Journal of Computational and Applied Mathematics 233 (2009) 473–481

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

A generalization of the three-dimensional Bernfeld–Haddock conjecture and its proof☆

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ARTICLE INFO

Article history:
Received 19 December 2008
Received in revised form 25 July 2009

Keywords:
Bernfeld–Haddock conjecture
Delay differential equation
Convergence

ABSTRACT

Consider the following system of delay differential equations

$$\begin{cases} x_1'(t) = -F(x_1(t)) + G(x_2(t-r_2)), \\ x_2'(t) = -F(x_2(t)) + G(x_3(t-r_3)), \\ x_3'(t) = -F(x_3(t)) + G(x_1(t-r_1)), \end{cases}$$

where r_1, r_2 and r_3 are positive constants, $F, G \in C(R^1)$, and F is nondecreasing on R^1. These systems have important practical applications and also are a three-dimensional generalization of the Bernfeld–Haddock conjecture. In this paper, by using comparative technique, we obtain the asymptotic behavior of solutions that each bounded solution of the systems tends to a constant vector under a desirable condition.

1. Introduction

In the 1976 international conference on nonlinear systems and their applications, Bernfeld and Haddock [1] proposed the following conjecture:

Conjecture ([1]). : *Every solution of the delay differential equation*

$$x'(t) = -x^{\frac{1}{3}}(t) + x^{\frac{1}{3}}(t-r), \tag{1.1}$$

where $r > 0$, tends to a constant as $t \longrightarrow \infty$.

Jehu [2] first confirmed the above conjecture, and Krisztin [3], Arino–Seguier [4] also asserted it independently. The higher-dimensional generalizations with applications to compartmental systems, including the non-smooth nonlinearity $x^{1/3}$, were given also in [5,6]. Recently, Ding [7–9], Yi and Huang [10] considered the following more general equation

$$x'(t) = -F(x(t)) + G(x(t-r)), \tag{1.2}$$

where $r > 0$ is a constant, $F, G : R^1 \to R^1$ are continuous functions satisfying either $G(x) \geq F(x)$ for all $x \in R^1$ or $G(x) \leq F(x)$ for all $x \in R^1$. It was shown in [7] (see also [8–10]) that if F is strictly increasing then each bounded solution of (1.2) tends to a constant as $t \to \infty$.

☆ This work was supported by grant (06JJ2063, 07JJ46001) from the Scientific Research Fund of Hunan Provincial Natural Science Foundation of China, and the Scientific Research Fund of Hunan Provincial Education Department of China (08C616).

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doi:10.1016/j.cam.2009.07.047

Moreover, Yi and Huang [11,12] consider a two-dimensional generalization of the Bernfeld–Haddock conjecture. More precisely, the system considered by [12] is

$$\begin{cases} x_1'(t) = -F(x_1(t)) + G(x_2(t-r_2)), \\ x_2'(t) = -F(x_2(t)) + G(x_1(t-r_1)), \end{cases} \tag{1.3}$$

where r_1 and r_2 are positive constants, $F, G \in C(R^1)$, and F is nondecreasing on R^1. Variants of system (1.3), which have been used as models for various phenomena such as some population growth, the spread of epidemics, the dynamics of capital stocks, etc. have recently received considerable attention in the literature (see, e.g., [13–23] and the references therein). Moreover, Yi and Huang [11] assumes that the following assumptions are satisfied:

($\mathbf{H}^+$) (i) $G \geq F$;

(ii) If $\alpha \in R^1$, $G(\alpha) = F(\alpha)$ and $\alpha = s(\alpha)$, then there exist $\varepsilon > 0$ and $L \in R^1$ such that $-F(x) + F(\alpha) \geq -L(x-\alpha)$ for all $x \in [\alpha, \alpha+\varepsilon]$, where $s(\alpha) = \sup\{\beta \in R^1 : F(\beta) = F(\alpha)\}$;

($\mathbf{H}^-$) (i) $G \leq F$;

(ii) If $\alpha \in R^1$, $G(\alpha) = F(\alpha)$ and $\alpha = i(\alpha)$, then there exist $\varepsilon > 0$ and $L \in R^1$ such that $-F(x) + F(\alpha) \leq -L(x-\alpha)$ for all $x \in [\alpha-\varepsilon, \alpha]$, where $i(\alpha) = \inf\{\beta \in R^1 : F(\beta) = F(\alpha)\}$.

By using monotonicity arguments, it is proved in [11] that every bounded solution of system (1.3) tends to a constant vector as $t \to \infty$ provided (1.3) satisfies one of the two assumptions ($\mathbf{H}^+$) and ($\mathbf{H}^-$). Unfortunately, the assumptions ($H_\pm$) exclude the situation of $F(x) = x^{\frac{1}{3}}$. Hence, a natural question arises: Does every bounded solution of system (1.3) tend to a constant vector as $t \to \infty$ provided either $G(x) \geq F(x)$ for all $x \in R^1$ or $G(x) \leq F(x)$ for all $x \in R^1$. Our goal in this paper is to answer this question about three dimension as following system:

$$\begin{cases} x_1'(t) = -F(x_1(t)) + G(x_2(t-r_2)), \\ x_2'(t) = -F(x_2(t)) + G(x_3(t-r_3)), \\ x_3'(t) = -F(x_3(t)) + G(x_1(t-r_1)). \end{cases} \tag{1.4}$$

The paper is organized as follows. In Section 2, we introduce some necessary notations and establish some preliminary results, which are important in the proofs of our main results. Based on the preparations in Section 2, we state and prove our main results in Section 3.

2. Preliminary results

In this section, some important properties of system (1.4) will be presented, which are of importance in proving our main results in Section 3.

