Singularly Perturbed Linear Boundary Value Problems*

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In this paper an alternative approach to the method of asymptotic expansions for the study of a singularly perturbed linear system with multiparameters and multiple time scales is developed. The method consists of developing a non-singular linear transformation that transforms an arbitrary n-time scale system into diagonal form. This fast and slow mode decomposition provides a modern technique to find an approximate solution of the original system in terms of the solution of an auxiliary system corresponding to the decoupled system. Furthermore, the decoupled system provides a useful mechanism to relate the asymptotic behavior of the solution of the original system and the solution of the degenerate system relative to the original system. © 1992 Academic Press, Inc.

1. INTRODUCTION

Singular perturbations of a two point boundary value problem are an active subject of research with a long history. By employing the asymptotic expansion of such systems under strong conditions on the coefficient matrices, existence, uniqueness, and approximations of solutions of such systems are studied in [11–13]. In [1, 6, 14, 15], under less demanding conditions on the coefficient matrices, boundary value problems for two-time scale linear systems are analyzed. Furthermore, the fast and slow
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mode decomposition approach [1, 9, 15] provides a modern alternative technique to study the singularly perturbed systems.

In this paper, by following a hierarchical order reduction scheme for a joint multiparameter and multi-time scale singular perturbation of a linear system [7, 8, 10], a linear non-singular transformation, which totally decouples an arbitrary n-time scale multiparameter linear singularly perturbed system, is developed. This fast and slow mode decomposition provides a technique to find an approximate solution of the original system in terms of the solution of an auxiliary system corresponding to the decoupled system. Furthermore, the decoupled system provides a useful mechanism to relate the asymptotic behavior of the solution of the original system and the solution of the degenerate system relative to the original system.

This paper is organized as follows: In Section 2, by following the argument of [7, 8, 10], a joint multi-time scale multiparameter singularly perturbed two point boundary value problem is formulated. For the sake of convenience and simplicity a few notations are defined in Section 3. These notations will be used throughout this paper. In Section 4, by following the decoupling procedure of [7], a totally decoupling process is briefly discussed. The validity of the transformation is investigated in Section 5. Moreover, the asymptotic behavior and the representation of the transformation in terms of the given coefficient matrices are given. In Section 6, a much simpler way to find a solution and an approximate solution of the boundary value problem is discussed. Furthermore, it is shown that the limit of the solution of the original boundary value problem is the same as the solution of the boundary value problem for the degenerate system. These results are motivated by the work of Chang [1] and Harris [6].

2. Problem Formulation

Consider a linear time-varying system of differential equations

\[ \varepsilon^i \dot{X}_{ij} = \sum_{p=1}^{n} \sum_{k=1}^{r_p} A_{jk}^{ip}(t) X_{pk} \quad (2.1) \]

with the boundary condition

\[ \sum_{p=1}^{n} \sum_{k=1}^{r_p} [Q_{jk}^{ip}(\varepsilon) X_{pk}(0) + R_{jk}^{ip}(\varepsilon) X_{pk}(1)] = B_{ij}(\varepsilon), \quad (2.2) \]

where \( \varepsilon = (\varepsilon_1^1 \varepsilon_1^2 \cdots \varepsilon_1^r_1 \cdots \varepsilon_n^1 \varepsilon_n^2 \cdots \varepsilon_n^r_n) \), \( X_{ij} \in \mathbb{R}^{n_y}, j \in J(1, r_i), i \in J(1, n) \); the
$r_i$'s are positive integers and $J(a, b) = \{a, a + 1, \ldots, b\}$; $a, b \in \mathbb{Z}^+ \cup \{0\}$, $n_i = \sum_{j=1}^{r_i} n_{ij}$; the dimension of the entire system (2.1) is $N = \sum_{i=1}^{n} n_i$. In (2.1) all the coefficient matrices are continuous on $\mathbb{R}^+$ and have appropriate dimensions. For $j \in J(1, r_i)$ and $i \in J(1, n)$, the parameters $\varepsilon_i^j$ are positive real numbers. For fixed $i \in J(1, n)$, the $\varepsilon_i^j$'s have the same order with respect to $j$. This means that the following inequality is valid:

$$\varepsilon_i^j \leq \frac{\varepsilon_i^k}{\varepsilon_i^{k+1}} \leq \bar{\varepsilon}_i, \quad j, k \in J(1, r_i),$$

where $\varepsilon_i$ and $\bar{\varepsilon}_i$ are positive numbers. Furthermore, for any $j \in J(1, r_i)$, the $\varepsilon_i^j$'s have different orders with respect to $i$.

**Assumption A.1.2.1.**

$$\varepsilon_1 = 1$$

$$\lim_{\varepsilon_i \to 0} \frac{\varepsilon_i+1}{\varepsilon_i} = 0 \quad \text{for each} \quad i \in J(2, n-1)$$

where $\varepsilon_i = (\varepsilon_i^1, \varepsilon_i^2, \ldots, \varepsilon_i^{r_i})^{1/r_i}$.

Using (2.4), we can rewrite the system (2.1)-(2.2) as

$$\varepsilon_i \dot{X}_i = \sum_{j=1}^{n} D_i A_{ji}(t) X_j,$$

with the boundary condition

$$\sum_{j=1}^{n} [Q_{ij}(\varepsilon) X_j(0) + R_{ij}(\varepsilon) X_j(1)] = B_i(\varepsilon), \quad i \in J(1, n),$$

where $X_i = (X_{i1}^T, X_{i2}^T, \ldots, X_{ir_i}^T)^T$; $i \in J(1, n)$; $A_{ij} = (A_{ij}(t))_{n_i \times n_p}$; $Q_{ij}(\varepsilon) = (Q_{ij}(\varepsilon))_{n_i \times n_p}$, $R_{ij}(\varepsilon) = (R_{ij}(\varepsilon))_{n_i \times n_p}$, $B_i(\varepsilon) = (B_i(\varepsilon))_{n_i \times 1}$; $i, p \in J(1, n)$.

For simplicity we omit the arguments of the matrix coefficient functions. The matrices in (2.5) are block matrices that are formed in an obvious way from the coefficient matrices in (2.1) with the $D_i$'s being defined by

$$D_i = \text{diag} \left\{ \left[ \begin{array}{c} \varepsilon_i^1 \\ \varepsilon_i^2 \\ \vdots \\ \varepsilon_i^{r_i} \end{array} \right] I_{11}, \left[ \begin{array}{c} \varepsilon_i^1 \\ \varepsilon_i^2 \\ \vdots \\ \varepsilon_i^{r_i} \end{array} \right] I_{22}, \ldots, \left[ \begin{array}{c} \varepsilon_i^1 \\ \varepsilon_i^2 \\ \vdots \\ \varepsilon_i^{r_i} \end{array} \right] I_{r_i} \right\}, \quad i \in J(1, n),$$

where the $I_{ij}$ are identity matrices of appropriate dimensions. In view of (2.3) and (2.4), the elements of the $D_i$ matrices are bounded, that is,

$$\varepsilon_i \leq \frac{\varepsilon_i}{\bar{\varepsilon}_i} \leq \bar{\varepsilon}_i, \quad j \in J(1, r_i), i \in J(1, n),$$

where the bounds in (2.8) depend on the corresponding bounds in (2.3).
Thus system (2.5)–(2.6) is equivalent to the multiparameter multiple time scale system (2.1)–(2.2).

3. NOTATIONS AND DEFINITIONS

For the sake of simplicity and convenience, let us introduce the following notations.

For \( m \in J(0, n-1) \), \( r \in J(0, m) \), \( l \in J(r, n) \)

\[
X_{lr}^m = (X_1^T(m), X_2^T(m), ..., X_{l-r}^T(m), X_{l-r+1}^T(m-1),
\]

\[
X_{l-r+2}^T(m-2), ..., X_{l}^T(0))^T,
\]

\[
X_{lr}^{m0} = X_{lr}^m(0), \quad X_{lr}^{m1} = X_{lr}^m(1),
\]

where \( (X_1^T(0), X_2^T(0), ..., X_{l}^T(0))^T = (X_1^T X_2^T, ..., X_{l}^T)^T, \)

\[
Q(\varepsilon) = (Q_{i,j}(\varepsilon))_{n \times n}, \quad R(\varepsilon) = (R_{i,j}(\varepsilon))_{n \times n}, \quad B(\varepsilon) = (B_{i}(\varepsilon))_{n \times 1},
\]

\[
I_r = I_{n \times n}, \quad I^r = I_{\sum_{i=1}^r n_i \times \sum_{i=1}^r n_i}, \quad I(r) = I_{\sum_{i=r}^n n_i \times \sum_{i=r}^n n_i},
\]

and for \( m \in J(0, n-2) \)

\[
L_m = (L_{n-m-1} L_{n-m-2}, ..., L_{n-m-1}), \quad M_m = (M_{1-n-m}^T M_{2-n-m}^T, ..., M_{n-m-1-n-m}^T)^T.
\]

Define

\[
\hat{A}_m = \begin{cases} 
D_i A_{ij}, & \text{for } m=0 \text{ and } i, j \in J(1, n) \\
A_{ij}^{m-1} + A_{i-n-m+1,j}^{m-1} L_{n-m+1,j}, & \text{for } m \in J(1, n-1) \text{ and } i, j \in J(1, n-m);
\end{cases}
\]

and

\[
\hat{A}_m = \begin{cases} 
D_i A_{ij}, & \text{for } m=0 \text{ and } i, j \in J(1, n) \\
\hat{A}_{ij}^{m-1} + \hat{A}_{i-n-m+1,j}^{m-1} [\hat{A}_{n-m+1,n-m+1}^{m-1} \hat{A}_{n-m+1,j}^{m-1}], & \text{for } m \in J(1, n-1) \text{ and } i, j \in J(1, n-m).
\end{cases}
\]

\( \hat{A}_m \) can be represented by

\[
\hat{A}_m = \begin{bmatrix} \hat{A}_{11}(m) & \hat{A}_{12}(m) \\ \hat{A}_{21}(m) & \hat{A}_{22}(m) \end{bmatrix},
\]
where \( A_{11}(m) = (\hat{A}^m_{ij})_{(n-m-1) \times (n-m-1)} \), \( A_{12}(m) = \hat{A}^m_{n-m-n-m} \), and \( m \in J(0, n-2) \). For \( m \in J(0, n-2) \), we denote

\[
A^m = \begin{bmatrix}
A_{11}(m) & A_{12}(m) \\
\frac{1}{\varepsilon_{n-m}}A_{12}(m) & \frac{1}{\varepsilon_{n-m}}A_{22}(m)
\end{bmatrix},
\]

where

\[
A_{11}(m) = \left( \frac{1}{\varepsilon_i} \hat{A}^m_{ij} \right)_{(n-m-1) \times (n-m-1)},
\]

\[
A_{12}(m) = \left( \frac{1}{\varepsilon_1} \hat{A}^m_{n-m-n-m} \frac{1}{\varepsilon_2} \hat{A}^m_{2n-m-n-m} \cdots \frac{1}{\varepsilon_{n-m-1}} \hat{A}^m_{n-m-1-n-m} \right)^T,
\]

\[
A_{21}(m) = (\hat{A}^m_{n-m-1-n-m-1} \hat{A}^m_{n-m-2-n-m-1} \cdots \hat{A}^m_{n-m-n-m-1}),
\]

\[ A_{22}(m) = \hat{A}^m_{n-m-n-m}. \]

