [13], we get

$$\left(\frac{\partial^n}{\partial p^n} N\left(\sum_{i=0}^{\infty} v_i p^i\right)\right)_{p=0} = \left(\frac{\partial^n}{\partial p^n} N\left(\sum_{i=0}^{n} v_i p^i\right)\right)_{p=0}. \tag{12}$$

Then

$$N\left(\sum_{i=0}^{\infty} v_i p^i\right) = \sum_{n=0}^{\infty}\left(\frac{1}{n!}\frac{\partial^n}{\partial p^n} N\left(\sum_{i=0}^{n} v_i p^i\right)\right)_{p=0} p^i. \tag{13}$$

We set

$$H_n(v_0, v_1, \ldots, v_n) = \left(\frac{1}{n!}\frac{\partial^n}{\partial p^n} N\left(\sum_{i=0}^{n} v_i p^i\right)\right)_{p=0}, \quad n = 0, 1, 2, \ldots, \tag{14}$$

where H_n s are the so-called He's polynomials [13]. Then;

$$N\left(\sum_{i=0}^{\infty} v_i p^i\right) = \sum_{n=0}^{\infty} H_i p^i. \tag{15}$$

Substituting (15) into (10), we drive;

$$\sum_{i=0}^{\infty} L(v_i) - L(u_0) = p\left[f(r) - L(u_0) - \sum_{n=0}^{\infty} H_i p^i\right]. \tag{16}$$

By equating the terms with the identical powers in p:

$$\begin{cases} p^0 : L(v_0) - L(u_0) = 0, \\ p^1 : L(v_1) = f(r) - L(u_0) - H_0, \\ p^2 : L(v_2) = -H_1, \\ \vdots \\ p^3 : L(v_{n+1}) = -H_n, \\ \vdots \end{cases} \tag{17}$$

So we derive

$$\begin{cases} v_0 = u_0, \\ v_1 = L^{-1}[f(r)] - u_0 - L^{-1}(H_0), \\ v_2 = -L^{-1}(H_1), \\ \vdots \\ v_{n+1} = -L^{-1}(H_n), \\ \vdots \end{cases} \tag{18}$$

Theorem 3.1. *Homotopy perturbation method used the solution of Eq. (1) is equivalent to determining the following sequence;*

$$s_n = v_1 + \ldots + v_n, \quad s_0 = 0, \tag{19}$$

by using the iterative scheme:

$$s_{n+1} = -L^{-1} N_n(s_n + v_0) - u_0 + L^{-1}(f(r)), \tag{20}$$

where

$$N_n\left(\sum_{i=0}^{n} v_i\right) = \sum_{n=0}^{n} H_i, \quad n = 0, 1, 2, \ldots. \tag{21}$$

Proof. For $n = 0$, from (20), we have;

$$\begin{aligned} s_1 &= -L^{-1} N_0(s_0 + v_0) - u_0 + L^{-1}(f(r)), \\ &= -L^{-1}(H_0) - u_0 + L^{-1}(f(r)). \end{aligned} \tag{22}$$

Then

$$v_1 = -L^{-1}(H_0) - u_0 + L^{-1}(f(r)). \tag{23}$$

For $n = 1$:

$$s_2 = -L^{-1}N_1(s_1 + v_0) - u_0 + L^{-1}(f(r)) = -L^{-1}(H_0 + H_1) - u_0 + L^{-1}(f(r)), = -L^{-1}(H_1) + v_1. \quad (24)$$

According to $s_2 = v_1 + v_2$, we get;

$$v_2 = -L^{-1}(H_1). \quad (25)$$

This theorem will be proved by strong induction. Let's assume that $v_{k+1} = -L^{-1}(H_k)$, for $k = 1, 2, \ldots, n-1$, so

$$s_{n+1} = -L^{-1}N_n(s_n + v_0) - u_0 + L^{-1}(f(r)) = -L^{-1}\left(\sum_{n=0}^{n} H_i\right) - u_0 + L^{-1}(f(r)), = -\sum_{n=0}^{n} L^{-1}(H_i) - u_0 + L^{-1}(f(r)) = v_1 + v_2 + \ldots + v_n - L^{-1}(H_n). \quad (26)$$

Then, from (19), it can be driven;

$$v_{n+1} = -L^{-1}(H_n). \quad (27)$$

Which is the same as the result of (18) from HPM, and the theorem is proved. □

Theorem 3.2. *Let B be a Banach space.*

(a) $\sum_{i=0}^{\infty} v_i$ *obtained by (18), convergence to $s \in B$, if*

$$\exists(0 \leqslant \lambda < 1),\ s.t\ (\forall n \in \mathbb{N} \Rightarrow \|v_n\| \leqslant \lambda\|v_{n-1}\|). \quad (28)$$

(b) $s = \sum_{n=1}^{\infty} v_n$, *satisfies in*

$$s = -L^{-1}N(s + v_0) - u_0 + L^{-1}(f(r)). \quad (29)$$

Proof.