Throughout this paper, we assume that $F, G \in C(R^1)$, and F is nondecreasing on R^1. We will use R^1_+ to denote the set of all nonnegative real numbers and R^3_+ denote the set of all nonnegative vectors in R^3. Define

$$C = C([-r_1, 0], R^1) \times C([-r_2, 0], R^1) \times C([-r_3, 0], R^1)$$

as the Banach space equipped with a supremum norm. Define

$$C_+ = C([-r_1, 0], R^1_+) \times C([-r_2, 0], R^1_+) \times C([-r_3, 0], R^1_+).$$

It follows that C_+ is an order cone in C and hence, C_+ induces a closed partial ordered relation on C. For any $\varphi, \psi \in C$ and $A \subseteq C$, the following notations will be used: $\varphi \leq \psi$ iff $\psi - \varphi \in C_+$, $\varphi < \psi$ iff $\varphi \leq \psi$ and $\varphi \neq \psi$, $\varphi \ll \psi$ iff $\psi - \varphi \in Int\ C_+$, $\varphi \leq A$ iff $\varphi \leq \psi$ for any $\psi \in A$, $\varphi < A$ iff $\varphi < \psi$ for any $\psi \in A$, $\varphi \ll A$ iff $\varphi \ll \psi$ for any $\psi \in A$. Notations such as "$\geq$", "$>$" and "$\gg$" have the natural meanings.

Furthermore, for the sake of convenience, we introduce the following auxiliary system

$$\begin{cases} x_1'(t) = -F(x_1(t)) + F(x_2(t-r_2)) \\ x_2'(t) = -F(x_2(t)) + F(x_3(t-r_3)) \\ x_3'(t) = -F(x_3(t)) + F(x_1(t-r_1)). \end{cases} \tag{2.1}$$

By using [24, Lemma 3.2], we can easily get by induction that both the initial value problems (1.4) and (2.1) have unique solutions on $[0, +\infty)$. Given $\varphi \in C$, we denote by $x_t(\varphi)$ $(x(t, \varphi))$ the solution of (1.4) with the initial data $x_0(\varphi) = \varphi$. Denote by $x_t(\varphi, F)$ $(x(t, \varphi, F))$ the solution of (2.1), together with the initial data $x_0(\varphi, F) = \varphi$. For any $x \in R^3$, we define $\hat{x} = ((\widehat{x})_1, (\widehat{x})_2, (\widehat{x})_3)$ by $(\widehat{x})_i(\theta) = x_i$, $\theta \in [-r_i, 0]$, $i = 1, 2, 3$.

Before continuing, it is convenient to introduce the following notations and establish some convention. Set

$$E_F = \{e \in R^3 : F(e_1) = F(e_2) = F(e_3)\}.$$

Define the positive semi-orbit by $O(\varphi) = \{x_t(\varphi) : t \geq 0\}$. If $O(\varphi)$ is bounded, then $\overline{O(\varphi)}$ is compact in C, where $\overline{O(\varphi)}$ denotes the closure of $O(\varphi)$. If $O(\varphi)$ is bounded, define

$$\omega(\varphi) = \bigcap_{t \geq 0} \overline{O(x_t(\varphi))},$$

i.e., $\omega(\varphi) = \{\psi \in C : \text{ there exists a subsequence } t_k \to +\infty \text{ such that } x_{t_k}(\varphi) \to \psi\}$. It follows that $\omega(x)$ is nonempty, compact, invariant and connected. Similarly, we can define the positive semi-orbit $O(\varphi, F)$ and the omega limit set $\omega(\varphi, F)$ of the solution $x_t(\varphi, F)$ of (2.1), respectively.

We make the following key definition.

Definition 2.1. $[a, b]$ is called an admitting closed super-interval with respect to F if $F(a) = F(b)$ and $a = i(a)$. $[a, b]$ is called an admitting closed sub-interval with respect to F if $F(a) = F(b)$ and $b = s(b)$.

Lemma 2.1. *Let $G \geq F$. Then for any $\varphi, \psi \in C$ with $\psi \geq \varphi$, we have $x_t(\psi) \geq x_t(\varphi, F)$ for all $t \in R^1_+$. Hence, $x_t(\psi, F) \geq x_t(\varphi, F)$ for all $t \in R^1_+$. Moreover, if $\varphi \in E_F$, then $x_t(\psi) \geq \varphi$ for all $t \in R^1_+$.*

Proof. Lemma 2.1 follows by applying [25, Proposition 1.1]. □

Lemma 2.2. *Assume that $G \geq F$, $\varphi \in C$, $\alpha \in E_F$ and $\varphi \geq \widehat{\alpha}$, then $x_t(\varphi) \geq \widehat{\alpha}$, $\forall\, t \in R^1_+$. In addition, the following conclusions hold:*

(i) *If $i \in \{1, 2, 3\}$, and $\alpha_i < s(\alpha_i)$, $\varphi_i(0) > \alpha_i$, then $x_i(t, \varphi) > \alpha_i$, $\forall\, t \in R^1_+$.*
(ii) *If $\alpha_1 = \alpha_2 = \alpha_3 = s(\alpha_1)$ and $\varphi_1(0) > \alpha_1$, then $x_3(r_1, \varphi) > \alpha_3$.*
(iii) *If $\alpha_1 = \alpha_2 = \alpha_3 = s(\alpha_1)$ and $\varphi_2(0) > \alpha_2$, then $x_1(r_2, \varphi) > \alpha_1$.*
(iv) *If $\alpha_1 = \alpha_2 = \alpha_3 = s(\alpha_1)$ and $\varphi_3(0) > \alpha_3$, then $x_2(r_3, \varphi) > \alpha_2$.*

Proof. It follows from Lemma 2.1 and $\varphi \geq \widehat{\alpha}$ that $x_t(\varphi) \geq x_t(\varphi, F) \geq x_t(\widehat{\alpha}, F) = \widehat{\alpha}$, $t \geq 0$. Now, we prove the remaining conclusions.

