ANOTHER SYSTEMATIC PHENOMENON IN THE
COHOMOLOGY OF THE STEENROD ALGEBRA

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§0. INTRODUCTION

The purpose of this paper is twofold. First, we describe a new operator in the cohomology of the mod 2 Steenrod algebra: $H^*(A)$. This operator increases the stem degree $t - s$ by 45 and the filtration degree $s$ by 6. It may be thought of as a kind of periodicity, and our main result is that under it the "wedge" subalgebra $[4]$ of $H^*(A)$ is repeated every 45 stems. Second, we regard the methods we use as further evidence of the efficacy of the technique of studying $H^*(A)$ by studying $H^*(B)$ for suitably chosen subalgebra $B$ of $A$. This technique has been used very successfully by Adams [1] in formulating his periodicity operator and more recently by Zachariou [7] to obtain part of the wedge. Using other subalgebras than that of this paper we have obtained information about other elements of $H^*(A)$ such as $n, t, N, G, R$, and $W$ (for the notation throughout the paper see [6]). These results will appear elsewhere.

As an application we show (§3) that the rank of $H^{a,i}(A)$ may be arbitrarily large.

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§1. THE MAIN THEOREM

The operator $M$ that we will be studying is defined in the following manner. Let $g_2 = \langle h_1 h_2^2, h_1, h_2, h_3 \rangle \in H^4,48(A)$; then $g_2 h_0^3 = 0$. If $x \in H^i(A)$ satisfies $h_0^3 x = 0$, we define $Mx = \langle g_2, h_0^3, x \rangle$, regarded as an element of $H^{i+2, i+3}(A)$ modulo the indeterminacy $h_0^3 h_5 d_0 x$ (since $h_0 h_5 d_0$ generates $H^6, 51(A)$ [6]).

Set $M' = M$ and $M'' = M(M'')^{-1}$; then if $Mx$ is defined, so is $M''x$. To see this, first note that $h_0^3(g_2 H^*(A) + h_5 d_0 H^*(A)) = 0$, since $h_0^3 g_2 = 0 = h_0^3 d_0$. Further $h_0^3 M''^{-1} x = h_0^3 \langle g_2, h_0^3, M''^{-2} x \rangle = h_0^3 \langle g_2, h_0^3 \rangle 2 M''^{-2} x = (h_0^3 \cup_1 h_0^3) g_2 M''^{-2} x = 0$ since the cup-one product is zero for dimensional reasons.

Let $W$ denote the wedge subalgebra. If $x \in W$ then $h_0 x = 0$ so $Mx$ is defined. We can now state our main result.

Theorem 1. If $x \neq 0$ in $W$ then $0 \notin M''x$ (for all $r \geq 1$).

We will also prove a somewhat more general result.
Theorem 2. If $w_o, w_1, \ldots, w_r \in W$ and if $0 \in w_o + Mw_1 + \cdots + M^rw$, then each $w_*$ is zero.

The proof of Theorem 1 may be outlined as follows. Let $B$ be the exterior subalgebra of $A$ generated by $Sq^{0.1}$, $Sq^{0.2}$ and $Sq^{0.0.0.1}$ (in the Milnor basis notation). Then the inclusion $i : B \to A$ induces a map $i^* : H^*(A) \to H^*(B) = \mathbb{Z}[x, \beta, \gamma]$ (where $x$, $\beta$, $\gamma$ correspond to $Sq^{0.1}$, $Sq^{0.2}$, and $Sq^{0.0.0.1}$ respectively). We first prove that $i^*$ is a monomorphism when restricted to $W$. Then we show that under $i^*$ the operator $M$ corresponds to multiplication by a fixed element of $H^*(B)$, and we determine this element to be $\alpha \beta^3 \gamma^3$ by considering a special case ($x = d_2$). Since $H^*(B)$ is a polynomial algebra, multiplication by any fixed element is monomorphic, and Theorem 1 follows.

The detailed proof of Theorem 1 is given in the next section. In Section 3 we obtain Theorem 2 and make some further remarks.

§2. PROOF OF THEOREM 1

Lemma. $i^*$ is monomorphic on the wedge subalgebra $W$ of $H^*A$.

Proof. Recall that $W$ has a $\mathbb{Z}_2$-basis given by $P^ig^j\lambda$ where $\lambda \in \{ d, l, d_m, e_m, gm, P^1g^2, d_0 e_0 g, d_0 g^2, e_0 g^2, P^1v, d_0 u, e_0 u, gu, P^1d_0 r, P^1e_0 r, P^1gr, d_0 e_0 r \}$ together with $g^0\tau$ where $\tau \in \{ g^2, v, w, d_0 r, e_0 r, gr \}$.

