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Quivers with relations arising from Koszul algebras of $\mathfrak{g}$-invariants ☆

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ABSTRACT

Let $\mathfrak{g}$ be a complex simple Lie algebra and let Ψ be an extremal set of positive roots. After Chari and Greenstein (2009) [9], one associates with Ψ an infinite dimensional Koszul algebra $\mathbf{S}_\Psi^\mathfrak{g}$ which is a graded subalgebra of the locally finite part of $((\operatorname{End}\mathbf{V})^{op}\otimes S(\mathfrak{g}))^\mathfrak{g}$, where $\mathbf{V}$ is the direct sum of all simple finite dimensional $\mathfrak{g}$-modules. We describe the structure of the algebra $\mathbf{S}_\Psi^\mathfrak{g}$ explicitly in terms of an infinite quiver with relations for $\mathfrak{g}$ of types A and C. We also describe several infinite families of quivers and finite dimensional associative algebras arising from this construction.

Introduction

One of the classical methods in the representation theory is to replace a category one wishes to study by an equivalent category of modules over an associative algebra. This approach was extensively used in the study of the category $\mathcal{O}$ (cf. for example [1–4,12,19–21]) and in many other situations and led to the introduction of highest weight categories in [4]. The associative algebra in question is usually the endomorphism algebra of a generator or a co-generator of the category. On the other hand, it is also known that endomorphism algebras often give rise to nice associative algebras (for example, in the case of the category $\mathcal{O}$ these algebras are Koszul). However, describing them in terms of generators and relations, or in terms of quivers with relations, is usually a rather involved task (cf. for example [20]).

In [8] the category $\mathcal{G}$ of graded finite dimensional modules over the polynomial current algebra $\mathfrak{g}[t]=\mathfrak{g}\otimes\mathbf{C}[t]$ of a finite dimensional complex Lie algebra $\mathfrak{g}$ was studied. That category can be perceived as a non-semisimple "deformation" of the semisimple category of finite dimensional $\mathfrak{g}$-modules. We proved that the category $\mathcal{G}$ is highest weight in the sense of [4]. We also studied a family

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of quivers arising from the endomorphism algebras of injective co-generators of certain Serre subcategories with finitely many simples in the cases when they are hereditary. For example, all Dynkin quivers can be realised in this way. We also considered an example where the endomorphism algebra was not hereditary and computed the relations in that algebra. However, it was already clear from that computation that describing quivers and relations for these algebras in general would be rather difficult.

The situation becomes more manageable if we pass to the truncated current algebra $\mathfrak{g}\otimes\mathbf{C}[t]/(t^2)$ which is isomorphic to the semidirect product $\mathfrak{g}\ltimes\mathfrak{g}$ of $\mathfrak{g}$ with its adjoint representation. The motivation for the study of graded representation of that algebra stems from the fact that several interesting families of indecomposable objects in $\mathcal{G}$ can be regarded as modules over $\mathfrak{g}\ltimes\mathfrak{g}$, namely the classical limits of Kirillov–Reshetikhin modules for $\mathfrak{g}$ of classical types [15] or more generally, of the minimal affinisations [5,6]. The category $\mathcal{G}_2$ of graded modules over $\mathfrak{g}\ltimes\mathfrak{g}$ was studied in [9]. In particular, we studied families of Serre subcategories of $\mathcal{G}_2$ associated with sets of roots maximising some linear functional. We call these sets extremal since they correspond to faces of the convex hull of roots of $\mathfrak{g}$. A study of these subcategories was motivated by the observation that, after [6,10], there is a natural extremal set of positive roots associated with a Kirillov–Reshetikhin module. Extremal sets have many interesting combinatorial properties and were studied in [7] (in particular, their complete list for $\mathfrak{g}$ of classical types was provided). After [9], given an extremal set Ψ contained in a fixed set of positive roots of $\mathfrak{g}$, one obtains a family of Serre subcategories which have enough projectives and for which the endomorphism algebra of a projective generator is Koszul. Then one constructs an infinite dimensional Koszul algebra $\mathbf{S}^{\mathfrak{g}}_{\Psi}$ which is "approximated" by these finite dimensional Koszul algebras. The advantage of this infinite dimensional algebra is that it allows us to study all these finite dimensional subalgebras simultaneously.

The aim of the present paper is to describe the structure of algebras $\mathbf{S}^{\mathfrak{g}}_{\Psi}$. We show that they can be realised as path algebras of quite nice quivers with relations. In some cases these quivers admit very explicit combinatorial realisations. We compute all relations in these algebras for $\mathfrak{g}$ of types A and C. Quite expectedly, that turns out to be rather difficult and uses monomial bases of the universal enveloping algebra of the lower triangular part of $\mathfrak{g}$. Due to very restrictive properties of extremal sets, in types A and C we can perform all computations using only the monomial bases in type A which are known very explicitly [16]. On the other hand, it is quite remarkable that to study the relations in $\mathbf{S}^{\mathfrak{g}}_{\Psi}$ we only need the most elementary properties of the extremal sets described in [9]. It should also be noted that, although an extremal set is conjugate under the action of the Weyl group to the set of roots of an abelian ideal in a suitable Borel subalgebra (cf. [7]), the algebras $\mathbf{S}^{\mathfrak{g}}_{\Psi}$ behave quite differently even for conjugate sets Ψ. For example, depending on whether the highest root of $\mathfrak{g}$ is contained in Ψ, all connected subalgebras of $\mathbf{S}^{\mathfrak{g}}_{\Psi}$ are infinite or finite dimensional.

The paper is organised as follows. In Section 1 we briefly review the construction of the algebras $\mathbf{S}^{\mathfrak{g}}_{\Psi}$ and present the main results. In Section 2 we develop the technique for computing relations, while in Section 3 we consider several relatively simple examples which illustrate how these methods are applied. In Section 4 we construct a family of elements in the universal enveloping algebra of a Borel subalgebra of $\mathfrak{g}$ corresponding to parabolic subalgebras with the Levi factor of type A which play the central role in our computations. Finally, in Sections 5 and 6 we undertake a systematic study of relations in the algebras $\mathbf{S}^{\mathfrak{g}}_{\Psi}$ for $\mathfrak{g}$ of types A and C. We also describe several infinite families of quivers arising from the study of connected subalgebras of $\mathbf{S}^{\mathfrak{g}}_{\Psi}$ when Ψ satisfies some "regularity" condition.

1. Main results

Throughout this paper we denote by $\mathbf{Z}_+$ the set of non-negative integers and by $\mathbf{C}$ the field of complex numbers. We consider $\mathbf{Z}_+\cup\{+\infty\}$ as a totally ordered semigroup with $+\infty>n$ and $+\infty+n=+\infty$ for all $n\in\mathbf{Z}_+$. All algebras and vector spaces are considered over $\mathbf{C}$. Tensor products and Hom spaces are taken over $\mathbf{C}$ unless specified otherwise. For an associative algebra A, A^{op} denotes its opposite algebra. For a vector space V, $V^*=\operatorname{Hom}(V,\mathbf{C})$. Given a Lie algebra $\mathfrak{a}$, we denote by $U(\mathfrak{a})$ its universal enveloping algebra and by $U(\mathfrak{a})_+$ the augmentation ideal in $U(\mathfrak{a})$. In particular, if $\mathfrak{a}$ is abelian, $U(\mathfrak{a})$ is the symmetric algebra $S(\mathfrak{a})$. Given an $\mathfrak{a}$-module V we denote by $V^{\mathfrak{a}}$ the subspace of $\mathfrak{a}$-invariant elements in V, that is $V^{\mathfrak{a}} = \{v \in V\colon\ xv = 0,\ \forall x \in \mathfrak{a}\}$.

1.1. Let $\mathfrak{g}$ be a finite dimensional simple complex Lie algebra and fix its Cartan subalgebra $\mathfrak{h}$. The Killing form of $\mathfrak{g}$ induces a non-degenerate bilinear form $(\cdot,\cdot)$ on $\mathfrak{h}^*$. Let $P \subset \mathfrak{h}^*$ be a weight lattice and let $R \subset P$ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Choose the set of simple roots $\alpha_i \in R$, $i \in I := \{1, \dots, \dim\mathfrak{h}\}$ and the corresponding fundamental weights $\varpi_i \in P$. Let $P^+ \subset P$ be the $\mathbf{Z}_+$-span of the ϖ_i and let R^+ be the intersection of R with the $\mathbf{Z}_+$-span of the α_i. Given $\beta \in R$, set for all $i \in I$,

$$\varepsilon_i(\beta) = \max\{t \in \mathbf{Z}_+\colon\ \beta + t\alpha_i \in R\}, \qquad \varphi_i(\beta) = \max\{t \in \mathbf{Z}_+\colon\ \beta - t\alpha_i \in R\}$$

and define

$$\varepsilon(\beta) := \sum_{i\in I} \varepsilon_i(\beta)\varpi_i, \qquad \varphi(\beta) := \sum_{i\in I} \varphi_i(\beta)\varpi_i.$$

Clearly, $\varepsilon(\beta), \varphi(\beta) \in P^+$. It is well known that $\varphi(\beta) = \varepsilon(\beta) + \beta$. For $\alpha \in R$ let $\mathfrak{g}_\alpha$ be the corresponding root subspace of $\mathfrak{g}$ and, given $\Psi \subset R^+$, let $\mathfrak{n}^\pm_\Psi = \bigoplus_{\alpha\in\Psi} \mathfrak{g}_{\pm\alpha}$. In particular, we write $\mathfrak{n}^\pm = \mathfrak{n}^\pm_{R^+}$ and set $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^-$.

We say that $\Psi \subset R$ is *extremal* if there exists $\xi \in P \setminus \{0\}$ such that

$$\Psi = \Big\{\alpha \in R\colon\ (\xi, \alpha) = \max_{\beta\in R}(\xi, \beta)\Big\}.$$

Geometrically, an extremal subset of R is the intersection with R of a face of the convex hull of R in the euclidean space spanned by R. Note that if $\xi \in P^+$ then $\Psi \subset R^+$. We will need the following property of extremal sets.

Lemma. *(See [9, Lemma 2.3].) Let $\Psi \subset R$ be extremal and suppose that*

$$\sum_{\alpha\in R} m_\alpha \alpha = \sum_{\beta\in\Psi} n_\beta \beta, \quad m_\alpha, n_\beta \in \mathbf{Z}_+.$$

Then

$$\sum_{\beta\in\Psi} n_\beta \leqslant \sum_{\alpha\in R} m_\alpha \tag{1.1}$$

with equality if and only if $m_\alpha = 0$ for all $\alpha \notin \Psi$.

We note the following

Corollary. *Let $\Psi \subset R$ be extremal. Then $\Psi + \Psi \cap (R \cup \{0\}) = \emptyset$ and*

$$\alpha, \beta \in R, \quad \alpha + \beta \in \Psi + \Psi \quad \Longrightarrow \quad \alpha, \beta \in \Psi.$$

Remark. It is shown in [7] that this property characterises extremal sets.

1.2. Let $\Psi \subset R^+$ be extremal. In [9] two infinite dimensional Koszul algebras $\mathbf{S}_\Psi^{\mathfrak g}$ and $\mathbf{E}_\Psi^{\mathfrak g}$ were constructed and it was shown that $(\mathbf{E}_\Psi^{\mathfrak g})^{op}$ is the quadratic dual of $\mathbf{S}_\Psi^{\mathfrak g}$ and the left global dimension of $\mathbf{S}_\Psi^{\mathfrak g}$ equals $|\Psi|$. This construction was motivated by the study of categories of graded representations of current algebras initiated in [8].

Given $\lambda \in P^+$, let $V(\lambda)$ be the unique, up to an isomorphism, simple finite dimensional $\mathfrak g$-module of highest weight λ. Let

$$\mathbf{V} = \bigoplus_{\lambda \in P^+} V(\lambda), \qquad \mathbf{V}^{\circledast} = \bigoplus_{\lambda \in P^+} V(\lambda)^*.$$

Then $\mathbf{V}^{\circledast} \otimes \mathbf{V}$ with the product given by

$$(f \otimes v)(g \otimes w) = g(v) f \otimes w, \quad f, g \in \mathbf{V}^{\circledast},\ v, w \in \mathbf{V}$$

is isomorphic to a subalgebra of the associative algebra $(\operatorname{End}_{\mathbf{C}} \mathbf{V})^{op}$ and hence for any associative algebra A, the space $\mathbf{A} = A \otimes \mathbf{V}^{\circledast} \otimes \mathbf{V}$ has a natural structure of an associative algebra. Moreover, if $A = \bigoplus_{n \in \mathbf{Z}_+} A[n]$ is a $\mathbf{Z}_+$-graded associative algebra, we obtain a grading on $\mathbf{A}$ by assigning to the elements of $\mathbf{V}^{\circledast} \otimes \mathbf{V}$ the grade zero, that is, $\mathbf{A}[k] = A[k] \otimes \mathbf{V}^{\circledast} \otimes \mathbf{V}$. In the rest of the paper, we identify the algebra $\mathbf{A}$ with $\mathbf{V}^{\circledast} \otimes A \otimes \mathbf{V}$ under the natural isomorphism of $\mathfrak g$-modules and with the induced algebra structure given by

$$(f \otimes a \otimes v)(g \otimes b \otimes w) = g(v) f \otimes ab \otimes w, \quad a, b \in A,\ f, g \in \mathbf{V}^{\circledast},\ v, w \in \mathbf{V}.$$

We write $\mathbf{T}$ (respectively, $\mathbf{S}$, $\mathbf{E}$) for $\mathbf{A}$ with A the tensor algebra $T(\mathfrak g)$ of $\mathfrak g$ (respectively, the symmetric algebra $S(\mathfrak g)$ and the exterior algebra $\bigwedge \mathfrak g$). In particular, in these cases A is a $\mathfrak g$-module with respect to the diagonal action, hence $\mathbf{A}$ is also a $\mathfrak g$-module and the multiplication is a homomorphism of $\mathfrak g$-modules. It follows that $\mathbf{A}^{\mathfrak g}$ is a subalgebra of $\mathbf{A}$. From now on, we let $\mathbf{A}$ be one of the algebras $\mathbf{T}$, $\mathbf{S}$ or $\mathbf{E}$. Given $\lambda \in P^+$, the algebra $\mathbf{A}^{\mathfrak g}$ contains a primitive idempotent 1_λ corresponding to the canonical $\mathfrak g$-invariant element in $V(\lambda)^* \otimes V(\lambda)$, or, equivalently to the identity element in $\operatorname{End} V(\lambda)$. Then we have

$$\mathbf{A}^{\mathfrak g} = \bigoplus_{\lambda, \mu \in P^+} 1_\lambda \mathbf{A}^{\mathfrak g} 1_\mu, \qquad 1_\lambda \mathbf{A}^{\mathfrak g} 1_\mu = \big(V(\lambda)^* \otimes A \otimes V(\mu)\big)^{\mathfrak g}.$$

1.3. Given $\Psi \subset R^+$, define a relation $\leqslant_\Psi$ on P by $\lambda \leqslant_\Psi \mu$ if $\mu - \lambda \in \mathbf{Z}_+\Psi$. It is straightforward to check that $\leqslant_\Psi$ is a partial order. In particular, $\leqslant := \leqslant_{R^+}$ is the standard partial order on P. If $\lambda \leqslant_\Psi \mu$ and $\lambda \neq \mu$ we write $\lambda <_\Psi \mu$. Note that for all $\lambda \in P$ and for all $\Psi \subset R^+$, the set $\{\mu \in P^+ \colon \mu \leqslant_\Psi \lambda\}$ is finite. Define a function $d_\Psi : \{(\lambda, \mu) \in P^+ \times P^+ \colon \lambda \leqslant_\Psi \mu\} \to \mathbf{Z}_+$ by

$$d_\Psi(\lambda, \mu) = \min\left\{ \sum_{\alpha \in \Psi} m_\alpha \colon \mu - \lambda = \sum_{\alpha \in \Psi} m_\alpha \alpha,\ m_\alpha \in \mathbf{Z}_+ \right\}.$$

Clearly, $d_\Psi(\lambda, \mu) = 0$ if and only if $\lambda = \mu$ and $d_\Psi(\lambda, \mu) + d_\Psi(\mu, \nu) \leqslant d_\Psi(\lambda, \nu)$ for all $\lambda \leqslant_\Psi \mu \leqslant_\Psi \nu$. Furthermore, if Ψ is extremal, we have

$$d_\Psi(\lambda, \mu) + d_\Psi(\mu, \nu) = d_\Psi(\lambda, \nu)$$

and if μ covers λ then $d_\Psi(\lambda, \mu) = 1$. In particular, in this case d_Ψ is the unique distance function for the poset $(P^+, \leqslant_\Psi)$.

Fix an extremal set $\Psi \subset R^+$. Given $F \subset P^+$, define

$$\mathbf{A}^{\mathfrak{g}}_{\Psi}(F) = \bigoplus_{\lambda,\mu \in F:\ \lambda \leqslant_{\Psi} \mu} 1_{\lambda} \mathbf{A}^{\mathfrak{g}}\big[d_{\Psi}(\lambda,\mu)\big] 1_{\mu}.$$

It is not hard to check that $\mathbf{A}^{\mathfrak{g}}_{\Psi}(F)$ is a subalgebra of $\mathbf{A}^{\mathfrak{g}}$. Let $\mathbf{A}^{\mathfrak{g}}_{\Psi} := \mathbf{A}^{\mathfrak{g}}_{\Psi}(P^+)$. Given $\lambda, \mu \in P^+$, define the following subsets of P^+

$$\leqslant_{\Psi} \lambda = \big\{\nu \in P^+ \colon \nu \leqslant_{\Psi} \lambda\big\}, \qquad \mu \leqslant_{\Psi} = \big\{\mu \leqslant_{\Psi} \nu \colon \nu \in P^+\big\}$$

and $[\mu, \lambda]_{\Psi} = (\leqslant_{\Psi} \lambda) \cap (\mu \leqslant_{\Psi})$. We say that $F \subset P^+$ is interval closed in the partial order $\leqslant_{\Psi}$ if $\lambda, \mu \in F$ implies that $[\lambda, \mu]_{\Psi} \subset F$.

The main properties of the algebras $\mathbf{S}^{\mathfrak{g}}_{\Psi}$ established in [9] are summarised below.

Theorem 1. *(See [9, Theorem 1].) Let Ψ be an extremal set of positive roots.*

(i) *Let $\mu, \nu \in P^+$. The subalgebras $\mathbf{S}^{\mathfrak{g}}_{\Psi}(\leqslant_{\Psi} \nu)$, $\mathbf{S}^{\mathfrak{g}}_{\Psi}(\mu \leqslant_{\Psi})$ and $\mathbf{S}^{\mathfrak{g}}_{\Psi}([\mu, \nu]_{\Psi})$ of $\mathbf{S}^{\mathfrak{g}}_{\Psi}$ are Koszul and have global dimension at most $|\Psi|$. The bound is attained for some $\mu', \nu' \in P^+$ with $\mu' \leqslant_{\Psi} \nu'$.*

(ii) *The algebra $\mathbf{S}^{\mathfrak{g}}_{\Psi}$ is Koszul and has left global dimension $|\Psi|$.*

Remark. The argument of [9, Proposition 4.5] actually proves that $\mathbf{S}^{\mathfrak{g}}_{\Psi}(F)$ is Koszul for any $F \subset P^+$ interval closed in the partial order $\leqslant_{\Psi}$.