(i) We only consider the case where $i = 1$ since the case where $i = 2, 3$ can be dealt with similarly. Assume, by way of contradiction, that conclusion (i) does not hold. Then

$$t_1 = \inf\{t > 0 : x_1(t, \varphi) = \alpha_1\} \in (0, \infty).$$

It follows from Lemma 2.1 that $x_1(t_1, \varphi) = \alpha_1$ and $x_1'(t_1, \varphi) = 0$. In view of (1.4), $G \geq F$ and $x_t(\varphi) \geq \widehat{\alpha}$ for all $t \geq 0$, we obtain

$$\begin{aligned} x_1'(t, \varphi) &= -F(x_1(t, \varphi)) + G(x_2(t - r_2, \varphi)) \\ &\geq -F(x_1(t, \varphi)) + F(x_2(t - r_2, \varphi)) \\ &\geq -F(x_1(t, \varphi)) + F(\alpha_2), \quad \forall\, t \geq 0. \end{aligned}$$

It follows from $x_1(t_1, \varphi) = \alpha_1 < s(\alpha_1)$ that there exists a constant δ such that $t_1 > \delta > 0$, and

$$\alpha_1 \leq x_1(t, \varphi) < s(\alpha_1), \quad \forall\, t \in (t_1 - \delta, t_1 + \delta).$$

Thus, for $t \in [t_1 - \delta, t_1]$,

$$x_1'(t, \varphi) \geq -F(s(\alpha_1)) + F(\alpha_2) = 0.$$

Then, for $t_1 - \delta < t \leq t_1$, we get

$$\alpha_1 \leq x_1(t, \varphi) \leq x_1(t_1, \alpha) = \alpha_1.$$

Therefore, $x_1(t_1 - \delta/2, \varphi) = \alpha_1$, a contradiction to the choice of t_1. Hence conclusion (i) follows.

(ii) If $\varphi_1(0) > \alpha_1, \alpha_1 = \alpha_2 = \alpha_3 = s(\alpha_1)$. We will show that $x_3(r_1, \varphi) > \alpha_3$. By way of contradiction, $x_3'(r_1, \varphi) = 0$ and $x_3(r_1, \varphi) = \alpha_3$. From (1.4), it follows that

$$\begin{aligned} x_3'(r_1, \varphi) &= -F(x_3(r_1, \varphi)) + G(x_1(0, \varphi)) \\ &\geq -F(\alpha_3) + F(\varphi_1(0)) \\ &> -F(\alpha_3) + F(\alpha_1) = 0. \end{aligned}$$

This contradiction implies that conclusion (ii) holds.

By using a similar argument as in the proof of conclusion (ii), we can prove that the conclusions (iii) and (iv) hold. This completes the proof. □

Lemma 2.3. *Let $[a, b]$ be an admitting closed super-interval with respect to F, $\alpha \in [a, b)$, $\beta = a$, then for any $M > 0$, there exists $\varepsilon_M > 0$ such that:*

(i) $\underline{\lim}_{t\to\infty} x(t, \varphi, F) \geq (\alpha, \beta, \beta)$, *where* $\varphi \geq (\alpha + M, \widehat{\beta - \varepsilon_M}, \beta - \varepsilon_M)$.
(ii) $\underline{\lim}_{t\to\infty} x(t, \varphi, F) \geq (\beta, \alpha, \beta)$, *where* $\varphi \geq (\beta - \varepsilon_M, \widehat{\alpha + M}, \beta - \varepsilon_M)$.
(iii) $\underline{\lim}_{t\to\infty} x(t, \varphi, F) \geq (\beta, \beta, \alpha)$, *where* $\varphi \geq (\beta - \varepsilon_M, \widehat{\beta - \varepsilon_M}, \alpha + M)$.

Proof. We only consider case (i) since case (ii) (case (iii)) can be dealt with similarly. Without loss of generality, let $F(a) = 0$ and $M \in (0, b - \alpha)$. If $r(x) = \alpha + M - 2x + F(\beta - x)(r_2 + r_3)$, then $\lim_{x \to 0^+} r(x) = \alpha + M$. Thus, there exists $\varepsilon_M > 0$ such that $r(\varepsilon_M) \geq \alpha + \varepsilon_M$. Next we will show that ε_M satisfies this lemma. Let $d_{\varepsilon_M} = (\alpha + M, \beta - \varepsilon_M, \beta - \varepsilon_M) \in R^3$ and $x(t) = x(\widehat{t, d_{\varepsilon_M}, F}), t \geq 0$. From Lemma 2.1, we obtain $x(t) \leq (\alpha + M, \beta, \beta)(\forall t \geq 0)$. Set

$$t_1 = \inf\{t \geq 0 : x_1(t) \leq \alpha\}.$$

Next we will show that $t_1 = +\infty$, otherwise $t_1 < +\infty$, $x_1(t_1) = \alpha$ and $x_1(t) > \alpha$, $\forall t \in [-r_1, t_1]$. From (2.1), for $t \in [0, t_1]$ we get

$$\begin{cases} x_1'(t) = F(x_2(t - r_2)), \\ x_2'(t) = -F(x_2(t)) + F(x_3(t - r_3)), \\ x_3'(t) = -F(x_3(t)). \end{cases} \tag{2.2}$$

We claim that $t_1 > r_2$. In fact, if $t_1 \leq r_2$, then, from $x_1(0) = \alpha + M$, $F(\beta - \varepsilon_M) < F(\beta) = 0$ and $t_1 < r_2 + r_3$, we obtain

$$\begin{aligned} \alpha &= x_1(t_1) \\ &= x_1(0) + \int_0^{t_1} F(x_2(s - r_2))\mathrm{d}s \\ &= x_1(0) + t_1 F(\beta - \varepsilon_M) \\ &\geq \alpha + M - 2\varepsilon_M + F(\beta - \varepsilon_M)(r_2 + r_3) + 2\varepsilon_M \\ &\geq r(\varepsilon_M) + 2\varepsilon_M \\ &\geq \alpha + 3\varepsilon_M. \end{aligned}$$

This contradiction implies that $t_1 > r_2$.

