International Journal of Solids and Structures 47 (2010) 539–548

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

Mechanically-based approach to non-local elasticity: Variational principles

M. Di Paola, A. Pirrotta *, M. Zingales

Dipartimento di Ingegneria Strutturale, Aerospaziale e Geotecnica (DISAG), University of Palermo, Viale delle Scienze, I-90128 Palermo, Italy

ARTICLE INFO

Article history:
Received 13 October 2008
Received in revised form 11 March 2009
Available online 1 October 2009

Keywords:
Non-local elasticity
Long-range interactions
Non-local elastic potential energy
Non-local state variables
Variational theorems

ABSTRACT

The mechanically-based approach to non-local elastic continuum, will be captured through variational calculus, based on the assumptions that non-adjacent elements of the solid may exchange central body forces, monotonically decreasing with their interdistance, depending on the relative displacement, and on the volume products. Such a mechanical model is investigated introducing primarily the dual state variables by means of the virtual work principle. The constitutive relations between dual variables are introduced defining a proper, convex, potential energy. It is proved that the solution of the elastic problem corresponds to a global minimum of the potential energy functional. Moreover, the Euler–Lagrange equations together with the natural boundary conditions associated to the total potential energy functional are established with variational calculus and they coincide with analogous relations already obtained by means of mechanical considerations. Numerical analysis of a tensile specimen has been introduced to show the capabilities of the proposed approach.

1. Introduction

The mechanics of elastic continuum has been widely used both in the scientific literature and engineering applications since the first apparition at the mid of the 18th century. The main equations of the linear elasticity theory represented by the equilibrium, the compatibility and the constitutive relations have been established both from mechanical and variational considerations. This latter approach makes it possible to derive important theorems of linear-Hookean solids, such as the minimum of strain energy functional, the Betti's, the Castigliano's and several other theorems on variational basis. Those theorems enjoy the symmetry of the elastic coefficients of the material and this character formed the basis for the formulation of widely-used approximate methods reverting, as limiting case, to exact solutions when available.

On the one hand, linear elasticity theory has been one of the mainly investigated methods to represent mechanical behaviour of solids; on the other hand several unexpected phenomena observed in experimental set ups, such as dispersion of elastic waves or the presence of shear bands in tensile specimen not predicted by classical elasticity theory led to abandon the classical Cauchy theory. In this framework different theories have been formulated to account for the presence of an inner microstructure responsible of the deviation of experimental data from the results of classical mechanics. Those deviations have been accounted introducing additional terms in the constitutive equations that involve gradients of the strain field (weak non-local theories) or weighted integrals of the strain field (strong non-local theories). Gradient non-local theories have been formulated at the beginning of the fifties introducing the micromorphic continuum theories (Eringen, 1967; Mindlin and Eshel, 1968) until more recent studies focusing on crack growth or damage accumulation (Aifantis, 1994; Gutkin and Aifantis, 1996, 1997; Aifantis, 2003; Ganghoffer and De Borst, 2000; Peerlings et al., 2001; Askes and Metrikine, 2002; Metrikine and Askes, 2002). Despite several similarities among different approaches in gradient theories of non-local elasticity some important differences are worthy to be discussed. In more detail, micromorphic theories may be considered as extensions of micropolar elasticity theory (Toupin, 1963) introducing a deformable elastic microstructure embedded in Cauchy space. The governing equations of the inner microstructure involves the presence of microstress and relative stresses between the micro–macro continuum. In such a theory, additional model of external forces, representing couple forces and double couple are also involved in the analysis of the microstructure interactions. Such considerations do not hold for the non-local gradient elasticity (Aifantis, 1994) that introduces non-local effects in the constitutive equations of the considered material by means of higher-order gradients of the strain field. In this context the equilibrium, as well as, the kinematic restraints of the non-local elastic continuum theory remain unchanged but the Navier governing equations of the elastic problem involves higher-order derivatives of the unknown displacement field. Some recent applications of the gradient theory of non-local elasticity has been recently proposed in nanoscale engineering applications (Aifantis, 2009; Kioseoglou et al., 2009). Some recent strategies to provide experimental measures of the elastic

* Corresponding author.
E-mail addresses: dipaola@diseg.unipa.it (M. Di Paola), pirrotta@diseg.unipa.it, antonina.pirrotta@unipa.it (A. Pirrotta), zingales@diseg.unipa.it (M. Zingales).

0020-7683/$ - see front matter © 2009 Published by Elsevier Ltd.
doi:10.1016/j.ijsolstr.2009.09.029

coefficients of the internal microstructure as well as a mathematically consistent formulation of the non-standard boundary conditions have been recently proposed (Maraganti and Sharma, 2007; Polizzotto, 2003).

Non-local integral theories have received growing interest in the last 60 s since the first pioneer studies (Kroner, 1967; Krumhanls, 1967; Eringen, 1972; Eringen and Edelen, 1972). These theories may explain some smoothing effects in stress singularities observed with local elasticity or in plastic behaviour. The non-local integral model mainly consists in postulating that the stress at a given point is related to the strain at that location and to a non-local contribution due to the surrounding medium given by convolution integral involving the strain field and a weighting attenuation function that accounts for long-distance effects. In this framework, recently, various contributions are available (Rogula, 1982; Polizzotto and Borino, 1998; Polizzotto, 2001; Bažant and Belytschko, 1984; Pijadier-Cabot and Bažant, 1987; Bažant and Jirásek, 2002; Borino et al., 2003) to cite just a few. It has to be stressed, that the Eringen stress–strain relation is not based on a proper mechanical model, and this produces some ambiguities in the correct definitions of the mechanical boundary conditions, for a bounded domain. Moreover, non-local elastic models, based on the introduction in stress–strain relations of the weighted strain field, do not fulfil the well-known theorems of elasticity such as the symmetry of the elastic operators and some, ad hoc, strategies have been proposed, recently, introducing suitable modifications of the weighting functions (Marotti De Sciarra, 2008).

A different approach, based upon the concept of long-range potential has been proposed, in the context of peridynamic theory of elasticity (Silling, 2000; Silling and Lehoucq, 2008) without the consideration of the contact forces between adjacent volume elements, that yields some drawbacks in presence of mechanical boundary conditions. Moreover, the absence of contact forces between interacting elements correspond to discontinuous displacement field for an 1D elastic solid with concentrated forces (Silling et al., 2003).

In a recent study a long-range cohesive interaction model has been proposed (Di Paola and Zingales, 2008) introducing non-local effects upon different considerations; This approach leads to physical model of non-local elasticity as: (i) A point-spring elastic network fully equivalent to discrete lattice models. (ii) A continuum model including the long-range interactions as central volume forces depending on the displacements of the volumes and on a distance-decaying function.

The relations of the physically-based approach to non-local elasticity coalesce, in unbounded domain, with the Eringen model of non-local elasticity, but, for a bounded bar, such equivalence could not be established and the constitutive law remains quite different with respect to the Eringen model. The physically-based non-local contribution in the constitutive law for a bounded bar is represented, in fact, by a double integral, instead of a simple convolution integral leading to the correct derivation of the mechanical boundary conditions (Di Paola et al., 2009). It was also shown that by selecting the distance-decaying function proportional to the absolute value of distance in power $-(1+\alpha)$; $\alpha \in \mathbb{R}$ the convolution integral, representing the non-local interactions, reverts to Marchaud fractional derivatives (Di Paola and Zingales, 2008; Cottone et al., 2009).

In this paper, the variational formulation of the problem is presented showing the mathematical consistency of the proposed model of the linearly elastic problem with long-range interactions. The virtual work theorem in presence of long-range interactions has been also reported to introduce the proper static-kinematic duality between the state variables of the elastic problem. The elastic potential energy of the solid has been introduced as a symmetric, convex and positive definite function of the state variables yielding the corresponding Euler–Lagrange equations, with the associate natural boundary conditions. A numerical application involving an 1D case in simple traction has been also reported to show the effects of the various parameters involved in the proposed, non-local elastic model.