1.4. Being Koszul, the algebras $\mathbf{S}^{\mathfrak{g}}_{\Psi}$ are quadratic and so to describe all relations in them it is enough to describe the quadratic relations. A convenient language for that is provided by quivers. We mostly follow the conventions from [18]. Let us briefly review the quiver terminology which will be used in the sequel.

Recall that a quiver Δ is a pair $\Delta = (\Delta_0, \Delta_1)$ where Δ_0 is the vertex set, Δ_1 is the set of arrows. *In this paper we only consider quivers without multiple arrows*, that is, for any pair $x, y \in \Delta_0$, there is at most one arrow $x \leftarrow y \in \Delta_1$ (in other words, Δ_1 identifies with a subset of $\Delta_0 \times \Delta_0$). A path of length k in such a quiver is a sequence $x_0, \dots, x_k \in \Delta_0$ such that for all $0 \leqslant i < k$, there is an arrow $x_i \leftarrow x_{i+1} \in \Delta_1$. Denote by $\Delta(x, y)$ the set of all paths in Δ from y to x. With every vertex $x \in \Delta_0$ we associate a trivial path 1_x of length 0.

The opposite quiver Δ^{op} of Δ is the quiver with the same vertex set obtained by reversing all arrows. The underlying graph $\bar{\Delta}$ of Δ is obtained from Δ by forgetting the orientation of the arrows. We say that Δ is connected if $\bar{\Delta}$ is connected. A connected quiver Δ is said to be of type $\mathbb{X}$ (respectively, of type $\tilde{\mathbb{X}}$), where $X = A, D, E$ if $\bar{\Delta}$ is the Dynkin diagram (respectively, extended Dynkin diagram) of a simple finite dimensional Lie algebra of type X.

A vertex x is said to be a direct successor (respectively, predecessor) of y if there is an arrow $x \leftarrow y$ (respectively, $y \leftarrow x$) in Δ_1. The set of direct successors (predecessors) of $x \in \Delta_0$ is denoted by x^+ (respectively, x^-). A vertex $x \in \Delta_0$ is called a source if $x^- = \emptyset$ and a sink if $x^+ = \emptyset$.

Given $\Delta'_0 \subset \Delta_0$, the *full subquiver* of Δ defined by Δ'_0 is $\Delta' = (\Delta'_0, \Delta'_1)$ where Δ'_1 is the set of all arrows in Δ_1 with starting and ending points in Δ'_0. A subquiver Δ' of Δ is called *convex* if for any vertices $x, y \in \Delta'_0$ we have $\Delta'(x, y) = \Delta(x, y)$, that is a path in Δ from y to x is completely contained in Δ'. In particular, a convex subquiver is full. A connected component of Δ is a full subquiver Δ' such that $\bar{\Delta}'$ is a connected component of $\bar{\Delta}$. Then Δ is a disjoint union of its connected components. Given $x \in \Delta_0$, we denote the connected component of Δ containing x by $\Delta[x]$.

A full embedding of quivers $\Delta \to \Delta'$ is a pair of injective maps $F_0 : \Delta_0 \to \Delta'_0$ and $F_1 : \Delta_1 \to \Delta'_1$ which are compatible in a natural way and such that $(F_0(\Delta_0), F_1(\Delta_1))$ is a full subquiver of Δ'. If both maps are bijective we say that Δ is isomorphic to Δ'.

Given a quiver $\Delta = (\Delta_0, \Delta_1)$, let $\mathbf{C}\Delta$ be the complex vector space with the basis consisting of all finite paths in Δ. The product of two paths is set to be their composition when they are composable, and zero otherwise. This defines on $\mathbf{C}\Delta$ a structure of a $\mathbf{Z}_+$-graded associative algebra, the grading being given by the length of paths. In particular, the 1_x, $x \in \Delta_0$ are primitive orthogonal idempotents and $\mathbf{C}\Delta[0]$ is commutative and semisimple. Clearly, $\mathbf{C}(\Delta^{op}) \cong (\mathbf{C}\Delta)^{op}$. The group $(\mathbf{C}^\times)^{\Delta_1}$ acts naturally on $\mathbf{C}\Delta[1]$ and for all $\boldsymbol{z} \in (\mathbf{C}^\times)^{\Delta_1}$ the action of $\boldsymbol{z}$ extends to an automorphism of $\mathbf{C}\Delta$ preserving the grading and the 1_x, $x \in \Delta_0$. Clearly, an isomorphism of quivers induces an isomorphisms of the corresponding path algebras.

A relation on Δ is a linear combination of paths from x to y for some $x, y \in \Delta_0$. In particular, a relation of the form p, where p is a path, is called a zero relation, while a relation of the form $p - p'$ is called a commutativity relation. Given a quiver Δ and a set of relations R, we can form an algebra $\mathbf{C}(\Delta, R) = \mathbf{C}(\Delta, V) := \mathbf{C}\Delta/\langle R\rangle$, where V is the vector subspace of $\mathbf{C}\Delta$ spanned by R. This algebra is often referred to as the path algebra of the quiver Δ with relations R.

1.5. We define the infinite quiver Δ_Ψ as

$$(\Delta_\Psi)_0 = P^+,$$
$$(\Delta_\Psi)_1 = \{(\lambda, \mu) \in P^+ \times P^+ \colon \mu - \lambda = \beta \in \Psi,\ \lambda - \varepsilon(\beta) = \mu - \varphi(\beta) \in P^+\}.$$

Thus, $(\Delta_\Psi)_1$ is a subset of the cover relation in $(P^+, \leqslant_\Psi)$. It is immediate that if there is a path from μ to λ in Δ_Ψ, then $\lambda \leqslant_\Psi \mu$ and the length of any such path is $d_\Psi(\lambda, \mu)$. In particular, the quiver Δ_Ψ has no loops or oriented cycles. Since for all $\lambda \in P^+$ the set $\leqslant_\Psi \lambda$ is finite, it follows that every vertex in Δ_Ψ is connected to a sink. Given $F \subset P^+$, denote $\Delta_\Psi(F)$ the full subquiver of Δ_Ψ defined by F. If F is interval closed in the partial order $\leqslant_\Psi$, $\Delta_\Psi(F)$ is convex. The following proposition is proved in 2.3.

Proposition. *Let $F \subset P^+$ be interval closed in the partial order $\leqslant_\Psi$. There exists a natural isomorphism of $\mathbf{Z}_+$-graded associative algebras $\mathbf{T}^{\mathfrak{g}}_\Psi(F) \to \mathbf{C}\Delta_\Psi(F)$. This isomorphism is unique up to an automorphism of $\mathbf{C}\Delta_\Psi(F)$ extending the natural action of $(\mathbf{C}^\times)^{(\Delta_\Psi(F))_1}$ on $\mathbf{C}\Delta_\Psi(F)[1]$.*

1.6. As proved in [9, Lemma 4.2], for each $F \subset P^+$ which is interval closed in the partial order $\leqslant_\Psi$, $\mathbf{S}^{\mathfrak{g}}_\Psi(F)$ is isomorphic to the quotient of $\mathbf{T}^{\mathfrak{g}}_\Psi(F)$ by a quadratic ideal and

$$\ker\big(\mathbf{T}^{\mathfrak{g}}_\Psi(F) \to \mathbf{S}^{\mathfrak{g}}_\Psi(F)\big) = \mathbf{T}^{\mathfrak{g}}_\Psi(F) \cap \ker\big(\mathbf{T}^{\mathfrak{g}}_\Psi \to \mathbf{S}^{\mathfrak{g}}_\Psi\big).$$

Fix an isomorphism $\Phi : \mathbf{T}^{\mathfrak{g}}_\Psi \to \mathbf{C}\Delta_\Psi$. Then Proposition 1.5 allows us to identify the idempotents (respectively, some fixed generators of degree 1) of $\mathbf{S}^{\mathfrak{g}}_\Psi$ with vertices (respectively, arrows) in the quiver Δ_Ψ. To describe the quadratic relations, we need to consider, for all $\lambda, \mu \in P^+$ with $\lambda \leqslant_\Psi \mu$ and $d_\Psi(\lambda, \mu) = 2$, that is, for all $\lambda, \lambda + \eta \in P^+$, $\eta \in \Psi + \Psi$, the subquivers $\Delta_\Psi([\lambda, \lambda + \eta]_\Psi)$ of Δ_Ψ and the subalgebras $\mathbf{S}^{\mathfrak{g}}_\Psi([\lambda, \lambda + \eta]_\Psi)$ of $\mathbf{S}^{\mathfrak{g}}_\Psi$.

Denote by $\mathfrak{R}_\Psi(\lambda, \lambda + \eta)$ the image of the canonical map $\ker(\mathbf{T}^{\mathfrak{g}}_\Psi([\lambda, \lambda + \eta]_\Psi) \twoheadrightarrow \mathbf{S}^{\mathfrak{g}}_\Psi([\lambda, \lambda + \eta]_\Psi))$ in $\mathbf{C}\Delta_\Psi([\lambda, \lambda + \eta]_\Psi)$ under Φ. We set $\mathfrak{R}_\Psi(\lambda, \lambda + \eta) = 0$ if $\lambda + \eta \notin P^+$.

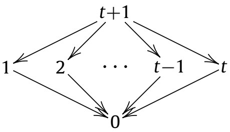
1.7. Let $\eta \in \Psi + \Psi$ and set

$$m_\eta = \big|\{(\beta, \beta') \in \Psi \times \Psi \colon \beta + \beta' = \eta\}\big|.$$

Note that $m_\eta = 1$ implies that $\eta \in 2\Psi$. For all $\lambda \in P^+$, let $t_{\lambda,\eta} = \dim 1_\lambda \mathbf{T}^{\mathfrak{g}}_\Psi 1_{\lambda+\eta}$ if $\lambda + \eta \in P^+$ and set $t_{\lambda,\eta} = 0$ otherwise. Since by Proposition 1.5, $t_{\lambda,\eta}$ equals the number of paths from $\lambda + \eta$ to λ in Δ_Ψ, it is immediate that $t_{\lambda,\eta} \leqslant m_\eta$ for all $\lambda \in P^+$.

Definition. An extremal set $\Psi \subset R^+$ is said to be *regular* if for all $\eta \in \Psi + \Psi$, $t_{\lambda,\eta} > 0 \Rightarrow t_{\lambda,\eta} = m_\eta$.

The quiver $\Delta_\Psi([\lambda, \lambda+\eta]_\Psi)$ identifies with the quiver

$\Gamma(t) =$  (1.2)

with t paths $\mathbf{p}_i = (0 \leftarrow r \leftarrow t+1)$, $1 \leqslant r \leqslant t$ of length 2, where $t = t_{\lambda,\eta}$.

Let V be a k-dimensional subspace of $\mathbf{C}\{\mathbf{p}_1, \dots, \mathbf{p}_t\}$. We say that V is generic if it is generic with respect to any coordinate flag corresponding to the basis $\mathbf{p}_1, \dots, \mathbf{p}_t$, that is for all $1 \leqslant i_1 < \dots < i_r \leqslant t$, $1 \leqslant r \leqslant t$ we have

$$\dim\bigl(V \cap \mathbf{C}\{\mathbf{p}_{i_1}, \dots, \mathbf{p}_{i_r}\}\bigr) = \begin{cases} 0, & 1 \leqslant r < t-k, \\ r+k-t, & t-k \leqslant r \leqslant t. \end{cases}$$

In particular, if $t = 1$, V is generic if and only if $\dim V = 1$. For instance, if $t = 2$ and $\dim V = 1$ then $\mathbf{C}(\Gamma(t), V)$ is of finite type. However, it has different isomorphism classes of indecomposable modules, depending on V being or not being generic. If $t = 3$, $\dim V = 1$ and V is generic then $\mathbf{C}(\Gamma(t), V)$ is unique up to an isomorphism, is canonical (cf. [18, §3.7]), of tubular type $\mathbb{D}_4$ and tame. If $t = 4$, V is generic and $\dim V = 2$ then we can assume, without loss of generality, that V is spanned by $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3$, $\mathbf{p}_1 + z\mathbf{p}_2 + \mathbf{p}_4$ for some $z \in \mathbf{C}^\times$. In particular, we have a family of algebras parametrised by elements of $\mathbb{P}^1$. The algebra $\mathbf{C}(\Gamma(t), V)$ is again canonical, of tubular type $\tilde{\mathbb{D}}_4$, and is tame (cf. [18]). In these cases the module categories of $\mathbf{C}(\Gamma(t), V)$ are described completely [18]. If V is not generic, $\mathbf{C}(\Gamma(t), V)$ it is still tame (cf. [13]). If $t > 4$, it is easy to see, using [11, Proposition 1.3], that $\mathbf{C}(\Gamma(t), V)$ is wild for all choices of V of dimension $\lfloor t/2 \rfloor$.

From now on, we identify $\xi \in \mathfrak{h}^*$ with the canonical algebra homomorphism $S(\mathfrak{h}) \to \mathbf{C}$ extending ξ. Let

$$\mathcal{N}_\eta = \bigl\{\lambda \in P^+\colon\ t_{\lambda,\eta} > 0,\ \mathfrak{R}_\Psi(\lambda, \lambda+\eta) \text{ is not generic}\bigr\}.$$

We can now formulate our main result.

Theorem 2. *Let Ψ be an extremal set of positive roots, $|\Psi| > 1$.*

(i) *The algebra $\mathbf{S}_\Psi^{\mathfrak{g}}$ is isomorphic to the quotient of the path algebra of the quiver Δ_Ψ by the ideal generated by the spaces $\mathfrak{R}_\Psi(\lambda, \lambda+\eta)$, $\eta \in \Psi + \Psi$, $\lambda, \lambda+\eta \in P^+$.*
Fix $\eta \in \Psi + \Psi$.

(ii) *If $m_\eta = 1$ then $\mathfrak{R}_\Psi(\lambda, \lambda+\eta) = 0$ for all $\lambda \in P^+$.*
Suppose that $\mathfrak{g}$ is of type A or C.

(iii) *If $t_{\lambda,\eta} > 1$, then $\dim \mathfrak{R}_\Psi(\lambda, \lambda+\eta) = \lfloor t_{\lambda,\eta}/2 \rfloor > 0$.*

(iv) *Suppose that $m_\eta > 1$. Then $\mathcal{N}_\eta$ is contained in a Zariski closed subset of $\mathfrak{h}^*$. Moreover $\mathcal{N}_\eta \cap \{\lambda \in P^+\colon t_{\lambda,\eta} = 2, 3\} = \emptyset$ and if Ψ is regular, then either $\mathcal{N}_\eta = \emptyset$ or there exists a linear polynomial $H_\eta \in S(\mathfrak{h})$ such that*

$$\mathcal{N}_\eta = P^+ \cap \bigl\{\xi \in \mathfrak{h}^*\colon\ \xi(H_\eta) = 0\bigr\}.$$

Analysis of other examples allows us to conjecture that the same result should hold for $\mathfrak{g}$ of all types. It might happen, though, that in some cases the Zariski closed set containing $\mathcal{N}_\eta$ cannot be described as a set of zeroes of a linear polynomial.

The above theorem is established in Propositions 5.6 and 5.7 for $\mathfrak{g}$ of type A and in Propositions 6.9, 6.10, 6.12 and 6.13 for $\mathfrak{g}$ of type C. In fact, we do not just establish the genericity of the spaces $\mathfrak{R}_\Psi(\lambda, \lambda+\eta)$ but also compute the relations explicitly. Needless to say, as we write the relations as linear combinations of paths, the specific coefficients we obtain depend on a fixed isomorphism Φ, or equivalently, on the choice of generators of degree one in $\mathbf{S}_\Psi^{\mathfrak{g}}$, which are unique up to non-zero scalars, while the genericity of the spaces $\mathfrak{R}_\Psi(\lambda, \lambda+\eta)$ is independent of that choice. We choose Φ so that the relations for $\eta \in \Psi + \Psi$ and $\lambda \in P^+$ satisfying $t_{\lambda,\eta} = m_\eta = 2$, are the commutativity relations.

Let us briefly explain how to compute relations in $\mathbf{E}_\Psi^{\mathfrak{g}}$ from those in $\mathbf{S}_\Psi^{\mathfrak{g}}$. There is a natural map $\langle\cdot,\cdot\rangle : \mathbf{C}\Delta_\Psi \otimes \mathbf{C}\Delta_\Psi^{op} \to \mathbf{C}$, such that $\langle(\mathbf{C}\Delta_\Psi)[k], (\mathbf{C}\Delta_\Psi^{op})[r]\rangle = 0$, $k \neq r$,

$$\langle 1_\lambda, 1_\mu\rangle = \delta_{\lambda,\mu}, \qquad \langle \lambda_1 \leftarrow \cdots \leftarrow \lambda_k, \mu_k \leftarrow \cdots \leftarrow \mu_1\rangle = \delta_{\lambda_1,\mu_1}\cdots\delta_{\lambda_k,\mu_k}.$$

It is not hard to see from [9, Proposition 5.3] that $\mathbf{E}_\Psi^{\mathfrak{g}}$ is isomorphic to the quotient of $\mathbf{C}\Delta_\Psi^{op}$ by the ideal generated by the spaces $\mathfrak{R}_\Psi(\lambda, \lambda+\eta)^! = \{x \in (\mathbf{C}\Delta_\Psi^{op})[2]\colon \langle \mathfrak{R}_\Psi(\lambda,\lambda+\eta), x\rangle = 0\}$.

1.8. We conclude this section with a description of an infinite family of quivers arising from this construction (see also 5.3).