Furthermore, we claim that $t_1 > r_2 + r_3$. In fact, if $t_1 \leq r_2 + r_3$, then

$$\begin{aligned} \alpha &= x_1(t_1) \\ &= x_1(0) + \int_0^{t_1} F(x_2(s - r_2))\mathrm{d}s \\ &= x_1(0) + \int_{-r_2}^{t_1 - r_2} F(x_2(s))\mathrm{d}s \\ &= x_1(0) + \int_{-r_2}^{0} F(x_2(s))\mathrm{d}s + \int_0^{t_1 - r_2} (-x_2'(s) + F(x_3(t - r_3)))\mathrm{d}s \\ &= x_1(0) + F(\beta - \varepsilon_M)r_2 + x_2(0) - x_2(t_1 - r_2) + (t_1 - r_2)F(\beta - \varepsilon_M) \\ &\geq \alpha + M + \beta - \varepsilon_M - \beta + F(\beta - \varepsilon_M)(r_2 + r_3) \\ &\geq r(\varepsilon_M) + \varepsilon_M \\ &\geq \alpha + 2\varepsilon_M. \end{aligned}$$

This contradiction implies that $t_1 > r_2 + r_3$.

Now, integrating (2.2), it results that

$$\begin{cases} x_1(t_1) = x_1(0) + \displaystyle\int_{-r_2}^{t_1 - r_2} F(x_2(s))\mathrm{d}s \\ x_2(t_1 - r_2) = x_2(0) - \displaystyle\int_0^{t_1 - r_2} F(x_2(s))\mathrm{d}s + \int_{-r_3}^{t_1 - r_2 - r_3} F(x_3(s))\mathrm{d}s \\ x_3(t_1 - r_2 - r_3) = x_3(0) - \displaystyle\int_0^{t_1 - r_2 - r_3} F(x_3(s))\mathrm{d}s \end{cases}$$

then

$$x_1(t_1) = \sum_{i=1}^{3} x_i(0) + \sum_{i=2}^{3} \int_{-r_i}^{0} F(x_i(s))\mathrm{d}s - x_2(t_1 - r_2) - x_3(t_1 - r_2 - r_3).$$

So $x_1(t_1) \geq \alpha + M + 2(\beta - \varepsilon_M) + F(\beta - \varepsilon_M)(r_2 + r_3) - 2\beta$, i.e., $x_1(t_1) \geq \alpha + \varepsilon_M > \alpha$, this is a contradiction. This implies that $t_1 = +\infty$. Moreover from (2.2) we know that $x_1(\cdot)(x_3(\cdot))$ separately is decreasing (nondecreasing) on $[0, \infty)$. Thus, for any $t > 0$, $x_3'(t) \geq 0$.

We claim that $x_2(t) \leq x_3(t - r_3)$ for all $t \geq r_3$. Otherwise, there exists $t^* > r_3$, such that $x_2(t^*) > x_3(t^* - r_3)$ and $x_2'(t^*) > x_3'(t^* - r_3) \geq 0$. From (2.1), we have $x_2'(t^*) = -F(x_2(t^*)) + F(x_3(t^* - r_3))$, then $x_2'(t^*) \leq 0$. This is a contradiction.

From the above claim and (2.1), we get $x_2'(t) \geq 0$ for all $t \geq r_3$, so there exists $d \in R^3$ such that $\lim_{t\to\infty} x(t) = d$, and $d_1 \geq \alpha$, $d_2 \leq \beta$, $d_3 \leq \beta$. Hence $d_2 = d_3 = \beta$ and $d \in E_F$. This completes the proof. □

From the compactness of $\omega(x)$ and the definition of E_F, we can show that:

Lemma 2.4. *Assume that $G \geq F$ and $\varphi \in C$ such that $O(\varphi)$ is bounded, then, there exists $\alpha^* \in R^3$ such that $\alpha^* = \sup\{e \in E_F : \widehat{e} \leq \omega(\varphi)\}$ and $\widehat{\alpha^*} \leq \omega(\varphi)$.*

Proof. Since $O(\varphi)$ is bounded, $\omega(\varphi)$ is compact. Hence, there exists $\alpha \in R^1$ such that

$$\widehat{(\alpha, \alpha, \alpha)} \leq \omega(\varphi).$$

Let

$$D_\varphi^+ = \{e \in E_F : \widehat{e} \leq \omega(\varphi)\}, \qquad D = \{e \in D_\varphi^+ : (\alpha, \alpha, \alpha) \leq e\} \subseteq R^3.$$

Then D is compact. By Zorn's lemma, D contains a maximal element and we denote it by $e^* = (e_1^*, e_2^*, e_3^*)$. Next we will show that $\sup D = e^*$. If not, then there exist $e_1, e_2, e_3 \in R^1$ such that $(e_1, e_2, e_3) \in D$ and $(e^* - (e_1, e_2, e_3)) \notin R_+^3$. Without loss of generality, we may assume that $e_1^* > e_1$, $e_2^* > e_2$ and $e_3^* < e_3$. By the definition of D, we obtain

$$\widehat{(e_1^*, e_2^*, e_3)} \leq \omega(\varphi) \quad \text{and} \quad F(e_1^*) = F(e_3).$$

Therefore,

$$\widehat{(e_1^*, e_2^*, e_3)} \in D \quad \text{and} \quad (e_1^*, e_2^*, e_3^*) < (e_1^*, e_2^*, e_3),$$

a contradiction to the definition of e^*. It follows that $\sup D_\varphi^+ = e^*$. This completes the proof. □

Lemma 2.5. *Assume that all the conditions of Lemma 2.4 hold, α^* be defined in Lemma 2.4, if $\omega(\varphi) \setminus \{\widehat{\alpha^*}\} \neq \emptyset$, then $\alpha_1^* = \alpha_2^* = \alpha_3^* = s(\alpha_1^*)$.*

Proof. Assume, by way of contradiction, that the conclusions do not hold. Then, there exits $i \in \{1, 2, 3\}$ such that $\alpha_i^* < s(\alpha_1^*)$. We shall consider seven cases as follows:

Case 1. $\alpha_i^* < s(\alpha_1^*)$, $\forall\, i = 1, 2, 3$.