2. The physically-based approach to non-local elasticity

In this section, some introductory remarks about fundamentals of non-local 1D mechanics, detailed in previous studies (Di Paola and Zingales, 2008; Di Paola et al., 2009), will be reported for clarity's sake and to introduce appropriate notations, as well. Extension to multi-dimensional elastic continuum is forthcoming in a separate study.

Let us discretize a simple bar into m volume elements: $\Delta V_j = A_j \Delta x$ as depicted in Fig. 1a, being $A_j = A(x_j)$ the cross-section area. Any volume element ΔV_j (Fig. 1b) is loaded by an external body-force field $f(x_j)\Delta V_j$, the contact forces N_j and N_{j+1} exerted by adjacent volume elements V_{j-1} and V_{j+1}, and by the resultant of the long-range forces Q_j from the non-adjacent volumes. The latter is the novel aspect of the model introduced in Di Paola et al. (2009), since N_j and N_{j+1} are the well-known actions in classical mechanics. Therefore, particular attention is devoted to this long-range interactions resultant Q_j, that is described by the sum of central forces applied on the centroids of the interacting volumes, depicted in Fig. 1c, and expressed as:

$$Q_j = \sum_{h=-\infty}^{\infty} Q^{(j,h)} \tag{1}$$

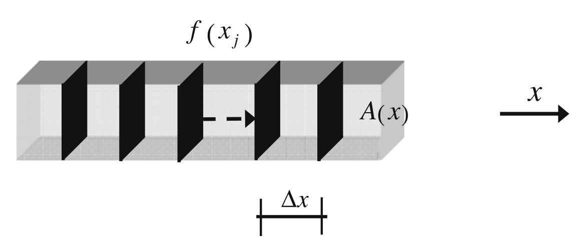

Fig. 1a. Discretized elastic bar loaded by an external volume force field $f(x)$.

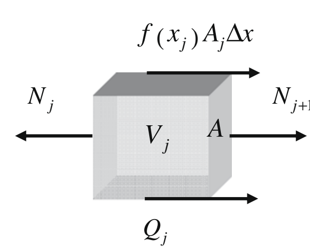

Fig. 1b. (b) Equilibrium of the volume element V_j including long-range interactions.

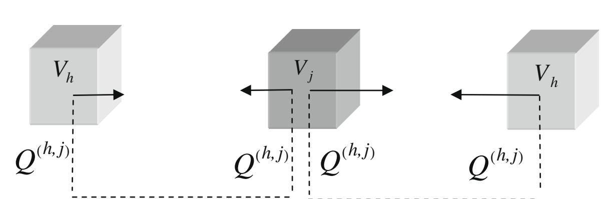

Fig. 1c. Long-range forces in the equilibrium of the volume element V_j.

with each component $Q^{(h,j)}$ represented, as commonly used in the context of lattice theories, by volume products of interacting elements (Krumhanls, 1963; Kunin, 1967; Lax, 1963):

$$Q^{(j,h)} = q^{(j,h)} V_h V_j = -q^{(h,j)} V_h V_j \tag{2}$$

and the specific long-range forces $q^{(h,j)}$ $\left([q^{(h,j)}] = FL^{-6}\right)$ has been assumed as in previous study:

$$q^{(j,h)} = g(x_h, x_j)(u_h - u_j) \tag{3}$$

where $u_j = u(x_j)$ and $u_h = u(x_h)$ are the axial displacements of the centroids of both volumes V_j, V_h and function $g(x_h, x_j) = g(x_j, x_h) = g(|x_h - x_j|)$ is a symmetric and positive function, monotonically decreasing with the distance $|x_j - x_h|$ with $([g(x,\xi)] = L^{-1})$. It is to be remarked that in a previous paper (Di Paola and Zingales, 2008) a slightly different, but fully equivalent expression for the definition reported in Eq. (3) was used in order to yield explicit fractional derivatives in the governing equation.

Thus, the equilibrium equation of a discretized bar framed into non-local continuum theory can be withdrawn from Fig. 1b as:

$$\Delta N_j + Q_j + f(x_j)A_j\Delta x = \Delta N_j + \sum_{h=0}^{m} q^{(j,h)} A_j A_h \Delta x^2 + f(x_j)A_j\Delta x = 0 \tag{4}$$

with $\Delta N_j = N_{j+1} - N_j$ in which Eqs. (2) and (3) have been accounted for.

Now the crucial point is to find out a physical point of view beside this theory. To aim at this, a close observation of Eq. (4) leads to consider a point spring model that will have an equivalent equilibrium equation such that this mechanical model restores the physical aspect to this theory.

In fact let us consider a discrete model of linearly elastic spring as shown in Fig. 2 (with only four points for clarity). Non-local effects may be captured considering linear-elastic springs of distance-decaying stiffness, in particular between two adjacent points P_j and P_h the interaction is depending on the stiffness: $K_j^l = \overline{E}A_j/\Delta x$ (where $\overline{E} = \beta_1 E$, E being the Young modulus, $0 \leqslant \beta_1 \leqslant 1$ a positive material constant) and $K_{jh}^{nl} = A_j A_h \Delta x^2 g(|x_j - x_h|)$. Between two non-adjacent points (j,h) interaction is only dependent on the stiffness: $K_{jh}^{nl} = A_j A_h \Delta x^2 g(|x_j - x_h|)$. Based on this assumption, it is apparent that the equilibrium equation of the point spring model is just Eq. (4). The mechanical scheme here discussed will be used to provide an approximate solution of the elastic problem with long-range interactions reported in Section 5.

Once the physical model of this proposed theory has been declared, we continue in determining all governing equations. Then, dividing Eq. (4) by Δx and letting $\Delta x \to 0$, Eq. (4) may be rewritten as:

$$\frac{d\sigma_l}{dx} + \int_0^L A(\xi)q(x,\xi)\,d\xi = -f(x) \tag{5}$$

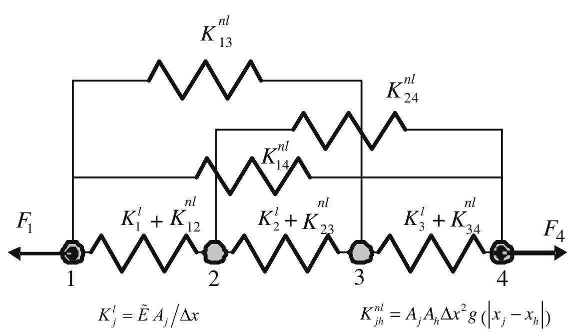

Fig. 2. Discrete non-local model with non-local linearly-elastic springs of distance-decaying stiffness.

Since $\sigma_l(x) = N(x)/A(x)$ is the local stress due to the contact forces, it follows that the integral terms in Eq. (5) are non-local contributions due to the long-range forces. Concisely it leads to:

$$\frac{d\sigma_l(x)}{dx} + \frac{d\sigma_{nl}(x)}{dx} = \frac{d\sigma(x)}{dx} = -f(x) \tag{6}$$

where function $\sigma_{nl}(x)$ is the non-local stress and $\sigma(x)$ is the overall stress at location x. Critical comparison between Eqs. (5) and (6) suggests to represent the non-local stress through the double integral:

$$\sigma_{nl}(x) = \int_{\xi_1=0}^{x}\int_{\xi_2=x}^{L} A(\xi_2)q(\xi_1,\xi_2)\,d\xi_2\,d\xi_1 \tag{7}$$

As in fact the following relation:

$$\frac{d}{dx}\int_{\xi_1=x}^{L}\int_{\xi_2=0}^{x} A(\xi_2)q(\xi_1,\xi_2)\,d\xi_2\,d\xi_1 = \int_0^L A(\xi)q(x,\xi)\,d\xi \tag{8}$$

holds true once the third Newton law for the long-range interactions have been accounted for as $q(\xi,x) = -q(x,\xi)$.

The governing equilibrium equation (5) of the 1D continuum with long-range forces may be written in terms of the displacement function introducing the kinematic compatibility $\varepsilon(x) = du(x)/dx$ and the stress–strain relation $\sigma_l(x) = \overline{E}\varepsilon(x)$:

$$\overline{E}\frac{d^2u(x)}{dx^2} + \int_0^L A(\xi)[u(\xi) - u(x)]g(x,\xi)\,d\xi = -f(x) \tag{9}$$

this integro-differential equation rules the displacement field along the bar accounting for the long-range forces $q(x,\xi)$ expressed as in Eq. (3).