(a) we have

$$\|s_{n+1} - s_n\| = \|v_{n+1}\| \leqslant \lambda\|v_n\| \leqslant \lambda^2\|v_{n-1}\| \leqslant \cdots \leqslant \lambda^{n+1}\|v_0\|. \quad (30)$$

For any $n, m \in \mathrm{N}$, $n \geqslant m$, we drive;

$$\begin{aligned}\|s_n - s_m\| &= \|(s_n - s_{n-1}) + (s_{n-1} - s_{n-2}) + \cdots + (s_{m+1} - s_m)\|,\\ &\leqslant \|s_n - s_{n-1}\| + \|s_{n-1} - s_{n-2}\| + \cdots + \|s_{m+1} - s_m\|,\\ &\leqslant \lambda^n\|v_0\| + \lambda^{n-1}\|v_0\| + \cdots + \lambda^{m+1}\|v_0\| \leqslant (\lambda^n + \lambda^{n-1} + \cdots + \lambda^{m+1})\|v_0\|,\\ &\leqslant (\lambda^{m+1} + \cdots + \lambda^n + \cdots)\|v_0\| \leqslant \lambda^{m+1}(1 + \lambda + \cdots + \lambda^n + \cdots)\|v_0\|,\\ &\leqslant \frac{\lambda^{m+1}}{1-\lambda}\|v_0\|.\end{aligned} \quad (31)$$

So

$$\lim_{n,m\to\infty}\|s_n - s_m\| = 0. \quad (32)$$

Then $\{s_n\}$, is Cauchy sequence in Banach space, and it is convergent, i.e.,

$$\exists s \in \mathrm{B}, s.t \lim_{n\to\infty} s_n = \sum_{n=1}^{\infty} v_n = s. \quad (33)$$

(b) From Eq. (20), we have;

$$\begin{aligned}\lim_{x\to\infty} s_{n+1} &= -L^{-1}\lim_{x\to\infty} N_n(s_n + v_0) - u_0 + L^{-1}(f(r))\\ &= -L^{-1}\lim_{x\to\infty} N_n\left(\sum_{i=0}^{n} v_i\right) - u_0 + L^{-1}(f(r)).\\ s &= -L^{-1}\lim_{x\to\infty}\sum_{i=0}^{n} H_i - u_0 + L^{-1}(f(r))\\ &= -L^{-1}\sum_{i=0}^{\infty} H_i - u_0 + L^{-1}(f(r)).\end{aligned} \quad (34)$$

But by Eqs. (21) and (15) for $p = 1$, we drive;

$$\sum_{i=0}^{\infty} H_i = N\left(\sum_{i=0}^{\infty} v_i\right). \quad (35)$$

So

$$s = -L^{-1}N\left(\sum_{i=0}^{\infty} v_i\right) - u_0 + L^{-1}(f(r)),$$
$$s = -L^{-1}N(s + v_0) - u_0 + L^{-1}(f(r)).$$ □

Lemma 1. *Eq. (29) is equivalent to;*

$$L(u) + N(u) - f(r) = 0. \quad (36)$$

Proof. We rewrite Eq. (29) as fallows;

$$s + u_0 = -L^{-1}N(s + v_0) + L^{-1}(f(r)). \quad (37)$$

By applying the operator L to Eq. (36) we derive;

$$L(s + u_0) = -N(s + v_0) + (f(r)).$$

But $u_0 = v_0$, then;

$$L(s + v_0) + N(s + v_0) = (f(r)). \quad (38)$$

By considering $u = s + v_0 = \sum_{n=0}^{\infty} v_i$, Eq. (36), has been derived which is the original equation. Then solution of Eq. (29) is the same as solution of $A(u) - f(r) = 0$. □

Definition 1. for every $i \in N$, we define;

$$\lambda_i = \begin{cases}\frac{\|v_{i+1}\|}{\|v_i\|}, & \|v_i\| \neq 0,\\ 0 & \|v_i\| = 0.\end{cases}$$

In Theorem 3.2, $\sum_{n=0}^{\infty} v_i$ converges to exact solution, when $0 \leqslant \lambda_i < 1$. If v_i and v'_i are obtained by two different homotopy, and $\lambda_i < \lambda'_i$ for each $i \in N$, the rate of convergence of $\sum_{n=0}^{\infty} v_i$ is higher than $\sum_{n=0}^{\infty} v'_i$.

Example 1. Consider the Lane–Emden equation in the following form

$$u'' + \frac{2}{x}u' + u = x^5 + 30x^3, u(0) = 0, u'(0) = 0. \quad (39)$$

With the exact solution $u(x) = x^5$.