By $\omega(\varphi) \setminus \{\widehat{\alpha^*}\} \neq \emptyset$ and the invariance of $\omega(\varphi)$, without loss of generality, we may assume that there exists $\psi \in \omega(\varphi)$ such that $\psi_1(0) > \alpha_1^*$. From Lemma 2.2(i), we obtain

$$x_1(t, \psi) > \alpha_1^*, \quad \forall\, t \in R_+^1.$$

Thus, there exists $M > 0$ such that

$$\alpha_1^* + 3M < s(\alpha_1^*) \;\; x_{r_1}(\psi) \geq (\alpha_1^* + \widehat{3M, \alpha_2^*}, \alpha_3^*).$$

Let $a = i(\alpha_1^*)$, $b = s(\alpha_1^*)$, $\alpha = M + \alpha_1^*$, $\beta = a$. From Lemma 2.3, there exists $\varepsilon_M > 0$ such that

$$\underline{\lim}_{t\to\infty} x(t, \eta) \geq (\alpha, \beta, \beta), \quad \text{where } \eta \geq (\alpha + M, \widehat{\beta - \varepsilon_M}, \beta - \varepsilon_M).$$

By the choice of $M > 0$, we obtain

$$x_{r_1}(\psi) \gg (\alpha + M, \widehat{\beta - \varepsilon_M}, \beta - \varepsilon_M).$$

In view of the definition of $\omega(\varphi)$, there exists $t_1 > 0$ such that

$$x_{t_1}(\varphi) \geq (\alpha + M, \widehat{\beta - \varepsilon_M}, \beta - \varepsilon_M).$$

Then

$$\frac{\lim}{t \to \infty} x(t, \varphi) \geq (\alpha, \beta, \beta).$$

Thus

$$\omega(\varphi) \geq (\alpha_1^* + \widehat{M, \alpha_2^*}, \alpha_3^*),$$

Again by the choice of $M > 0$, we obtain$(\alpha_1^* + M, \alpha_2^*, \alpha_3^*) \in E_F$, a contradiction to the choice of α^*.

Case 2. $\alpha_1^* < s(\alpha_1^*)$, $\alpha_2^* < s(\alpha_1^*)$ and $\alpha_3^* = s(\alpha_1^*)$.

By using a similar argument as in the proof of Case (i), we can prove: $\forall \psi \in \omega(\varphi)$, $\psi_i(\theta) = \alpha_i^*$, $\forall\, \theta \in [-r_i, 0]$, $i = 1, 2$. Again from $\omega(\varphi) \setminus \{\widehat{\alpha^*}\} \neq \emptyset$, we get that there exists $\psi \in \omega(\varphi)$ such that $\psi_3(0) > \alpha_3^*$. By (1.4) and the above claim, we have

$$0 = x_2'(r_3, \psi) = -F(x_2(r_3, \psi)) + G(x_3(0, \psi)),$$

i.e. , $F(\alpha_2^*) = G(\psi_3(0)) \geq F(\psi_3(0)) > F(\alpha_3^*)$, a contradiction to $\alpha^* \in E_F$.

Case 3. $\alpha_1^* = s(\alpha_1^*)$, $\alpha_2^* < s(\alpha_1^*)$ and $\alpha_3^* < s(\alpha_1^*)$. By using a similar argument as in the proof of Case 2, we can derive contradictions.

Case 4. $\alpha_1^* < s(\alpha_1^*)$, $\alpha_2^* = s(\alpha_1^*)$ and $\alpha_3^* < s(\alpha_1^*)$. By using a similar argument as in the proof of Case 2, we can derive contradictions.

Case 5. $\alpha_1^* = s(\alpha_1^*)$, $\alpha_2^* = s(\alpha_1^*)$ and $\alpha_3^* < s(\alpha_1^*)$.

By using a similar argument as in the proof of Case (i), We can prove: $\forall \psi \in \omega(\varphi)$, $\psi_3(\theta) = \alpha_3^*$, $\forall \theta \in [-r_3, 0]$. Again from 1.4, we obtain

$$0 = x_3'(t, \psi) = -F(x_3(t, \psi)) + G(x_1(t - r_1, \psi)), \quad \forall t \in R_+^1.$$

Thus,

$$F(\alpha_3^*) = G(x_1(t - r_1, \psi)) \geq F(x_1(t - r_1, \psi)).$$

From $\alpha_1^* = s(\alpha_1^*)$, $\omega(\varphi) \geq \widehat{\alpha^*}$ and $\alpha^* \in E_F$, we get

$$x_1(t - r_1, \psi) = \alpha_1^*, \quad \forall t \in R_+^1.$$

By using a similar argument we can show that $x_2(t, \psi) = \alpha_2^*$, $\forall t \in [-r_2, +\infty)$, a contradiction to $\omega(\varphi) \setminus \{\widehat{\alpha^*}\} \neq \emptyset$.

Case 6. $\alpha_1^* = s(\alpha_1^*)$, $\alpha_2^* < s(\alpha_1^*)$ and $\alpha_3^* = s(\alpha_1^*)$. By using a similar argument as in the proof of Case 5, we can derive contradictions.

Case 7. $\alpha_1^* < s(\alpha_1^*)$, $\alpha_2^* = s(\alpha_1^*)$ and $\alpha_3^* = s(\alpha_1^*)$. By using a similar argument as in the proof of Case 5, we can derive contradictions.