For \( m \in J(0, n-2) \), we denote

\[
\bar{D}_m = \text{diag}(\varepsilon_1, \varepsilon_2, I_{n-m-1}),
\]

\[
\bar{f}_m = -\hat{A}_{22}(m)^{-1} \hat{A}_{21}(m);
\]

\[
\hat{M}_m = \hat{A}_{12}(m) \hat{A}_{22}(m)^{-1}.
\]

4. **Diagonalization Process**

In this section our aim is to develop a procedure to totally decouple the original system (2.5). This can be done by carrying over the procedure exactly as in [7]. The only difference is that the boundary conditions are changed due to the application of the transformation. By following a procedure as in Section 4 in [7], at the \( m \)th step, we will have

\[
S_{n-m}: \dot{X}^m_{n-m0} = A^mX^m_{n-m0},
\]

\[
e_{n-i}X_{n-i}(i+1) = (A_{22}(i) - e_{n-i}L_iA_{12}(i)) X_{n-i}(i+1), \quad \text{for} \quad i \in J(0, m-1).
\]

\[
Q(m, \varepsilon) X^m_{nm-1} + R(m, \varepsilon) X^m_{nm-1} = B(\varepsilon), \quad m \in J(0, n-2).
\]
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Coupled system (4.1) \((S_{n-m}, \varepsilon)\) can be rewritten as

\[
\begin{align*}
\dot{x}_{n-m-1}^m &= A_{11}(m) x_{n-m-1}^m + A_{12}(m) x_{n-m}(m), \\
\varepsilon_{n-m} \dot{x}_{n-m}(m) &= A_{21}(m) x_{n-m-1}^m + A_{22}(m) x_{n-m}(m), \\
\varepsilon_{n-i} x_{n-i}(i+1) &= (A_{22}(i) - \varepsilon_{n-i} \lambda_i A_{12}(i)) x_{n-i}(i+1), \\
&\text{for } i \in \mathbb{J}(0, m-1).
\end{align*}
\]

(4.3)

Assumption A1.4.3. \(A_{22}(m)\) is non-singular.

Now we apply Chang's transformation [1] defined by

\[
\dot{x}_{n-m+1}^m = \tilde{T}_{n-m} x_{n-m}^m
\]

(4.5)

to system (4.3) with (4.4), and

\[
\tilde{T}_n = T_n,
\]

where

\[
\tilde{T}_{n-m} = \begin{bmatrix} T_{n-m} & 0 \\ 0 & I(n-m) \end{bmatrix}, \quad \text{for } m \in \mathbb{J}(1, n-2)
\]

and

\[
T_{n-m} = \begin{bmatrix} I^{n-m-1} + \varepsilon_{n-m} M_m L_m & -\varepsilon_{n-m} M_m \\ -L_m & I_{n-m} \end{bmatrix}, \quad \text{for } m \in \mathbb{J}(0, n-2),
\]

(4.6)

and \(L_m, M_m\) satisfy the following differential equations:

\[
\begin{align*}
\varepsilon_{n-m} \dot{L}_m &= -\varepsilon_{n-m} L_m (A_{11}(m) + A_{12}(m) L_m) \\
&\quad + A_{21}(m) + A_{22}(m) L_m, \\
\varepsilon_{n-m} \dot{M}_m &= -M_m (A_{22}(m) - \varepsilon_{n-m} L_m A_{12}(m)) \\
&\quad + \varepsilon_{n-m} (A_{11}(m) + A_{12}(m) L_m) M_m + A_{12}(m).
\end{align*}
\]

(4.7)

Then we will have

\[
\begin{align*}
\dot{x}_{n-m-1}^{m+1} &= (A_{11}(m) + A_{12}(m) L_m) x_{n-m-1}^{m+1}, \\
\varepsilon_{n-i} \dot{x}_{n-i}(i+1) &= (A_{22}(i) - \varepsilon_{n-i} \lambda_i A_{12}(i)) x_{n-i}(i+1), \\
Q(m+1, \varepsilon) x_{n-m+1}^{m+1} &= R(m+1, \varepsilon) x_{n-m+1}^{m+1} = B(\varepsilon),
\end{align*}
\]

(4.8)

where

\[
Q(m+1, \varepsilon) = Q(m, \varepsilon) \tilde{T}_{n-m}^{-1}, \quad R(m+1, \varepsilon) = R(m, \varepsilon) \tilde{T}_{n-m}^{-1}.
\]

(4.9)
Furthermore, this fast and slow mode decoupling procedure terminates after \((n - 1)\) steps. In the light of this, \(m\) in (4.1) and (4.2) belongs to \(J(1, n - 1)\). At the end of \((n - 1)\) steps the original system (2.4) with (2.5) will be completely decoupled.

This totally decoupled system is rewritten as

\[
\dot{U}_1 = A_{11}(n - 1) U_1, \tag{4.10}
\]

\[
\varepsilon_{n-m} \dot{U}_{n-m} = [A_{22}(m) - \varepsilon_{n-m} L_m A_{12}(m)] U_{n-m}, \quad \text{for} \; m \in J(0, n - 2).
\]

\[
Q(n - 1, \varepsilon) U^0 + R(n - 1, \varepsilon) U^1 = B(\varepsilon). \tag{4.11}
\]

**Remark 4.1.** From (4.7), for \(m \in J(0, n - 2)\), \(T_{n-m}\) is determined by the following differential equations:

\[
\varepsilon_{n-m} \dot{L}_m = - \varepsilon_{n-m} L_m (A_{11}(m) + A_{12}(m) L_m) + A_{21}(m) + A_{22}(m) L_m,
\]

\[
\varepsilon_{n-m} \dot{M}_m = - \bar{M}_m (A_{23}(m) - \varepsilon_{n-m} L_m A_{12}(m)) + \varepsilon_{n-m} \bar{D}_m (A_{11}(m) + A_{12}(m) L_m) + A_{12}(m) L_m \bar{D}_m^{-1} \bar{M}_m + \bar{D}_m A_{12}(m), \tag{4.12}
\]

where \(\bar{M}_m = \bar{D}_m M_m\).

**Remark 4.2.** We remark that instead of applying transformations \(T_n, T_{n-1}, \ldots, T_{n-m}\) successively, to the original system \((m + 1)\) times, we can apply a composition of transformations

\[
T^{n-m} = T_{n-m} \circ T_{n-m+1} \circ \ldots \circ T_n
\]

to the original system (2.5). An application of this composite transformation \(T^{n-m}\) to (2.5) gives rise to (4.8). The structure of \(T^{n-m}\) is

\[
T^{n-m} = \begin{bmatrix} T_{n-m} & 0 \\ 0 & I(n-m) \end{bmatrix} \circ \begin{bmatrix} P^1_{n-m+1}(1) & C_{n-m+1}(1) \\ R^1_{n-m+1}(1) & P^2_{n-m+1}(1) \end{bmatrix} \circ \ldots \circ \begin{bmatrix} P^1_n & C_n \\ R_n & P^2_n \end{bmatrix},
\]

where, for all \(m \in J(0, n - 2)\) and \(q \in J(0, m - 1)\), \(P^1_{n-q}(k), P^2_{n-q}(k), C_{n-q}(k), R_{n-q}(k)\) are given by the following: for \(1 \leq k \leq n - q - 1\),

\[
[P^1_{n-q}(k)]_{i,j} = \begin{cases} I + \varepsilon_{n-q} M_i \cdot A_{n-q} L_{n-q} j, & \text{for } i - j, 1 \leq i, j \leq n - q - k, \\ \varepsilon_{n-q} M_i \cdot A_{n-q} L_{n-q} j, & \text{for } i \neq j, 1 \leq i, j \leq n - q - k, \end{cases}
\]
\begin{align*}
[R_n(1)]_{i,j} &= -L_{nj}, \quad \text{for } i = 1, 1 \leq j \leq n - q - 1, \\
[R_{n-q}(1)]_{i,j} &= \begin{cases} -L_{n-q} & \text{for } i = 1, 1 \leq j \leq n - q - 1, \\
0, & \text{for } 2 \leq i \leq q + 1, 1 \leq n - q - 1. \end{cases}
\end{align*}

For $2 \leq k \leq n - q - 1$

\begin{align*}
[R_{n-q}(k)]_{i,j} &= \begin{cases} \varepsilon_{n-q}M_{n-q-k+i-n-q}L_{n-q-j}, & \text{for } i = j, 1 \leq i, j \leq n - q - k, \\
[R_{n-q}(k-1)]_{i,j}, & \text{for } 2 \leq i \leq k + q, 1 \leq j \leq n - q, \\
[C_n(1)]_{i,1} &= -\varepsilon_nM_{in}, \quad \text{for } 1 \leq i \leq n - q - 1, \\
[C_{n-q}(1)]_{i,j} &= \begin{cases} -\varepsilon_{n-q}M_{in-q}, & \text{for } 1 \leq i \leq n - q - 1, j = 1, \\
0, & \text{for } 1 \leq i \leq n - q + 1, 2 \leq j \leq q + 1. \end{cases}
\end{cases}
\end{align*}

\begin{align*}
P_n^2(1) = I_n, \quad P_{n-q}(1) &= I(n-q), \\
[P_n^2(2)]_{i,j} &= \begin{cases} I_{n-1} + \varepsilon_nM_{n-1}nL_{n-n-1}, & \text{for } i = 1, j = 1, \\
-L_{n-n-1}, & \text{for } i = 2, j = 1, \\
-\varepsilon_nM_{n-1}n, & \text{for } i = 1, j = 2, \\
I_n, & \text{for } i = 2, j = 2, \end{cases}
\end{align*}