Given $\boldsymbol{x} = (x_1, \ldots, x_r) \in \mathbf{Z}_+^r$, let $|\boldsymbol{x}| = \sum_{j=1}^r x_j$. Set $\boldsymbol{e}_i^{(r)} = (\delta_{i,j})_{1\leqslant j\leqslant r} \in \mathbf{Z}_+^r$. Given $\boldsymbol{m} = (m_1, \ldots, m_r) \in (\mathbf{Z}_+ \cup \{+\infty\})^r$, we define the quiver $\Xi(\boldsymbol{m})$ as follows. The vertices of $\Xi(\boldsymbol{m})$ are the lattice points in the r-dimensional rectangular parallelepiped $[0, m_1] \times \cdots \times [0, m_r]$. Given $\boldsymbol{x} = (x_1, \ldots, x_r) \in \Xi(\boldsymbol{m})_0$, the arrows ending at $\boldsymbol{x}$ are

$$\boldsymbol{x} \leftarrow \boldsymbol{x} + 2\boldsymbol{e}_j^{(r)}, \qquad x_j < m_j - 1, \quad 1 \leqslant j \leqslant r$$

and

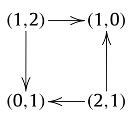
$$\boldsymbol{x} \leftarrow \boldsymbol{x} + \boldsymbol{e}_j^{(r)} + \boldsymbol{e}_k^{(r)}, \qquad x_i < m_i, \qquad x_j < m_j, \quad 1 \leqslant i < j \leqslant r.$$

Let $\Xi_a(\boldsymbol{m})$, $a = 0, 1$ be the full subquiver of $\Xi(\boldsymbol{m})$ defined by the set

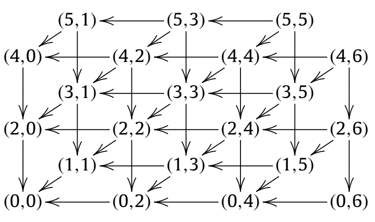
$$\bigl\{\boldsymbol{x} \in \Xi(\boldsymbol{m})_0\colon |\boldsymbol{x}| = a \ (\mathrm{mod}\ 2)\bigr\}.$$

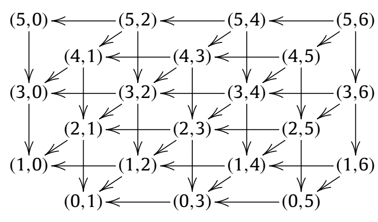
It is immediate that $\Xi_a(\boldsymbol{m})$ is a convex subquiver of $\Xi(\boldsymbol{m})$.

For instance, for $r = 2$ and $m_1 = m_2 = 1$, $\Xi_0(\boldsymbol{m})$ is the quiver of type $\mathbb{A}_2$ with the linear orientation and $\Xi_1(\boldsymbol{m})$ has two isolated vertices (in fact, this is the only case when $\Xi_1(\boldsymbol{m})$ is not connected). For $m_1 = m_2 = 2$, $\Xi_0(\boldsymbol{m})$ is the quiver (1.2) with $t = 3$, while $\Xi_1(\boldsymbol{m})$ is



An example with $m_1 = 6$, $m_2 = 5$ is shown below

$\Xi_0(\boldsymbol{m}) =$ 

$\Xi_1(\boldsymbol{m}) =$ 

Note that in this case $\Xi_1(\boldsymbol{m}) \cong \Xi_0(\boldsymbol{m})^{op}$ (cf. Proposition 6.3). For $r=3$ and $m_1=m_2=m_3=1$, $\Xi_0(\boldsymbol{m})$ (respectively, $\Xi_1(\boldsymbol{m})$) is the quiver of type $\mathbb{D}_4$ where the triple node is the unique sink (source). Finally, $\Xi_0((2,1,1))$ is the quiver (1.2) with $t=4$ where $(0,0,0)$ is the sink and $(2,1,1)$ is the source, while $\Xi_0((1,1,1,1))$ is the quiver (1.2) with $t=6$, where $(0,0,0,0)$ is the sink and $(1,1,1,1)$ is the source. We prove (cf. Proposition 6.3) that the isomorphism classes of quivers $\Xi_a(\boldsymbol{m})$ with $r>1$ are parametrised by partitions.

Proposition. *Suppose that $\mathfrak{g}$ is of type C and Ψ is regular. Let $\lambda \in P^+$ and suppose that $|\lambda^- \cup \lambda^+| > 0$. Then the connected component $\Delta_\Psi[\lambda]$ of Δ_Ψ is isomorphic to $\Xi_a(\boldsymbol{m})$ for some $\boldsymbol{m} \in (\mathbf{Z}_+ \cup \{+\infty\})^r$, $r>0$ and $a \in \{0,1\}$.*

In particular, our isomorphism $\mathbf{T}_\Psi^{\mathfrak{g}} \to \mathbf{C}\Delta_\Psi$ induces an isomorphism of a subalgebra of $\mathbf{T}_\Psi^{\mathfrak{g}}$ corresponding to an interval closed set onto $\Xi_a(\boldsymbol{m})$. Therefore, we can define a family of relations on $\Xi_a(\boldsymbol{m})$, depending on positive integer parameters, which yields an infinite family of finite dimensional Koszul algebras.

2. Relations in $\mathbf{S}_\Psi^{\mathfrak{g}}$

2.1. Let V be a $\mathfrak{g}$-module. Given $\mu \in \mathfrak{h}^*$, let

$$V_\mu = \{v \in V\colon\ hv = \mu(h)v,\ h \in \mathfrak{h}\}.$$

If V is finite dimensional, then $V = \bigoplus_{\mu \in P} V_\mu$. Moreover, V is isomorphic to a direct sum of simple finite dimensional modules $V(\lambda)$, $\lambda \in P^+$. In particular, the adjoint representation $\mathfrak{g}$ is isomorphic to $V(\theta)$ where θ is the highest root of $\mathfrak{g}$.

Fix Chevalley generators $e_i \in \mathfrak{g}_{\alpha_i}$, $f_i \in \mathfrak{g}_{-\alpha_i}$ and $h_i \in \mathfrak{h}$, $i \in I$ of $\mathfrak{g}$. The module $V(\lambda)$ is generated by a highest weight vector $v_\lambda \in V(\lambda)_\lambda$ satisfying

$$\operatorname{Ann}_{U(\mathfrak{g})} v_\lambda = U(\mathfrak{g})\big(\mathfrak{n}^+ + \ker\lambda\big) + \sum_{i \in I} U(\mathfrak{g}) f_i^{\lambda(h_i)+1}.$$

For each $\lambda \in P^+$, we fix v_λ once for all and then we fix $\xi_{-\lambda} \in V(\lambda)^*_{-\lambda}$ such that $\xi_{-\lambda}(v_\lambda)=1$. Then we have

$$\operatorname{Ann}_{U(\mathfrak{g})} \xi_{-\lambda} = U(\mathfrak{g})\big(\mathfrak{n}^- + \ker(-\lambda)\big) + \sum_{i \in I} U(\mathfrak{g}) e_i^{\lambda(h_i)+1}.$$

In particular,

$$\operatorname{Ann}_{U(\mathfrak{n}^-)} v_\lambda = \sum_{i \in I} U\big(\mathfrak{n}^-\big) f_i^{\lambda(h_i)+1}, \qquad \operatorname{Ann}_{U(\mathfrak{n}^+)} \xi_{-\lambda} = \sum_{i \in I} U\big(\mathfrak{n}^+\big) e_i^{\lambda(h_i)+1}.$$

Given $\lambda \in P^+$ and a finite dimensional $\mathfrak{g}$-module M, let

$$M^\lambda = \{m \in M\colon\ \operatorname{Ann}_{U(\mathfrak{n}^+)} m \supset \operatorname{Ann}_{U(\mathfrak{n}^+)} \xi_{-\lambda}\}.$$

If N is a subspace of M, let $N^\mu = N \cap M^\mu$. We will need the following results (cf. [17]; we use some of them in the form in which they are presented in [14], where the corresponding statements are established in the case of integrable modules over quantised enveloping algebras of Kac–Moody algebras).

Proposition. *Let $\mu, \nu \in P^+$ and let M be a finite dimensional $\mathfrak{g}$-module.*

(i) $\operatorname{Hom}_{\mathfrak{g}}(V(\lambda), M) \cong M^{\mathfrak{n}^+} \cap M_\lambda$.

(ii) $V(\mu)\otimes V(\nu)^* = U(\mathfrak{g})(v_\mu \otimes \xi_{-\nu})$.
(iii) *There exist canonical isomorphisms of vector spaces*

$$\operatorname{Hom}_{\mathfrak{g}}\bigl(V(\mu)\otimes V(\nu)^*, M\bigr) \cong \operatorname{Hom}_{\mathfrak{g}}\bigl(V(\mu), M\otimes V(\nu)\bigr) \cong \bigl(V(\mu)^*\otimes M\otimes V(\nu)\bigr)^{\mathfrak{g}} \\ \cong \bigl(V(\nu)^*\otimes M^*\otimes V(\mu)\bigr)^{\mathfrak{g}} \cong \operatorname{Hom}_{\mathfrak{g}}\bigl(V(\nu), M^*\otimes V(\mu)\bigr).$$

(iv) $M^{\nu}_{\mu-\nu} = \{m \in M_{\mu-\nu}\colon \operatorname{Ann}_{U(\mathfrak{n}^-)} m \supset \operatorname{Ann}_{U(\mathfrak{n}^-)} v_\mu\}$.
(v) *The linear map* $M^{\nu}_{\mu-\nu} \to \operatorname{Hom}_{\mathfrak{g}}(V(\mu)\otimes V(\nu)^*, M)$ *given by* $m \mapsto \chi_m$, *where*

$$\chi_m\bigl(a(v_\mu\otimes\xi_{-\nu})\bigr) = am, \quad a\in U(\mathfrak{g})$$

is an isomorphism of vector spaces. In particular, all vector spaces in (iii) *are isomorphic to* $M^{\nu}_{\mu-\nu}$.

2.2. Let $K = \bigoplus_{x\in J}\mathbf{C}e_x$ be a semisimple commutative algebra with primitive pairwise orthogonal idempotents e_x and let V be a K-bimodule. Assume that $\dim e_x V e_y < \infty$ for all $x, y \in J$ and that $V = \bigoplus_{x,y\in J} e_x V e_y$ (which is always the case if J is finite). Let $T^0_K(V) = K$, $T^r_K(V)$ be the r-fold tensor product of V over K and set $T_K(V) = \bigoplus_{r\in\mathbf{Z}_+} T^r_K(V)$. This is a $\mathbf{Z}_+$-graded associative algebra. In particular, if A is a $\mathbf{Z}_+$-graded associative algebra and $A[0]$ is commutative semisimple, we have a canonical homomorphism of associative algebras $T_{A[0]}(A[1]) \to A$ (cf. [2]).

Let Δ be the quiver with $\Delta_0 = J$ and with $\dim e_x V e_y$ arrows $x \leftarrow y$ for all $x, y \in J$. We have an isomorphism of algebras $K \to \bigoplus_{x\in J}\mathbf{C}1_x \subset \mathbf{C}\Delta$. In particular, we can regard the subspace of $\mathbf{C}\Delta$ spanned by all arrows as a K-bimodule and for any choice of basis in $e_x V e_y$, $x, y\in J$ this subspace is naturally isomorphic to V as an K-bimodule. This isomorphism extends canonically to an isomorphism of graded associative algebras $T_K(V) \to \mathbf{C}\Delta$. Then, if A is a quotient of $T_K(V)$ by an ideal which has the trivial intersection with $T^r_K(V)$, $r = 0, 1$, then A is isomorphic to the path algebra $\mathbf{C}(\Delta, R)$ where R is the image of $\ker(T_K(V) \to A)$ in $\mathbf{C}\Delta$.

An associative algebra A is said to be connected if $A = A_1 \oplus A_2$ where the A_j are subalgebras implies that $A_1 = 0$ or $A_2 = 0$. Clearly, $\mathbf{C}(\Delta, R)$ is connected if and only if Δ is connected.

2.3. Let $\Psi \subset R^+$ be a fixed extremal set.

Proposition. *Let* $F \subset P^+$ *be interval closed in the partial order* $\leqslant_\Psi$. *Then the algebra* $\mathbf{T}^{\mathfrak{g}}_{\Psi}(F)$ *is isomorphic, as a* $\mathbf{Z}_+$*-graded algebra, to the path algebra of the quiver* $\Delta_\Psi(F)$. *In particular, for all* $\lambda, \mu \in F$,

$$\bigl|\Delta_\Psi(F)(\lambda,\mu)\bigr| = \begin{cases} \dim(V(\lambda)^*\otimes T^{d_\Psi(\lambda,\mu)}(\mathfrak{g})\otimes V(\mu))^{\mathfrak{g}}, & \lambda \leqslant_\Psi \mu,\\ 0, & \textit{otherwise},\end{cases}$$

and if $F' \subset F$ *is interval closed, then* $\Delta_\Psi(F')$ *is a convex subquiver of* $\Delta_\Psi(F)$. *Furthermore,* $\mathbf{S}^{\mathfrak{g}}_\Psi(F)$ *is isomorphic to the quotient of* $\mathbf{C}\Delta_\Psi(F)$ *by an ideal generated by paths of length* 2.

Proof. By [9, Proposition 4.4], $\mathbf{T}^{\mathfrak{g}}_\Psi(F)$ is isomorphic to $T_{\mathbf{T}^{\mathfrak{g}}_\Psi(F)[0]}(\mathbf{T}^{\mathfrak{g}}_\Psi(F)[1])$ as a $\mathbf{Z}_+$-graded associative algebra. Since $\mathbf{T}^{\mathfrak{g}}_\Psi(F)[0] = \bigoplus_{\lambda\in F}\mathbf{C}1_\lambda$, it is enough to prove that for all $\lambda,\mu\in F$, the number of arrows $\lambda \leftarrow \mu$ equals $\dim 1_\lambda \mathbf{T}^{\mathfrak{g}}_\Psi(F)[1]1_\mu$. The latter is zero unless $\mu = \lambda+\beta$, $\beta\in\Psi$. Since by Proposition 2.1,

$$1_\lambda \mathbf{T}^{\mathfrak{g}}_\Psi(F)[1]1_{\lambda+\beta} = \bigl(V(\lambda)^*\otimes\mathfrak{g}\otimes V(\lambda+\beta)\bigr)^{\mathfrak{g}} \cong \mathfrak{g}^{\lambda}_{\beta}$$

and $\dim\mathfrak{g}_\beta = 1$, it is enough to prove that $\lambda \leftarrow \lambda+\beta \in (\Delta_\Psi)_1$ if and only if $\mathfrak{g}^\lambda_\beta \neq 0$. Observe first that $\lambda - \varepsilon(\beta)\in P^+$ implies that $\lambda+\beta = \lambda-\varepsilon(\beta)+\varphi(\beta)\in P^+$. Since $\gamma, \gamma+\alpha_i \in R$ implies that $e_i\mathfrak{g}_\gamma \neq 0$, it follows that $e_i^t\mathfrak{g}_\beta \neq 0$ for all $0 \leqslant t \leqslant \varepsilon_i(\beta)$. Therefore, $\mathfrak{g}^\lambda_\beta \neq 0$ if and only if $\lambda(h_i) \geqslant \varepsilon_i(\beta)$ for all $i\in I$. The remaining assertions are straightforward. □

2.4. For all $\beta \in \Psi$ and for all $\lambda \leftarrow \lambda+\beta \in (\Delta_\Psi)_1$, fix $0 \neq \boldsymbol{a}_{\lambda,\beta} \in 1_\lambda \mathbf{T}^{\mathfrak{g}}_\Psi 1_{\lambda+\beta} = 1_\lambda \mathbf{S}^{\mathfrak{g}}_\Psi 1_{\lambda+\beta}$. This choice is unique up to a non-zero scalar. It follows from [9, Proposition 4.4] that the elements 1_λ, $\lambda \in P^+$ and $\boldsymbol{a}_{\lambda,\beta}$, $\lambda \leftarrow \lambda+\beta \in (\Delta_\Psi)_1$ generate $\mathbf{T}^{\mathfrak{g}}_\Psi$ and $\mathbf{S}^{\mathfrak{g}}_\Psi$. In particular, for all $\lambda \leqslant_\Psi \mu$ with $d_\Psi(\lambda,\mu)=2$ the set

$$\left\{\boldsymbol{a}_{\lambda,\beta}\boldsymbol{a}_{\lambda+\beta,\beta'}\colon\ \beta,\beta' \in \Psi,\ \mu = \lambda+\beta+\beta',\ \lambda \leftarrow \lambda+\beta, \lambda+\beta \leftarrow \mu \in (\Delta_\Psi)_1\right\}$$

is a basis of $1_\lambda \mathbf{T}^{\mathfrak{g}}_\Psi 1_\mu$. By Proposition 2.1 and Corollary 1.1,

$$1_\lambda \mathbf{T}^{\mathfrak{g}}_\Psi 1_\mu \cong \left(T^2(\mathfrak{g})\right)^\lambda_{\mu-\lambda} = \left(T^2\left(\mathfrak{n}^+_\Psi\right)\right)^\lambda_{\mu-\lambda}.$$

Let $\Pi_\lambda(\beta,\beta')$ be the image of $\boldsymbol{a}_{\lambda,\beta}\boldsymbol{a}_{\lambda+\beta,\beta'}$ under this isomorphism. Using [9, Lemma 4.2] we obtain the following

Proposition. *Let $\eta \in \Psi+\Psi$, $\lambda, \lambda+\eta \in P^+$ and assume that $\Delta_\Psi(\lambda,\lambda+\eta) \neq \emptyset$. The elements $\Pi_\lambda(\beta,\beta')$ where $\beta,\beta' \in \Psi$, $\beta+\beta'=\eta$ and $\lambda \leftarrow \lambda+\beta \leftarrow \lambda+\eta \in \Delta_\Psi(\lambda,\lambda+\eta)$ form a basis of $T^2(\mathfrak{n}^+_\Psi)^\lambda_\eta$. In particular, we have a relation*

$$\sum_{\beta\in\Psi\colon \lambda\leftarrow\lambda+\beta\leftarrow\lambda+\eta\in\Delta_\Psi(\lambda,\lambda+\eta)} x_\beta \boldsymbol{a}_{\lambda,\beta}\boldsymbol{a}_{\lambda+\eta,\eta-\beta} = 0$$

in $\mathbf{S}^{\mathfrak{g}}_\Psi$ if and only if

$$\sum_{\beta\in\Psi\colon \lambda\leftarrow\lambda+\beta\leftarrow\lambda+\eta\in\Delta_\Psi(\lambda,\lambda+\eta)} x_\beta \Pi_\lambda(\beta,\eta-\beta) \in \bigwedge^2 \mathfrak{n}^+_\Psi.$$

2.5. Thus, to describe the relations, it remains to find a way for describing the elements $\Pi_\lambda(\beta,\beta')$. It turns out that the most convenient language is provided by $\mathfrak{g}$-module maps.