In view of all the discussions above, we conclude that $\alpha_1^* = \alpha_2^* = \alpha_3^* = s(\alpha_1^*)$. This completes the proof. □

Lemma 2.6. *Assume that all the conditions of Lemma 2.5 hold. Then, we obtain*

(i) $\forall \psi \in \omega(\varphi) \setminus \{\widehat{\alpha^*}\}$, $i = \{1, 2, 3\}$, *there exists* $s^* \in [0, r_1 + r_2 + r_3]$ *such that*

$$x_i(s^*, \psi) = \alpha_i^*;$$

(ii) $\forall \psi \in \omega(\varphi) \setminus \{\widehat{\alpha^*}\}$, *there exists* $t^* \in [0, r_1 + r_2 + r_3]$ *such that*

$$\begin{cases} x_1(t^* + k(r_1 + r_2 + r_3), \psi) = \alpha_1^*, \\ x_2(t^* + k(r_1 + r_2 + r_3) + r_1 + r_3, \psi) = \alpha_2^*, \\ x_3(t^* + k(r_1 + r_2 + r_3) + r_1, \psi) = \alpha_3^*, \end{cases}$$

for all nonnegative integers $k \geq 0$.

Proof. From $\omega(\varphi) \setminus \{\widehat{\alpha^*}\} \neq \emptyset$ and Lemma 2.5, we have

$$\alpha_1^* = \alpha_2^* = \alpha_3^* = s(\alpha_1^*).$$

(i) Assume, by way of contradiction, that conclusion (i) does not hold. Then, there exits $\psi \in \omega(\varphi) \setminus \{\widehat{\alpha^*}\}$ such that

$$x_1(t, \psi) > \alpha_1^*, \quad \forall t \in [0, r_1 + r_2 + r_3].$$

By Lemma 2.2 (ii), we have $x_3(t, \psi) > \alpha_3^*$, $\forall t \in [r_1, r_1 + r_2 + r_3]$, again by Lemma 2.2 (iv), we get $x_2(t_1, \psi) > \alpha_2^*$, $\forall t \in [r_1 + r_3, r_1 + r_2 + r_3]$, and from the definition of "$\gg$", then

$$x_{r_1+r_2+r_3}(\psi) \gg \widehat{\alpha^*}.$$

Together with the definition of $\omega(\varphi)$ and Lemma 2.2, we obtain that $\omega(\varphi) \gg \widehat{\alpha^*}$, a contradiction to the choice of α^*. This implies that conclusion (i) holds.

(ii) Set

$$\begin{cases} A_k = \{t \in [0, r_1 + r_2 + r_3] : x_1(t + k(r_1 + r_2 + r_3), \psi) = \alpha_1^*\}, \\ B_k = \{t \in [0, r_1 + r_2 + r_3] : x_2(t + k(r_1 + r_2 + r_3) + r_1 + r_3, \psi) = \alpha_2^*\}, \\ C_k = \{t \in [0, r_1 + r_2 + r_3] : x_3(t + k(r_1 + r_2 + r_3) + r_1, \psi) = \alpha_3^*\}, \end{cases}$$

where $k \geq 0$. From Lemma 2.2(ii), (iii) and (iv), we get $A_k \subseteq B_{k-1}$, $B_k \subseteq C_k$ and $C_k \subseteq A_k$, $\forall k \geq 1$. Thus

$$A_k \subseteq A_{k-1} \qquad B_k \subseteq B_{k-1} \qquad C_k \subseteq C_{k-1}, \quad \forall k \geq 1.$$

By the compactness of $[0, r_1 + r_2 + r_3]$, we have

$$\bigcap_{k \geq 1} A_k = \bigcap_{k \geq 1} B_k = \bigcap_{k \geq 1} C_k \neq \emptyset.$$

Choose $t^* \in \bigcap_{k \geq 1} A_k$, then t^* meet the requirement of conclusion (ii). This completes the proof. □

3. Main results

With the preparations in Section 2, we are ready to state and prove our main results.

Theorem 3.1. *Assume that $G \geq F$, $\varphi \in C$ such that $O(\varphi)$ is bounded, then, there exists $\alpha^* \in R^3$ such that $\omega(\varphi) = \{\widehat{\alpha^*}\}$.*

Proof. Let $\alpha^* = \sup\{\alpha \in E_F : \widehat{\alpha} \leq \omega(\varphi)\}$. We shall show that $\omega(\varphi) = \{\widehat{\alpha^*}\}$. If not, then $\omega(\varphi) \setminus \{\widehat{\alpha^*}\} \neq \emptyset$. From Lemma 2.5, we obtain $\alpha_1^* = \alpha_2^* = \alpha_3^* = s(\alpha_1^*)$. In view of Lemma 2.6, we can suppose that there exists $\psi \in \omega(\varphi) \setminus \{\widehat{\alpha^*}\}$ such that

$$\begin{cases} x_1(k(r_1+r_2+r_3), \psi) = \alpha_1^*, \\ x_2(k(r_1+r_2+r_3)+r_1+r_3, \psi) = \alpha_2^*, \\ x_3(k(r_1+r_2+r_3)+r_1, \psi) = \alpha_3^*, \end{cases}$$

where $k \geq 1$.