To solve Eq. (39) by homotopy perturbation method, we construct the following Homotopy;

$$u'' - U''_0 = p(x^5 + 30x^3 - \frac{2}{x}u' - u - u''_0). \quad (40)$$

Let's consider the solution u as the summation of a series;

$$u = \sum_{i=0}^{\infty} u_i. \tag{41}$$

Substituting (41) into (40) leads to;

$$\sum_{i=0}^{\infty} u_i'' - U_0'' = p(x^5 + 30x^3 - \frac{2}{x}\sum_{i=0}^{\infty} u_i' - \sum_{i=0}^{\infty} u_i - u_0'').$$

Beginning with $u_0 = 0$, we get ;

$u_1 = \frac{1}{42}x^7 + \frac{3}{2}x^5;$
$u_2 = -\frac{11}{252}x^7 - \frac{3}{4}x^5 - \frac{1}{4024}x^9;$
$u_3 = \frac{25}{36288}x^9 + \frac{7}{216}x^7 + \frac{3}{8}x^5 + \frac{1}{332640}x^{11};$
$u_4 = -\frac{137}{19958400}x^{11} - \frac{271}{435456}x^9 - \frac{179}{9072}x^7 - \frac{3}{16}x^5 - \frac{1}{51891840}x^{13};$
$u_5 = \frac{7}{148262400}x^{13} + \frac{8419}{1197504000}x^{11} + \frac{2245}{5225472}x^9 + \frac{601}{54432}x^7 + \frac{3}{32}x^5 + \frac{1}{10897286400}x^{15};$

By considered $\|f(x)\| = \max_{0 \leqslant x \leqslant 1} |f(x)|$, we have;

$\lambda_1 = 0.5210503471, \quad \lambda_2 = 0.5139910140,$
$\lambda_3 = 0.5093374003, \quad \lambda_4 = 0.5062439696,$
$\lambda_5 = 0.5041785188, \quad \lambda_6 = 0.5027965117.$

If the linear part of equation is consider as follows;

$$Lu = u'' + \frac{2}{x}u' = x^{-2}\frac{d}{dx}\left(x^2\frac{du}{dx}\right).$$

Then we construct the following homotopy

$$\left(v'' + \frac{2}{x}v'\right) - \left(u_0'' + \frac{2}{x}u_0'\right) = p(x^5 + 30x^3 - v - u_0'' - \frac{2}{x}u_0'), \tag{42}$$

where

$$L^{-1}(u) = \int_0^x x^{-2}\int_0^x x^2 u(x)dx\, dx.$$

Suppose $u = v_0 + pv_1 + p^2v_2 + \cdots$, and $v_0 = u_0 = 0$. So;

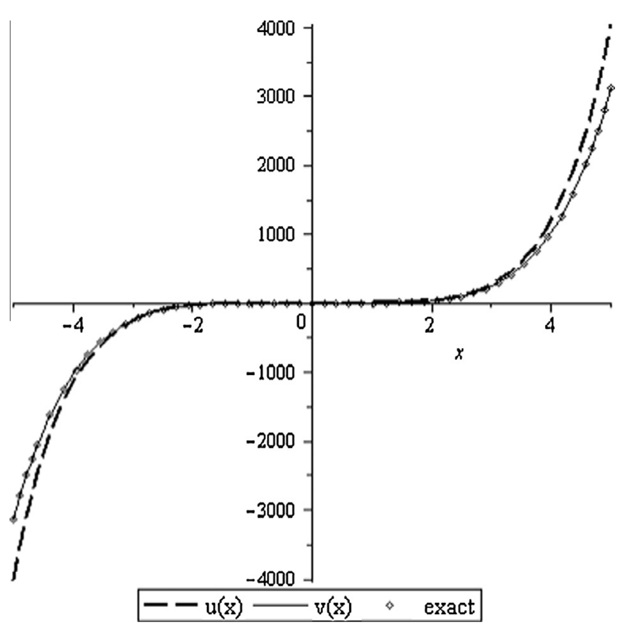

Fig. 1 Plots of solution of HPM and exact solution for Ex. 3.2.1.

$v_1 = \frac{1}{56}x^7 + x^5;$
$v_2 = -\frac{1}{5040}x^9 - \frac{1}{56}x^7;$
$v_3 = \frac{1}{665280}x^{11} + \frac{1}{5040}x^9;$
$v_4 = -\frac{1}{121080960}x^{13} - \frac{1}{665280}x^{11};$
$v_5 = \frac{1}{29059430400}x^{15} + \frac{1}{121080960}x^{13};$

Then;

$\lambda_1 = 0.0177387914, \quad \lambda_2 = 0.0110722610,$
$\lambda_3 = 0.00756010906, \quad \lambda_4 = 0.00548724954,$
$\lambda_5 = 0.00416293765, \quad \lambda_6 = 0.00326590091.$

By comparison between the obtained results in Example 3.2.1, it can be concluded that the rate of convergence of homotopy (42) is higher than homotopy (41) (see Fig. 1).

4. Conclusion

The Homotopy perturbation method is a simple and powerful tool for obtaining the solution of functional equations. In this article, we have proved convergence for Homotopy perturbation method. As a result from this paper, under assumption of Theorem 3.2, homotopy perturbation method is convergence to exact solution of the problem. The problem of study of convergence conditions for differential equations, integral equations, integro-differential equations and theirs systems is also under study in our research group.

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