Let V be a finite dimensional $\mathfrak{g}$-module. Given $f \in \operatorname{Hom}_{\mathfrak{g}}(V(\mu), V\otimes V(\lambda))$, note that f is uniquely determined by $f(v_\mu)$. Using Proposition 2.1 we obtain an isomorphism of vector spaces

$$\operatorname{Hom}_{\mathfrak{g}}\left(V(\mu), V\otimes V(\lambda)\right) \to V^\lambda_{\mu-\lambda}$$

given by

$$f \mapsto v_f := (1\otimes \xi_{-\lambda}) f(v_\mu).$$

In particular, we have

$$f(v_\mu) = v_f \otimes v_\lambda \left(\operatorname{mod} U\left(\mathfrak{n}^+\right)_+ v_f \otimes U\left(\mathfrak{n}^-\right)_+ v_\lambda\right). \tag{2.1}$$

Let $\beta \in \Psi$, $\lambda \in P^+$ and assume that $\lambda \leftarrow \lambda+\beta \in (\Delta_\Psi)_1$ and so $\mathfrak{g}_\beta = \mathfrak{g}^\lambda_\beta$. Fix root vectors $e_\gamma \in \mathfrak{g}_\gamma \setminus \{0\}$, $\gamma \in R^+$. Then by (2.1) we have a unique $0 \neq p_{\lambda,\beta} \in \operatorname{Hom}_{\mathfrak{g}}(V(\lambda+\beta), \mathfrak{g}\otimes V(\lambda))$ satisfying

$$p_{\lambda,\beta}(v_{\lambda+\beta}) = e_\beta \otimes v_\lambda + \sum_{\beta<\gamma} e_\gamma \otimes \mathbf{u}_{\beta,\gamma}(\lambda) v_\lambda, \tag{2.2}$$

where $\mathbf{u}_{\beta,\gamma}(\lambda) \in U(\mathfrak{n}^-)_{\beta-\gamma}$. Clearly, $p_{\lambda,\beta}(v_{\lambda+\beta})$ spans $(\mathfrak{g}\otimes V(\lambda))^{\mathfrak{n}^+}_{\lambda+\beta}$. Note that the elements $\mathbf{u}_{\beta,\gamma}(\lambda)$ are uniquely determined modulo $\operatorname{Ann}_{U(\mathfrak{n}^-)} v_\lambda$.

2.6. Let $F(\mathfrak{h})$ be the field of fractions of $S(\mathfrak{h})$. Given $\beta \in \Psi$, let

$$F_\beta(\mathfrak{h}) = \{ fg^{-1} \in F(\mathfrak{h}):\ \lambda \in P^+,\ \lambda \leftarrow \lambda + \beta \in (\Delta_\Psi)_1 \ \Rightarrow\ \lambda(g) \neq 0 \}.$$

Clearly, $F_\beta(\mathfrak{h})$ is a subring of $F(\mathfrak{h})$. Given $\lambda \leftarrow \lambda + \beta \in (\Delta_\Psi)_1$, note that $\lambda : S(\mathfrak{h}) \to \mathbf{C}$ extends canonically to a homomorphism $F_\beta(\mathfrak{h}) \to \mathbf{C}$ which we also denote by λ.

Furthermore, regard $U(\mathfrak{b})$ as a right $S(\mathfrak{h})$-module via the right multiplication and a left $U(\mathfrak{n}^-)$ module via the left multiplication. Then $U(\mathfrak{b}) \otimes_{S(\mathfrak{h})} F_\beta(\mathfrak{h})$ is a right $S(\mathfrak{h})$-module and is isomorphic to $U(\mathfrak{n}^-) \otimes F_\beta(\mathfrak{h})$ as a left $U(\mathfrak{n}^-)$-module by the PBW theorem. Thus, λ induces a surjective homomorphism of left $U(\mathfrak{n}^-)$-modules $\pi_{\lambda,\beta} : U(\mathfrak{b}) \otimes_{S(\mathfrak{h})} F_\beta(\mathfrak{h}) \to U(\mathfrak{n}^-)$.

Let $\lambda \in \mathfrak{h}^*$. The quotient of $U(\mathfrak{b})$ by the left ideal generated by the kernel of $\lambda : S(\mathfrak{h}) \to \mathbf{C}$ is isomorphic to $U(\mathfrak{n}^-)$ as a left $U(\mathfrak{n}^-)$-module and so we have a surjective homomorphism of left $U(\mathfrak{n}^-)$-modules $\pi_\lambda : U(\mathfrak{b}) \to U(\mathfrak{n}^-)$. Clearly, the restriction of π_λ to $U(\mathfrak{n}^-)$ is the identity map. Furthermore, if V is a finite dimensional $\mathfrak{g}$-module, $v \in V_\mu$ and $x \in U(\mathfrak{b})$, then $x - \pi_\mu(x) \in \operatorname{Ann}_{U(\mathfrak{g})} v$.

Lemma. *Suppose that $x \in U(\mathfrak{b})$, $y \in U(\mathfrak{b})_{-\eta}$, $\eta \in \mathbf{Z}_+ R^+$. Then $\pi_\lambda(xy) = \pi_{\lambda-\eta}(x)\pi_\lambda(y)$. Furthermore, if $\lambda \leftarrow \lambda + \beta \in (\Delta_\Psi)_1$ then $\pi_{\lambda,\beta}(x \otimes f) = \pi_\lambda(x) \otimes \pi_{\lambda,\beta}(f)$ for all $f \in F_\beta(\mathfrak{h})$.*

Proof. Note that $\pi_\lambda(xy) = \pi_\lambda(x)\pi_\lambda(y)$ for all $x \in U(\mathfrak{n}^-)$, $y \in S(\mathfrak{h})$. Since $U(\mathfrak{b}) \cong U(\mathfrak{n}^-) \otimes S(\mathfrak{h})$ by the PBW theorem, it is enough to show that $\pi_\lambda(hy) = \pi_{\lambda-\eta}(h)\pi_\lambda(y)$ for all $h \in \mathfrak{h}$, $y \in U(\mathfrak{b})_{-\eta}$. We have $\pi_\lambda(hy) = \pi_\lambda(yh) - \eta(h)\pi_\lambda(y) = \pi_\lambda(y)(\lambda - \eta)(h) = \pi_{\lambda-\eta}(h)\pi_\lambda(y)$. The second assertion is obvious. □

Given $\beta \in \Psi$, we have a group homomorphism $F_\beta(\mathfrak{h})^\times \to (\mathbf{C}^\times)^{(\Delta_\Psi)_1}$ defined by $h \mapsto (z_{\lambda,\gamma}(h):\ \gamma \in \Psi, \lambda \leftarrow \lambda + \gamma \in (\Delta_\Psi)_1)$, where

$$z_{\lambda,\gamma}(h) = \begin{cases} \lambda(h), & \gamma = \beta, \\ 1, & \gamma \neq \beta. \end{cases}$$

This yields a natural group homomorphism $\prod_{\beta\in\Psi} F_\beta(\mathfrak{h})^\times \to (\mathbf{C}^\times)^{(\Delta_\Psi)_1}$. We denote its image by G_Ψ.

2.7.

Definition. Let $\beta \in \Psi$. We call a tuple

$$(\mathbf{u}_{\beta,\gamma} \in U(\mathfrak{b})_{\beta-\gamma} \otimes_{S(\mathfrak{h})} F_\beta(\mathfrak{h}):\ \beta \leqslant \gamma, \gamma \in R^+)$$

an *adapted family for β* if $\mathbf{u}_{\beta,\beta} = 1$ and for all $\lambda \leftarrow \lambda + \beta \in (\Delta_\Psi)_1$, the vector

$$\sum_{\gamma\in R^+:\ \beta\leqslant\gamma} e_\gamma \otimes \pi_{\lambda,\beta}(\mathbf{u}_{\beta,\gamma}) v_\lambda \tag{2.3}$$

spans $(\mathfrak{g} \otimes V(\lambda))^{\mathfrak{n}^+}_{\lambda+\beta}$.

Proposition. *Let $\beta \in \Psi$ and suppose that $(\mathbf{u}_{\beta,\gamma} \in U(\mathfrak{b}) \otimes_{S(\mathfrak{h})} F_\beta(\mathfrak{h}):\ \beta \leqslant \gamma, \gamma \in R^+)$ is an adapted family for β. Then for all $\beta' \in \Psi$ and for all $\lambda \in P^+$ such that $\lambda \leftarrow \lambda + \beta, \lambda + \beta \leftarrow \lambda + \beta + \beta' \in (\Delta_\Psi)_1$ we have, up to a non-zero scalar,*

$$\Pi_\lambda(\beta', \beta) = e_\beta \otimes e_{\beta'} + \sum_{\beta<\gamma:\ \gamma, \beta+\beta'-\gamma\in\Psi} e_\gamma \otimes \mathbf{u}_{\beta,\gamma}(\lambda + \beta') e_{\beta'}, \tag{2.4}$$

where

$$\mathbf{u}_{\beta,\gamma}(\nu) = \pi_{\nu,\beta}(\mathbf{u}_{\beta,\gamma}) \pmod{\operatorname{Ann}_{U(\mathfrak{n}^-)} v_\nu}, \qquad \nu \leftarrow \nu+\beta \in (\Delta_\Psi)_1.$$

In particular, if $\beta \in \Psi$ is maximal, $\Pi(\beta',\beta) = e_\beta \otimes e_{\beta'}$.

Proof. Let $\nu_1 \leqslant_\Psi \nu_2$. Since $\mathfrak{g} \cong \mathfrak{g}^*$, by Proposition 2.1(iii) we have the following canonical isomorphisms of vector spaces

$$\begin{aligned} 1_{\nu_1}\mathbf{T}_\Psi^{\mathfrak{g}} 1_{\nu_2} &= \big(V(\nu_1)^* \otimes T^{d_\Psi(\nu_1,\nu_2)}(\mathfrak{g}) \otimes V(\nu_2)\big)^{\mathfrak{g}} \\ &\cong \big(V(\nu_2)^* \otimes T^{d_\Psi(\nu_1,\nu_2)}(\mathfrak{g}) \otimes V(\nu_1)\big)^{\mathfrak{g}} \\ &\cong \operatorname{Hom}_{\mathfrak{g}}\big(V(\nu_2), T^{d_\Psi(\nu_1,\nu_2)}(\mathfrak{g}) \otimes V(\nu_1)\big). \end{aligned}$$

Moreover, this isomorphism is compatible with products and compositions, that is, if $x \in 1_{\nu_1}\mathbf{T}_\Psi^{\mathfrak{g}} 1_{\nu_2}$, $y \in 1_{\nu_2}\mathbf{T}_\Psi^{\mathfrak{g}} 1_{\nu_3}$, $\nu_1 \leqslant_\Psi \nu_2 \leqslant_\Psi \nu_3$ and

$$x \mapsto f \in \operatorname{Hom}_{\mathfrak{g}}\big(V(\nu_2), T^{d_\Psi(\nu_1,\nu_2)}(\mathfrak{g}) \otimes V(\nu_1)\big), \qquad y \mapsto g \in \operatorname{Hom}_{\mathfrak{g}}\big(V(\nu_3), T^{d_\Psi(\nu_2,\nu_3)}(\mathfrak{g}) \otimes V(\nu_2)\big),$$

then

$$xy \mapsto (1 \otimes f) \circ g \in \operatorname{Hom}_{\mathfrak{g}}\big(V(\nu_3), T^{d_\Psi(\nu_1,\nu)}(\mathfrak{g}) \otimes V(\nu_1)\big).$$

In particular, $\beta \in \Psi$ and $\lambda \leftarrow \lambda+\beta \in (\Delta_\Psi)_1$, we may assume, without loss of generality, that $p_{\lambda,\beta} \in \operatorname{Hom}_{\mathfrak{g}}(V(\lambda+\beta), \mathfrak{g} \otimes V(\lambda))$ is the image of $\boldsymbol{a}_{\lambda,\beta}$ under the above isomorphism. Then $\Pi_\lambda(\beta',\beta)$ is the image of

$$(1 \otimes p_{\lambda,\beta'}) \circ p_{\lambda+\beta',\beta}$$

under the isomorphism

$$\operatorname{Hom}_{\mathfrak{g}}\big(V(\lambda+\beta+\beta'), T^2(\mathfrak{g}) \otimes V(\lambda)\big) \to T^2\big(\mathfrak{n}_\Psi^+\big)^\lambda_{\beta+\beta'}.$$

It is now immediate from (2.1), (2.3) and Proposition 2.1(i) that

$$\Pi_\lambda(\beta',\beta) = e_\beta \otimes e_{\beta'} + \sum_{\gamma \in R^+:\ \beta<\gamma} e_\gamma \otimes \mathbf{u}_{\beta,\gamma}(\lambda+\beta')e_{\beta'}.$$

Since $\mathbf{u}_{\beta,\gamma}(\lambda+\beta')e_{\beta'} \in \mathfrak{g}_{\beta+\beta'-\gamma}$, it follows from Corollary 1.1 that $\mathbf{u}_{\beta,\gamma}(\lambda+\beta')e_\beta \neq 0$ implies that $\gamma, \beta+\beta'-\gamma \in \Psi$. $\square$

The following elementary corollary establishes part (ii) of Theorem 2.

Corollary. *Let $\beta \in \Psi$, $\lambda \in P^+$. Then $(\lambda \leftarrow \lambda+\beta \leftarrow \lambda+2\beta) \notin \mathfrak{R}_\Psi(\lambda, \lambda+2\beta)$.*

3. First examples

The aim of this section is to provide the reader with relatively simple examples of quivers and relations arising from algebras $\mathbf{S}_{\Psi}^{\mathfrak{g}}$, before we undertake a complete study of all possible relations in these algebras for $\mathfrak{g}$ of types A and C. We begin with the infinite dimensional example announced in [9] which is independent of type of $\mathfrak{g}$. The same computation allows us to obtain a complete description of relations in $\mathbf{S}_{\Psi}^{\mathfrak{g}}$ for $\mathfrak{g}$ of type A_2. Then we describe the relations in the algebras corresponding to $\mathfrak{g}$ of type G_2. The remaining rank 2 case is postponed until 6.11.

Throughout the rest of the paper, given $\lambda \in P^+$ and $i \notin I$, we set $\lambda(h_i) = +\infty$.

3.1. We begin by excluding the case $|\Psi| = 1$. In this case the algebra $\mathbf{S}_{\Psi}^{\mathfrak{g}}$ is hereditary and we have two possibilities. If $\Psi = \{\theta\}$ then every connected component of Δ_{Ψ} is isomorphic to the quiver $\mathbb{A}_{\infty}^{op}$, where

$$\mathbb{A}_{\infty} = 0 \to 1 \to 2 \to \cdots$$

If $\Psi = \{\beta\}$ with $\beta \neq \theta$ then $\beta \notin P^+$ (it is easy to check that if the highest short root is contained in Ψ then $|\Psi| > 1$) and so the connected components of Δ_{Ψ} are either simple one-dimensional or of type $\mathbb{A}_n$ with the subspace orientation.

3.2. Suppose that $\mathfrak{g}$ is not of type A or C (in fact, the computation of the relations works for the type A as well, but the quiver is more complicated, as we will see below; the corresponding construction for the type C will be discussed later). Then there exists a unique $i_0 \in I$ such that $\theta - \alpha_{i_0} \in R^+$ and it is not hard to see that $\Psi = \{\theta, \theta - \alpha_{i_0}\}$ is extremal.

Recall (cf. [18]) that a pair (Δ, τ) where $\Delta = (\Delta_0, \Delta_1)$ is a quiver without multiple arrows and $\tau : \Delta'_0 \to \Delta_0$, $\Delta'_0 \subseteq \Delta_0$ is an injective map, is called a translation quiver (and τ is called the translation map) if $(\tau(z))^+ = z^-$ for all $z \in \Delta'_0$. A full embedding of translation quivers $(\Delta, \tau) \to (\Delta', \tau')$ is a full embedding of quivers $\Delta \to \Delta'$ which maps the domain of τ into the domain of τ' and is compatible with the maps τ, τ'. If (Δ, τ) is a translation quiver and has no multiple arrows, a relation of the form $\sum_{y \in x^-} (x \leftarrow y)(y \leftarrow \tau(x))$, $x \in \Delta_0$, is called a mesh relation.

Given a quiver Δ, a translation quiver $\mathbf{Z}\Delta$ is defined by

$$(\mathbf{Z}\Delta)_0 = \mathbf{Z} \times \Delta_0, \quad (\mathbf{Z}\Delta)_1 = \big\{(n, x) \leftarrow (n, y),\ (n+1, y) \leftarrow (n, x)\colon\ x \leftarrow y \in \Delta_1\big\},$$
$$\tau\big((n, x)\big) = (n-1, x).$$

If Δ is a Dynkin quiver, $\mathbf{Z}\Delta$ depends only on $\bar{\Delta}$ (cf. [18, §2.1]).

Proposition. *Every connected subalgebra of $\mathbf{S}_{\Psi}^{\mathfrak{g}}$ is isomorphic to the path algebra of the translation quiver*

$$\Gamma = \begin{array}{ccccccccccc} & & & & & & \vdots & & \vdots & & \\ & & & & & & \downarrow & & \downarrow & & \\ & & & & (0,2) & \leftarrow & (1,2) & \leftarrow & (2,2) & \leftarrow & \cdots \\ & & & & \downarrow & & \downarrow & & \downarrow & & \\ & & (0,1) & \leftarrow & (1,1) & \leftarrow & (2,1) & \leftarrow & (3,1) & \leftarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ (0,0) & \leftarrow & (1,0) & \leftarrow & (2,0) & \leftarrow & (3,0) & \leftarrow & (4,0) & \leftarrow & \cdots \end{array} \tag{3.1}$$

with the translation map $\tau(m, n) = (m, n+1)$, $m, n \in \mathbf{Z}_+$ and with the mesh relations.

Proof. Suppose that $\lambda \in P^+$ is a sink in Δ_Ψ. Since $\varphi(\theta) = \varpi_{i_0}$, we must have $\lambda(h_{i_0}) = 0$. Suppose that $\mu \in P^+$ is a sink in $\Delta_\Psi[\lambda]$, $\mu \neq \lambda$. Since $\Delta_\Psi[\lambda]_0 \subset (\lambda + \mathbf{Z}\Psi) \cap P^+$, $\mu = \lambda + m\theta + k\beta$ for some $m, k \in \mathbf{Z}$. Interchanging the role of λ and μ, if necessary, we may assume that $m \geqslant 0$. Since $\theta(h_{i_0}) = 1$, $\beta(h_{i_0}) = -1$, we have $m = k > 0$. On the other hand, for all $j \neq i_0$ we have $\beta(h_j) = \varphi_j(\beta)$ and so $\mu(h_j) - \lambda(h_j) = k\varphi_j(\beta)$. Since $\lambda(h_j) \geqslant 0$ and $\varphi_{i_0}(\beta) = 0$, this implies that $\mu - \varphi(\beta) \in P^+$ which is a contradiction since μ is a sink. Thus, every connected component of Δ_Ψ contains a unique sink λ hence

$$\Delta_\Psi[\lambda]_0 \subset (\lambda \leqslant_\Psi) = \{\lambda + r\theta + s\beta\colon\ 0 \leqslant s \leqslant r\}.$$

Note that $\lambda + r\theta + s\beta$, $0 \leqslant s \leqslant r$ is connected to λ by a path

$$\lambda \leftarrow \lambda + \theta \leftarrow \cdots \leftarrow \lambda + r\theta \leftarrow \lambda + r\theta + \beta \leftarrow \cdots \leftarrow \lambda + r\theta + s\beta.$$

Thus, $\Delta_\Psi[\lambda]_0 = (\lambda \leqslant_\Psi)$. Define a map $\Delta_\Psi[\lambda]_0 \to \Gamma_0 = \mathbf{Z}_+ \times \mathbf{Z}_+$ by $\lambda + r\theta + s\beta \mapsto (r - s, s)$. This map is clearly a bijection. Furthermore, we have an arrow $\lambda + r\theta + s\beta \leftarrow \lambda + (r+1)\theta + s\beta$ and an arrow $\lambda + r\theta + s\beta \leftarrow \lambda + r\theta + (s+1)\beta$ provided that $s < r$. Since in the quiver Γ we have an arrow $(m, n) \leftarrow (m+1, n)$ for all $m, n \in \mathbf{Z}_+$ and an arrow $(m, n) \leftarrow (m-1, n+1)$ for all $m > 0$, it follows that $\Delta_\Psi[\lambda] \cong \Gamma$. Finally, if we define $\tau : \Delta_\Psi[\lambda]_0 \to \Delta_\Psi[\lambda]_0$ by $\tau(\mu) = \mu + \theta + \beta$, we conclude that our isomorphism is in fact an isomorphism of translation quivers.