\begin{align*}
[P_{n-q}^2(2)]_{i,j} &= \begin{cases} I_{n-q} + \varepsilon_{n-q}M_{n-q-1}n-Q_{n-q-1}, & \text{for } i = 1, j = 1, \\
L_{n-q-n-1}, & \text{for } i = 2, j = 1, \\
-\varepsilon_{n-q}M_{n-q-1}n, & \text{for } i = 1, j = 2, \\
[P_{n-q}(1)]_{i,j}, & \text{for } 3 \leq i \leq q + 2, j = 1 \\
0, & \text{and } i = 1, 3 \leq j \leq q + 2, \end{cases}
\end{align*}

\begin{align*}
[P_n^2(k)]_{i,j} &= \begin{cases} I_{n-k+1} + \varepsilon_nM_{n-k+1}nL_{n-n-k+1}, & \text{for } i = 1, j = 1, \\
\varepsilon_nM_{n-k+i}L_{n-n-k+i}, & \text{for } 2 \leq i \leq k - 1, j = 1, \\
-\varepsilon_nM_{n-k+1}, & \text{for } i = k, j = 1, \\
-\varepsilon_nM_{n-k+1}n, & \text{for } i = 1, j = k, \\
[P_n^2(k-1)]_{i,j}, & \text{for } 2 \leq i, j \leq k, \end{cases}
\end{align*}
\[ [P_{n-q}^2(k)]_{ij} = \begin{cases} I_{n-q-k+1} + e_{n-q} M_{n-q-k+1}, & \text{for } i = 1, j = 1, \\ L_{n-q-n-q-k+1}, & \text{for } 2 \leq i \leq k - 1, j = 1, \\ e_{n-q} M_{n-q-k+i} L_{n-q-n-q-k+j}, & \text{for } i = 1, 2 \leq j \leq k - 1, \\ -L_{n-q-n-q-k+1}, & \text{for } i = k, j = 1, \\ -e_{n-q} M_{n-q-k+i} n-q, & \text{for } i = 1, j = k, \\ [P_{n-q}^2(k-1)]_{ij}, & \text{for } 2 \leq i, j \leq k, \\ 0, & \text{for } k+1 \leq i \leq k+q, j = 1 \\ \end{cases} \]

We note that \( T_2 \) transforms the original system (2.5)-(2.6) into the totally decoupled system (4.10)-(4.11). Furthermore, we observe that (4.13) is the inverse of \( T_{n-m} \), for \( m \in J(0, n-2) \), and hence \( T_{n-m} \) is invertible. Denoting \( S_{n-m} \) as the inverse of \( T_{n-m} \), \( S^2 = S_{n-m} \circ S_{n-m} \circ \cdots \circ S_2 \) is an inverse of \( T^2 \). Therefore the solution of the original system can be given in terms of the solution of the totally decoupled multi-time scale system (4.10)-(4.11) by

\[ X(t) = S^2(t) U(t), \]

where \( U(t) \) is the solution of (4.10)-(4.11).

5. VALIDITY OF TRANSFORMATIONS

In order to establish the validity of the transformations, we will establish the existence, boundedness, and other fundamental properties of (4.12) for \( m \in J(0, n-2) \).

**Assumption A5.1.** For every \( m \in J(0, n-2) \), the absolute value of the real part of all eigenvalues of \( \tilde{A}_{22}(m) \) are greater than or equal to \( 2\alpha_m \), where \( \alpha_m > 0 \).

**Assumption A5.2.** Coefficient functions \( A_{ij} \), \( i, j \in J(1, n) \) in system (2.5) and \( \tilde{A}_{22}^{-1}(m) \), \( m \in J(0, n-2) \) are bounded.

**Assumption A5.3.** \( \tilde{A}_{22}^{-1}(m) \tilde{A}_{21}(m), \tilde{A}_{12}(m) \tilde{A}_{22}^{-1}(m), m \in J(0, n-2) \), are bounded and satisfy the Lipschitz condition

\[ \| \tilde{A}_{22}^{-1}(m, t) \tilde{A}_{21}(m, t) - \tilde{A}_{22}^{-1}(m, s) \tilde{A}_{21}(m, s) \| \leq l_m |t-s| \]  

for \( 0 \leq t, s \leq 1 \),

\[ \| \tilde{A}_{12}(m, t) \tilde{A}_{22}^{-1}(m, t) - \tilde{A}_{12}(m, s) \tilde{A}_{22}^{-1}(m, s) \| \leq m_m |t-s| \]  

for \( 0 \leq t, s \leq 1 \),

for some positive real numbers \( l_m \) and \( m_m \).
Remark 5.1. From the boundedness of the coefficient matrices $A_{i,j}$, $i, j \in J(1, n)$, and the definition of $A_{11}(0)$, $A_{12}(0)$, $A_{21}(0)$, and $A_{22}(0)$, it is clear that $\varepsilon_n A_{11}(0)$, $\varepsilon_n A_{12}(0)$, $A_{21}(0)$, and $A_{22}(0)$ are bounded. If $L_0$ is bounded then it is clear that $\varepsilon_{n-1} A_{11}(1)$, $\varepsilon_{n-1} A_{12}(1)$, $A_{21}(1)$, and $A_{22}(1)$ are bounded. Continuing this process, one concludes that if $L_{m-1}$ is bounded, then it follows that $\varepsilon_{n-m} A_{11}(m)$, $\varepsilon_{n-m} A_{12}(m)$, $A_{21}(m)$, and $A_{22}(m)$ are bounded.

Moreover, if $L_0 = \hat{L}_0 + O(\varepsilon_n/\varepsilon_{n-1})$ then

$$A_{22}(1) = A_{22}(1) + O\left(\frac{\varepsilon_n}{\varepsilon_{n-1}}\right), \quad A_{21}(1) = A_{21}(1) + O\left(\frac{\varepsilon_n}{\varepsilon_{n-1}}\right),$$

$$A_{12}(1) = A_{12}(1) + D_1 O\left(\frac{\varepsilon_n}{\varepsilon_{n-1}}\right), \quad A_{11}(1) = A_{11}(1) + D_1 O\left(\frac{\varepsilon_n}{\varepsilon_{n-1}}\right).$$

Continuing in this manner, one can conclude that if $L_m = \hat{L}_m + \sum_{k=n-m}^{n} O(\varepsilon_k/\varepsilon_{k-1})$, then

$$A_{22}(m+1) = A_{22}(m+1) + \sum_{k=n-m}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right),$$

$$A_{21}(m+1) = A_{21}(m+1) + \sum_{k=n-m}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right),$$

$$A_{12}(m+1) = A_{12}(m+1) + D_{m+1} \sum_{k=n-m}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right),$$

$$A_{11}(m+1) = A_{11}(m+1) + D_{m+1} \sum_{k=n-m}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right).$$

In the following we state and prove the boundedness and an approximation of $L_m(t)$ for each $m \in J(0, n-2)$.

Theorem 5.1. Under the assumptions A2.1, A5.1, A5.2, and A5.3, for each $m \in J(0, n-2)$, there exist $\varepsilon_{mk} > 0$ such that for all $\varepsilon_k$'s satisfying

$$\frac{\varepsilon_k}{\varepsilon_{k-1}} \leq \frac{\varepsilon_{mk}}{\varepsilon_{mk-1}}, \quad k \in J(2, n-m)$$

the solutions $L_m(t)$ of (4.12) are bounded, and $L_m(t)$ can be approximated by

$$L_m(t) = \hat{L}_m(t) + \sum_{k=n-m}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right), \quad \text{for all } 0 \leq t \leq 1. \quad (5.1)$$

Proof. The proof of the theorem requires a sequential proof. This means that first we prove the existence of a bounded solution $L_m$ of (4.12) for $m = 0$. By using this and Remark 5.1, we prove the existence of a
bounded solution of (4.12) for \( m = 1 \). We continue this sequential process until \( m = n - 2 \). To avoid the repetition in the argument of the proof, in the following we present the proof of existence of bounded solution of \( L_m \) of (4.12) for any \( m \in I(0, n-2) \).

Let \( \phi_1(t) \) be a fundamental matrix solution of

\[
\dot{X} = A_{11}(m, t) X.
\]

Then there exist positive real numbers \( \gamma_m, K_m \) such that

\[
\|\phi_1(t) \phi_1^{-1}(s)\| \leq K_m \exp(\gamma_m |t-s|), \quad 0 \leq t, s \leq 1.
\] (5.2)

where \( \gamma_m = O(\sum_{i=1}^{n-m-1} (1/\varepsilon_i)) \). From Remark 5.1 and Lemma 1 in [3],

\[
\varepsilon_{n-m} \dot{Y} = A_{22}(m, t) Y
\] (5.3)

has a fundamental matrix solution \( \phi_2(t) \), a projection \( P_m \), and a positive real number \( K_{m2} \) such that

\[
\|\phi_2(t) P_m \phi_2^{-1}(s)\| \leq K_{m2} \exp\left[ -\frac{\alpha_m}{\varepsilon_{n-m}} (t-s) \right], \quad 0 \leq s \leq t \leq 1,
\] (5.4)

\[
\|\phi_2(t) (I_{n-m} - P_m) \phi_2^{-1}(s)\| \leq K_{m2} \exp\left[ -\frac{\alpha_m}{\varepsilon_{n-m}} (s-t) \right], \quad 0 \leq t \leq s \leq 1.
\]

Let

\[
\Omega = [T | T: [0, 1] \to \mathbb{R}^{n_{n-m} \times s(n-m-1)}, T \text{ is continuous and } \|T\|_0 < \rho],
\]

where \( s(n-m-1) = \sum_{i=n-m-1}^{n} n_{ij} \), \( \| \cdot \|_0 \)-sup norm.