Summing up the elastic problem of the non-local 1D continuum may be written in terms of equilibrium, compatibility and constitutive law as follows:

$$\begin{cases} \dfrac{d\sigma(x)}{dx} = \dfrac{d\sigma_l}{dx} + \displaystyle\int_0^L A(\xi)[u(\xi) - u(x)]g(x,\xi)\,d\xi \\ \dfrac{du(x)}{dx} = \varepsilon(x) \\ \sigma(x) = \sigma_l(x) + \sigma_{nl}(x) = \overline{E}\varepsilon(x) + \displaystyle\int_{\xi_1=0}^{x}\int_{\xi_2=x}^{L} A(\xi_2) \\ \qquad \times[u(\xi_2) - u(\xi_1)]g(\xi_1,\xi_2)\,d\xi_2\,d\xi_1 \end{cases} \tag{10a – c}$$

Together with the kinematic and static boundary conditions

$$u(0) = u_0; \quad u(L) = u_L \tag{11a}$$

$$\sigma_l(0)A(0) = -F_0; \quad \sigma_l(L)A(L) = F_L \tag{11b}$$

It is worth noting, that conditions in Eq. (11b) are valid, since the double integral in Eq. (10c) vanishes at the boundaries. In fact the external forces are equilibrated, at the boundaries of the bar, only by the local internal stresses, such that the static boundary conditions may be enforced without difficulties, since they involve the first derivative of the displacement field, as in classic mechanics.

Moreover, the constitutive relation reported in Eq. (10c) reverts to the well-known Eringen model of strong non-local elasticity (Eringen, 1972):

$$\sigma(x) = \overline{E}\varepsilon(x) + \int_{-\infty}^{\infty} A\bar{g}(x,\xi)\varepsilon(\xi)\,d\xi \tag{12}$$

being $\bar{g}(x,\xi)$ the model attenuation function, under the conditions:

(i) Uniform cross-section of the 1D model $A(\xi) = A =$ const. (ii) Unbounded domain that implies no impending boundaries. (iii) An exponential or fractional-type distance-decaying function (Di Paola and Zingales, 2008; Di Paola et al., 2009). Those conditions ensures that the proposed model of long-range forces coalesces with the Eringen model in the unbounded domain under restrictive assumption on the attenuation function. For a bounded domain the constitutive law must be expressed as in Eq. (10c) that does not coalesce with Eringen model being (see Di Paola et al., 2009 for details):

$$\int_0^L A(\xi)\bar{g}(x,\xi)\varepsilon(\xi)\,d\xi \neq \int_{\xi_1=x}^{L}\int_{\xi_2=0}^{x} A(\xi)g(\xi_1,\xi_2)[u(\xi_2) - u(\xi_1)]\,d\xi_2\,d\xi_1 \quad (13)$$

On this solid ground in the next sections the fundamental identity and the variational theorems applied to the proposed model of long-range forces will be addressed and discussed in detail.

3. The fundamental relations in presence of long-range central interactions

The fundamental relations of the continuum mechanics with long-range interactions may be obtained, as in classical continuum mechanics, by evaluating the internal work done by the contact forces and by long-range interactions. By looking at Fig. 3, where there is depicted a simplified model with only four elements for clearness sake, the resultants of the long range forces, namely Q_j $(j = 1,\dots,4)$ take the following form:

$$\begin{aligned} Q_1 &= Q^{(1,2)} + Q^{(1,3)} + Q^{(1,4)}; \quad Q_2 = Q^{(2,3)} + Q^{(2,4)} - Q^{(1,2)} \\ Q_3 &= Q^{(3,4)} - Q^{(2,3)} - Q^{(1,3)}; \quad Q_4 = -Q^{(1,4)} - Q^{(2,4)} - Q^{(3,4)} \end{aligned} \quad (14)$$

the internal work associated to the displacement $u_j = u(x_j)$ $(j = 1,\dots,4)$ is given as:

$$\begin{aligned} L_{\text{int}} &= N_1u_1 + (N_2 - N_1)u_2 + (N_3 - N_2)u_3 - N_3u_4 \\ &+ \left(Q^{(1,2)} + Q^{(1,3)} + Q^{(1,4)}\right)u_1 + \left(Q^{(2,3)} + Q^{(2,4)}\right)u_2 \\ &- Q^{(1,2)}u_2 + Q^{(3,4)}u_3 - \left(Q^{(2,3)} + Q^{(1,3)}\right)u_3 \\ &- \left(Q^{(2,4)} + Q^{(3,4)} + Q^{(1,4)}\right)u_4 \end{aligned} \quad (15)$$

Which, extended to the case of an arbitrary number m of volumes, is written as:

$$L_{\text{int}} = \sum_{j=1}^{m}(N_j - N_{j-1})u_j + \sum_{j=1}^{m}\left(\sum_{r=j+1}^{m} Q^{(j,r)} - \sum_{h=1}^{j-1} Q^{(h,j)}\right)u_j \quad (16)$$

Introducing Eq. (2) into Eq. (16) it yields:

$$\begin{aligned} L_{\text{int}} &= \sum_{j=1}^{m}\frac{(N_j - N_{j-1})}{\Delta x}u_j\Delta x \\ &+ \sum_{j=1}^{m}\left(\sum_{r=j+1}^{m} q^{(j,r)}A_jA_r - \sum_{h=1}^{j-1} q^{(h,j)}A_jA_h\right)u_j\Delta x^2 \end{aligned} \quad (17)$$

whose limit for $\Delta x \to 0$ is:

$$\begin{aligned} L_{\text{int}} &= \int_0^L \frac{dN(x)}{dx}u(x)\,dx \\ &+ \int_0^L \left(\int_x^L A(x)A(\xi)q(x,\xi)\,d\xi - \int_0^x A(x)A(\xi)q(\xi,x)\,d\xi\right)u(x)\,dx \\ &= \int_0^L \frac{dN(x)}{dx}u(x)\,dx + \int_0^L\int_0^L A(x)A(\xi)q(x,\xi)u(x)\,dx \end{aligned} \quad (18)$$

where the latter equality has been established with $q(\xi,x) = -q(x,\xi)$. The latter contains two terms: (i) the contribution related to the work done by the contact forces $N(x)$ and (ii) the contribution due to the work done by the long-range interactions $q(x,\xi)$. The expression of the internal work done by long-range interactions and contact forces reported in Eq. (18) has been simply derived on the basis of physical model at hand but it does not contain the state variables of the elastic problem.

Moreover, in order to write Eq. (18) in terms of the state variable, that is a fundamental step in the formulation of a linear theory of elasticity accounting for long-range interactions, it needs to recast Eq. (17) in the equivalent form:

$$L_{\text{int}} = -\sum_{j=1}^{m} N_j\frac{(u_{j+1} - u_j)}{\Delta x}\Delta x - \sum_{j=1}^{m}\left(\sum_{i=j+1}^{m} q^{(j,i)}A_iA_j\right)(u_j - u_i)\Delta x^2 \quad (19)$$

yielding at the limit, the expression of the internal work as:

$$L_{\text{int}} = -\int_0^L N(x)\frac{du}{dx}\,dx + \int_0^L \left(\int_x^L A(x)A(\xi)q(x,\xi)(u(\xi) - u(x))\,d\xi\right)dx \quad (20)$$