It remains to compute the relations in our algebra. Since $\beta < \theta$, by Proposition 2.7 we have $\Pi_\lambda(\beta, \theta) = e_\theta \otimes e_\beta$. Assuming that $[e_{i_0}, e_\beta] = e_\theta$ we can easily check that $\mathbf{u}_{\beta,\beta} = 1$ and $\mathbf{u}_{\beta,\theta} = -f_{i_0} \otimes h_{i_0}^{-1} \in U(\mathfrak{b})_{-\alpha_{i_0}} \otimes_{S(\mathfrak{h})} F_\beta(\mathfrak{h})$ form an adapted family for β. Since $f_{i_0} \notin \operatorname{Ann}_{U(\mathfrak{n}^-)} v_\nu$ if $\nu(h_{i_0}) > 0$, we conclude that $\Pi_\lambda(\theta, \beta) = e_\beta \otimes e_\theta - (\lambda(h_{i_0}) + 1)^{-1} e_\theta \otimes e_\beta$.

Suppose that $\lambda(h_{i_0}) > 0$. Then $t_{\lambda,\theta+\beta} = 2$ and, clearly, $\lambda(h_{i_0})\Pi_\lambda(\beta, \theta) - (\lambda(h_{i_0}) + 1)\Pi_\lambda(\theta, \beta) \in \bigwedge^2 \mathfrak{n}_\Psi^+$. Fix the isomorphism $\Phi : \mathbf{T}_\Psi^{\mathfrak{g}} \to \mathbf{C}\Delta_\Psi$ by assigning

$$\boldsymbol{a}_{\lambda,\theta} \mapsto (\lambda \leftarrow \lambda + \theta), \quad \lambda \in P^+$$

and

$$\boldsymbol{a}_{\lambda,\beta} \mapsto (-1)^{\lambda(h_{i_0})} \big(\lambda(h_{i_0})\big)^{-1} (\lambda \leftarrow \lambda + \beta), \quad \lambda(h_{i_0}) > 0.$$

Then it is easy to see that $\mathfrak{R}_\Psi(\lambda, \lambda + \theta + \beta)$ is spanned by the mesh relation with respect to our translation map. If $\lambda(h_{i_0}) = 0$ then $\Pi_\lambda(\theta, \beta) \in \bigwedge^2 \mathfrak{n}_\Psi^+$, so the unique path is a zero relation and is again the mesh relation with respect to our translation map. □

Note that we have a full embedding of translation quivers $\Gamma \hookrightarrow \mathbf{Z}\mathbb{A}_\infty$ given on the vertices by $(r, s) \mapsto (-r - s, r)$. The quiver Γ^{op} identifies with the Auslander–Reiten quiver for $\mathbb{A}_\infty$ and so a connected subalgebra of $\mathbf{S}_\Psi^{\mathfrak{g}}$ can be regarded as an infinite dimensional analogue of the Auslander algebra of $\mathbf{C}\mathbb{A}_\infty$.

3.3. In the remainder of the section we will consider $\mathfrak{g}$ of types A_2 and G_2. Identify P with $\mathbf{Z} \times \mathbf{Z}$ and write $(\lambda(h_1), \lambda(h_2))$ for $\lambda \in P$.

Let $\mathfrak{g}$ be of type A_2. Then R^+ contains two extremal sets with $|\Psi| > 1$, namely $\Psi_i = \{\alpha_i, \theta\}$, $i \in I$. Clearly, it is enough to analyse one of them, say $\Psi = \Psi_1$.

Suppose that $(m, n), (m', n') \in P^+$ are in the same connected component. Then $(m', n') \in ((m, n) + \mathbf{Z}\Psi) \cap P^+$, that is, $(m', n') = (m + r + 2s, n + r - s)$ for some $r, s \in \mathbf{Z}$. This implies that

$m' - n' = m - n \pmod 3$. Since $\varphi(\alpha_1) = (2,0)$, $\varphi(\theta) = \theta = (1,1)$, the sinks in Δ_Ψ are $(0,m)$, $m \in \mathbf{Z}_+$ and $(1,0)$. Let $0 \leqslant r < 3$. Then we have

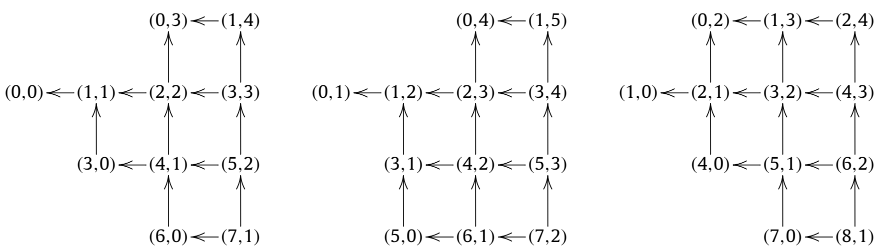
$$(0,r) \leftarrow (1, r+1) \leftarrow \cdots \leftarrow (2k, 2k+r) \rightarrow \big(2(k-1), 2k+1+r\big) \rightarrow \cdots \rightarrow (0, 3k+r),$$

hence all sinks $(0, 3k+r)$ lie in $\Delta_\Psi[(0,r)]$. Finally, we have $(1,0) \leftarrow (2,1) \rightarrow (0,2)$ hence $(1,0)$ belongs to $\Delta_\Psi[(0,2)]$. Thus, Δ_Ψ has three connected components given by

$$\Delta_\Psi\big[(0,r)\big]_0 = \big\{(m,n) \in \mathbf{Z}_+ \times \mathbf{Z}_+ \colon m - n = r \pmod 3\big\}$$

the arrows being $(m,n) \leftarrow (m+1, n+1)$ and $(m,n) \leftarrow (m+2, n-1)$, $n > 0$. The translation structure is given by $\tau(m,n) = (m+3, n)$. The computation of relations performed in 3.2 implies that all relations are the mesh relations.

It is easy to see that the quivers $\Delta_\Psi[(0,r)]$, $r \in \{0,1,2\}$, and hence the corresponding connected subalgebras of $\mathbf{S}_\Psi^{\mathfrak{g}}$, are not isomorphic. For that, note that $\Delta_\Psi[(0,r)]$ has a unique sink λ_r such that $|\lambda^-| = 1$ (indeed, clearly $\lambda_0 = (0,0)$, $\lambda_1 = (0,1)$ and $\lambda_2 = (1,0)$ have this property). It follows that any full map of quivers $\Delta_\Psi[(0,r)] \rightarrow \Delta_\Psi[(0,s)]$ must send λ_r to λ_s and λ_r^- to λ_s^-. On the other hand, λ_r belongs to the following full connected subquivers of $\Delta_\Psi[(0,r)]$, respectively



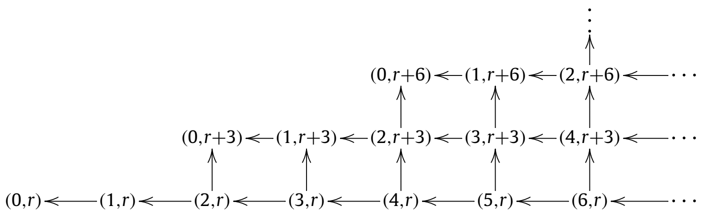
These quivers are obviously non-isomorphic.

3.4. Let $\mathfrak{g}$ be of type G_2. Let α_1 (respectively, α_2) be the long (respectively, the short) simple root. Then $R^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2, \theta = 2\alpha_1 + 3\alpha_2\}$. It is not hard to show that there are only two extremal sets of positive roots containing more than one element, namely $\Psi_1 = \{\theta - \alpha_1, \theta\}$ and $\Psi_2 = \{\alpha_1, \theta\}$, which correspond to the two one-dimensional faces of the convex hull of R having trivial intersection with $-R^+$. The set Ψ_1 has already been considered in Proposition 3.2. We should only note that since $\varphi(\theta) = (1,0)$ and $\varphi(\theta - \alpha_1) = (0,3)$, $(0,r)$, $0 \leqslant r < 3$ are the only sinks in Δ_Ψ and hence by Proposition 2.3 Δ_Ψ has three isomorphic connected components.

The situation is rather different if $\Psi = \Psi_2$. Since $\varphi(\alpha_1) = (2,0)$ and $\varphi(\theta) = (1,0)$, it follows that (m,n) is a sink in Δ_Ψ if and only if $m = 0$. Furthermore, since $\varepsilon(\alpha_1) = (0,3)$, we have an arrow $(m,n) \leftarrow (m+2, n-3)$ if and only if $n \geqslant 3$. Suppose that we have two sinks $(0,x)$, $(0,y)$ in the same connected component of Δ_Ψ. Then we must have $(m + 2n, x - 3n) = (0,y)$ for some $m, n \in \mathbf{Z}$, hence $x = y \pmod 3$. Furthermore, let $0 \leqslant r \leqslant 2$. Then we have in Δ_Ψ

$$(0,r) \leftarrow (1,r) \leftarrow \cdots \leftarrow (2n, r) \rightarrow \big(2(n-1), r+3\big) \rightarrow \cdots \rightarrow \big(2, 3(n-1)+r\big) \rightarrow (0, 3n+r).$$

Thus, every sink $(0, 3n+r)$, $n \in \mathbf{Z}_+$ lies in $\Delta_\Psi[(0,r)]$. Therefore, Δ_Ψ has three isomorphic connected components and the quiver $\Delta_\Psi[(0,r)]$ is



that is, $\Delta_\Psi[(0,r)]_0 = \{(m, 3n+r)\colon\ m,n \in \mathbf{Z}_+\}$ and the arrows are $(m, 3n+r) \leftarrow (m+1, 3n+r)$, $m,n \in \mathbf{Z}_+$, $(m, 3n+r) \leftarrow (m+2, 3(n-1)+r)$, $n > 0$. This is clearly a translation quiver with $\tau((m, 3k+r)) = (m+3, 3(k-1)+r)$, $m \in \mathbf{Z}_+$, $k > 0$. In particular, Ψ is our first example of a regular extremal set. Clearly, there is a full embedding of $\Delta_\Psi[(0,r)]$ into any of the infinite connected quivers considered in 3.3.

3.5. It remains to describe the relations. We write $x^{(p)} = x^p/p! \in U(\mathfrak{g})$, $x \in \mathfrak{g}$, $p \in \mathbf{Z}_+$. Fix root vectors in $\mathfrak{g}$ so that $e_{\alpha_1+p\alpha_2} = (\operatorname{ad} e_2)^{(p)} e_1$, $1 \leqslant p \leqslant 3$ and $[e_1, e_{\alpha_1+3\alpha_2}] = e_\theta$. We have only one non-trivial case to consider, namely $\eta = \theta + \alpha_1 = (3, -3)$. If $\Delta_\Psi((m,n),(m+3,n-3))$ is non-empty it always contains two paths. Proposition 2.7 immediately implies that

$$\Pi_\lambda(\alpha_1, \theta) = e_\theta \otimes e_{\alpha_1}.$$

To find $\Pi_\lambda(\theta, \alpha_1)$, note that $\{\gamma \in R^+\colon\ \alpha_1 \leqslant \gamma\} = \{\alpha_1, \alpha_1+\alpha_2, \alpha_1+2\alpha_2, \alpha_1+3\alpha_2, \theta\}$. Since $\dim U(\mathfrak{n}^-)_{-(\alpha_1+3\alpha_2)} = 4$, the monomials

$$f_2^{(a)} f_1 f_2^{(3-a)}, \quad 0 \leqslant a \leqslant 3$$

which are of course all possible monomials in the f_i of weight $-\alpha_1 - 3\alpha_2$, form a basis of $U(\mathfrak{n}^-)_{-(\alpha_1+3\alpha_2)}$. It is not hard to see that the element

$$\begin{aligned} U = {} & f_1 f_2^{(3)} (h_2+1) h_2 (h_2-1) - f_2 f_1 f_2^{(2)} (h_2+1) h_2 (h_2-2) \\ & + f_2^{(2)} f_1 f_2 (h_2+1)(h_2-1)(h_2-2) - f_2^{(3)} f_1 h_2 (h_2-1)(h_2-2) \in U(\mathfrak{b}) \end{aligned}$$

satisfies

$$e_1 U = 6 f_2^{(3)} (h_1 + h_2 + 1) + U(\mathfrak{g})\mathfrak{n}^+, \qquad e_2 U \in U(\mathfrak{g})\mathfrak{n}^+.$$

Define $\mathbf{u}_{\alpha_1,\gamma} \in U(\mathfrak{b})_{\alpha_1-\gamma} \otimes_{S(\mathfrak{h})} F_{\alpha_1}(\mathfrak{h})$, $\alpha_1 \leqslant \gamma$ as

$$\mathbf{u}_{\alpha_1,\alpha_1+p\alpha_2} = (-1)^p f_2^p \otimes \prod_{t=0}^{p-1} (h_2 - t)^{-1}, \quad 0 \leqslant p \leqslant 3,$$

$$\mathbf{u}_{\alpha_1,\theta} = U \otimes \big((h_1+h_2+1) h_2 (h_2-1)(h_2-2)\big)^{-1}.$$

Then it is easy to see that $(\mathbf{u}_{\alpha_1,\gamma}\colon\ \alpha_1 \leqslant \gamma)$ is an adapted family for α_1, hence by Proposition 2.7,

$$\Pi_\lambda(\theta, \alpha_1) = e_{\alpha_1} \otimes e_\theta + \big((\lambda_1+\lambda_2+2)\lambda_2(\lambda_2-1)(\lambda_2-2)\big)^{-1} e_\theta \otimes \pi_{\lambda+\theta}(U) e_\theta.$$

Clearly, $\pi_{\lambda+\theta}(U) = -\lambda_2(\lambda_2-1)(\lambda_2-2)f_2^{(3)}f_1 + \operatorname{Ann}_{U(\mathfrak{g})}e_\theta$. Since $(\lambda+\theta)(h_1) > 0$, we conclude using finite dimensional $\mathfrak{sl}_2$ theory that $f_2^{(3)}f_1 \notin \operatorname{Ann}_{U(\mathfrak{n}^-)}v_{\lambda+\theta}$. Thus, we get

$$\Pi_\lambda(\theta,\alpha_1) = e_{\alpha_1}\otimes e_\theta - (\lambda_1+\lambda_2+2)^{-1}e_\theta\otimes e_{\alpha_1}.$$

It follows that none of the two paths is a relation and that the relations can be chosen to be the mesh relations.

4. A recursive family of elements in $U(\mathfrak{b})$

In this section we construct a family of elements of $U(\mathfrak{b})$ which will play the crucial role in constructing adapted families for $\mathfrak{g}$ of type A and C.

4.1. Suppose that $\mathfrak{g}$ is of type A_ℓ. After [16], a monomial

$$f_1^{a_{1,1}}\bigl(f_2^{a_{2,2}}f_1^{a_{2,1}}\bigr)\cdots\bigl(f_\ell^{a_{\ell,\ell}}\cdots f_1^{a_{\ell,1}}\bigr)\in U\bigl(\mathfrak{n}^-\bigr)$$

where $a_{j,i+1}\geqslant a_{j,i}$ for all $1\leqslant j\leqslant\ell$, $1\leqslant i\leqslant j-1$, is called standard. Furthermore, let $\lambda\in P^+$. A standard monomial that satisfies

$$\lambda(h_i)\geqslant a_{j,i}-a_{j,i-1}+\sum_{r=j+1}^{\ell}(2a_{r,i}-a_{r,i-1}-a_{r,i+1}),\quad 1\leqslant j\leqslant\ell,\ 1\leqslant i\leqslant j \tag{4.1}$$

is called λ-standard [16, Definition 22]. In the above we adopt the convention that $a_{j,s}=0$ if $s<0$ or $s>j$. We have the following

Theorem 3. *(See [16, Theorems 17 and 25].) Standard monomials form a basis of $U(\mathfrak{n}^-)$. Moreover, for all $\lambda\in P^+$, the set*

$$\{Fv_\lambda:\ F \text{ is a } \lambda\text{-standard monomial}\}$$

is a basis of $V(\lambda)$.

Assume now that $\mathfrak{g}$ is a simple Lie algebra of rank ℓ, $J=\{i,i+1,\dots,j\}$, $1\leqslant i\leqslant j\leqslant\ell$. Suppose that the Lie subalgebra $\mathfrak{g}_J$ of $\mathfrak{g}$ generated by the e_r, f_r, $r\in J$ is of type A_{j-i+1}. Let $\mu\in P^+$ and let $\eta=\sum_{r\in J}k_r\alpha_r$, $k_r\in\mathbf{Z}_+$. Set

$$\mathcal{J}(\eta)=\left\{\boldsymbol{a}=(a_{s,r})_{i\leqslant s\leqslant j,\,i\leqslant r\leqslant s}:\ a_{s,r+1}\geqslant a_{s,r},\ i+1\leqslant s\leqslant j,\ i\leqslant r\leqslant s-1,\ \sum_{s=r}^{j}a_{s,r}=k_r,\ i\leqslant r\leqslant j\right\}$$

and

$$\mathcal{J}(\eta,\mu)=\left\{\boldsymbol{a}\in J(\eta):\ \mu(h_k)\geqslant a_{s,k}-a_{s,k-1}+\sum_{r=s+1}^{j}(2a_{r,k}-a_{r,k-1}-a_{r,k+1}),\ i\leqslant s,k\leqslant j\right\},$$

where we assume that $a_{s,k}=0$ if $k<i$ or $k>s$. Using Theorem 3, we immediately obtain

Proposition. *The monomials*

$$f_i^{a_{i,i}}\big(f_{i+1}^{a_{i+1,i+1}}f_i^{a_{i+1,i}}\big)\cdots\big(f_j^{a_{j,j}}\cdots f_i^{a_{j,i}}\big),\qquad \boldsymbol{a}=(a_{s,k})_{i\leqslant k\leqslant s\leqslant j}\in\mathcal{J}(\eta)$$

form a basis of $U(\mathfrak{n}^-)_{-\eta}$ *and the vectors*

$$f_i^{a_{i,i}}\big(f_{i+1}^{a_{i+1,i+1}}f_i^{a_{i+1,i}}\big)\cdots\big(f_j^{a_{j,j}}\cdots f_i^{a_{j,i}}\big)v_\mu,\qquad \boldsymbol{a}=(a_{s,k})_{i\leqslant k\leqslant s\leqslant j}\in\mathcal{J}(\eta,\mu)$$

form a basis of $V(\mu)_{\mu-\eta}$. *In particular, if* $\mu(h_i)>0$ (*respectively,* $\mu(h_j)>0$) *then* $f_j\cdots f_i v_\mu\neq 0$ (*respectively,* $f_i\cdots f_j v_\mu\neq 0$).