Define

\[
S: \Omega \to C[[0, 1], \mathbb{R}^{n_{n-m} \times s(n-m-1)}]
\] (5.5)

such that

\[
S(T(t)) = \int_0^t \phi_2(t) P_m \phi_2^{-1}(s) \left[ -T(s) A_{12}(m, s) T(s) + \frac{1}{\varepsilon_{n-m}} A_{21}(m, s) \right] \phi_1(s) \phi_1^{-1}(t) ds
\]

\[
- \int_t^1 \phi_2(t) [I_{n-m} - P_m] \phi_2^{-1}(s)
\]

\[
\left[ -T(s) A_{12}(m, s) T(s) + \frac{1}{\varepsilon_{n-m}} A_{21}(m, s) \right] \phi_1(s) \phi_1^{-1}(t) ds
\]

\[
- \phi_2(t) P_m \phi_2^{-1}(0) A_{22}^{-1}(m, 0) A_{21}(m, 0) \phi_1(0) \phi_1^{-1}(t)
\]

\[
- \phi_2(t) [I_{n-m} - P_m] \phi_2^{-1}(1) A_{22}^{-1}(m, 1) A_{21}(m, 1) \phi_1(1) \phi_1^{-1}(t).
\] (5.6)
Choose $\varepsilon_{mk}/\varepsilon_{mk-1}$, $k \in J(2, n-m)$, and $\rho$ such that the following inequalities are satisfied:

$$
\varepsilon_{n-m} \gamma_m \leq \frac{\gamma_m}{2},
$$

$$
\varepsilon_{n-m} K_{m1}^2 K_{m2}^2 \frac{1}{\gamma_m} \left( \frac{2}{\gamma_m} \|A_{12}(m)\|_0 + \|A_{22}^{-1}(m) A_{21}(m)\|_0 \right) \|A_{12}(m)\|_0 \leq \frac{1}{64},
$$

(5.7)

and

$$
\rho = 4K_{m1} K_{m2} \left( \frac{2}{\gamma_m} \|A_{12}(m)\|_0 + \|A_{22}^{-1}(m) A_{21}(m)\|_0 \right).
$$

With this choice, as in the proof of Theorem in [2], one can show that $S$ maps $\Omega$ into $\Omega$ and $S$ is a contraction mapping. By the Banach Fixed Point theorem, $S$ has a fixed point $L_m$. Moreover

$$
L_m(t) = \int_0^t \phi_2(t) P_m \phi_2^{-1}(s) \left[ -L_m(s) A_{12}(m, s) L_m(s) + \frac{1}{\varepsilon_{n-m}} A_{21}(m, s) \right] ds
$$

$$
\times \phi_1(s) \phi_1^{-1}(t) ds
$$

$$
- \int_0^t \phi_2(t) \left[ I_{n-m} - P_m \right] \phi_2^{-1}(s)
$$

$$
\times \left[ -L_m(s) A_{12}(m, s) L_m(s) + \frac{1}{\varepsilon_{n-m}} A_{21}(m, s) \right] \phi_1(s) \phi_1^{-1}(t) ds
$$

$$
- \phi_2(t) P_m \phi_2^{-1}(0) A_{22}^{-1}(m, 0) A_{21}(m, 0) \phi_1(0) \phi_1^{-1}(t)
$$

$$
- \phi_2(t) \left[ I_{n-m} - P_m \right] \phi_2^{-1}(1) A_{22}^{-1}(m, 1) A_{21}(m, 1) \phi_1(1) \phi_1^{-1}(t)
$$

satisfies (4.12).

This proves the existence of a bounded solution $L_m(t)$ of (4.12).

Now let us find an approximate value of $L_m(t)$ for $0 \leq t \leq 1$. The first term of (5.8) can be rewritten as

$$
\int_0^t \phi_2(t) P_m \phi_2^{-1}(s) \left[ -L_m(s) A_{12}(m, s) L_m(s) + \frac{1}{\varepsilon_{n-m}} A_{21}(m, s) \right] ds
$$

$$
\times \phi_1(s) \phi_1^{-1}(t) ds
$$

$$
= \int_0^t \phi_2(t) P_m \phi_2^{-1}(s) \left[ -L_m(s) A_{12}(m, s) L_m(s) \right] \phi_1(s) \phi_1^{-1}(t) ds
$$

$$
+ \int_0^t \phi_2(t) P_m \phi_2^{-1}(s) \frac{1}{\varepsilon_{n-m}} A_{21}(m, s) \phi_1(s) \phi_1^{-1}(t) ds.
$$

(5.9)
From (5.2), (5.4), and the first term of (5.9), we have

$$\left\| \int_0^t \phi_2(t) P_m \phi_2^{-1}(s) \left[ -L_m(s) A_{12}(m, s) L_m(s) \right] \phi_1(s) \phi_1^{-1}(t) \, ds \right\| = O\left( \frac{e_{n-m}}{e_{n-m-1}} \right). \quad (5.10)$$

Now we will prove that for every $0 \leq t \leq 1$, the second term of (5.9) can be expressed as

$$\int_0^t \phi_2(t) P_m \phi_2^{-1}(s) \frac{1}{e_{n-m}} A_{21}(m, s) \phi_1(s) \phi_1^{-1}(t) \, ds = -\phi_2(t) P_m \phi_2^{-1}(t) A_{22}^{-1}(m, t) A_{21}(m, t)$$

$$+ \phi_2(t) P_m \phi_2^{-1}(0) A_{22}^{-1}(m, 0) A_{21}(m, 0) \phi_1^{-1}(0) \phi_1(t) + O\left( \frac{e_{n-m}}{e_{n-m-1}} \right).$$

Let us rewrite the above integral as

$$\int_0^t \phi_2(t) P_m \phi_2^{-1}(s) \frac{1}{e_{n-m}} A_{21}(m, s) \phi_1(s) \phi_1^{-1}(t) \, ds$$

$$= \int_0^t \phi_2(t) P_m \phi_2^{-1}(s) \frac{1}{e_{n-m}} A_{22}(m, s) A_{22}^{-1}(m, t)$$

$$\times A_{21}(m, t) \phi_1(s) \phi_1^{-1}(t) \, ds - \int_0^t \phi_2(t) P_m \phi_2^{-1}(s) \frac{1}{e_{n-m}} A_{22}(m, s)$$

$$\times \left[ A_{22}^{-1}(m, t) A_{21}(m, t) - A_{22}^{-1}(m, 0) A_{21}(m, 0) \right] \phi_1(s) \phi_1^{-1}(t) \, ds. \quad (5.11)$$

Applying the integration by parts formula to the first term of (5.11), we will have

$$\int_0^t \phi_2(t) P_m \phi_2^{-1}(s) \frac{1}{e_{n-m}} A_{22}(m, s) A_{22}^{-1}(m, t) A_{21}(m, t) \phi_1(s) \phi_1^{-1}(t) \, ds$$

$$= -\phi_2(t) P_m \phi_2^{-1}(t) A_{22}^{-1}(m, t) A_{21}(m, t)$$

$$+ \phi_2(t) P_m \phi_2^{-1}(0) A_{22}^{-1}(m, t) A_{21}(m, t) \phi_1(0) \phi_1^{-1}(t)$$

$$+ \int_0^t \phi_2(t) P_m \phi_2^{-1}(s) A_{22}^{-1}(m, t) A_{21}(m, t) A_{11}(m, s) \phi_1(s) \phi_1^{-1}(t) \, ds. \quad (5.12)$$
From (5.2) and (5.4), the third term of (5.12) can be approximated as

$$\int_0^t \phi_2(t) P_m \phi_2^{-1}(s) A_{22}^{-1}(m, t) A_{21}(m, t) \times A_{11}(m, s) \phi_1(s) \phi_1^{-1}(t) \, ds = O\left(\frac{\varepsilon_{n-m}}{\varepsilon_{n-m-1}}\right). \quad (5.13)$$

Considering the second term in (5.11), from the assumption A5.3 and the inequalities (5.2) and (5.4), we get

$$\left\| \int_0^t \phi_2(t) P_m \phi_2^{-1}(s) \frac{1}{\varepsilon_{n-m}} A_{22}(m, s) [A_{22}^{-1}(m, t) A_{21}(m, t) - A_{22}^{-1}(m, s) A_{21}(m, s)] \phi_1(s) \phi_1^{-1}(t) \, ds \right\| \leq \int_0^t K_{m1} \exp\left[ -\frac{\gamma_m}{\varepsilon_{n-m}} (t-s) \right] l_m(t-s) K_{m2} \exp[\gamma_m(t-s)] \, ds = O(\varepsilon_{n-m}). \quad (5.14)$$

From (5.9)-(5.14) and from the third term of (5.8), we have

$$\int_0^t \phi_2(t) P_m \phi_2^{-1}(s) \left[ -L_m(s) A_{12}(m, s) L_m(s) + \frac{1}{\varepsilon_{n-m}} A_{21}(m, s) \right] \phi_1(s) \phi_1^{-1}(t) \, ds$$

$$- \phi_2(t) P_m \phi_2^{-1}(0) A_{22}^{-1}(m, 0) A_{21}(m, 0) \phi_1(0) \phi_1^{-1}(t)$$

$$= -\phi_2(t) P_m \phi_2^{-1}(t) A_{22}^{-1}(m, t) A_{21}(m, t) + O\left(\frac{\varepsilon_{n-m}}{\varepsilon_{n-m-1}}\right)$$

for every $0 \leq t \leq 1$. \quad (5.15)

Similarly one can prove that

$$\int_0^t \phi_2(t) \left[ I_{n-m} - P_m \right] \phi_2^{-1}(s) \left[ -L_m(s) A_{12}(m, s) L_m(s) + \frac{1}{\varepsilon_{n-m}} A_{21}(m, s) \right] \phi_1(s) \phi_1^{-1}(t) \, ds$$

$$- \phi_2(t) \left[ I_{n-m} - P_m \right] \phi_2^{-1}(1) A_{22}^{-1}(m, 1) A_{21}(m, 1) \phi_1(1) \phi_1^{-1}(t)$$

$$= -\phi_2(t) \left[ I_{n-m} - P_m \right] \phi_2^{-1}(t) A_{22}^{-1}(m, t) A_{21}(m, t) + O\left(\frac{\varepsilon_{n-m}}{\varepsilon_{n-m-1}}\right)$$

for every $0 \leq t \leq 1$. 


This together with (5.15) yields
\[
L_m(t) = -A_{22}^{-1}(m, t) A_{21}(m, t) + O\left(\frac{\varepsilon_{n-m}}{\varepsilon_{n-m-1}}\right) \quad \text{for every } 0 \leq t \leq 1.
\] (5.16)

From this fact, Corollary 2.5.1, and Remark 5.1, we have
\[
A_{21}(m, t) = \hat{A}_{21}(m, t) + \sum_{k=n-m}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right)
\]
\[
A_{22}(m, t) = \hat{A}_{22}(m, t) + \sum_{k=n-m}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right).
\]
Relation (5.1) follows from (5.16) and the above expression. This completes the proof of the theorem.