At this stage, it is worthy mentioning that the internal work represented in Eq. (19) and derived by means of mechanical considerations involves a double sum with recursive index, yielding the x-dependence of the inner integration boundary, reported in Eq. (20). On the other hand, Eq. (19) may also be rewritten in the equivalent form (adding all the contributions from the right) as:

$$L_{\text{int}} = -\sum_{j=1}^{m} N_j\frac{(u_{j+1} - u_j)}{\Delta x}\Delta x + \sum_{j=1}^{m}\left(\sum_{i=1}^{j+1} q^{(i,j)}A_iA_j\right)(u_i - u_j)\Delta x^2 \quad (21)$$

and summing Eqs. (19) and (21) and taking the limit for $\Delta x \to 0$ yields the equivalent expression of the internal work in the following form:

$$L_{\text{int}} = -\sum_{j=1}^{m} N_j\frac{(u_{j+1} - u_j)}{\Delta x}\Delta x + \frac{1}{2}\sum_{j=1}^{m}\left(\sum_{i=1}^{m} q^{(j,i)}A_iA_j\right)(u_i - u_j)\Delta x^2 \quad (22a)$$

$$\begin{aligned} L_{\text{int}} &= -\int_0^L N(x)\frac{du}{dx}\,dx \\ &+ \frac{1}{2}\int_0^L \left(\int_0^L A(x)A(\xi)q(x,\xi)(u(\xi) - u(x))\,d\xi\right)dx \end{aligned} \quad (22b)$$

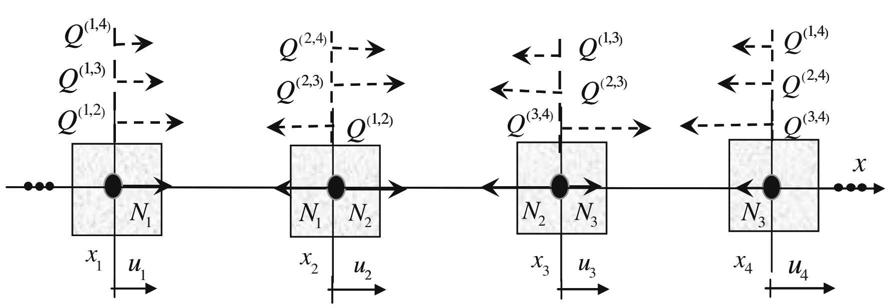

Fig. 3. Evaluation of the internal work in presence of long-range interactions.

that does not involve moving integration boundaries and it still involves relative displacements between volume elements of the bar. Eq. (22b) is the more convenient expression for the extension to multidimensional case as well as for the development of approximate solutions of elastic problems with long-range interactions.

The equivalent expressions reported in Eqs. (18) and (20) and in Eq. (22) are valid for any class of long-range interactions with the only requirement to be central forces proportional to the volumes of the interacting elements.

The first term in Eq. (20) and in Eq. (22) could also be obtained integrating by parts the first integral in Eq. (18), whereas the second integral term is the internal work done by the long-range interactions by relative displacement of the interacting volumes. These concepts are the two crucial steps necessary to formulate the fundamental relations of the non-local continuum mechanics with long-range forces.

To this aim let us suppose that $\tilde{u}(x)$ is a compatible displacement field satisfying the kinematic boundary conditions of the 1D solid and let us define $\hat{\sigma}(x) = \hat{\sigma}_l(x) + \hat{\sigma}_{nl}(x)$ an equilibrated stress field with respect to volume forces $\hat{f}(x)$ and forces $\widehat{F}_0$, $\widehat{F}_L$ at the borders.

The work done by the body force field $\hat{f}(x)$ through the axial displacement $\tilde{u}(x)$ is written in the left-hand side of the following identity:

$$\int_0^L A(x)\hat{f}(x)\tilde{u}(x)\,dx = -\int_0^L A(x)\frac{d\hat{\sigma}}{dx}\tilde{u}(x)\,dx = -\left[\int_0^L A(x)\tilde{u}(x)\frac{d\hat{\sigma}_l}{dx}\,dx + \int_0^L A(x)\tilde{u}(x)\frac{d\hat{\sigma}_{nl}}{dx}\,dx\right] \tag{23}$$

since Eq. (6) holds true. Integrating by parts the first term at the right-hand side of Eq. (23) after some straightforward algebra, it leads to the fundamental identity of mechanics $L_{\text{ext}} = L_{\text{int}}$ being external L_{ext} and internal L_{int} virtual work expressed, respectively:

$$L_{\text{ext}} = \tilde{u}(0)\widehat{F}_0 + \tilde{u}(L)\widehat{F}_L + \int_0^L A(x)\hat{f}(x)\tilde{u}(x)\,dx \tag{24}$$

$$L_{\text{int}} = \int_0^L A(x)\hat{\sigma}_l(x)\frac{d\tilde{u}}{dx}\,dx - \frac{1}{2}\int_0^L\left(\int_0^L A(x)A(\xi)\hat{q}(\xi,x)\tilde{\eta}(x,\xi)\,d\xi\right)dx \tag{25}$$

Interestingly Eq. (25) shows that, the internal virtual work, in presence of long-range cohesive interactions, is composed by two different contributions: (i) the virtual work done by the local stress $\hat{\sigma}_l(x)$ by the first derivative $d\tilde{u}(x)/dx$ that is perfectly analogous to the classical mechanics case and (ii) the virtual work done by the long-range interactions $\hat{q}(x,\xi)$ by the relative displacement $\tilde{\eta}(x,\xi) = \tilde{u}(\xi) - \tilde{u}(x)$.

At this stage it is necessary to point out the importance of the fundamental identity in the context of the proposed, physically-based, approach to non-local continuum, introducing the long-range interactions $\hat{q}(x,\xi)$. The equivalence $L_{\text{ext}} = L_{\text{int}}$ stands the basis to represent, consistently, the static-kinematic duality and henceforth to lead to the correct representation of the state variable, $\tilde{\eta}(x,\xi) = \tilde{u}(\xi) - \tilde{u}(x)$, dual to the long-range forces $\hat{q}(x,\xi)$.

The fundamental identity for the 1D continuum including long-range central interactions may be particularized in three different forms as follows.

3.1. Principle of virtual displacements

Let us assume that $\sigma(x)$, $f(x)$ and F_0, F_L are the (equilibrated) real stress and external loads applied on the 1D solid, while $\tilde{u}(x) = \delta u(x)$ and $\tilde{\varepsilon}(x) = \delta\varepsilon(x) = \delta\frac{du(x)}{dx} = \frac{d}{dx}\delta u(x)$ are arbitrary variation $\delta(\cdot)$ of a kinematically admissible displacement field. In this context the relation $L_{\text{ext}} = L_{\text{int}}$ is rewritten in the following form:

$$\begin{aligned}\delta u(0)F_0 + \delta u(L)F_L + \int_0^L A(x)f(x)\delta u(x)\,dx &= \int_0^L A(x)\sigma_l(x)\delta\frac{du(x)}{dx}\,dx - \frac{1}{2}\int_0^L\left(\int_0^L A(x)A(\xi)q(\xi,x)\delta(u(\xi)-u(x))\,d\xi\right)dx\\ &= \int_0^L A(x)\sigma_l(x)\frac{d}{dx}\delta u(x)\,dx - \int_0^L\int_0^L A(x)A(\xi)q(\xi,x)\delta u(\xi)\,d\xi\,dx\end{aligned} \tag{26}$$

Eq. (26) holds true for any displacement field $\delta u(x)$ under the kinematic condition $\frac{d}{dx}\delta u(x) = \delta\frac{d}{dx}u(x) = \delta\varepsilon$ so that the equilibrium equations in Eq. (11) are satisfied for the real stress field $\sigma(x)$ and the external loads $f(x)$ and F_0, F_L.