Let $x(t) = x(t, \psi)$. For $s \in [0, r_1+r_2+r_3]$ and $k \geq 0$, integrating (1.4), we obtain

$$\begin{cases} \int_{k(r_1+r_2+r_3)+s}^{(k+1)(r_1+r_2+r_3)} x_1'(t)\mathrm{d}t = \int_{k(r_1+r_2+r_3)+s}^{(k+1)(r_1+r_2+r_3)} [-F(x_1(t)) + G(x_2(t-r_2))]\mathrm{d}t \\ \int_{k(r_1+r_2+r_3)+r_1+r_3+s}^{(k+1)(r_1+r_2+r_3)+r_1+r_3} x_2'(t)\mathrm{d}t = \int_{k(r_1+r_2+r_3)+r_1+r_3+s}^{(k+1)(r_1+r_2+r_3)+r_1+r_3} [-F(x_2(t)) + G(x_3(t-r_3))]\mathrm{d}t \\ \int_{k(r_1+r_2+r_3)+r_1+s}^{(k+1)(r_1+r_2+r_3)+r_1} x_3'(t)\mathrm{d}t = \int_{k(r_1+r_2+r_3)+r_1+s}^{(k+1)(r_1+r_2+r_3)+r_1} [-F(x_3(t)) + G(x_1(t-r_1))]\mathrm{d}t. \end{cases}$$

Set

$$\begin{cases} a_k(s) = \int_{k(r_1+r_2+r_3)+s}^{(k+1)(r_1+r_2+r_3)} F(x_1(t))\mathrm{d}t \\ A_k(s) = \int_{k(r_1+r_2+r_3)+s}^{(k+1)(r_1+r_2+r_3)} G(x_1(t))\mathrm{d}t \\ b_k(s) = \int_{k(r_1+r_2+r_3)+r_1+r_3+s}^{(k+1)(r_1+r_2+r_3)+r_1+r_3} F(x_2(t))\mathrm{d}t \\ B_k(s) = \int_{k(r_1+r_2+r_3)+r_1+r_3+s}^{(k+1)(r_1+r_2+r_3)+r_1+r_3} G(x_2(t))\mathrm{d}t \\ c_k(s) = \int_{k(r_1+r_2+r_3)+r_1+s}^{(k+1)(r_1+r_2+r_3)+r_1} F(x_3(t))\mathrm{d}t \\ C_k(s) = \int_{k(r_1+r_2+r_3)+r_1+s}^{(k+1)(r_1+r_2+r_3)+r_1} G(x_3(t))\mathrm{d}t \end{cases}$$

and

$$\begin{cases} y_1(k(r_1+r_2+r_3)+s) = x_1(k(r_1+r_2+r_3)+s) - \alpha_1^*, \\ y_2(k(r_1+r_2+r_3)+s) = x_2(k(r_1+r_2+r_3)+r_1+r_3+s) - \alpha_1^*, \\ y_3(k(r_1+r_2+r_3)+s) = x_3(k(r_1+r_2+r_3)+r_1+s) - \alpha_1^*, \end{cases}$$

then

$$\begin{cases} 0 \leq y_1(k(r_1+r_2+r_3)+s) = a_k(s) - B_{k-1}(s) \leq a_k(s) - b_{k-1}(s), \\ 0 \leq y_2(k(r_1+r_2+r_3)+s) = b_k(s) - C_k(s) \leq b_k(s) - c_k(s), \\ 0 \leq y_3(k(r_1+r_2+r_3)+s) = c_k(s) - A_k(s) \leq c_k(s) - a_k(s), \end{cases}$$

hence

$$\sum_{k=1}^{n} \sum_{i=1}^{3} y_i(k(r_1+r_2+r_3)+s) \leq \sum_{k=1}^{n} (b_k(s) - b_{k-1}(s)) \leq b_n(s) - b_0(s).$$

Together with the compactness and invariance of $\omega(\varphi)$, it follows that there exists $M > 0$ such that $|b_k(s)| \leq M$ for any $k \geq 0$, $s \in [0, r_1+r_2+r_3]$. So

$$\sum_{k=1}^{n} \sum_{i=1}^{3} y_i(k(r_1+r_2+r_3)+s) \leq 2M, \quad \forall\, n \geq 1.$$

Thus, $y_i(k(r_1+r_2+r_3)+s)$ uniformly converges to 0 for $s\in[0,r_1+r_2+r_3]$. Then,

$$\lim_{k\to\infty} a_k(0)=\lim_{k\to\infty} b_k(0)=\lim_{k\to\infty} c_k(0)=F(\alpha_1^*)(r_1+r_2+r_3).$$

Again from the discussions above and the choice of ψ, we obtain

$$\begin{cases} a_k(0)\geq b_{k-1}(0),\\ b_k(0)\geq c_k(0),\\ c_k(0)\geq a_k(0)\end{cases}$$

where $k\geq 0$. Hence

$$a_k(0)\geq a_0(0)\qquad b_k(0)\geq b_0(0)\quad\text{and}\quad c_k(0)\geq c_0(0),\quad \forall\, k\geq 0.$$

In view of $\psi\in\omega(\varphi)\setminus\{\widehat{\alpha^*}\}$ and $\alpha_1^*=\alpha_2^*=\alpha_3^*=s(\alpha_1^*)$, we get

$$\max\{a_k(0),b_k(0),c_k(0)\}\geq \max\{a_0(0),b_0(0),c_0(0)\}>F(\alpha_1^*)(r_1+r_2+r_3),$$

a contradiction to $\lim_{k\to\infty}(\max\{a_k(0),b_k(0),c_k(0)\})=F(\alpha_1^*)(r_1+r_2+r_3)$. Thus $\omega(\varphi)=\{\widehat{\alpha^*}\}$. This completes the proof. □

Theorem 3.2. *Assume that $G\leq F$, $\varphi\in C$ such that $O(\varphi)$ is bounded, then, there exists $\alpha^*\in R^3$ such that $\omega(\varphi)=\{\widehat{\alpha^*}\}$.*

Proof. Let $f(-x)=-F(x)$, $g(x)=-G(-x)$, then f is nondecreasing, $g\geq f$. Set $y_i(t)=-x_i(t,\varphi)$, $\forall t\geq -r_i$, then

$$\begin{cases} y_1'(t)=-f(y_1(t))+g(y_2(t-r_2)),\\ y_2'(t)=-f(y_2(t))+g(y_3(t-r_3)),\\ y_3'(t)=-f(y_3(t))+g(y_1(t-r_1)).\end{cases}$$

It follows from Theorem 3.1 that there exists $\beta^*\in R^3$ such that $\lim_{t\to\infty}(y_1(t),y_2(t),y_3(t))=\beta^*$. Set $\alpha^*=-\beta^*$, then $\omega(\varphi)=\{\widehat{\alpha^*}\}$. This completes the proof. □

Corollary 3.1. *If F is nondecreasing, then, each solution of*

$$\begin{cases} x_1'(t)=-F(x_1(t))+F(x_2(t-r_2)),\\ x_2'(t)=-F(x_2(t))+F(x_1(t-r_1)),\end{cases}$$

tends to a constant vector as $t\longrightarrow\infty$.