**Theorem 5.2.** Under the assumptions of Theorem 5.1, for each \(m \in J(0, n - 2)\), there exist \(\varepsilon_{km} > 0\) such that for \(\varepsilon_k\) satisfying
\[
\frac{\varepsilon_k}{\varepsilon_{k-1}} \leq \frac{\varepsilon_{km}}{\varepsilon_{k-m}}, \quad k \in J(2, n-m)
\]
the solutions \(\hat{M}_m\) of (4.12) are bounded for all \(0 \leq t \leq 1\), and
\[
\hat{M}_m(t) = \hat{M}_m(t) + \sum_{k=n-m}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right), \quad \text{for every } 0 \leq t \leq 1. \] (5.17)

**Proof.** Let \(\phi_1(t), \phi_2(t)\) be the fundamental matrix solutions of
\[
\dot{X} = \tilde{D}_m[A_{11}(m) + A_{12}(m) L_m] \tilde{D}_m^{-1} X,
\]
\[
\varepsilon_{n-m} \dot{Y} = [A_{22}(m) - \varepsilon_{n-m} L_m A_{12}(m)] Y, \quad \text{respectively.}
\]
By using Lemma 1 in [3] the following inequalities are satisfied:
\[
\|\phi_1(t) \phi_1^{-1}(s)\| \leq K_{m1} \exp(\gamma_m |t-s|), \quad 0 \leq t, s \leq 1.
\]
\[
\|\phi_2(t) P_m \phi_2^{-1}(s)\| \leq K_{m2} \exp\left(\frac{-\gamma_m}{2\varepsilon_{n-m}} (t-s)\right), \quad 0 \leq s \leq t \leq 1,
\]
\[
\|\phi_2(t)(I_{n-m} - P_m) \phi_2^{-1}(s)\| \leq K_{m2} \exp\left(\frac{-\gamma_m}{2\varepsilon_{n-m}} (s-t)\right), \quad 0 \leq t \leq s \leq 1,
\] (5.18)

where \(K_{m1}, K_{m2}, \gamma_m\) are positive real numbers and \(P_m\) is a projection matrix. Moreover, \(\gamma_m = O(\sum_{k=2}^{n-m-1} (1/\varepsilon_k))\).
Define

$$
\bar{M}_m(t) = \int_0^t \phi_1(t) \phi_1^{-1}(s) \frac{1}{\varepsilon_{n-m}} \bar{D}_m A_{12}(m, s) \phi_2(s) [I_{n-m} - P_m] \phi_2^{-1}(t) \, ds
$$

$$
- \int_t^1 \phi_1(t) \phi_1^{-1}(s) \frac{1}{\varepsilon_{n-m}} \bar{D}_m A_{12}(m, s) \phi_2(s) P_m \phi_2^{-1}(t) \, ds
$$

$$
+ \phi_1(t) \phi_1^{-1}(0) \frac{1}{\varepsilon_{n-m}} \bar{D}_m A_{12}(m, 0) A_{12}^{-1}(m, 0) \phi_2(0)
$$

$$
\times [I_{n-m} - P_m] \phi_2^{-1}(t) \, ds
$$

$$
- \phi_1(t) \phi_1^{-1}(1) \frac{1}{\varepsilon_{n-m}} \bar{D}_m A_{12}(m, 1) \phi_2(1) P_m \phi_2^{-1}(t).
$$

Clearly (5.19) satisfies (4.12).

By choosing $2\varepsilon_{n-m} / m \leq \alpha_m / 2$,

$$
\|\bar{M}(t)\| \leq \frac{2K_{m1}K_{m2}}{\alpha_m} (\|\bar{D}_m A_{12}(m)\|_0 + \|\bar{D}_m A_{12}(m) A_{22}^{-1}(m)\|_0).
$$

This proves the boundedness of $\bar{M}_m$. The proof of (5.17) is analogous to the proof of (5.1). This completes the proof of the theorem.

**Corollary 5.1.** Let the assumptions of Theorem 5.1 be satisfied. Then the differential equation described by

$$
\varepsilon_{n-i} \bar{L}_i(\varepsilon^m) = -\varepsilon_{n-i} L_i(\varepsilon^m)[A_{11}(i, \varepsilon^m) + A_{12}(i, \varepsilon^m)] + A_{21}(i, \varepsilon^m) + A_{22}(i, \varepsilon^m) L_i(\varepsilon^m), \quad \text{for} \quad i > m, \ m \in J(0, n-2)
$$

has a bounded solution, where $\varepsilon^m = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_{n-m-1}, 0, ..., 0)$.

Moreover,

$$
\|I_i(\varepsilon) - L_i(\varepsilon^m)\| \leq \sum_{k=n-m}^n O\left(\frac{\varepsilon_k}{\varepsilon_k - 1}\right),
$$

where $I_i$ is the solution of (4.12).

**Proof.** From Theorem 5.1 and the definition of $A_{kl}$,

$$
A_{kl}(i, \varepsilon) = A_{kl}(i, \varepsilon^m) + \sum_{k=n-m}^n O\left(\frac{\varepsilon_k}{\varepsilon_k - 1}\right), \quad \text{for} \quad 1 \leq k, l \leq 2.
$$

Using the above statement the proof of existence of a bounded solution $L_i(\varepsilon^m)$ is analogous to the proof of Theorem 5.1. To prove the relation
(5.22), consider \((L_i(\varepsilon) - L_i(\varepsilon'))\). By using inequalities (5.2) and (5.4), the rest of the proof is analogous to the proof of Theorem 5.1.

**Corollary 5.2.** Let the assumptions of Theorem 5.2 be satisfied. Then the differential equation described by

\[
\varepsilon_{n-i}\dot{\bar{M}}(\varepsilon^m) = -\bar{M}_i(\varepsilon^m)[A_{22}(i, \varepsilon^m) - \varepsilon_{n-i}L_i(\varepsilon^m)A_{12}(i, \varepsilon^m)] \\
+ \varepsilon_{n-i}\bar{D}^{-1}_i\left[A_{11}(i, \varepsilon^m) + A_{12}(i, \varepsilon^m)L_i(\varepsilon^m)\right]\bar{D}^{-1}_i\bar{M}_i(\varepsilon^m) \\
+ \bar{D}_iA_{12}(i, \varepsilon^m),
\]

has a bounded solution for \(i > m, m \in J(0, n-2)\).

Moreover

\[
\|\bar{M}_i(\varepsilon^m) - \bar{M}_i(\varepsilon)\| \leq \sum_{k=n-m}^n O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right),
\]

where \(\bar{M}_i(\varepsilon)\) is the solution of (4.12).

**Proof.** The proof is similar to the proof of Theorem 5.2 and Corollary 5.1.

### 6. Existence and Approximation Theorems

This section deals with the existence of solutions of the original problem (2.5)-(2.6) by establishing the existence of a solution of the totally decoupled system (4.10) with boundary condition (4.11). Furthermore, the solution of the original problem is analyzed as \(\varepsilon_i \to 0\) for fixed \(i \in J(2, n)\) and it is shown that it converges to a solution of the corresponding degenerate system. An approximate solution of (2.5)-(2.6) is obtained by considering the auxiliary system corresponding to the totally decoupled system (4.10) with boundary condition (4.11). These results are an extension and generalization of earlier results [1, 6, 11, 14].

In the following, we establish the existence of a solution of the totally decoupled system (4.10) with boundary condition (4.11) and the original boundary value problem (2.5)-(2.6). For this purpose, we present the following lemma.

**Lemma 6.1.** Let the assumptions of Theorem 5.1 be satisfied. Then there exist fundamental matrix solutions \(\phi_{n-m}(t)\) of

\[
\varepsilon_{n-m}\dot{U}_{n-m} = [A_{22}(m) - \varepsilon_{n-m}L_mA_{12}(m)]U_{n-m}, \quad m \in J(0, n-2),
\]

where \(U_{n-m}\) is the solution of (4.12).
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and projection matrices $P_m$ such that

$$
\| \phi_{n-m}(t) P_m \phi_{n-m}^{-1}(s) \| \leq K_m \exp \left[ \frac{-2\varepsilon_m}{\varepsilon_{n-m}} (t - s) \right], \quad t \geq s,
$$

(6.2)

$$
\| \phi_{n-m}(t)(I_{n-m} - P_m) \phi_{n-m}^{-1}(s) \| \leq K_m \exp \left[ \frac{-2\varepsilon_m}{\varepsilon_{n-m}} (s - t) \right], \quad s \geq t,
$$

for some positive number $K_m$ which is independent of $\varepsilon_{n-m}$.

Moreover, if $\phi(t) = \phi_1(t, \varepsilon)$ is a fundamental matrix solution of

$$
\dot{U}_{1} = A_{11}(n-1) U_1,
$$

(6.3)

then,

$$
U_i(t) = \mathbb{I}_i(t) \eta_i, \quad i \in J(1, n),
$$

(6.4)

is a bounded solution of (4.10) for any $(\eta_1^T \eta_2^T \cdots \eta_n^T) \in \mathbb{R}^n$, where

$$
\mathbb{I}_i(t) = \phi_i(t),
$$

$$
\mathbb{I}_i(t) = \phi_i(t) P_{n-i} \phi_i^{-1}(0) + \phi_i(t)(I_i - P_{n-i}) \phi_i^{-1}(1), \quad \text{for} \quad i \in J(2, n).
$$

(6.5)

**Proof.** Existence of fundamental matrix solutions $\phi_{n-m}(t)$ for $m \in J(0, n-2)$ of (6.1) satisfying inequalities (6.2) is obvious from Lemma 1 in [3]. From (6.5), it is clear that expression (6.4) satisfies the system of differential equations (4.10). From the boundedness of $A_{11}(n-1)$, it is known that the fundamental matrix solution $\phi_1(t)$ of (6.3) is bounded. The boundedness of solution (6.4) follows from (6.2), (6.3), and the above statement. This completes the proof of the lemma.

**Theorem 6.1.** Let the hypothesis of Lemma 6.1 be satisfied. Let $A(\varepsilon)$ be defined by

$$
A(\varepsilon) = Q(n-1, \varepsilon) \text{diag}\{U_1(0), U_2(0), \ldots, U_n(0)\}
$$

$$
+ R(n-1, \varepsilon) \text{diag}\{U_1(1), U_2(1), \ldots, U_n(1)\},
$$

(6.6)

where the $U_i$'s are as defined in Lemma 6.1. Assume that $A(\varepsilon)$ is non-singular. Then

(i) the boundary value problem (4.10) with boundary condition (4.11) has a solution which can be expressed as

$$
U_i(t) = \mathbb{I}_i(t) \eta_i, \quad i \in J(1, n),
$$

(6.7)

where $(\eta_1^T \eta_2^T \cdots \eta_1^T)^T = A(\varepsilon)^{-1} B(\varepsilon)$,
(ii) the original boundary value problem (2.5)–(2.6) has a bounded solution and it is given by

\[ X(t) = S^2(t) \, U(t), \]  

(6.8)

where \( S^2 \) is as in Remark 4.2 and \( U(t) = (U_1^T(t), U_2^T(t), \ldots, U_n^T(t))^T \) is as defined in (i).