3.2. Principle of virtual forces

Let us assume that $u(x)$ and $\varepsilon(x)$ are the real displacement field, the axial strains and relative displacement field $\eta(x,\xi)$ and that $\hat{\sigma}_l(x) = \delta\sigma_l(x)$, $\hat{f}(x) = \delta f(x)$, $\widehat{F}_0 = \delta F_0$, $\widehat{F}_L = \delta F_L$ and $\hat{q}(x,\xi) = \delta q(x,\xi)$ are arbitrary, but equilibrated local stress and long-range forces under variations of the applied loads $\delta f(x)$, δF_0, δF_L. The fundamental relation $L_{\text{ext}} = L_{\text{int}}$ is represented as:

$$u_0\delta F_0 + u_L\delta F_L + \int_0^L A(x)\delta f(x)u(x)\,dx = \int_0^L A(x)\delta\sigma_l(x)\varepsilon(x)\,dx - \frac{1}{2}\int_0^L\left(\int_0^L A(x)A(\xi)\delta q(\xi,x)\eta(x,\xi)\,d\xi\right)dx \tag{27}$$

may be transformed accounting for the equilibrium equations:

$$\frac{d\delta\sigma_l(x)}{dx} + \int_0^L A(\xi)\delta q(x,\xi)\,d\xi = -\delta f(x)\delta\sigma_l(0)A(0) = -\delta F_0; \quad \delta\sigma_l(L)A(L) = \delta F_L \tag{28a, b}$$

and multiplying Eq. (28a) by $A(x)u(x)$ it yields, after integration in $[0, L]$, the following identity:

$$u(x)A(x)\delta\sigma_l(x)|_0^L - \int_0^L A(x)\frac{du(x)}{dx}\delta\sigma_l(x)\,dx + \int_0^L A(x)u(x)\delta f(x)dx + \int_0^L\left(\int_0^L A(x)A(\xi)\delta q(x,\xi)\,d\xi\right)u(x)\,dx = 0 \tag{29}$$

that added to the right-hand side of Eq. (27) and accounting for Eqs. (18) and (22) and Eqs. (28b) it yields

$$(u_0 - u(0))\delta F_0 + (u_L - u(L))\delta F_L = \int_0^L A(x)\left(\varepsilon(x) - \frac{du(x)}{dx}\right)\delta\sigma_l(x)\,dx + \frac{1}{2}\int_0^L\int_0^L A(x)A(\xi)[\eta(x,\xi) - (u(\xi) - u(x))]\delta q(x,\xi)\,d\xi\,dx \tag{30}$$

and since the equilibrium equations are satisfied for any static variables $\hat{\sigma}_l(x) = \delta\sigma_l(x)$, $\hat{f}(x) = \delta f(x)$, $\widehat{F}_0 = \delta F_0$, $\widehat{F}_L = \delta F_L$ and $\hat{q}(x,\xi) = \delta q(x,\xi)$ that satisfy the equilibrium relation, Eq. (30) the field conditions $\varepsilon(x) = du(x)/dx$, $\eta(x,\xi) = u(x) - u(\xi)$ as well as the essential boundary conditions $u_0 = u(0)$; $u_L = u(L)$.

3.3. Energy balance

The energy balance between the applied loads and the corresponding strain energy stored in the bar is obtained under the assumption that in the fundamental equation $(L_{\text{ext}} = L_{\text{int}})$ the kinematic variables $u(x)$ and $\varepsilon(x)$ and the static variables $\sigma_l(x)$ and $q(x,\xi)$ are the actual ones (compatible and equilibrated solutions). Under these assumptions:

$$L_{ext} = u(0)F_0 + u(L)F_L + \int_0^L A(x)f(x)u(x)\,dx = \int_0^L A(x)\sigma_l(x)\frac{du(x)}{dx}\,dx + \frac{1}{2}\int_0^L\left(\int_0^L A(x)A(\xi)q(x,\xi)(u(x)-u(\xi))\,d\xi\right)dx = L_{int} \quad (31)$$

Eq. (31) coalesces with Eq. (22b) since $A(x)\sigma_l(x) = N(x)$. In the energy balance the first term is the well-known local contribution to the strain energy, while the second one is the contribution of the long-range terms to the internal stored energy of the body.

4. The variational theorems of 1D continuum with long-range forces

The elastic potential energy and the complementary elastic potential energy stored in the 1D continuum with long-range forces may now easily derived from the results obtained in the previous section.

Let us introduce the constitutive relations $\sigma_l(x) = \overline{E}\varepsilon(x)$ and those about long-range forces introduced in Eq. (3a) into the energy balance reported in Eq. (31)

$$\frac{1}{2}\left[u(0)F_0 + u(L)F_L + \int_0^L A(x)f(x)u(x)\,dx\right] = \frac{1}{2}\left[\int_0^L A(x)\overline{E}\varepsilon(x)^2\,dx + \frac{1}{2}\int_0^L\left(\int_0^L g_A(x,\xi)(u(x)-u(\xi))^2\,d\xi\right)dx\right] \quad (32)$$

where we set the symmetric distance-decaying function $g_A(x,\xi) = A(x)A(\xi)g(x,\xi)$ for simplicity of notation. We define the right-hand side of Eq. (32) as the potential elastic energy $\Phi(\varepsilon,\eta)$ in the following form:

$$\Phi(\varepsilon,\eta) = \Phi_l(\varepsilon) + \Phi_{nl}(\eta) = \int_0^L \varphi_l(\varepsilon)\,dx + \frac{1}{2}\int_0^L\int_0^L \varphi_{nl}(\eta(x,\xi))\,d\xi\,dx \quad (33)$$

with $\eta(x,\xi)$ the relative displacement field and the local $\varphi_l(\varepsilon)$ and non-local $\varphi_{nl}(\eta(x,\xi))$ potential energy density (for unitary length) defined as:

$$\varphi_l(\varepsilon) = \frac{1}{2}A(x)\overline{E}\varepsilon^2; \quad \varphi_{nl}(\eta) = \frac{1}{2}g_A(x,\xi)(\eta(x,\xi))^2 \quad (34a, b)$$

The potential densities defined in Eq. (34a) and (34b) have been introduced upon the kinematic variables $\varepsilon(x) = du/dx$ and $\eta(x,\xi)$ and are composed of two contributions: (i) a local contribution represented by the first term in Eq. (33) and (ii) a non-local contribution depending on the state variable $\eta(x,\xi) = u(x) - u(\xi)$, consistently associated to the long-range cohesive interaction $q(x,\xi)$.

Similar arguments may be used to define the dual elastic potential energy of the 1D solid including long-range effects, dubbed Ψ, as:

$$\Psi = \Psi_l(\sigma_l) + \Psi_{nl}(q) = \int_0^L \psi_l(\sigma_l)\,dx + \frac{1}{2}\int_0^L\int_0^L \psi_{nl}(q(x,\xi))\,d\xi\,dx \quad (35)$$

With the local complementary energy density and the non-local counterpart, namely $\psi_l(\sigma_l)$ and $\psi_{nl}(q(x,\xi))$ defined, respectively:

$$\psi_l(\sigma_l) = \frac{A(x)\sigma_l^2}{2\overline{E}}; \quad \psi_{nl}(q) = (A(x)A(\xi))^2\frac{q(x,\xi)^2}{2g_A(x,\xi)} \quad (36a, b)$$

The mathematical consistency, of the introduced elastic potential energy density, may be assessed deriving the densities $\varphi_l(\varepsilon)$ and $\varphi_{nl}(\eta(x,\xi))$ with respect to the state variables as:

$$\sigma_l = \frac{1}{A(x)}\frac{\partial\varphi_l}{\partial\varepsilon} = \overline{E}\varepsilon; \quad q(x,\xi) = \frac{1}{A(x)A(\xi)}\frac{\partial\varphi_{nl}}{\partial\eta} = g(x,\xi)\eta(x,\xi) \quad (37a, b)$$

that are fully correspondent to the used constitutive relations for the local Cauchy stress σ_l and the non-local central forces $q(x,\xi)$. The inverse constitutive relations may be obtained by differentiating the complementary densities $\psi_l(\sigma_l)$ and $\psi_{nl}(q)$ with respect to the state variables yielding:

$$\varepsilon = \frac{1}{A(x)}\frac{\partial\psi_l}{\partial\sigma_l} = \frac{\sigma_l}{\overline{E}}; \quad \eta(x,\xi) = \frac{1}{A(x)A(\xi)}\frac{\partial\psi_{nl}}{\partial q} = \frac{q(x,\xi)}{g(x,\xi)} \quad (38a, b)$$

Corresponding to the inverse constitutive relations that can also be obtained from simple manipulation of Eq. (3).