Remark. The last assertion can of course be established by a simple induction on $j-i$ from the elementary theory of finite dimensional $\mathfrak{sl}_2$-modules.

4.2. Let $J\subset I$ and assume that $\mathfrak{g}_J$ is of type $A_{|J|}$. Let $\Sigma(i,j)$, $i\leqslant j\in J$ be the set of all bijective maps $\sigma:\{i,i+1,\dots,j\}\to\{1,\dots,j-i+1\}$ satisfying

$$\sigma(r+1)<\sigma(r)\quad\Longrightarrow\quad\sigma(r+1)=\sigma(r)-1,\quad i\leqslant r<j.$$

Given $\sigma\in\Sigma(i,j)$, let $\mathbf{f}_\sigma=f_{\sigma^{-1}(1)}\cdots f_{\sigma^{-1}(j-i-1)}$. Let $\alpha_{i,j}=\sum_{r=i}^{j}\alpha_r\in R^+$.

Lemma. *The set* $\{\mathbf{f}_\sigma\colon\ \sigma\in\Sigma(i,j)\}$ *is a basis of* $U(\mathfrak{n}^-)_{-\alpha_{i,j}}$.

Proof. Clearly, if $\sigma\in\Sigma(i,j)$ then $\mathbf{f}_\sigma$ is a standard monomial, and if $\sigma\neq\sigma'$ then the monomials $\mathbf{f}_\sigma$, $\mathbf{f}_{\sigma'}$ are distinct. Now, we prove by induction on $j-i$ that every standard monomial of weight $-\alpha_{i,j}$ is of the form $\mathbf{f}_\sigma$. If $j=i$ there is nothing to prove. If $j>i$, let F be a standard monomial of weight $-\alpha_{i,j}$. Removing f_j from F we obtain a standard monomial of weight $-\alpha_{i,j-1}$ which is equal to $\mathbf{f}_\tau$, $\tau\in\Sigma(i,j-1)$ by the induction hypothesis. Now, since F is standard and every f_r, $i\leqslant r\leqslant j$ occurs in F exactly once, it follows that either f_j occurs in the $(j-i+1)$th position or f_{j-1} occurs immediately after f_j. In the first case, set $\sigma(r)=\tau(r)$, $r<j$, $\sigma(j)=j-i+1$. In the second case, set for all $r<j$

$$\sigma(r)=\begin{cases}\tau(r), & \tau(r)<\tau(j-1),\\ \tau(r)+1, & \tau(r)\geqslant\tau(j-1)\end{cases}$$

and let $\sigma(j)=\tau(j-1)$. Then it is easy to see that $F=\mathbf{f}_\sigma$ and $\sigma\in\Sigma(i,j)$. □

4.3. Given $\eta\in\mathfrak{h}^*$, the assignment $h\mapsto h-\eta(h)$, $h\in\mathfrak{h}$, $x\mapsto x$, $x\in U(\mathfrak{n}^-)$ extends to an algebra automorphism $\psi_\eta:U(\mathfrak{b})\to U(\mathfrak{b})$. Clearly, $\psi_\eta\psi_{\eta'}=\psi_{\eta+\eta'}$ for all $\eta,\eta'\in\mathfrak{h}^*$ and

$$xy=\psi_\eta(y)x,\quad \forall x\in U(\mathfrak{g})_\eta,\ y\in S(\mathfrak{h}).$$

Observe also that $\pi_\lambda\circ\psi_\eta=\pi_{\lambda-\eta}$, $\lambda,\eta\in\mathfrak{h}^*$.

Given $r\leqslant s\in J$ and $\lambda\in\mathfrak{h}^*$, set

$$\mathcal{H}_{r,s}:=h_r+\dots+h_s+s-r\in S(\mathfrak{h}).$$

We use the convention that $\mathcal{H}_{r,s}=0$ if $r>s$. Note that $\lambda(\mathcal{H}_{r,s})\in\mathbf{Z}_+$ for all $\lambda\in P^+$ and $\lambda(\mathcal{H}_{r,s})=0$, $r\leqslant s$ if and only if $r=s$ and $\lambda(h_s)=0$. Define

$$\mathcal{X}^{\pm}_{i,j,k} := \sum_{\sigma\in\Sigma(i,j)} \mathbf{f}_\sigma c^{\pm}_\sigma(k), \qquad \mathcal{X}^-_{i,j} := \mathcal{X}^-_{i,j,j},\ \mathcal{X}^+_{i,j} := \mathcal{X}^+_{i,j,i},$$

where the $c^{\pm}_\sigma \in S(\mathfrak{h})$ are given by the following formulae

$$c^-_\sigma(k) = \prod_{s=i+1}^{j} (-1)^{\delta_{\sigma(s),\sigma(s-1)-1}}(\mathcal{H}_{s,k} + 1 - \delta_{\sigma(s),\sigma(s-1)-1}), \quad j \leqslant k \in J,$$

$$c^+_\sigma(l) = \prod_{r=i}^{j-1}(-1)^{1+\delta_{\sigma(r+1),\sigma(r)-1}}(\mathcal{H}_{l,r} + \delta_{\sigma(r+1),\sigma(r)-1}), \quad l \leqslant i \in J.$$

We let $\mathcal{X}^{\pm}_{j+1,j,k} = 1$, $1 \leqslant j \leqslant \ell - 1$, $\mathcal{X}^{\pm}_{i,j,k} = 0$, $i > j+1$.

Lemma. *Let $i \leqslant j \in J$, $\eta \in P$ and $r \in I$. Then*

$$e_r\psi_\eta(\mathcal{X}^-_{i,j}) = \psi_\eta(\delta_{r,i}\mathcal{X}^-_{i+1,j}\mathcal{H}_{i,j} + \mathcal{X}^-_{r+1,j}\mathcal{X}^-_{i,r-1,j}\eta(h_r)) + \psi_{\eta+\alpha_r}(\mathcal{X}^-_{i,j})e_r, \tag{4.2a}$$

$$e_r\psi_\eta(\mathcal{X}^+_{i,j}) = \psi_\eta(\delta_{r,j}\mathcal{X}^+_{i,j-1}\mathcal{H}_{i,j} + \mathcal{X}^+_{i,r-1}\mathcal{X}^+_{r+1,j,i}\eta(h_r)) + \psi_{\eta+\alpha_r}(\mathcal{X}^+_{i,j})e_r. \tag{4.2b}$$

In particular,

$$e_r\mathcal{X}^-_{i,j} = \delta_{r,i}\mathcal{X}^-_{i+1,j}\mathcal{H}_{i,j} + \psi_{\alpha_r}(\mathcal{X}^-_{i,j})e_r, \tag{4.2c}$$

$$e_s\mathcal{X}^+_{i,j} = \delta_{s,j}\mathcal{X}^+_{i,j-1}\mathcal{H}_{i,j} + \psi_{\alpha_s}(\mathcal{X}^+_{i,j})e_s. \tag{4.2d}$$

Proof. We only establish (4.2a), the proof of (4.2b) being similar. The argument is by induction on $j - i$. Note that the induction begins since $\mathcal{X}^-_{i,i} = f_i$ and so

$$e_r\psi_\eta(\mathcal{X}^-_{i,i}) = \mathcal{X}^-_{i,i}e_r + \delta_{r,i}h_i = \psi_{\eta+\alpha_r}(\mathcal{X}^-_{i,i})e_r + \delta_{r,i}\psi_\eta(\mathcal{X}^-_{i+1,i}(\mathcal{H}_{i,i} + \eta(h_i))).$$

We claim that the $\mathcal{X}^-_{p,q,k}$, $p < q \leqslant k \in J$ satisfy

$$\mathcal{X}^-_{p,q,k} = f_p\mathcal{X}^-_{p+1,q,k}(\mathcal{H}_{p+1,k} + 1) - \mathcal{X}^-_{p+1,q,k}f_p\mathcal{H}_{p+1,k}. \tag{4.3}$$

Indeed, since f_p commutes with the f_t, $p+1 < t$, $t \in J$, a standard monomial F of weight $-\alpha_{p,q}$ equals either $f_p\mathbf{f}_\tau$ or $\mathbf{f}_\tau f_p$, $\tau \in \Sigma(p+1,q)$. Let $\sigma, \sigma' \in \Sigma(p,q)$ be the elements corresponding, respectively, to $f_p\mathbf{f}_\tau$ and $\mathbf{f}_\tau f_p$. Then

$$c^-_\sigma(k) = (\mathcal{H}_{p+1,k} + 1)c^-_\tau(k), \qquad c^-_{\sigma'}(k) = -\mathcal{H}_{p+1,k}c^-_\tau(k).$$

Note that f_i commutes with $\mathcal{X}^-_{p,j,k}$, $p > i+1$. If $r \neq i$, we immediately obtain from (4.3), the induction hypothesis and the properties of ψ that

$$\begin{aligned} e_r\psi_\eta(\mathcal{X}^-_{i,j}) &= \psi_\eta(f_i\mathcal{X}^-_{r+1,j}\mathcal{X}^-_{i+1,r-1,j}(\mathcal{H}_{i+1,j} + 1) - \mathcal{X}^-_{r+1,j}\mathcal{X}^-_{i+1,r-1,j}f_i\mathcal{H}_{i+1,j})\eta(h_r) \\ &\quad + \delta_{r,i+1}\psi_\eta(f_i\mathcal{X}^-_{i+2,j}\mathcal{H}_{i+1,j}(\mathcal{H}_{i+1,j} + 1) - \mathcal{X}^-_{i+2,j}\mathcal{H}_{i+1,j}f_i\mathcal{H}_{i+1,j}) + \psi_{\eta+\alpha_r}(\mathcal{X}^-_{i,j})e_r \\ &= \psi_\eta(\mathcal{X}^-_{r+1,j}\mathcal{X}^-_{i,r-1,j})\eta(h_r) + \psi_{\eta+\alpha_r}(\mathcal{X}^-_{i,j})e_r. \end{aligned}$$

Suppose now that $r = i$. Then we obtain from (4.3) and the induction hypothesis

$$\begin{aligned} e_i \psi_\eta(\mathcal{X}_{i,j}^-) &= h_i \psi_\eta(\mathcal{X}_{i+1,j}^-(\mathcal{H}_{i+1,j}+1)) - \psi_\eta(\mathcal{X}_{i+1,j}^- \mathcal{H}_{i+1,j}) + \psi_{\eta+\alpha_i}(\mathcal{X}_{i,j}^-) e_i \\ &= \psi_\eta(\mathcal{X}_{i+1,j}^-((h_i + \eta(h_i) + 1)(\mathcal{H}_{i+1,j}+1) - (h_i + \eta(h_i))\mathcal{H}_{i+1,j})) + \psi_{\eta+\alpha_i}(\mathcal{X}_{i,j}^-) e_i \\ &= \psi_\eta(\mathcal{X}_{i+1,j}^-(\mathcal{H}_{i,j} + \eta(h_i))) + \psi_{\eta+\alpha_i}(\mathcal{X}_{i,j}^-) e_i. \qquad \square \end{aligned}$$

5. Type A_ℓ, $\ell > 1$

5.1. We have $R^+ = \{\alpha_{i,j}\colon 1 \leqslant i \leqslant j \leqslant \ell\}$. In particular, $\theta = \alpha_{1,\ell}$. In terms of fundamental weights, $\alpha_{i,j} = \varpi_i + \varpi_j - \varpi_{i-1} - \varpi_{j+1}$, where we set $\varpi_0 = \varpi_{\ell+1} = 0$. Since $\varepsilon(\alpha_{i,j}) = \varpi_{i-1} + \varpi_{j+1}$, we immediately obtain

Lemma. *Let $\alpha_{i,j} \in \Psi$, $\lambda \in P^+$. Then $\lambda \leftarrow \lambda + \alpha_{i,j} \in (\Delta_\Psi)_1$ if and only if $\lambda(h_{i-1}), \lambda(h_{j+1}) > 0$.*

5.2. We now proceed to describe the set of paths of length 2 in Δ_Ψ. Suppose that $\alpha_{i,m}, \alpha_{j,k} \in \Psi$, $i \leqslant j$. If $m + 1 < j$ we have $\alpha_{i,m} + \alpha_{j,k} = \alpha_{i,k} - \alpha_{m+1,j-1}$ which is a contradiction by Corollary 1.1, while $j = m + 1$ implies that $\alpha_{i,m} + \alpha_{j,k} = \alpha_{i,k} \in R^+$ which is again a contradiction. Thus, we must have $j \leqslant m$. If $j = i$ or $m = k$, there is only one way of writing $\alpha_{i,m} + \alpha_{j,k}$ as a sum of roots. Otherwise we may assume without loss of generality that $i < j \leqslant k < m$ and so we have

$$\alpha_{i,m} + \alpha_{j,k} = \alpha_{i,k} + \alpha_{j,m}.$$

It is easy to check that the sets

$$\begin{aligned} &\{\alpha_{i,j}, \alpha_{i,k}\}, && i \leqslant j < k, \\ &\{\alpha_{i,k}, \alpha_{j,k}\}, && i < j \leqslant k, \\ &\{\alpha_{i,m}, \alpha_{j,k}, \alpha_{i,k}, \alpha_{j,m}\}, && i < j \leqslant k < m \end{aligned}$$

are extremal and so all cases listed above actually occur.

Now we can list all paths of length 2 in Δ_Ψ. First, let $\eta = \alpha_{i,j} + \alpha_{i,k}$, $i \leqslant j < k$. Suppose that $\lambda + \eta \in P^+$. Then $\lambda(h_{i-1}) > 1$ and either $\lambda(h_{j+1}), \lambda(h_{k+1}) > 0$ or $\lambda(h_{k+1}) > 0$, $j = k + 1$ and $\lambda(h_k) = \lambda(h_{j+1}) = 0$. Using Lemma 5.1 we see that $\Delta_\Psi(\lambda, \lambda + \eta)$ is non-empty only if $\lambda(h_{i-1}) > 1$, $\lambda(h_{k+1}) > 0$ and we have

$$\Delta_\Psi(\lambda, \lambda+\eta) = \begin{cases} \{\lambda \leftarrow \lambda + \alpha_{i,j} \leftarrow \lambda + \eta, \lambda \leftarrow \lambda + \alpha_{i,k} \leftarrow \lambda + \eta\}, & \lambda(h_{j+1}) > 0, \\ \{\lambda \leftarrow \lambda + \alpha_{i,k} \leftarrow \lambda + \eta\}, & k = j+1,\ \lambda(h_{j+1}) = 0. \end{cases} \tag{5.1}$$

Similarly, if $\eta = \alpha_{i,k} + \alpha_{j,k}$, $i < j \leqslant k$, then $\Delta_\Psi(\lambda, \lambda + \eta)$ is non-empty only if $\lambda(h_{k+1}) > 1$, $\lambda(h_{i-1}) > 0$ and

$$\Delta_\Psi(\lambda, \lambda+\eta) = \begin{cases} \{\lambda \leftarrow \lambda + \alpha_{i,k} \leftarrow \lambda + \eta, \lambda \leftarrow \lambda + \alpha_{j,k} \leftarrow \lambda + \eta\}, & \lambda(h_{j-1}) > 0, \\ \{\lambda \leftarrow \lambda + \alpha_{i,k} \leftarrow \lambda + \eta\}, & i = j-1,\ \lambda(h_i) = 0. \end{cases} \tag{5.2}$$

Finally, let $\eta = \alpha_{i,m} + \alpha_{j,k} = \alpha_{i,k} + \alpha_{j,m}$, $i < j \leqslant k < m$. If $\lambda + \eta \in P^+$, we must have $\lambda(h_{i-1})$, $\lambda(h_{m+1}) > 0$. Using Lemma 5.1 again we see that $\Delta_\Psi(\lambda, \lambda + \eta)$, if non-empty, has one of the following forms:

$$\{\lambda \leftarrow \lambda + \alpha_{r,s} \leftarrow \lambda + \eta : (r,s) \in \{(i,m),(i,k),(j,m),(j,k)\}\}, \quad \lambda(h_{j-1}), \lambda(h_{k+1}) > 0, \tag{5.3}$$

$$\{\lambda \leftarrow \lambda + \alpha_{i,m} \leftarrow \lambda + \eta, \lambda \leftarrow \lambda + \alpha_{i,k} \leftarrow \lambda\}, \quad i = j-1,\ \lambda(h_{j-1}) = 0,\ \lambda(h_{k+1}) > 0, \tag{5.4}$$

$$\{\lambda \leftarrow \lambda + \alpha_{i,m} \leftarrow \lambda + \eta, \lambda \leftarrow \lambda + \alpha_{j,m} \leftarrow \lambda\}, \quad k = m-1,\ \lambda(h_{j-1}) > 0,\ \lambda(h_{k+1}) = 0, \tag{5.5}$$

$$\{\lambda \leftarrow \lambda + \alpha_{i,m} \leftarrow \lambda + \eta\}, \quad i = j-1,\ k = m-1,\ \lambda(h_{j-1}) = \lambda(h_{k+1}) = 0. \tag{5.6}$$

In particular, we have the following

Lemma. *An extremal set $\Psi \subset R^+$ is regular if and only if $\alpha_{i,j}, \alpha_{i,k} \in \Psi$, $j < k$, (respectively, $\alpha_{i,k}, \alpha_{j,k} \in \Psi$, $i < j$), implies that $k > j+1$ (respectively, $j > i+1$).*

5.3. Fix $r, s > 0$. Given $\boldsymbol{m} = (m_1, \dots, m_r) \in (\mathbf{Z}_+ \cup \{+\infty\})^r$, $\boldsymbol{n} = (n_1, \dots, n_s) \in (\mathbf{Z}_+ \cup \{+\infty\})^s$ and $a \in \mathbf{Z}$, $-|\boldsymbol{n}| \leqslant a \leqslant |\boldsymbol{m}|$, we define a quiver $\Gamma_a(\boldsymbol{m}, \boldsymbol{n})$ as follows. We set

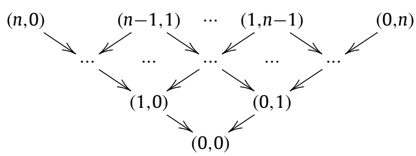
$$\Gamma_a(\boldsymbol{m}, \boldsymbol{n})_0 = \{(\boldsymbol{x}, \boldsymbol{y}) = ((x_1, \dots, x_r), (y_1, \dots, y_s)) \in \mathbf{Z}_+^r \times \mathbf{Z}_+^s\colon x_i \leqslant m_i,\ 1 \leqslant i \leqslant r,$$
$$y_j \leqslant n_j,\ 1 \leqslant j \leqslant s,\ |\boldsymbol{x}| = |\boldsymbol{y}| + a\}.$$

In other words, $\Gamma(\boldsymbol{m}, \boldsymbol{n})$ is just the set of lattice points in the $(r+s)$-dimensional rectangular parallelepiped $[0, m_1] \times \cdots \times [0, m_r] \times [0, n_1] \times \cdots \times [0, n_s]$ which lie on the hyperplane $z_1 + \cdots + z_r - z_{r+1} - \cdots - z_{r+s} = a$. The arrows are

$$(\boldsymbol{x}, \boldsymbol{y}) \leftarrow (\boldsymbol{x} + \boldsymbol{e}_i^{(r)}, \boldsymbol{y} + \boldsymbol{e}_j^{(s)}), \qquad x_i < m_i,\ y_j < m_j,\ 1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant s.$$

Note that the map $(\boldsymbol{x}, \boldsymbol{y}) \mapsto (\boldsymbol{m} - \boldsymbol{x}, \boldsymbol{n} - \boldsymbol{y})$ yields an isomorphism of quivers $\Gamma_a(\boldsymbol{m}, \boldsymbol{n}) \cong \Gamma_{|\boldsymbol{m}|-|\boldsymbol{n}|-a}(\boldsymbol{m}, \boldsymbol{n})^{op}$.