**Proof.** From Lemma 6.1, (6.7) satisfies the system of differential equation (4.10). Also, it can be verified that (6.7) satisfies the boundary condition (4.11). This completes the proof of (i). Let us prove (ii). Since \( S^* = (T^*)^{-1} \) and \( U \) is the solution of the transformed system (4.10) with boundary condition (4.11), \( X(t) = S^2(t) \, U(t) \) is a solution of (2.5)–(2.6). Boundedness of solution (6.8) follows from the boundedness of \( U(t) \) and \( S^2 \). This completes the proof of the theorem.

In the following we present a theorem which deals with convergence of decoupled system (4.10) and the original problem (2.5) with corresponding degenerate systems, as \( \varepsilon_{n-m} \to 0 \), for \( m \in J(0, n-2) \). For this purpose we present two lemmas which are useful in our further discussion.

**Lemma 6.2.** Let the assumptions of Theorem 6.1 be satisfied. Then the matrix solutions \( \mathbb{U}_i(t), \mathbb{W}_i(t); i \in J(m+1, n-2), \) in (6.5) converge to matrix solutions \( \mathbb{V}_i(t), \mathbb{W}_n(t); i \in J(m+1, n-2), \) of the following system of differential equation, respectively, as \( \varepsilon_{n-m} \to 0: \)

\[
W_i = A_{11}(n-1, \varepsilon^m) \, W_1, 
\]

(6.9)

\[
\varepsilon_{n-i} W_{n-i} = [A_{22}(i, \varepsilon^m) - \varepsilon_{n-i} L_i(\varepsilon^m) A_{12}(i, \varepsilon^m)] \, W_{n-i}, \quad i \in J(m+1, n-2), 
\]

(6.10)

where \( \varepsilon^m = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-m-1}, 0, \ldots, 0) \).

**Proof.** From Assumption A2.1, \( \varepsilon_i \to 0 \) for \( i \in J(n-m+1, n) \) as \( \varepsilon_{n-m} \to 0 \). From this and Corollary 5.1, we have

\[
\|L_i(\varepsilon^m) - L_i(\varepsilon)\| \leq \sum_{k=n-m}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right). \]  

(6.11)

From the definition of \( A_{22}(i, \varepsilon), A_{12}(\varepsilon) \) for \( i \in J(m+1, n-2) \), and (6.11), we obtain

\[
A_{11}(n-1, \varepsilon^m) = A_{11}(n-1, \varepsilon) + \sum_{k=n-m}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right), \]  

(6.12)

\[
A_{22}(i, \varepsilon^m) - \varepsilon_{n-i} L_i A_{12}(i, \varepsilon^m) = A_{22}(i, \varepsilon) - \varepsilon_{n-i} L_i(\varepsilon) A_{12}(i, \varepsilon) + \sum_{k=n-m}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right), \]  

(6.13)
From (6.3), (6.9), and (6.12), the fundamental matrix solution $U_i(t)$ of (6.3) as defined in (6.5) converges uniformly to a fundamental matrix solution $W_i(t)$ of (6.9) and satisfies the relation

$$\|U_i(t) - W_i(t)\| \leq \sum_{k=n-m}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right).$$

From (6.1), (6.2), (6.10), (6.13), Lemma 1 in [3], and [4], there exists a fundamental matrix solution $\phi_{n-i}(t), i \in J(m+1, n-2)$ of (6.10) such that

$$\|\phi_{n-i}(t) P_i \phi_{n-i}^{-1}(s) - \phi_{n-i}(t) P_i \phi_{n-i}^{-1}(s)\| \leq K_{i1} \exp\left[\frac{(\sigma_m - O_m)}{\varepsilon_{n-m}} (t-s)\right], \quad t \geq s,$$

where $K_{i1}$ is a positive number independent of $\varepsilon_{n-i}$, $P_i$ is a projection matrix as in Lemma 6.1, and $O_m = \sum_{k=n-m}^{n} O(\varepsilon_k/\varepsilon_{k-1})$.

Furthermore $\phi_{n-i}(t)$ satisfies the expression

$$\|\phi_{n-i}(t) P_i \phi_{n-i}^{-1}(s) - \phi_{n-i}(t) P_i \phi_{n-i}^{-1}(s)\| \leq \sum_{k=n-m}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right), \quad t \geq s,$$

$$\|\phi_{n-i}(t)(I_n - P_i) \phi_{n-i}^{-1}(s) - \phi_{n-i}(t)(I_n - P_i) \phi_{n-i}^{-1}(s)\| \leq \sum_{k=n-m}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right), \quad s \geq t,$$

where $\phi_{n-i}(t)$ as in Lemma 6.1.

Now let us choose $W_{n-i}, i \in J(m+1, n-2)$, as

$$W_{n-i}(t) = \phi_{n-i}(t) P_{n-i} \phi_{n-i}^{-1}(0) + \phi_{n-i}(t)(I_n - P_{n-i}) \phi_{n-i}^{-1}(1).$$

One can verify that $W_i(t), W_{n-i}(t); i \in J(m+1, n-2)$, are matrix solutions of (6.9) and (6.10), respectively. Moreover the convergence of $U_{n-i}(t)$ to $W_{n-i}(t)$ for $i \in J(m+1, n-1)$ as $\varepsilon_{n-m} \to 0$, follows from (6.5), (6.16), and (6.17). This completes the proof of the lemma.

**Assumption A6.1.** Assume that $Q(\varepsilon), R(\varepsilon),$ and $B(\varepsilon)$ have the representations

$$Q(\varepsilon) = Q(0) + \sum_{k=2}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right), \quad R(\varepsilon) = R(0) + \sum_{k=2}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right)$$

$$B(\varepsilon) = B(0) + \sum_{k=2}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right),$$

respectively.
Assumption A6.2. $\Delta(\varepsilon^m)$ is non-singular, where

$$
\Delta(\varepsilon^m) = Q(n-1, \varepsilon^m) \text{ diag}\{\mathbb{W}_1(0), ..., \mathbb{W}_{n-m-1}(0), \phi_{n-m}(0) \}
\times P_m\phi_{n-m}^{-1}(0), ..., \phi_n(0) P_0\phi_n^{-1}(0) \}
+ R(n-1, \varepsilon^m) \text{ diag}\{\mathbb{W}_1(1), ..., \mathbb{W}_{n-m-1}(1), \phi_{n-m}(1)\}
- m^2 P_m
- m(1) \phi_{n-m}^{-1}(1), ..., \phi_n(1)(I_n - P_0)\phi_n^{-1}(1)\}.
$$

(6.18)

**Lemma 6.3.** Let the assumptions of Lemma 6.2 and A6.1, A6.2 be satisfied. Then $\Delta(\varepsilon)$ converges to $\Delta(\varepsilon^m)$ as $\varepsilon_{n-m} \to 0$, where $\Delta(\varepsilon)$ is given as in Theorem 6.1. Furthermore, $\Delta(\varepsilon)$ is non-singular and $\|\Delta^{-1}(\varepsilon) - \Delta^{-1}(\varepsilon^m)\|$ converges to zero as $\varepsilon_{n-m} \to 0$.

**Proof.** From Theorems 5.1 and 5.2, Corollaries 5.1 and 5.2, and the definition of $S^2$, we have

$$
\|S^2(\varepsilon) - S^2(\varepsilon^m)\| \leq \sum_{k=n-m}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right).
$$

(6.19)

From Assumption A6.1 and the expressions of $Q(n-1, \varepsilon)$ and $R(n-1, \varepsilon)$, we have

$$
\|Q(n-1, \varepsilon) - Q(n-1, \varepsilon^m)\| \leq \sum_{k=n-m}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right),
$$

(6.20)

$$
\|R(n-1, \varepsilon) - RQ(n-1, \varepsilon^m)\| \leq \sum_{k=n-m}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right).
$$

From (6.6), (6.16), (6.20), and Lemma 6.2, we arrive at

$$
\|\Delta - \Delta(\varepsilon^m)\| \leq \sum_{k=n-m}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right).
$$

(6.21)

Non-singularity of $\Delta(\varepsilon)$ follows from Assumption A6.2, (6.21) and it is clear that

$$
\|\Delta^{-1}(\varepsilon) - \Delta^{-1}(\varepsilon^m)\| \leq \sum_{k=n-m}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right).
$$

(6.22)

This completes the proof of the lemma.

**Theorem 6.2.** Let the assumption of Lemma 6.3 be satisfied. Then,

(i) solution of the decoupled system (4.10) with (4.11) in (6.7) con-
verges to a function \( \bar{U}(t) \) for \( 0 < t < 1 \) as \( \varepsilon_{n-m} \to 0 \). Moreover, the function \( \bar{U}(t) \) satisfies the following degenerate system of boundary value problems:

\[
\dot{W}_i = A_{11}(n-1, \varepsilon^m) W_1, \\
\varepsilon_{n-i} \dot{W}_{n-i} = \left[ A_{zz}(i, \varepsilon^m) - \varepsilon_{n-i} L_{i}(\varepsilon^m) A_{zz}(i, \varepsilon^m) \right] W_{n-i}, \\
\text{for } i \in J(m+1, n-2), \\
0 = W_i(t), \quad \text{for } i \in J(n-m, n)
\]

(6.23)

with the first \( \sum_{i=m+1}^{m+1} n_i \) equations of

\[
A^{-1}(\varepsilon^m) Q(n-1, \varepsilon^m) W^0 + A^{-1}(\varepsilon^m) R(n-1, \varepsilon^m) W^1 = A^{-1}(\varepsilon^m) B(\varepsilon^m).
\]

(6.24)

(ii) Solution of the original boundary value problem (2.5)-(2.6) converges to a function \( \bar{X} \), for \( 0 < t < 1 \), as \( \varepsilon_{n-m} \to 0 \). Furthermore, the function \( \bar{X} \) satisfies the following degenerate system of boundary value problems:

\[
\varepsilon_i \dot{X}_i = \sum_{j=1}^{n} D_{ij} A_{ij} \bar{X}_j, \quad \text{for } i \in J(1, n-m-1), \\
0 = \sum_{j=1}^{n} D_{ij} A_{ij} \bar{X}_j, \quad \text{for } i \in J(n-m, n)
\]

(6.25)

with the first \( \sum_{i=m+1}^{m+1} n_i \) equations of

\[
A^{-1}(\varepsilon^m) Q(n-1, \varepsilon^m) \bar{X}^0 + A^{-1}(\varepsilon^m) R(n-1, \varepsilon^m) \bar{X}^1 = A^{-1}(\varepsilon^m) B(\varepsilon^m).
\]

(6.26)

Proof. From Assumption A1.2.1, \( \varepsilon_i \to 0 \) for \( i \in J(n-m, n) \) as \( \varepsilon_{n-m} \to 0 \). From (6.2), (6.5), and (6.7), for \( i \in J(n-m, n) \), we have

\[
\|U_i(t)\| \leq \left\{ \exp \left[ -\frac{\alpha_i}{\varepsilon_{n-i}} t \right] + \exp \left[ -\frac{\alpha_i}{\varepsilon_{n-i}} (1-t) \right] \right\} \|\eta_i(\varepsilon)\|. \tag{6.27}
\]