The constitutive relations between the dual variables η and q, reported in Eqs. (37b) and (38b) deserves some further considerations about the requirements of the function $g(x,\xi)$. Such a distance decaying function has been introduced, based upon the mechanical consideration that the long-range cohesive interactions oppose to relative displacements so that function $g(x,\xi)$ was assumed symmetric and positive definite. Once the elastic potential energy $\varphi_{nl}(\eta)$ or the complementary elastic energy $\psi_{nl}(q)$ has been introduced the same requirements may be withdrawn from energetic considerations. In this context, in fact, the reversible transformations of linear elasticity may be guaranteed *if and only if* the elastic potential energy of the elastic body is a symmetric, convex and positive definite functional of the state variables. This consideration leads to conclude that the distance-decaying function $g(x,\xi)$ must be symmetric and positive definite yielding convex elastic potentials in Eqs. (34b) and (36b). Moreover it has been proved that the Drucker stability criterion is locally fulfilled only if the decaying function is strictly positive in the whole domain (Di Paola et al., 2009).

Some further comments must be reported, at this stage, about the differences between the proposed model of elastic potential energy provided by the physically-based approach and the elastic potential energy of the strong non-local theories (Kroner, 1967). In this latter models the elastic potential energy of the generalized continuum has been obtained in the following form (with the appropriate symbols, see Eq. (22) Kroner, 1967):

$$\Phi_K = \frac{1}{2}\int_0^L EA(x)\varepsilon(x)^2\,dx + \frac{1}{4}\int_0^L\int_0^L A^2\bar{g}(x,\xi)\varepsilon(x)\varepsilon(\xi)\,d\xi\,dx + \Phi_r \quad (39)$$

where Φ_r is some residual energy that vanishes by enforcing some conditions in the interior domain and the border of the solid (see Eqs. (13)–(15), Kroner, 1967) that *de facto* are limitations on the function $\bar{g}(x,\xi)$. By enforcing such conditions in Eq. (39) and properly manipulating the double integral, the Eringen model expressed in Eq. (12) is restored also for a bounded domain. This means that, in order to set $\Phi_r = 0$, the mathematical restrictions on the kernel to derive the Eringen model require the knowledge of the specific surface and specific volume at microstructure level close to the borders that are seldom available. On the other hand, in the authors' opinion the distance-decaying function must only depend on the material properties of the body and not on its boundary conditions and this condition may be achieved with proper definition of the elastic potential energy obtained by combination of Eqs. (33) and (34) yielding:

$$\Phi = \frac{1}{2}\int_0^L A(x)\overline{E}\varepsilon(x)^2\,dx + \frac{1}{4}\int_0^L\int_0^L g_A(x,\xi)(u(x)-u(\xi))^2\,d\xi\,dx \quad (40)$$

By comparing Eq. (39) to Eq. (40) we may state that in the proposed model the residual energy Φ_r is not present and the ingredients of the first term at the right-hand side of Eq. (39) are the same, that is contact forces are related to the strain. Moreover, the main difference between Eqs. (39) and (40) remains in the presence of strains products in the integral term that can be achieved only under severe restrictions about the functional class of the attenuation function that must be the solution of a proper boundary value problem involving also, as parameters, the specific surface and the specific volume of the solid material. This difference is substantial since Eq. (40), obtained with the aid of physical model, allows a consistent formulation of the elastic problem involving long-range forces in bounded domain. As in fact from Eq. (40) the governing Eq. (10a) and (10c) and the boundary conditions both kinematic and static (Eqs. (11a) and (11b)) may be fully restored, without any mathematical assumption on the class of the distance-decaying functions, as it will be shown in the following. The previous considerations lead us to conclude that the inconsistencies in the Eringen model for a bounded domain arises from the assumption that in the potential energy the non-local contribution is postulated without an underlying mechanical model.

4.1. The minimum of the total potential energy functional

Once the potential energy densities have been defined, the variational approach to the mechanics of 1D solid with long-range forces may be reported. To this aim let us suppose that the external force field $f(x)$ can be derived by an opportune potential function $P(u)$ as $f = -dP(u)/du$ so that they are conservative. The total potential energy $\Pi(u,\varepsilon,\eta)$ is represented by the relation $\Pi(u,\varepsilon,\eta) = \Phi(\eta,\varepsilon) + P(u)$ that is a function of the strain field and of the displacement field of the body. Under the assumption that $u(x)$ and $\varepsilon(x)$ are, respectively, the displacement and the strain field solution of the elastic problem with long-range forces, the first variation of $\Pi(u,\varepsilon,\eta)$ reads:

$$\delta\Pi(u,\varepsilon,\eta) = \delta\Phi(\varepsilon,\eta) - \left[\int_0^L Af(x)\delta u(x)\,dx + F_L\delta u(L) + F_0\delta u(0)\right] \tag{41}$$

with the first variation of the elastic potential energy defined as:

$$\delta\Phi(\varepsilon,\eta) = \int_0^L A(x)\varepsilon(x)\overline{E}\delta\varepsilon\,dx + \frac{1}{2}\int_0^L\left(\int_0^L g_A(x,\xi)\eta(x,\xi)\delta\eta\,d\xi\right)dx \tag{42}$$

The first variation of the total elastic potential energy in Eq. (42) vanishes as $\delta\Pi(u,\varepsilon,\eta) = 0$ in correspondence of the solution of the elastic problem, since it represents the equivalence between the work done by external loads and by internal elastic forces, as predicted by the two sides of the energy balance in Eq. (31).

The introduction of the functional relation represented by the total elastic potential energy $\Pi(u,\varepsilon,\eta)$ yields a minimum theorem for the total potential energy function $\Pi(u,\varepsilon,\eta)$, totally analogous to the case of classical elasticity.

The evidence of such a theorem may be performed evaluating the difference $\Pi(u+\delta u,\varepsilon+\delta\varepsilon,\eta+\delta\eta) - \Pi(u,\varepsilon,\eta)$ as:

$$\begin{aligned}
&\Pi(u+\delta u,\varepsilon+\delta\varepsilon,\eta+\delta\eta) - \Pi(u,\varepsilon,\eta)\\
&\quad= \int_0^L[\varphi_l(\varepsilon+\delta\varepsilon) - \varphi_l(\varepsilon)]\,dx\\
&\qquad+\frac{1}{2}\int_0^L\int_0^L[\varphi_{nl}(\eta+\delta\eta) - \varphi_{nl}(\eta)]\,d\xi\,dx\\
&\qquad-\left[\int_0^L A(x)f(x)\delta u(x)\,dx + F_L\delta u(L) + F_0\delta u(0)\right]
\end{aligned} \tag{43}$$

and introducing Taylor series expansion of local and non-local potential energies truncated to the second-order terms as:

$$\begin{aligned}
\varphi_l(\varepsilon) &\cong \varphi_l(\varepsilon) + \frac{\partial\varphi_l(\varepsilon)}{\partial\varepsilon}\delta\varepsilon + \frac{1}{2}\frac{\partial^2\varphi_l(\varepsilon)}{\partial\varepsilon^2}(\delta\varepsilon)^2\\
\varphi_{nl}(\eta) &\cong \varphi_{nl}(\eta) + \frac{\partial\varphi_{nl}(\eta)}{\partial\eta}\delta\eta + \frac{1}{2}\frac{\partial^2\varphi_l(\eta)}{\partial\eta^2}(\delta\eta)^2
\end{aligned} \tag{44a, b}$$

in Eq. (40), and accounting for $\delta\Pi(u,\varepsilon,\eta) = 0$ as from Eq. (38), it yields:

$$\begin{aligned}
&\Pi(u+\delta u,\varepsilon+\delta\varepsilon,\eta+\delta\eta) - \Pi(u,\varepsilon,\eta)\\
&\quad\cong \frac{1}{2}\int_0^L A(x)\overline{E}(\delta\varepsilon)^2\,dx + \frac{1}{4}\int_0^L\left(\int_0^L g_A(x,\xi)(\delta\eta)^2\,d\xi\right)dx \geqslant 0
\end{aligned} \tag{45}$$

leading to conclude that the solution of the elastic problem of the 1D solid with non-local interactions corresponds to a minimum of the total elastic potential energy.