First we show that $i^*e_0 = \alpha \beta^3$. This can be seen in a number of ways. First, $e_0$ already appears in $H^*(A_2)$ ($A_2$ the subalgebra of $A$ generated by $Sq^1$, $Sq^2$, $Sq^3$) and using the resolutions described in [5] we can read off $i^*e_0 = \alpha \beta^3$. Second, this is also a consequence of Zachariou’s work [7], i.e. if $C$ is the subalgebra generated by $Sq^{0.1}$, $Sq^{0.2}$ then we have $C \to B \to A$ and he shows that $i^*e_0 = \alpha \beta^3$. Third, this result can be obtained by looking at representatives on the $E_2$-level of the May spectral sequences of $A$ and $B$.

Now we have the following relations in $H^*(A)$: $d_0 g = e_0^2$, $uw = v^2$, $rl = gv = e_0 w$ [6], $mv = e_0 g^3$ [4]. From the first of these, $i^*(d_0 g) = (i^*e_0)^2 = \alpha^2 \beta^2$. It follows that $i^*d_0 = \alpha^2 \beta^2$ and $i^*g = \beta^4$ (no other possibility is consistent with the bigrading conditions). Then the last relation implies that $i^*(mv) = \alpha \beta^15$ from which it follows that $i^*m = \beta^7$, $i^*v = \alpha \beta^8$. From $uw = v^2$ we get $i^*u = \alpha^2 \beta^7$, $i^*w = \beta^3$. Then from $rl = gv$ we get $i^*r = \beta^6$, $i^*l = \alpha \beta^6$.

Finally we claim that the periodicity operators $P^x$ satisfy $i^*P^x = \alpha \beta^4 i^*x$. This is a consequence of Adams’ work [1, Lemma 4.3].

It now follows that $i^*x \neq 0$ for $x \in W$. Since all the basis elements of $W$ have distinct bigradings, the lemma is proved.

Lemma. $i^* : H^*(A) \to H^*(B)$ takes the indeterminacy of $Mx$ to 0.

Proof. Since $H^*(B) = 0$ for $t < 3$, $i^*h_2 = 0$. As for $g_2$, using the Steenrod operations in $H^*(A)$ and $H^*(B)$, we have $i^*g_2 = i^*Sq^0g = Sq^0i^*g = Sq^0\beta^4 = (Sq^0\beta)^4 = 0$ since $Sq^0\beta = 0$ (for dimensional among other reasons).

Main Lemma. If $Mx$ is defined, then $i^*Mx = (\alpha \beta^3 \gamma^2)i^*x$. 

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Proof. To prove this we need a canonical construction of the Massey product \( M_x \); we use the cobar construction. We have a map \( i^*: C^*A \rightarrow C^*B \). For \( m \in H^*(\ ) \) let \( m \) denote a representative cocycle in \( C^*(\ ) \) and for a cocycle \( n \in C^*(\ ) \) let \([n]\) denote the cohomology class. Then a representative for \( M_x \) has the form \( \tilde{g}_2a + b\xi \) where \( \delta a = \tilde{h}_3\xi \) and \( \delta b = \tilde{g}_2\tilde{h}_3 \), and \( i^*Mx = [i^*(\tilde{g}_2a + b\xi)] \) since the indeterminacy goes to 0. But \( i^*\tilde{h}_3 = 0 \) implies that \( \delta i^*a = 0 \) and \( \delta i^*b = 0 \), further \( i^*\tilde{g}_2 = 0 \). Therefore \([i^*(\tilde{g}_2a + b\xi)] = [i^*\tilde{g}_2][i^*a] + [i^*b][i^*\tilde{g}_2] = [i^*b][i^*x] = [i^*b][i^*x] \). It remains to determine \([i^*b]\), which we do by computing \( i^*Mx \) for \( x = d_0 \).

Sublemma. \( M(d_0) = B_{21} \) (here \( B_{21} \) is the generator of \( H^{10,69}(A) \) [6]).

Proof. First note that the indeterminacy of \( M(d_0) \) is \( g_2H^{6,21}(A) + d_0H^{6,51}(A) = d_0h_3h_5d_6^2 = 0 + 0 \).

The proof consists in verifying the Massey product in the May spectral sequence. At \( E_2 \), \( \langle b_2^2, h_3, d_0 \rangle \) has the form \( \langle b_2^2, h_3, h_2(1)^2 \rangle \). And we have that \( h_0^2h_1(1)^2 = d_2(b_0^2^2h_0(1)) \) and \( b_2b_3h_3^3 = d_6K \) where \( K = h_0(1)(b_0^2b_1 + b_0b_3b_1) \) [6]. Therefore the triple product for \( M(d_0) \) can be formed in \( E_2 \) and equals \( d_6K \). But \( d_6K \) survives in the May spectral sequence and represents \( B_{21} \). Our result now follows from a convergence theorem of May [13, Theorem 4.1].