From the boundedness of \( \|A^{-1}(\varepsilon)\| \), which is proved in Lemma 6.3, and the boundedness of \( B(\varepsilon) \) and (6.27), we conclude that \( U_i(t) \) converges to \( \bar{U}(t) = 0 \), for \( 0 < t < 1 \) as \( \varepsilon_{n-m} \to 0 \) for \( i \in J(n-m, n) \). This proves that the limit function \( U_i(t), i \in J(n-m, n) \), satisfies the algebraic system in (6.23) for \( 0 < t < 1 \) as \( \varepsilon_{n-m} \to 0 \). Now we prove that the limit function of \( U_i(t) \), for \( i \in J(1, n-m-1) \), can be expressed by

\[
\bar{U}_i(t) = \mathbb{W}_i(t) \bar{\eta}_i, \tag{6.28}
\]
where \( \tilde{\eta}_i \) is the \( i \)th component of \( A^{-1}(\varepsilon^m) B(\varepsilon^m) \), and \( \bar{U}_i(t) \) satisfies differential system (6.23). To prove \( U_i(t) \) in (6.4) converges to \( \bar{U}_i(t) \) in (6.28), let us consider

\[
\| U_i(t) - \bar{U}_i(t) \| = \| U_i(t) \eta_i - W_i(t) \tilde{\eta}_i(t) \|. \tag{6.29}
\]

From (6.14), (6.16), (6.21), and Assumption A6.1, we have

\[
\| U_i(t) \eta_i - W_i(t) \tilde{\eta}_i(t) \| \leq \| U_i(t) - W_i(t) \| \| \eta_i(t) \| + \| W_i(t) \| \| \eta_i - \tilde{\eta}_i \|. \tag{6.30}
\]

Convergence of \( U_i(t) \) to \( \bar{U}_i(t) \) for \( i \in J(1, n - m - 1) \) as \( \varepsilon_{n-m} \to 0 \) follows from (6.29) and (6.30). Proof of \( \bar{U}_i(t) \), \( i \in J(1, n - m - 1) \), satisfies the differential system (6.23) and follows from Lemma 6.2. This completes the proof that \( U(t) \) converges to function \( \eta(t) \) and \( a(t) \) satisfies (6.23). To prove the boundary condition, let us notice that the function \( \eta(t) \) can be defined as

\[
c_i(\eta) = p_i(\eta) \pi^{-1} \phi_i(\eta), \quad i \in J(n - m, n), \tag{6.31}
\]

where \( r_{ji} = 0, i \in J(n - m, n) \).

Then the right hand side of the boundary condition (6.24) can be expressed as

\[
A^{-1}(\varepsilon^m) [Q(n - 1, \varepsilon^m) \bar{U}^0 + R(n - 1, \varepsilon^m) \bar{U}^1] = A^{-1}(\varepsilon^m) \begin{bmatrix} \eta_1^T, \ldots, \eta_n^T \end{bmatrix}^T.
\]

From this and (6.28), the first \( (n - m - 1) \) components of the above expression \( \tilde{\eta}_1, \ldots, \tilde{\eta}_{n-m-1} \) are the first \( (n - m - 1) \) components of \( A^{-1}(\varepsilon^m) B(\varepsilon^m) \).

This completes the proof of (i). Let us illustrate the proof of (ii). Let us analyze \( S^2 \) as \( \varepsilon_{n-m} \to 0 \). From Theorem 5.1, by using the convergence of \( L_i \) to \( \bar{L}_i \) for \( i \in J(0, m) \), as \( \varepsilon_{n-m} \to 0 \), and from Theorem 5.2, by using the boundedness of \( M_i \), for \( i \in J(0, m) \), we observe that \( S_{n-i} \), defined in Remark 4.2, converges to \( \bar{S}_{n-i} \) as \( \varepsilon_{n-m} \to 0 \), where

\[
\bar{S}_{n-i} = \begin{bmatrix} I_{n-i-1} & 0 & 0 \\ \bar{L}_i & I_{n-i} & 0 \\ 0 & 0 & I(n-i-1) \end{bmatrix}, \quad i \in J(0, m). \tag{6.32}
\]
From Corollaries 5.1 and 5.2, it is clear that for \( i \in J(m + 1, n - 2) \), \( S_{n-i} \), defined in Remark 4.2, converges to \( \bar{S}_{n-i} \) as \( \epsilon_{n-m} \to 0 \), where

\[
\bar{S}_{n-i} = \begin{bmatrix}
I_{n-i} - 1 & \tau_{n-i} M_i(\epsilon^m) & 0 \\
0 & I_{n-i} + \tau_{n-i} L_i(\epsilon^m) M_i(\epsilon^m) & 0 \\
0 & 0 & I(n-i-1)
\end{bmatrix},
\]

for \( i \in J(0, m) \). (6.33)

Let us define \( S^2 \) as

\[
S^2 = \bar{S}_{n} \circ \bar{S}_{n-1} \circ \cdots \circ \bar{S}_{2},
\]

and \( \bar{X} = S^2 \bar{U} \).

Now we will prove that the solution \( X(t) \) of the original problem (2.5)--(2.6) converges to the function \( \bar{X}(t) \) in (6.34). From Corollaries 5.1 and 5.2, and (6.29), (6.30), (6.32), and (6.33), we have

\[
\|X(t) - \bar{X}(t)\| \leq \|S^2 U(t) - \bar{X}^2 \bar{U}(t)\|
\]

\[
\leq \|S^2 - \bar{S}^2\| \|U(t)\| + \|\bar{S}^2\| \|U(t) - \bar{U}(t)\|
\]

\[
\leq \sum_{k=n-m}^{n} O\left(\frac{\epsilon_k}{\epsilon_{k-1}}\right).
\]

By direct calculation it can be verified that, \( \bar{T}_{n-i}, i \in J(0, m) \), is an inverse of \( \bar{S}_{n-i} \), where

\[
\bar{T}_{n-i} = \begin{bmatrix}
I_{n-i} - 1 & 0 & 0 \\
- \bar{L}_i & I_{n-i} & 0 \\
0 & 0 & I(n-m-1)
\end{bmatrix}, \quad i \in J(0, m).
\]

From (6.34) and (6.35), we can deduce that

\[
(\bar{T}_{n-m} \circ \bar{T}_{n-m+1} \circ \cdots \circ \bar{T}_n) \bar{X} = (S_{n-m-1} \circ S_{n-m-2} \circ \cdots \circ S_2) \bar{U}.
\]

From the fact that \( \bar{U}(t) = 0 \) for \( i \in J(n-m, n) \) and from the structure of \( \bar{T}_n, \ldots, \bar{T}_{n-m}, \bar{S}_{n-m-1}, \ldots, \bar{S}_2 \), we note that the components \( (n-m) \text{th} - nth \) of \( (\bar{T}_{n-m} \circ \bar{T}_{n-m+1} \circ \cdots \circ \bar{T}_n) \bar{X} \) are identically equal to zero. That is,

\[
\sum_{j=1}^{n-i-1} \bar{L}_{ij} \bar{X}_j + \bar{X}_{n-i} = 0 \quad \text{for} \quad i \in J(0, m).
\]

From (6.37) and Theorem 5.1, for \( i = 0 \), we have

\[
\sum_{j=1}^{n-1} \bar{A}_{nn}^{0-1} \bar{A}_{nj} \bar{X}_j + \bar{X}_n = 0.
\]
By definition $\hat{A}_{nj}^0 = D_n A_{nj}$; $i \in J(1, n)$, the above equation is equivalent to

$$\sum_{j=1}^{n} D_n A_{nj} \bar{X}_j = 0. \quad (6.38)$$

From (6.37) and Theorem 5.1, for $i = 1$, we have

$$\sum_{j=1}^{n-2} \hat{A}_{n-1, j}^1 \hat{A}_{n-1, j}^1 \bar{X}_j + \bar{X}_{n-1} = 0. \quad (6.39)$$

The above equation is equivalent to

$$\sum_{j=1}^{n-1} \hat{A}_{n-1, j}^1 \bar{X}_j = 0. \quad (6.39)$$

From the definition of $\hat{A}_{n-1, j}^1$, $j \in J(1, n-1)$, and (6.38), it follows that the above equation can be rewritten as

$$\sum_{j=1}^{n} D_{n-1} A_{n-1, j} \bar{X}_j = 0. \quad (6.40)$$

Similar to (6.38) and (6.39), from (6.37), one can derive an equivalent system of the following type: for $i \in J(0, m)$

$$\sum_{j=1}^{n} D_{n-i} A_{n-i, j} \bar{X}_j = 0. \quad (6.40)$$

This completes the proof that the components $\bar{X}_j$, $j \in J(1, n)$, satisfy the algebraic system in (6.25). Now we will prove that $\bar{X}_i$, $i \in J(1, n-m-1)$, satisfies the differential system in (6.25). Notice that $\bar{X}_i$, $i \in J(1, n-m-1)$ is the $i$th component of $(S_{n-m-1} \circ S_{n-m-2} \circ \cdots \circ S_2) \bar{U}$. Let us denote $\bar{\xi}(t) = \bar{S}_2 \bar{U}(t)$. Then,

$$\begin{bmatrix} \bar{\xi}_1 \\ \bar{\xi}_2 \end{bmatrix} = \bar{S}_2 \begin{bmatrix} \bar{U}_1 \\ \bar{U}_2 \end{bmatrix}, \quad (6.41)$$

and $(\bar{\xi}_3, \ldots, \bar{\xi}_n)^T = (\bar{U}_3^T, \ldots, \bar{U}_n^T)$, where

$$\bar{S}_2 = \begin{bmatrix} I^1 \\ L_{n-2}(e^m) \\ e_2 M_{n-2}(e^m) \\ L_{n-2}(e^m) + e_2 L_{n-2}(e^m) M_{n-2}(e^m) \end{bmatrix}. \quad (6.42)$$

We note that $L_{n-2}(e^m)$ and $M_{n-2}(e^m)$ satisfy differential equations (5.21) and (5.23), respectively. $(\bar{U}_1^T, \bar{U}_2^T)^T$ satisfies differential equations in (6.23).
Further notice that for \( i = n - 2 \) one can verify that \( \left( \xi_1^T, \xi_2^T \right) \) satisfies the system of differential equations

\[
\begin{bmatrix}
\dot{\xi}_1 \\
\dot{\xi}_2
\end{bmatrix} = \begin{bmatrix}
A_{11}(n-2, \varepsilon^m) & A_{12}(n-2, \varepsilon^m) \\
\frac{1}{\varepsilon_2} A_{21}(n-2, \varepsilon^m) & \frac{1}{\varepsilon_2} A_{22}(n-2, \varepsilon^m)
\end{bmatrix} \begin{bmatrix}
\xi_1 \\
\xi_2
\end{bmatrix}.
\]