Similar arguments may also be invoked for the dual functional, the total complementary energy as well as for all the others theorems of linear elasticity as the Betti's and Clapeyron work theorems and they are not reported for shortness sake.

4.2. The Euler–Lagrange equations of the elastic problem with long-range forces

The total elastic potential energy proposed in Section 4.1 provides, with the usual rules of variational calculus, the Euler–Lagrange equations and the natural boundary conditions of the elastic continuum with long-range forces. This formulation serves to validate the physically-based model of non-local elasticity also on variational basis, that means by directly postulating the potential elastic energy functional $\Phi(\varepsilon,\eta)$ as defined from Eq. (33).

To this aim we perform the first variation $\delta\Pi(u,\varepsilon,\eta) = 0$ with respect to the arguments namely $\delta u(x), \delta\varepsilon(x)$ and $\delta\eta(x)$ yielding:

$$\begin{aligned}
\delta\Pi(u,\eta,\varepsilon) = &\left(\int_0^L A(x)\overline{E}\varepsilon(x)\delta\varepsilon\,dx + \frac{1}{2}\int_0^L\left(\int_0^L g_A(x,\xi)\eta(x,\xi)\delta\eta\,d\xi\right)dx\right)\\
&-\int_0^L f(x)A(x)\delta u(x)\,dx + F_0\delta u(0) - F_L\delta u(L)
\end{aligned} \tag{46}$$

And, after some straightforward algebra, accounting for Eqs. (18) and (22) representing the internal work the following equality holds:

$$\begin{aligned}
&\frac{1}{2}\int_0^L\int_0^L g_A(x,\xi)\eta(x,\xi)\delta\eta\,d\xi\,dx\\
&\quad= \int_0^L\int_0^{L-x} g_A(x,\xi)\eta(x,\xi)\delta\eta\,d\xi\,dx\\
&\quad= \int_0^L\left(\int_0^L g_A(x,\xi)(u(\xi)-u(x))\,d\xi\right)\delta u(x)\,dx
\end{aligned} \tag{47}$$

The left-hand side is the first variation of the non-local contribution to strain energy and it can be derived as $\delta\Phi_{nl} = \Phi_{nl}(\eta+\delta\eta) - \Phi_{nl}(\eta)$ and the right-hand side of Eq. (47) is the variation of the internal virtual work due to variation of the displacement field $\delta u(x)$. Eq. (47) may be introduced into Eq. (43) yielding:

$$\begin{aligned}
\delta\Pi(u,\eta,\varepsilon) = \overline{E}&\left(-\int_0^L A(x)\frac{d\varepsilon}{dx}\delta u(x)\,dx + A(x)\varepsilon(x)\delta u(x)|_0^L\right)\\
&+\int_0^L\int_0^L g_A(x,\xi)[u(\xi)-u(x)]\,d\xi\delta u(x)\,dx\\
&-\int_0^L f(x)A\delta u(x)\,dx + F_0\delta u(0) - F_L\delta u(L)
\end{aligned} \tag{48}$$

in which the local contribution has been integrated by parts. Eq. (48) may be recast in more convenient form:

$$\delta\Pi(\eta,\varepsilon)=\int_0^L\left(-\overline{E}A(x)\frac{d^2u}{dx^2}-\int_0^L[u(\xi)-u(x)]g_A(x,\xi)\,d\xi\right.$$
$$\left.-f(x)A(x)\right)\delta u(x)\,dx+\left(F_0+\overline{E}A(0)\varepsilon(x)\right)\delta u(0)$$
$$-\left(F_L-\overline{E}A(L)\varepsilon(x)\right)\delta u(L) \tag{49}$$

yielding $\delta\Pi=0\;\forall\delta u(x)$ only if:

$$\overline{E}\frac{d^2u}{dx^2}+\int_0^L A(\xi)[u(\xi)-u(x)]g(x,\xi)\,d\xi=f(x) \tag{50a}$$

$$\overline{E}A(0)\varepsilon(0)=A\sigma_I(0)=-F_0;\quad \overline{E}A(L)\varepsilon(L)=A\sigma_I(L)=F_L \tag{50b}$$

that is Eqs. (50a) and (50b) are the Euler–Lagrange equations of the posed mechanical problem together with the natural boundary conditions. It may be observed, by direct comparison, that Eqs. (50a) and (50b) coalesce with the governing integro-differential equations and mechanical boundary conditions of the proposed 1D continuum with long-range interactions already derived in Eq. (9) on mechanical considerations.

Summing up the total potential energy function of the kinematic state variables of the problem, involving the convex potential elastic energy and the potential of external load has been introduced. The first variation of such a functional provides, in the context of variational approach, the Euler–Lagrange differential equations and the natural boundary conditions associated to the total elastic potential energy. It has been proved that such Euler–Lagrange equations and the associated natural boundary conditions coalesce with the governing integro-differential equation and the static boundary conditions already introduced on physical bases. This consideration yields the conclusion that the elastic equilibrium problem with the introduction of long-range interactions on physical considerations is well-posed and hence it provides an unique solution.

5. Numerical application

In this section, the proposed model of non-local elastic solid with long-range interactions is applied to a simple 1D case showing the class of displacement field obtained with the proposed model.

Let us consider a 1D bar acted upon by two self-equilibrated point forces F applied at the bar ends. For the long-range forces in Eq. (3), the exponential form

$$g(|x-\xi|)=\frac{(1-\beta_1)\overline{E}}{2A^2\lambda}\exp(-|x-\xi|/\lambda) \tag{51}$$

is taken for the distance-decaying function $g(|x-\xi|)$, where λ is the internal length corresponding to the influence distance beyond which the non-local effects may be neglected. Assume the following values of the bar geometrical and mechanical parameters: $A = 1\ \text{cm}^2$, $\overline{E}=2.1\times10^6\ \text{daN cm}^{-2}$, $L=100$ cm, $F = 100$ daN. The solution of the boundary value integrodifferential problem reported in Eqs. (50a) and (50b) will be obtained introducing a finite difference operator $d/dx\cong\Delta/\Delta x$ and, introducing a discrete grid with abscissas $x_j=(j-1)\Delta x$ $(j=1,2,\ldots,m+1)$ with $\Delta x=L/m$ the following system of algebraic equation is obtained:

$$\begin{aligned}&K^lu_1-K^lu_2+\sum_{h=2}^{m}K^{nl}_{1h}(u_1-u_h)=F_1\Delta x\\&\ldots\\&-K^lu_{j-1}+2K^lu_j-K^lu_{j+1}+\sum_{\substack{h=1\\h\neq j}}^{m}K^{nl}_{jh}(u_j-u_h)=F_j\Delta x\\&\ldots\\&K^lu_m-K^lu_{m+1}+\sum_{h=2}^{m}K^{nl}_{mh}(u_m-u_h)=F_m\Delta x\end{aligned} \tag{52}$$

with $K^l=\overline{E}A/\Delta x$, and the springs connecting non-adjacent points in the position h and j possess a distance-decaying stiffness $K^{nl}_{jh}=A^2\Delta x^2 g(|x_j-x_h|)$. The first terms in Eq. (52) are the contact forces between adjacent points while the summations are non-local forces exerted on points x_j by the surrounding elements located at abscissas x_h. Also, in the right-hand side of Eq. (52) $F_j=Af(x_j)$ are the external nodal forces per unit length. At the limit, when $\Delta x\to0$ Eq. (52) reverts exactly to Eq. (50) and this enables us to affirm that the point-spring model is equivalent to that proposed in Section 2. Be $\mathbf{K}^l$ the tridiagonal matrix:

$$\mathbf{K}^l=\begin{bmatrix}K^l & -K^l & 0 & 0 & \cdots & 0\\ & 2K^l & -K^l & 0 & \cdots & 0\\ & & \cdots & \cdots & \cdots & \cdots\\ & \text{SYM} & & & 2K^l & -K^l\\ & & & & & K^l\end{bmatrix} \tag{53}$$