Sublemma. \( i^*M(d_0) = \alpha^3\beta^5\gamma^2 \).

Proof. We prove this by examining the situation on the \( E_2 \)-level of the May spectral sequence. Filter \( A \) by powers of its augmentation ideal \( IA \), i.e. \( F_pA = A \), \( F_pA = (IA)^p \) for \( p < 0 \). Then filter \( B \) by \( F_pB = B \cap F_pA \). Following May [2], the filtrations induce filtrations of the cobar constructions and give rise to spectral sequences \( E_*^*,*^* \) converging to \( H^*(A) \) and \( H^*(B) \) respectively. (For \( A \) this is just the usual May spectral sequence.) Then at the \( E_2 \)-level we have \( B_{21} \) represented by \( h_1(1)^2(b_0^2b_1 + b_0b_3b_1) \); but \( i_2^*h_1(1) = \alpha\beta, z^2b_4 = \gamma^2, i_2^*b_1 = \beta^2 \) and \( i_2^*b_3 = 0 \), therefore \( i_2^*(h_1(1)^2(b_0^2b_1 + b_0b_3b_1)) = \alpha^2\beta^3\gamma^2 \). Since \( B \) is an exterior algebra it is easy to verify that \( E_2(B) = E_\infty(B) = B \) (as algebras), therefore \( i^*B_{21} = \alpha^3\beta^5\gamma^2 \).

Since \( i^*d_0 = \alpha^2\beta^2 \), it follows that \([i^*b] = \alpha\beta^3\gamma^2 \), and this completes the proof of the Main Lemma and of Theorem 1.

§3. FURTHER REMARKS

A closer examination of the calculations of Section 2 yields some stronger results.

Lemma. The image of \( W \) under \( i^* \) is the ideal generated by \( \beta^8 \) in the subalgebra \( Z_2[\alpha, \beta] \) of \( H^*(B) \).

In other words, the image has as a \( Z_2 \)-basis all elements of the form \( \alpha^i\beta^j \) where \( i \geq 0, j \geq 8 \).

This is proved simply by checking the images of the elements given previously as a \( Z_2 \)-basis for \( W \). We leave the verification to the reader.
Proof of Theorem 2. If $0 \in w_n + Mw_1 + \cdots + M'w_r$ then $i^*(w_n + Mw_1 + \cdots + M'w_r) = 0$. But $i^*(M'w_n)$ contains $\gamma$ to the power $2n$, since $i^*(M'w_n) = \alpha^\gamma \beta^{2n} \gamma^{2^k} i^*(w_n)$ and $i^*(w_n) \in \mathbb{Z}_2[\alpha, \beta]$. Thus no cancellation can occur and $i^*$ must vanish on each term. By Theorem 1, this implies that each $w_n$ is zero.

Before making our final application, we put in evidence some obvious multiplicative properties of the operator $M$.

If $Mx$ and $My$ are defined then obviously $M(xy)$ is defined; moreover we have the relation $(Mx)(My) = M^2(xy)$. This follows from standard identities on triple products: $x(My) = x(g_2, h_3, y) = (g_2, h_3, xy) = M(xy)$ and therefore $(Mx)(My) \subset M((Mx)y) \subset M(M(xy))$.

In particular $(Mx)^r \subset M^r(x')$. If $x$ is a wedge element then all its powers are non-zero [4] and thus $Mx$ (or $M'x$) generates a polynomial subalgebra of $H^*A$.

More generally, if $x$ is any element of $H^*A$ such that $i^*x \neq 0$ then $i^*(x') = (i^*x)^r \neq 0$, so that all powers of $x$ are non-zero. This applies to such elements as $i, j, k, x'$ and $B_{21}$ in addition to the wedge.

Proposition. The rank of $H^*\cdot(A)$ as a $\mathbb{Z}_2$-vector space may be arbitrarily large.

Proof. We observe that there exist $x, y \in W$ such that $Mx$ and $y$ have the same bigrading. Then the exponent of $\gamma$ in $i^*[(Mx)^m y^n m]$ is just $2m$, because $i^*x$ and $i^*y$ do not contain $\gamma$ while each “factor” of $M$ contributes a factor of $\gamma^2$. Hence $\{(Mx)^m y^n m : 0 \leq m \leq n\}$ are all distinct elements with the same bigrading.

For example, both $MP^1e_0 r$ and $g^5$ lie in $H^{20,120}(A)$, and $i^*((MP^1e_0 r)^m (g^5)^n m) = \alpha^6m \beta^{30n - 8m} \gamma^2m$.

Thus in the $(100n)$-stem there is a bigrading of rank at least $n + 1$.

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