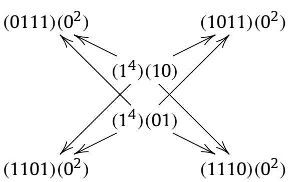
For example, $\Gamma_0((n,n),(n)) \cong \Gamma_n((n,n),(n))^{op}$ is isomorphic to the following quiver

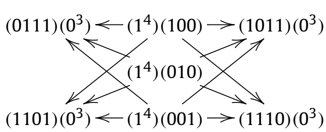


(5.7)

This is a translation quiver, with $\tau((x,y)) = (x+1, y+1)$, $0 \leqslant x + y \leqslant n-2$. It is easy to see that there is a full embedding of translation quivers of the above quiver into $\mathbf{Z}\Gamma_{2n+1}$ where Γ_{2n+1} is any quiver of type $\mathbb{A}_{2n+1}$. On the vertices, that embedding is given by $(x,y) \mapsto (-y, n+y-x)$, where we assume that the vertices of Γ_{2n+1} are numbered from 0 to $2n$. There is also a full embedding of the above quiver into the Auslander–Reiten quiver of the hereditary algebra of type $\mathbb{A}_{2n+1}$ where the nth node is the unique source.

Clearly $\Gamma_2((1^3),(1))$ is the quiver of type $\mathbb{D}_4$ in which the triple node is the unique source. Two more small examples (respectively, $\Gamma_3((1^4),(1^2))$ and $\Gamma_3((1^4),(1^3))$) are shown below





Lemma. *The quiver $\Gamma_a(\boldsymbol{m}, \boldsymbol{n})$ is connected.*

Proof. Clearly, every vertex in $\Gamma_a(\boldsymbol{m},\boldsymbol{n})$ is connected to a sink and a vertex $(\boldsymbol{x},\boldsymbol{y})\in\Gamma_a(\boldsymbol{m},\boldsymbol{n})$ is a sink if and only if either $\boldsymbol{x}=\boldsymbol{0}\in\mathbf{Z}_+^r$ or $\boldsymbol{y}=\boldsymbol{0}\in\mathbf{Z}_+^s$. In particular, if $a=0$ then $\Gamma_0(\boldsymbol{m},\boldsymbol{n})$ has a unique sink and hence is connected. If $a>0$ (respectively, $a<0$) the sinks in $\Gamma_a(\boldsymbol{m},\boldsymbol{n})$ are the vertices $(\boldsymbol{x},\boldsymbol{0})$ (respectively, $(\boldsymbol{0},\boldsymbol{y})$) with $|\boldsymbol{x}|=a$ (respectively, $|\boldsymbol{y}|=-a$). Suppose that $a>0$, the other case being similar. If $a=|\boldsymbol{m}|$ then we have a unique sink which is also a source. Otherwise, let $S=\{\boldsymbol{x}\in\mathbf{Z}_+^r\colon x_i\leqslant m_i,\ |\boldsymbol{x}|=a\}$ and let $\prec$ be the lexicographic order on S. Let $(\boldsymbol{x},\boldsymbol{0})$, $\boldsymbol{x}\in S$ be a sink and suppose that $\boldsymbol{x}$ is not the minimal element of S. Let $1\leqslant j\leqslant r$ be maximal such that $x_j<m_j$. If there is $1\leqslant i<j$ minimal such that $x_i>0$, we have $(\boldsymbol{x},\boldsymbol{0})\leftarrow(\boldsymbol{x}+\boldsymbol{e}_j^{(r)},\boldsymbol{e}_1^{(s)})\to(\boldsymbol{x}-\boldsymbol{e}_i^{(r)}+\boldsymbol{e}_j^{(r)},\boldsymbol{0})$ and $\boldsymbol{x}-\boldsymbol{e}_i^{(r)}+\boldsymbol{e}_j^{(r)}\prec\boldsymbol{x}$. Suppose that $x_i=0$ for all $i<j$. Since $\boldsymbol{x}$ is not minimal, there exists $\boldsymbol{x}'\in S$ such that $\boldsymbol{x}'\prec\boldsymbol{x}$, that is $x'_i=0$, $1\leqslant i<j$ and $x'_j<x_j$. Since $|\boldsymbol{x}'|=a=|\boldsymbol{x}|$, we must have $x'_k>x_k$ for some $j<k\leqslant r$, which is a contradiction by the choice of j. Thus, the connected component of $(\boldsymbol{x},\boldsymbol{0})$ contains a sink $(\boldsymbol{x}',\boldsymbol{0})$ with $\boldsymbol{x}'\prec\boldsymbol{x}$. The assertion is now immediate. □

5.4. Fix $1\leqslant i_1<\cdots<i_r<j_1<\cdots<j_s\leqslant\ell\in I$ and consider $\Psi=\{\alpha_{i_p,j_q}\colon 1\leqslant p\leqslant r,\ 1\leqslant q\leqslant s\}$. It is easy to see that Ψ is extremal. Assume further that $i_{p+1}\neq i_p+1$, $j_{q+1}\neq j_q+1$ for all $1\leqslant p<r$, $1\leqslant q<s$, and so by Lemma 5.2, Ψ is regular.

Proposition. *Let $\lambda\in P^+$. Then the quiver $\Delta_\Psi[\lambda]$ is isomorphic to $\Gamma_a(\boldsymbol{m},\boldsymbol{n})$ where $\boldsymbol{m}=(\lambda(h_{i_p-1})+\lambda(h_{i_p}))_{1\leqslant p\leqslant r}$, $\boldsymbol{n}=(\lambda(h_{j_q})+\lambda(h_{j_q+1}))_{1\leqslant q\leqslant s}$ and $a=\sum_{p=1}^r\lambda(h_{i_p})-\sum_{q=1}^s\lambda(h_{j_q})$.*

Proof. Let $J=\{i_p,i_p-1\colon\ 1\leqslant p\leqslant r\}\cup\{j_q,j_q+1\colon\ 1\leqslant q\leqslant s\}$. Suppose that $\mu\in\Delta_\Psi[\lambda]_0$. Since $\Delta_\Psi[\lambda]_0\subset(\lambda+\mathbf{Z}\Psi)\cap P^+$, we have $\mu(h_j)=\lambda(h_j)$, $j\notin J$, and

$$\mu(h_{i_p})=\lambda(h_{i_p})+\sum_{q=1}^s x_{p,q},\qquad \mu(h_{i_p-1})=\lambda(h_{i_p-1})-\sum_{q=1}^s x_{p,q},\quad 1\leqslant p\leqslant r,$$
$$\mu(h_{j_q})=\lambda(h_{j_q})+\sum_{p=1}^r x_{p,q},\qquad \mu(h_{j_q+1})=\lambda(h_{j_q+1})-\sum_{p=1}^r x_{p,q},\quad 1\leqslant q\leqslant s,$$

where $x_{p,q}\in\mathbf{Z}$, $1\leqslant p\leqslant r$, $1\leqslant q\leqslant s$. It follows that $\Delta_\Psi[\lambda]_0$ is contained in the set $S(\lambda)$ of $\mu\in P^+$ satisfying the following conditions

$$\mu(h_{i_p-1})+\mu(h_{i_p})=\lambda(h_{i_p-1})+\lambda(h_{i_p}),\qquad \mu(h_{j_q})+\mu(h_{j_q+1})=\lambda(h_{j_q})+\lambda(h_{j_q+1}),$$
$$\sum_{p=1}^r\mu(h_{i_p})-\sum_{p=1}^s\mu(h_{j_q})=\sum_{p=1}^r\lambda(h_{i_p})-\sum_{p=1}^s\lambda(h_{j_q}).$$

Clearly, if $\mu\in S(\lambda)$ then $\mu^-\subset S(\lambda)$ and so $S(\lambda)$ defines a convex subquiver of Δ_Ψ containing $\Delta_\Psi[\lambda]$ as a full connected subquiver. Define a map $S(\lambda)\to\Gamma_a(\boldsymbol{m},\boldsymbol{n})_0$ by

$$\mu\mapsto\big(\big(\mu(h_{i_1}),\dots,\mu(h_{i_r})\big),\big(\mu(h_{j_1}),\dots,\mu(h_{j_s})\big)\big).$$

This map is clearly a bijection and it is easy to see that it induces an isomorphism of quivers. Since by Lemma 5.3 the quiver $\Gamma_a(\boldsymbol{m},\boldsymbol{n})$ is connected, the assertion follows. □

5.5. For $1\leqslant i<j\leqslant\ell$, we fix root vectors $e_{i,j}\in\mathfrak{g}_{\alpha_{i,j}}$ such that

$$[e_r,e_{p,q}]=\delta_{r,p-1}e_{r,q}-\delta_{r,q+1}e_{p,r},\qquad [f_r,e_{p,q}]=\delta_{r,p}e_{r+1,q}-\delta_{r,q}e_{p,r-1}.\tag{5.8}$$

For example, the standard basis of the matrix realisation of $\mathfrak{sl}_{\ell+1}$ has these properties.

Fix $\alpha_{i,j} \in \Psi$. Clearly $\{\gamma \in R^+ \colon \alpha_{i,j} \leqslant \gamma\} = \{\alpha_{p,q} \colon 1 \leqslant p \leqslant i, j \leqslant q \leqslant \ell\}$. If $\lambda, \lambda + \alpha_{i,j} \in P^+$, we have $\lambda(h_{i-1}), \lambda(h_{j+1}) > 0$ and so

$$\lambda(\mathcal{H}_{t,i-1}) \geqslant \lambda(h_{i-1}) > 0, \qquad \lambda(\mathcal{H}_{j+1,t}) \geqslant \lambda(h_{j+1}) > 0, \quad 1 \leqslant t \leqslant i-1, \ j+1 \leqslant t \leqslant \ell.$$

Therefore

$$\mathcal{H}_{r,i-1}, \mathcal{H}_{j+1,s} \in F_{\alpha_{i,j}}(\mathfrak{h})^\times, \quad 1 \leqslant r \leqslant i-1, \ j+1 \leqslant s \leqslant \ell.$$

For all $1 \leqslant p \leqslant i, \ j \leqslant q \leqslant \ell$, define $\mathbf{u}_{\alpha_{i,j},\alpha_{p,q}} \in U(\mathfrak{b})_{\alpha_{i,j}-\alpha_{p,q}} \otimes_{S(\mathfrak{h})} F_{\alpha_{i,j}}(\mathfrak{h})$ by

$$\mathbf{u}_{\alpha_{i,j},\alpha_{p,q}} = (-1)^{i-p} \mathcal{X}^-_{p,i-1} \mathcal{X}^+_{j+1,q} \otimes \prod_{t=p}^{i-1} \mathcal{H}^{-1}_{t,i-1} \prod_{t=j+1}^{q} \mathcal{H}^{-1}_{j+1,t}. \tag{5.9}$$

Lemma. *Let $\alpha_{i,j} \in \Psi$, $1 \leqslant i < j \leqslant \ell$. Then $(\mathbf{u}_{\alpha_{i,j},\alpha_{p,q}} \colon 1 \leqslant p \leqslant i, \ j \leqslant q \leqslant \ell)$ is an adapted family for $\alpha_{i,j}$.*

Proof. We have $\pi_{\lambda,\alpha_{i,j}}(\mathbf{u}_{\alpha_{i,j},\alpha_{p,q}}) = \mathcal{X}^-_{p,i-1}(\lambda)\mathcal{X}^+_{j+1,q}(\lambda) B_{p,q}(i,j,\lambda)$, where

$$B_{p,q}(i,j,\lambda) := (-1)^{i-p} \prod_{t=p}^{i-1} \big(\lambda(\mathcal{H}_{t,i-1})\big)^{-1} \prod_{t=j+1}^{q} \lambda(\mathcal{H}_{j+1,t})^{-1}. \tag{5.10}$$

Write $B_{p,q} = B_{p,q}(i,j,\lambda)$ to shorten the notation. It is immediate that the $B_{p,q}$ satisfy

$$B_{p+1,q} = -\lambda(\mathcal{H}_{p,i-1}) B_{p,q}, \qquad B_{p,q-1} = \lambda(\mathcal{H}_{j+1,q}) B_{p,q}. \tag{5.11}$$

Let

$$v = \sum_{\alpha_{i,j} \leqslant \gamma} e_\gamma \otimes \pi_{\lambda,\alpha_{i,j}}(\mathbf{u}_{\alpha_{i,j},\gamma}) v_\lambda.$$

Since $v \in (\mathfrak{g} \otimes V(\lambda))_{\lambda+\beta}$ and $\mathbf{u}_{\alpha_{i,j},\alpha_{i,j}} = 1$, it remains to prove that $e_r v = 0$ for all $1 \leqslant r \leqslant \ell$. It follows immediately from Lemma 4.3 and (5.8) that $e_r v = 0$, $i \leqslant r \leqslant j$. Note that $\mathcal{X}^-_{p,i-1}(\lambda)$ and $\mathcal{X}^+_{j+1,q}(\lambda)$ commute for all $1 \leqslant p \leqslant i, \ j+1 \leqslant q \leqslant \ell$. Let $1 \leqslant r \leqslant i-1$. By Lemma 4.3,

$$e_r \mathcal{X}^-_{p,i-1}(\lambda)\mathcal{X}^+_{j+1,q}(\lambda) v_\lambda = \delta_{r,p} \lambda(\mathcal{H}_{r,i-1}) \mathcal{X}^-_{r+1,i-1}(\lambda)\mathcal{X}^+_{j+1,q}(\lambda) v_\lambda,$$

hence, using (5.8) and (5.11), we obtain

$$e_r v = \sum_{q=j+1}^{\ell} \big(B_{r+1,q} + B_{r,q}\lambda(\mathcal{H}_{r,i-1})\big) e_{r,q} \otimes \mathcal{X}^-_{r+1,i-1}(\lambda)\mathcal{X}^+_{j+1,q}(\lambda) v_\lambda = 0.$$

Finally, suppose that $j+1 \leqslant r \leqslant \ell$. By Lemma 4.3,

$$e_r \mathcal{X}^-_{p,i-1}(\lambda)\mathcal{X}^+_{j+1,q}(\lambda) v_\lambda = \delta_{r,q} \lambda(\mathcal{H}_{j+1,r}) \mathcal{X}^-_{p,i-1}(\lambda)\mathcal{X}^+_{j+1,r-1}(\lambda) v_\lambda$$

and so by (5.8) and (5.11)

$$e_r v = \sum_{p=1}^{i} \big(-B_{p,r-1} + B_{p,r}\lambda(\mathcal{H}_{j+1,r})\big) e_{p,r} \otimes \mathcal{X}^{-}_{p,i-1}(\lambda)\mathcal{X}^{+}_{j+1,r-1}(\lambda) v_\lambda = 0. \qquad \square$$

5.6. To describe the relations without ambiguity, we need to fix an isomorphism $\mathbf{T}^{\mathfrak{g}}_{\Psi} \to \mathbf{C}\Delta_{\Psi}$ which amounts to fixing an element $\boldsymbol{z} \in (\mathbf{C}^{\times})^{(\Delta_\Psi)_1}$. Given $\alpha_{i,j} \in \Psi$, let

$$\mathcal{Z}_{\alpha_{i,j},\Psi} = \prod_{1 \leqslant t < i : \alpha_{t,j} \in \Psi} \mathcal{H}_{t,i-1} \prod_{j < t \leqslant \ell :\ \alpha_{i,t} \in \Psi} \mathcal{H}_{j+1,t}. \tag{5.12}$$

Since $\lambda + \alpha_{i,j} \in P^+$ implies that $\lambda(h_{i-1}), \lambda(h_{j+1}) > 0$, $\mathcal{Z}_{\alpha_{i,j},\Psi} \in F_{\alpha_{i,j}}(\mathfrak{h})^\times$. Let $\boldsymbol{z}$ be the image of $(\mathcal{Z}^{-1}_{\beta,\Psi})_{\beta\in\Psi} \in \prod_{\beta\in\Psi} F_\beta(\mathfrak{h})^\times$ in G_Ψ. In other words, we fix the isomorphism Φ by requiring

$$\boldsymbol{a}_{\lambda,\alpha_{i,j}} \mapsto \lambda(\mathcal{Z}_{\alpha_{i,j},\Psi})^{-1}(\lambda \leftarrow \lambda + \alpha_{i,j}), \quad \lambda, \lambda + \alpha_{i,j} \in P^+.$$

Now we are ready to compute all relations in the algebra $\mathbf{S}^{\mathfrak{g}}_{\Psi}$. For readers convenience, we describe different cases in separate propositions.