and $\mathbf{K}^{nl}$ the fully populated non-local stiffness matrix given as

$$\mathbf{K}^{nl}=\begin{bmatrix}K^{nl}_{11} & -K^{nl}_{12} & -K^{nl}_{13} & \cdots & \cdots & -K^{nl}_{1m}\\ & K^{nl}_{22} & -K^{nl}_{23} & \cdots & \cdots & -K^{nl}_{2m}\\ & & \cdots & \cdots & \cdots & \cdots\\ & \text{SYM} & & & K^{nl}_{m-1m-1} & -K^{nl}_{m-1m}\\ & & & & & K^{nl}_{mm}\end{bmatrix} \tag{54}$$

where we denoted $K^{nl}_{jj}=\sum_{h=1,h\neq j}^{m}K^{nl}_{jh}$. The equilibrium relation reported in Eq. (52) may be rewritten in the following matrix form:

$$\mathbf{Ku}=(\mathbf{K}^l+\mathbf{K}^{nl})\mathbf{u}=\mathbf{f} \tag{55}$$

where the load vector reads: $\mathbf{f}^T=\Delta x[F_1\quad F_2\quad\cdots\quad F_m]$. The matrix equation in Eq. (55) corresponds to the governing equations of the point-spring equivalent model introduced in Section 3, as physically equivalent to the proposed elastic model with long-range interactions.

In Fig. 4a, the strain energy in Eq. (42) has been reported as the number N of volume elements increases, for selected values of parameters λ, β_1. Convergence up to the first three digits is generally encountered for $N\geqslant6000$. Therefore, solutions obtained for $N=9000$ of Eq. (52) will be taken as sufficiently accurate solutions for all the response variables shown in the following.

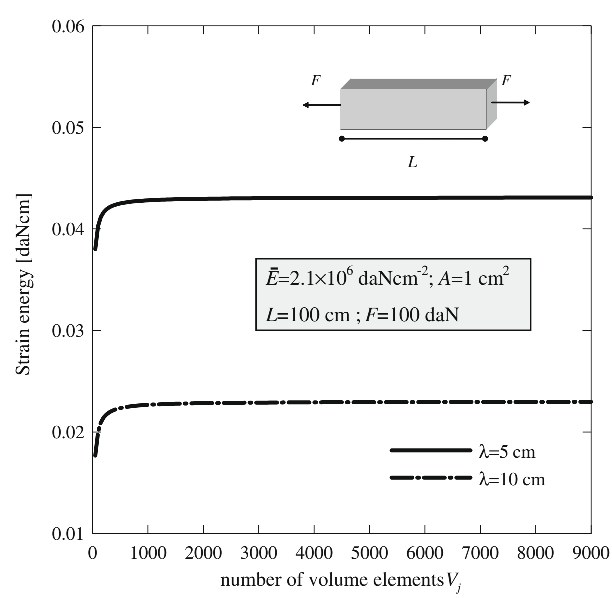

Fig. 4a. Exponential decaying function (51): strain energy response for $\beta_1=0.7$ and different internal lengths λ.

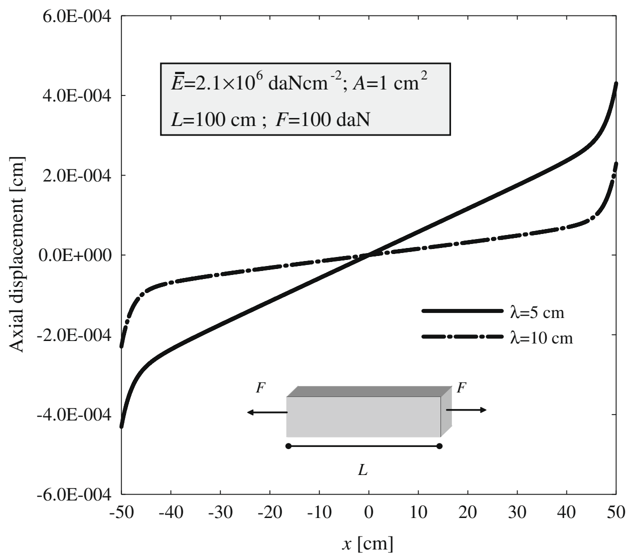

Fig. 4b. Exponential attenuation function (51): displacement response for $\beta_1 = 0.7$ and different internal lengths λ.

Sample of the axial displacement has been reported in Fig. 4b for different values of the internal length scale to highlight the deviation of the displacement field from the linear behaviour predicted by continuum mechanics as the internal length λ increases. Observation of Fig. 4b shows also that there is a central core of the bar that maintains a linear displacement function and the size of the core is strictly influenced by the values of internal length λ. At the limit for $\lambda \to 0$ no edge-effects are detected and the displacement field is linear with vanishing non-local effects. Some further details about parametric effects may be found in other studies (Di Paola et al., 2009).

The proposed approach to the solution of the integrodifferential boundary value problem expressed in Eqs. (50a) and (50b) has been developed from the physically-based model of non-local elasticity proposed in Section 2. Some other solution strategies for the proposed boundary value problem has been reported in the recent literature (Failla et al., 2009).

6. Conclusions

In this paper, a physically-based approach to mechanics of non-local elastic continuum has been investigated within variational context. This approach is the intermediate step in the formulation of the non-local elasticity theory with long-range interactions for tridimensional solids, since it provides the static and kinematic variables of the proposed non-local model.

Despite formulation of such a problem may be provided on mechanical ground the need for variational approaches that involve weak formulations of the elastic problem in terms of elastic potential energy is a necessary step toward the development of analytical approximate solutions and of reliable numerical methods. As in fact numerical stability and convergence of such methods may be assessed only for well-posed elastic problems. To this aim it has been shown that, by the particularization of the virtual work principle, all the well-known theorems of continuum mechanics may be restored introducing additional variables related to the work done by the long-range interactions by the relative displacement of the interacting volumes. The provided expressions of the virtual work theorems allow to introduce the, mathematically and mechanically consistent, state variables of the elastic problem. The state variables of the elastic problem have been used in the context of energy balance, to establish the functional form of the elastic potential and complementary elastic potential of the solid model with long-range interactions. The assumption of quadratic functionals of the state variables of the elastic problem yields the constitutive equations between static-kinematic variables that coincide with the relations postulated on mechanical considerations. The specific class of the distance decaying function that relates the strength of the interactions between non-adjacent volumes, already postulated on physical considerations, has also been discussed in energetic context to yield the reversible transformations of linear elasticity. It has been shown that, as soon as the distance-decaying function is symmetric and positive definite, both the elastic potential and the complementary elastic potential are convex functionals of the state variables of the problems. Thus an unique elastic solution exists as in the context of classical elasticity theory corresponding to minimize the total potential energy of the solid.

The proposed approach has been further investigated, with respect to the governing equations and the position of natural boundary conditions, by the Euler–Lagrange equation of the problem associated to the introduced functional of total elastic potential energy. As in fact the Euler–Lagrange equations of the problem, and the associated natural boundary conditions coincide with the governing equations obtained by means of mechanical considerations. The assessment of a convex potential energy functional of the elastic continuum problem with long-range forces yields to conclude, in addition, that the elastic operators introduced in the model are symmetric functional of the state variables. This consideration is worthy to be reported since all the well-known theorems such as the Betti–Maxwell's reciprocity theorems, Clapeyron work theorem among others, still hold for the proposed physically-based model of non-local elasticity theory. Analogous theorems could not be established, in general, for other integral models non-local elasticity in which symmetry of elastic operators could not be guaranteed in general frameworks. A numerical application has been also reported to show the capabilities of the proposed approach providing also a numerical solution strategy of the governing equations of the elastic problem. It may be shown that the non-local elasticity in conjunction with the introduction of long-range interactions in the elastic model allows for the introduction of a proper length scale, material dependent, that rules the decay of the long-range internal forces beside classical parameters of elasticity theory.

Acknowledgments

The authors are very grateful to the Italian Minister of Educations and University (MIUR) for research Grant PRIN2006, National coordinator Prof. L. Materazzi and to research Grant PRIN2008, national coordinator Prof. A. Carpinteri. Those financial support have been gratefully acknowledged.

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