(6.43)

In a similar fashion if we continue this, then we observe that \( \left( \bar{X}_{1}^T, ..., \bar{X}_{n-m-1}^T \right)^T \) satisfies the system of differential equations

\[
\begin{bmatrix}
\dot{\bar{X}}_{n-m-2}(m) \\
\dot{\bar{X}}_{n-m-1}
\end{bmatrix} = \begin{bmatrix}
A_{11}(n-2, \varepsilon^m) & A_{12}(n-2, \varepsilon^m) \\
\frac{1}{\varepsilon_2} A_{21}(n-2, \varepsilon^m) & \frac{1}{\varepsilon_2} A_{22}(n-2, \varepsilon^m)
\end{bmatrix} \begin{bmatrix}
\bar{X}_{n-m-2}(m) \\
\bar{X}_{n-m-1}
\end{bmatrix},
\]

where

\[
\bar{X}_{n-m-2}(m)^T = (\bar{X}_{1}^T, \bar{X}_{2}^T, ..., \bar{X}_{n-m-2}^T)^T.
\]

The above system can be rewritten as

\[
e_i \dot{\bar{X}}_i = \sum_{j=1}^{n-m-1} \bar{A}_{ij}^{m+1} \bar{X}_j, \quad i \in J(1, n-m-1).
\]

(6.44)

From the definition of \( \bar{A}_{ij}^m \), (6.44) reduces to

\[
e_i \dot{\bar{X}}_i = \sum_{j=1}^{n-m-1} \bar{A}_{ij}^m \bar{X}_j + \bar{A}_{i n-m}^m \sum_{j=1}^{n-m-1} \bar{L}_{ij} \bar{X}_j, \quad i \in J(1, n-m-1).
\]

From (6.40), the above system is equivalent to the differential system

\[
e_i \dot{\bar{X}}_i = \sum_{j=1}^{n-m} \bar{A}_{ij}^m \bar{X}_j, \quad i \in J(1, n-m-1).
\]

Continuing in this manner, the above system is equivalent to the differential system

\[
e_i \dot{\bar{X}}_i = \sum_{j=1}^{n} D_i \bar{A}_{ij} \bar{X}_j, \quad i \in J(1, n-m-1).
\]

This completes the proof that the solution \( X(t) \) of the original boundary value problem (2.5)–(2.6) converges to the function \( \bar{X}(t) \) in (6.34) as
Now we prove that $X(t)$ satisfies the boundary condition (6.26). From the fact that $Q(n - 1, \varepsilon) = Q(\varepsilon) S^2$ and $R(n - 1, \varepsilon) = R(\varepsilon) S^2$, Assumption A6.1, and the definition of $S^2$, it is clear that $Q(n - 1, \varepsilon_n) = Q(\varepsilon_n) S^2$ and $R(n - 1, \varepsilon_n) = R(\varepsilon_n) S^2$.

From (6.24), (6.34), and (6.43), we conclude that $X(t)$ satisfies the boundary condition (6.26). This completes the proof of the theorem.

An auxiliary system corresponding to the totally decoupled system (4.10) with boundary condition (4.11) is given by

$$
\dot{V}_1 = \tilde{A}_{11}(n - 1) V_1, \quad (6.45)
$$

$$
\varepsilon_{n-i} \dot{V}_{n-i} = \tilde{A}_{22}(i) V_{n-i}, \quad i \in J(0, n-2)
$$

$$
\dot{Q}(n-1) V^0 + \tilde{R}(n-1) V^1 = B(0). \quad (6.46)
$$

In the following, we establish an approximate solution of the decoupled system (4.10) with boundary condition (4.11) by developing condition (6.46). Moreover, we obtain an approximate solution of the original boundary value problem (2.5)-(2.6) by using an approximate solution of the decoupled system (4.10) with boundary condition (4.11). For this purpose we present the following lemma.

**Lemma 6.4.** Let the assumption of Theorem 6.1 be satisfied. Then there exist matrix solutions $V_m(t)$ of (6.45) such that

$$
\|U_m(t) - V_m(t)\| \leq \sum_{k=n-m}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right), \quad (6.47)
$$

where $U_m(t)$ is as in (6.5).

**Proof.** From Lemma 1 in [3, 4], and the fact that

$$
\|A_{22}(m) - \varepsilon_{n-m} L_m A_{12}(m) - \tilde{A}_{22}(m)\| \leq \sum_{k=n-m}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right),
$$

there exists a fundamental matrix solution $\tilde{\phi}_{n-m}(t)$ of

$$
\varepsilon_{n-m} \dot{V}_{n-m} = \tilde{A}_{22}(m) V_{n-m}, \quad m \in J(0, n-2),
$$

where $\tilde{A}_{22}(m)$ is the $2 \times 2$ matrix in $L_2(0, n-2)$ with entries $A_{22}(m)$, $\varepsilon_{n-m} L_m A_{12}(m)$, and $\tilde{A}_{22}(m)$.
such that

\[
\| \hat{\phi}_{n-m}(t) P_m \hat{\phi}_{n-m}^{-1}(s) - \phi_{n-m}(t) P_m \phi_{n-m}^{-1}(s) \| \\
\leq \sum_{k=n-m}^{n} O \left( \frac{\varepsilon_k}{\varepsilon_{k-1}} \right) (t-s), \quad t \geq s,
\]

(6.48)

\[
\| \hat{\phi}_{n-m}(t) (I_{n-m} - P_m) \hat{\phi}_{n-m}^{-1}(s) - \phi_{n-m}(t) (I_{n-m} - P_m) \phi_{n-m}^{-1}(s) \| \\
\leq \sum_{k=n-m}^{n} O \left( \frac{\varepsilon_k}{\varepsilon_{k-1}} \right) (s-t), \quad s \geq t.
\]

(6.49)

where \( \phi_{n-m} \) and \( P_m \) are as in Lemma 6.1.

Define

\[
\forall_{1}(t) = \hat{\phi}_{1}(t),
\]

\[
\forall_{n-m}(t) = \hat{\phi}_{n-m}(t) P_m \hat{\phi}_{n-m}^{-1}(0) + \hat{\phi}_{n-m}(t) \\
x (I_{n-m} - P_m) \hat{\phi}_{n-m}^{-1}(1), \quad \text{for } i \in J(2, n).
\]

(6.50)

where \( \hat{\phi}_{n-m}(t) \) are as in (6.48) and \( \phi_{1}(t) \) is a fundamental matrix solution of the first equation in (6.45) with \( \phi_{1}(0) = \phi_{1}(0) \). Then from the fact that

\[
A_{11}(n-1) = \hat{A}_{11}(n-1) + \sum_{k=2}^{n} O \left( \frac{\varepsilon_k}{\varepsilon_{k-1}} \right),
\]

it is clear that

\[
\| \phi_{1}(t) - \hat{\phi}_{1}(t) \| \leq \sum_{k=2}^{n} O \left( \frac{\varepsilon_k}{\varepsilon_{k-1}} \right).\]

(6.51)

Proof of (6.47) follows from (6.48), (6.49), and (6.50). This completes the proof of the lemma.

**Assumption A6.3.** Assume that \( \hat{A}(\varepsilon) \) is non singular and \( \hat{A}^{-1}(\varepsilon) \) is bounded where

\[
\hat{A}(\varepsilon) = \hat{M}(0) \text{ diag}\{ \forall_{1}(0), \ldots, \forall_{n}(0) \} + \hat{R}(1) \text{ diag}\{ \forall_{1}(1), \ldots, \forall_{n}(1) \}.\]

(6.51)

**Theorem 6.3.** Let the assumptions of Theorem 6.1 and Assumptions A6.1, A6.2, and A6.3 be satisfied. Then there exists a bounded solution \( V(t) \) of auxiliary system (6.45) with (6.46) which approximates the solution \( U(t) \) in (6.7). Furthermore, \( \hat{S}^2(t) V(t) \) approximates the solution \( X(t) \) in (6.8), of the original boundary value problem (2.5)–(2.6).
Proof. Let us develop a solution of (6.45) with boundary condition (6.46). From Lemma 6.4 and Assumption A6.3, solution of (6.45) with boundary condition (6.46) can be expressed as

$$V_m(t) = \mathcal{V}_m(t) \hat{\eta}_m,$$  
(6.52)

where

$$\hat{\eta}_m = \hat{A}^{-1}(\varepsilon) B(0).$$

From Assumption A6.1 and (6.6), (6.48), and (6.51), we have

$$\|A(\varepsilon) - \hat{A}(\varepsilon)\| \leq \sum_{k=2}^{n} O \left( \frac{\varepsilon_k}{\varepsilon_{k-1}} \right).$$  
(6.53)

From Assumption A6.3 and (6.53), we observe that $A(\varepsilon)$ is non-singular and

$$\|A^{-1}(\varepsilon) - \hat{A}^{-1}(\varepsilon)\| \leq \sum_{k=2}^{n} O \left( \frac{\varepsilon_k}{\varepsilon_{k-1}} \right).$$  
(6.54)

From (6.7), (6.52), and (6.54), we have

$$\|U_m(t) - V_m(t)\| \leq \|U_m(t) \eta_m - \mathcal{V}_m(t) \hat{\eta}_m\|$$
$$\leq \sum_{k=2}^{n} O \left( \frac{\varepsilon_k}{\varepsilon_{k-1}} \right).$$  
(6.55)

This proves that the solution $V(t)$ in (6.52) of auxiliary system (6.45) with boundary condition (6.46) approximates the solution $U(t)$ in (6.7) of the decoupled system (4.10) with boundary condition (4.11). Now we will prove that $\hat{S}^2 \hat{V}$ approximates the solution $X(t)$ in (6.8) of the original boundary value problem (2.5)–(2.6). From (6.8), (6.52), (6.55), and the fact that

$$\|S^2 - \hat{S}^2\| \leq \sum_{k=2}^{n} O \left( \frac{\varepsilon_k}{\varepsilon_{k-1}} \right),$$

we have

$$\|X - \hat{S}^2 V\| = \|S^2 U - \hat{S}^2 V\|$$
$$\leq \sum_{k=2}^{n} O \left( \frac{\varepsilon_k}{\varepsilon_{k-1}} \right).$$

This completes the proof of the theorem.
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REFERENCES