Corollary 3.2. *If F is nondecreasing, $G\geq F$ or $G\leq F$, then, each bounded solution of*

$$x'(t)=-F(x(t))+G(x(t-r))$$

tends to a constant as $t\longrightarrow\infty$.

Proof. Consider the synchronization solution of (1.4), together with Theorem 3.1. or Theorem 3.2., we can prove that the conclusions hold. □

Remark 1. Corollary 3.2 also gives an improvement of the results in Ding [7–9] and a new form of proof on the Bernfeld–Haddock conjecture.

Acknowledgement

The authors would like to express their sincere appreciation to the reviewer for his/her helpful comments for improving the presentation and quality of this paper.

References

[1] S.R. Bernfeld, J.R.A. Haddock, Variation of Razumikhin's method for retarded functional equations, in: Nonlinear Systems and Applications, An International Conference, Academic Press, New York, 1977, pp. 561–566.
[2] C. Jehu, Comportement asymptotique des solutions de equation $\dot{x}(t)=-f(t,x(t))+f(t,x(t-1))+h(t)$, Ann. Soc. Sci. Brux. I 92 (4) (1979) 263–269. (in French).
[3] T. Krisztin, On the convergence of solutions of functional–differential equations, Acta Sci. Math. (Szeged) 43 (1981) 45–54.
[4] O. Arino, P. Seguier, About the behavior at infinity of solutions $\dot{x}(t)=f(t-1,x(t-1))-f(t,x(t))$, J. Math. Anal. Appl. 96 (1983) 420–436.
[5] T. Krisztin, Convergence of solutions of a nonlinear integro-differential equation arising in compartmental systems, Acta Sci. Math. (Szeged) 47 (1984) 471–485.
[6] T. Krisztin, An invariance principle of Lyapunov–Razumikhin type and compartmental systems, in: World Congress of Nonlinear Analysts'92, Vol. I–IV. Tampa, FL, 1992, de Gruyter, Berlin, 1996, pp. 1371–1379.
[7] T. Ding, Asymptotic behavior of solutions of some retarded differential equations, Sci. China Ser. A 8 (1981) 939–945 (in Chinese).
[8] T. Ding, Asymptotic behavior of solutions of some retarded differential equations, Sci. China Ser. A 25 (1982) 263–371.

[9] T. Ding, Applications of the Qualitative Methods in Ordinary Differential Equations, China Higher Education Press, Peking, 2004, pp. 155–163 (in Chinese).
[10] T.S. Yi, L.H. Huang, A generalization of the Bernfeld–Haddock conjecture and its proof, Acta Math. Sinica 50 (2) (2007) 210–261 (in Chinese).
[11] T.S. Yi, L.H. Huang, Asymptotic behavior of solutions to a class of systems of delay differential equations, Acta Math. Sinica 23 (8) (2007) 1375–1384.
[12] T.S. Yi, L.H. Huang, Convergence of solution to a class of systems of delay differential equations, Nonlinear Dyn. Syst. Theory 5 (2005) 189–200.
[13] O. Arino, E. Haourigui, On the asymptotic behavior of solutions of some delay differential systems which have a first integral, J. Math. Anal. Appl. 122 (1987) 36–46.
[14] O. Arino, P. Seguier, About the behavior at infinity of solutions of: $x'(t) = f(t - x(t-1)) - f(t, x(t))$, J. Math. Anal. Appl. 96 (1983) 420–436.
[15] F. Atkinson, J. Haddock, Criteria for asymptotic constancy of solutions of functional differential equations, J. Math. Anal. Appl. 91 (1983) 410–423.
[16] F. Atkinson, J. Haddock, O. Staffans, Integral inequalities and exponential convergence of differential equations with bounded delay, in: W. Everitt, B. Sleeman (Eds.), Ordinary and Partial Differential Equations, in: Lecture Notes in Math., Springer-Verlag, Berlin, 1982, pp. 56–68.
[17] K.L. Cooke, J.L. Kaplan, A periodicity threshold theorem for epidemic and population growth, Math. Biosci. 31 (87–104) (1976) 410–423.
[18] K.L. Cooke, J. Yorke, Some equations modelling growth process and gonorrhea epidemics, Math. Biosci. 16 (75–107) (1973) 155–163.
[19] I. Györi, Connections between compartmental systems with pipes and integrodifferential equations, Math. Modelling 7 (1986) 1215–1238.
[20] J.R. Haddock, M.N. Nkashama, J. Wu, Asymptotic constancy for pseudo monotone dynamical systems on function spaces, J. Differential Equations 100 (1992) 292–311.
[21] J.R. Haddock, J. Terjeki, Lyapunov–Razumikhin functions and invariance principle for functional differential equations, J. Differential Equations 48 (1983) 95–122.
[22] S.H. Hu, L.H. Huang, T.S. Yi, Convergence of bounded solutions to a class of systems of delay differential equations, Nonlinear Anal. TMA 61 (2005) 543–549.
[23] J. Kaplan, M. Sorg, J. Yorke, Solutions of $x'(t) = f(x(t), x(t - L))$ have limits when f is an order relation, Nonlinear Anal. TMA 3 (1979) 53–58.
[24] T.S. Yi, L.H. Huang, Convergence for pseudo monotone semiflows on product ordered topological spaces, J. Differential Equations 214 (2005) 429–456.
[25] H.L. Smith, Monotone semiflows generated by functional differential equations, J. Differential Equations 66 (1987) 420–442.