Proposition. *Let Ψ be an extremal set, $|\Psi| > 1$.*

- (i) *Let $\alpha_{i,j}, \alpha_{i,k} \in \Psi$, $1 \leqslant i \leqslant j < k \leqslant \ell$. Then for all $\lambda \in P^+$ such that $t_{\lambda,\alpha_{i,j}+\alpha_{i,k}} = 2$, $\mathfrak{R}_\Psi(\lambda, \lambda + \alpha_{i,j} + \alpha_{i,k})$ is spanned by the commutativity relation. If $t_{\lambda,\alpha_{i,j}+\alpha_{i,k}} = 1$, $\dim \mathfrak{R}_\Psi(\lambda, \lambda + \alpha_{i,j} + \alpha_{i,k}) = 1$.*
- (ii) *Let $\alpha_{i,k}, \alpha_{j,k} \in \Psi$, $1 \leqslant i < j \leqslant k \leqslant \ell$. Then for all $\lambda \in P^+$ such that $t_{\lambda,\alpha_{i,k}+\alpha_{j,k}} = 2$, $\mathfrak{R}_\Psi(\lambda, \lambda + \alpha_{i,k} + \alpha_{j,k})$ is spanned by the commutativity relation. If $t_{\lambda,\alpha_{i,k}+\alpha_{j,k}} = 1$, $\dim \mathfrak{R}_\Psi(\lambda, \lambda + \alpha_{i,k} + \alpha_{j,k}) = 1$.*

In particular, if $\eta \in \Psi + \Psi$ satisfies $m_\eta = 2$, then $\mathcal{N}_\eta = \emptyset$.

Proof. We present a detailed argument here since the computations of this kind will be used repeatedly in the rest of this paper and in the future we will omit most of the details.

Retain the notations of the proof of Lemma 5.5. To prove (i), note that $\alpha_{i,j} < \alpha_{i,k}$. It follows from Proposition 2.7 and Lemma 5.5 that

$$\Pi_\lambda(\alpha_{i,j}, \alpha_{i,k}) = e_{i,k} \otimes e_{i,j},$$
$$\Pi_\lambda(\alpha_{i,k}, \alpha_{i,j}) = e_{i,j} \otimes e_{i,k} + e_{i,k} \otimes \mathbf{u}_{\alpha_{i,j},\alpha_{i,k}}(\mu) e_{i,k}, \qquad \mu = \lambda + \alpha_{i,k},$$

and

$$\mathbf{u}_{\alpha_{i,j},\alpha_{i,k}}(\mu) = \pi_{\mu,\alpha_{i,j}}(\mathbf{u}_{\alpha_{i,j},\alpha_{i,k}}) = B_{i,k}(i,j,\mu)\mathcal{X}^{+}_{j+1,k}(\mu).$$

Note that $\mu(h_k) = \lambda(h_k) + 1 > 0$. Let $\sigma \in \Sigma(j+1,k)$ and suppose that $\mathbf{f}_\sigma \notin \mathrm{Ann}_{U(\mathfrak{g})} e_{i,k}$. Then (5.8) we must have $\sigma(k) = k - j$ and so $\mathbf{f}_\sigma e_{i,k} = -\mathbf{f}_{\sigma'} e_{i,k-1}$, where $\sigma' \in \Sigma(j+1,k-1)$ is the restriction of σ. Following this way we conclude that $\sigma(r) = r - j$, $j+1 \leqslant r \leqslant k$, that is $\mathbf{f}_\sigma = f_{j+1} \cdots f_k$. Since $\mu(h_k) > 0$, using Proposition 4.1 and Lemma 4.3 we conclude that

$$\mathcal{X}^{+}_{j+1,k}(\mu) = (-1)^{k-j-1} \prod_{t=j+1}^{k-1} \mu(\mathcal{H}_{j+1,t}) f_{j+1} \cdots f_k + \mathrm{Ann}_{U(\mathfrak{g})} v_\mu \cap \mathrm{Ann}_{U(\mathfrak{g})} e_{i,k} \tag{5.13}$$

and $f_{j+1}\cdots f_k \notin \mathrm{Ann}_{U(\mathfrak{n}^-)}\, v_\mu$. Therefore,

$$\mathbf{u}_{\alpha_{i,j},\alpha_{i,k}}(\mu)e_{i,k} = -\big(\mu(\mathcal{H}_{j+1,k})\big)^{-1}e_{i,j}$$

hence

$$\Pi_\lambda(\alpha_{i,k},\alpha_{i,j}) = e_{i,j}\otimes e_{i,k} - \big(\lambda(\mathcal{H}_{j+1,k})+1\big)^{-1}e_{i,k}\otimes e_{i,j}. \tag{5.14}$$

It is easy to see that the intersection of $\mathbf{C}\Pi_\lambda(\alpha_{i,j},\alpha_{i,k}) + \mathbf{C}\Pi_\lambda(\alpha_{i,k},\alpha_{i,j})$ with $\bigwedge^2 \mathfrak{n}_\Psi^+$ is spanned by

$$\lambda(\mathcal{H}_{j+1,k})\Pi_\lambda(\alpha_{i,j},\alpha_{i,k}) - \big(\lambda(\mathcal{H}_{j+1,k})+1\big)\Pi_\lambda(\alpha_{i,k},\alpha_{i,j}).$$

Thus, $\mathfrak{R}_\Psi(\lambda,\lambda+\eta)$, $\eta = \alpha_{i,j}+\alpha_{i,k}$ is spanned by

$$\begin{aligned}&\lambda(\mathcal{H}_{j+1,k})\big(\lambda(\mathcal{Z}_{\alpha_{i,j},\Psi})(\lambda+\alpha_{i,j})(\mathcal{Z}_{\alpha_{i,k},\Psi})\big)^{-1}(\lambda\leftarrow\lambda+\alpha_{i,j}\leftarrow\lambda+\eta)\\ &\quad-\big(\lambda(\mathcal{H}_{j+1,k})+1\big)\big(\lambda(\mathcal{Z}_{\alpha_{i,k},\Psi})(\lambda+\alpha_{i,k})(\mathcal{Z}_{\alpha_{i,j},\Psi})\big)^{-1}(\lambda\leftarrow\lambda+\alpha_{i,k}\leftarrow\lambda+\eta).\end{aligned}$$

Since $\alpha_{t,j}+\alpha_{i,k} = \alpha_{t,k}+\alpha_{i,j}$, $1\leqslant t<i$, Corollary 1.1 implies that for all $1\leqslant t<i$, $\alpha_{t,j}\in\Psi$ if and only if $\alpha_{t,k}\in\Psi$. Then

$$\begin{aligned}&\lambda\big(\mathcal{Z}_{\alpha_{i,k},\Psi}\mathcal{Z}^{-1}_{\alpha_{i,j},\Psi}\big)(\lambda+\alpha_{i,k})(\mathcal{Z}_{\alpha_{i,j},\Psi})(\lambda+\alpha_{i,j})\big(\mathcal{Z}^{-1}_{\alpha_{i,k},\Psi}\big)\\ &\quad=\frac{(\lambda(\mathcal{H}_{j+1,k})+1)\prod_{1\leqslant t<i:\ \alpha_{t,k}\in\Psi}\lambda(\mathcal{H}_{t,i-1})\prod_{1\leqslant t<i:\ \alpha_{t,j}\in\Psi}(\lambda(\mathcal{H}_{t,i-1})-1)}{\lambda(\mathcal{H}_{j+1,k})\prod_{1\leqslant t<i:\ \alpha_{t,j}\in\Psi}\lambda(\mathcal{H}_{t,i-1})\prod_{1\leqslant t<i:\alpha_{t,k}\in\Psi}(\lambda(\mathcal{H}_{t,i-1})-1)}\\ &\quad=\frac{\lambda(\mathcal{H}_{j+1,k})+1}{\lambda(\mathcal{H}_{j+1,k})},\end{aligned}$$

and so $\mathfrak{R}_\Psi(\lambda,\lambda+\eta)$ is spanned by the commutativity relation. If $t_{\lambda,\eta}=1$, by (5.1) we have $\lambda(\mathcal{H}_{j+1,k}) = \lambda(h_{j+1}) = 0$ and it follows from (5.14) that $\Pi_\lambda(\alpha_{i,k},\alpha_{i,j})\in\bigwedge^2\mathfrak{n}_\Psi^+$. Thus, the unique path $(\lambda\leftarrow\lambda+\alpha_{i,j+1}\leftarrow\lambda+\eta)$ in $\Delta_\Psi(\lambda,\lambda+\eta)$ is a relation.

To prove (ii), note that $\alpha_{j,k}<\alpha_{i,k}$ and so

$$\begin{aligned}&\Pi_\lambda(\alpha_{j,k},\alpha_{i,k}) = e_{i,k}\otimes e_{j,k},\\ &\Pi_\lambda(\alpha_{i,k},\alpha_{j,k}) = e_{j,k}\otimes e_{i,k} + e_{i,k}\otimes\mathbf{u}_{\alpha_{j,k},\alpha_{i,k}}(\nu)e_{i,k},\quad \nu=\lambda+\alpha_{i,k},\end{aligned}$$

where

$$\mathbf{u}_{\alpha_{j,k},\alpha_{i,k}}(\nu) = \pi_{\nu,\alpha_{j,k}}(\mathbf{u}_{\alpha_{j,k},\alpha_{i,k}}) = B_{i,k}(j,k,\nu)\mathcal{X}^-_{i,j-1}(\nu).$$

An argument similar to the above shows that

$$\mathcal{X}^-_{i,j-1}(\nu) = (-1)^{j-i-1}\prod_{t=i+1}^{j-1}\nu(\mathcal{H}_{t,j-1})f_{j-1}\cdots f_i + \mathrm{Ann}_{U(\mathfrak{g})}\,v_\nu\cap\mathrm{Ann}_{U(\mathfrak{g})}\,e_{i,k}, \tag{5.15}$$

hence

$$\Pi_\lambda(\alpha_{i,k},\alpha_{j,k}) = e_{j,k}\otimes e_{i,k} - \big(\lambda(\mathcal{H}_{i,j-1})+1\big)^{-1} e_{i,k}\otimes e_{j,k}. \tag{5.16}$$

To finish the computation, we observe that $\alpha_{i,k}+\alpha_{j,t}=\alpha_{j,k}+\alpha_{i,t}$, $k<t\leqslant\ell$, hence by Corollary 1.1 $\alpha_{i,t}\in\Psi$ if and only if $\alpha_{j,t}\in\Psi$ for all $k<t\leqslant\ell$. This implies that

$$\lambda\big(\mathcal{Z}_{\alpha_{i,k},\Psi}\mathcal{Z}^{-1}_{\alpha_{j,k},\Psi}\big)(\lambda+\alpha_{i,k})(\mathcal{Z}_{\alpha_{j,k},\Psi})(\lambda+\alpha_{j,k})\big(\mathcal{Z}^{-1}_{\alpha_{i,k},\Psi}\big) = \big(\lambda(\mathcal{H}_{i,j-1})+1\big)\big(\lambda(\mathcal{H}_{i,j-1})\big)^{-1}.$$

Finally, if $t_{\lambda,\alpha_{i,k}+\alpha_{j,k}}=1$, (5.2) implies that $\lambda(\mathcal{H}_{i,j-1})=0$ hence $\Pi_\lambda(\alpha_{i,k},\alpha_{j,k})\in\bigwedge^2\mathfrak{n}^+_\Psi$ and so the corresponding path is a relation. $\square$

Example. Fix $i<j<k\in I$ with $k\neq i+1$ and either $i\neq 1$ or $k\neq\ell$. Let $\lambda=m(\varpi_{i-1}+\varpi_{j+1}+\varpi_{k+1})$. Then by Proposition 5.3 and by the above, $\mathbf{S}^{\mathfrak{g}}_\Psi(\lambda\leqslant_\Psi)$ has global dimension 2 and is isomorphic to the path algebra of the translation quiver (5.7) with the mesh relations. In particular, it is isomorphic to a subalgebra of the Auslander algebra of the path algebra of the quiver of type $\mathbb{A}_{2m+1}$, where the node preserved by the diagram automorphism is the unique sink.

5.7. The following proposition completes the proof of Theorem 2 for $\mathfrak{g}$ of type A.

Proposition. *Let $\Psi\subset R^+$, $|\Psi|\geqslant 4$ be extremal. Suppose that*

$$\{\alpha_{i,k},\alpha_{j,k},\alpha_{i,m},\alpha_{j,m}\}\subset\Psi,\quad i<j\leqslant k<m$$

and let $\eta=\alpha_{i,k}+\alpha_{j,m}=\alpha_{i,m}+\alpha_{j,k}$. Let $x_\lambda=\lambda(\mathcal{H}_{i,j-1})$, $y_\lambda=\lambda(\mathcal{H}_{k+1,m})$.

(i) *Suppose that $t_{\lambda,\eta}=4$, $x_\lambda\neq y_\lambda$. Then $\mathfrak{R}_\Psi(\lambda,\lambda+\eta)$ is spanned by*

$$(x_\lambda+1)(y_\lambda+2)(\lambda\leftarrow\lambda+\alpha_{i,k}\leftarrow\lambda+\eta)-(x_\lambda+2)(y_\lambda+1)(\lambda\leftarrow\lambda+\alpha_{j,m}\leftarrow\lambda+\eta)$$
$$-(x_\lambda-y_\lambda)(\lambda\leftarrow\lambda+\alpha_{i,m}\leftarrow\lambda+\eta)$$

and

$$x_\lambda(y_\lambda+1)(\lambda\leftarrow\lambda+\alpha_{i,k}\leftarrow\lambda+\eta)-(x_\lambda+1)y_\lambda(\lambda\leftarrow\lambda+\alpha_{j,m}\leftarrow\lambda+\eta)$$
$$-(x_\lambda-y_\lambda)(\lambda\leftarrow\lambda+\alpha_{j,k}\leftarrow\lambda+\eta).$$

(ii) *Suppose that $t_{\lambda,\eta}=4$ and $x_\lambda=y_\lambda$. Then $\mathfrak{R}_\Psi(\lambda,\lambda+\eta)$ is spanned by*

$$(\lambda\leftarrow\lambda+\alpha_{i,k}\leftarrow\lambda+\eta)-(\lambda\leftarrow\lambda+\alpha_{j,m}\leftarrow\lambda+\eta)$$

and

$$2(\lambda\leftarrow\lambda+\alpha_{i,k}\leftarrow\lambda+\eta)-x_\lambda(\lambda\leftarrow\lambda+\alpha_{i,m}\leftarrow\lambda+\eta)-(x_\lambda+2)(\lambda\leftarrow\lambda+\alpha_{j,k}\leftarrow\lambda+\eta).$$

(iii) *Suppose that* $t_{\lambda,\eta} = 2$. *Then* $x_\lambda \neq y_\lambda$ *and either* $i = j-1$, $x_\lambda = 0$ *and the relation is*

$$y_\lambda(\lambda \leftarrow \lambda + \alpha_{i,m} \leftarrow \lambda + \eta) + (y_\lambda + 2)(\lambda \leftarrow \lambda + \alpha_{i,k} \leftarrow \lambda + \eta),$$

or $m = k+1$, $y_\lambda = 0$ *and the relation is*

$$x_\lambda(\lambda \leftarrow \lambda + \alpha_{i,m} \leftarrow \lambda + \eta) + (x_\lambda + 2)(\lambda \leftarrow \lambda + \alpha_{j,m} \leftarrow \lambda + \eta).$$

(iv) *Suppose that* $t_{\lambda,\eta} = 1$. *Then* $i = j-1$, $m = k+1$, $x_\lambda = y_\lambda = 0$ *and* $\mathfrak{R}_\Psi(\lambda, \lambda + \eta) = 0$.

Thus, $\mathcal{N}_\eta$ *is contained in the set* $P^+ \cap \{\xi \in \mathfrak{h}^*\colon\ \xi(\mathcal{H}_{i,j-1} - \mathcal{H}_{k+1,m}) = 0\}$ *and coincides with this set if* Ψ *is regular.*

Proof. We have $\alpha_{j,k} < \alpha_{i,k}, \alpha_{j,m} < \alpha_{i,m}$ while $\alpha_{i,k}$, $\alpha_{j,m}$ are not comparable in the standard partial order.

To prove (i) we compute using Lemma 5.5, Proposition 2.7 and (5.13), (5.15)

$$\Pi_\lambda(\alpha_{j,k}, \alpha_{i,m}) = e_{i,m} \otimes e_{j,k}, \tag{5.17a}$$

$$\begin{aligned}\Pi_\lambda(\alpha_{i,m}, \alpha_{j,k}) = {} & e_{j,k} \otimes e_{i,m} - (x_\lambda + 1)^{-1} e_{i,k} \otimes e_{j,m} - (y_\lambda + 1)^{-1} e_{j,m} \otimes e_{i,k} \\ & + (x_\lambda + 1)(y_\lambda + 1)^{-1} e_{i,m} \otimes e_{j,k},\end{aligned} \tag{5.17b}$$

$$\Pi_\lambda(\alpha_{j,m}, \alpha_{i,k}) = e_{i,k} \otimes e_{j,m} - (y_\lambda + 1)^{-1} e_{i,m} \otimes e_{j,k}, \tag{5.17c}$$

$$\Pi_\lambda(\alpha_{i,k}, \alpha_{j,m}) = e_{j,m} \otimes e_{i,k} - (x_\lambda + 1)^{-1} e_{i,m} \otimes e_{j,k}. \tag{5.17d}$$

In particular, we see that none of the paths in $\Delta_\Psi(\lambda, \lambda + \eta)$ is a relation. Furthermore, we have

$$\begin{aligned}\alpha_{t,k} \in \Psi \quad &\Longleftrightarrow \quad \alpha_{t,m} \in \Psi, \quad 1 \leqslant t < j, \\ \alpha_{i,t} \in \Psi \quad &\Longleftrightarrow \quad \alpha_{j,t} \in \Psi, \quad k < t \leqslant \ell.\end{aligned}$$

Indeed this follows from Corollary 1.1 by observing that $\alpha_{j,m} + \alpha_{t,k} = \alpha_{t,m} + \alpha_{j,k}$, $\alpha_{i,t} + \alpha_{j,k} = \alpha_{i,k} + \alpha_{j,t}$. Then if we set $z = \lambda(\mathcal{Z}_{\alpha_{j,k},\Psi})(\lambda + \alpha_{j,k})(\mathcal{Z}_{\alpha_{i,m},\Psi})$,

$$\begin{aligned}\lambda(\mathcal{Z}_{\alpha_{i,m},\Psi})(\lambda + \alpha_{i,m})(\mathcal{Z}_{\alpha_{j,k},\Psi}) &= (x_\lambda + 1)(y_\lambda + 1) x_\lambda^{-1} y_\lambda^{-1} z, \\ \lambda(\mathcal{Z}_{\alpha_{j,m},\Psi})(\lambda + \alpha_{j,m})(\mathcal{Z}_{\alpha_{i,k},\Psi}) &= (y_\lambda + 1) y_\lambda^{-1} z, \\ \lambda(\mathcal{Z}_{\alpha_{i,k},\Psi})(\lambda + \alpha_{i,k})(\mathcal{Z}_{\alpha_{j,m},\Psi}) &= (x_\lambda + 1) x_\lambda^{-1} z.\end{aligned}$$

The relations in (i) and in (ii) are now straightforward.

To prove (iii) observe that in these cases we have, respectively,

$$\lambda\big(\mathcal{Z}_{\alpha_{i,m},\Psi} \mathcal{Z}^{-1}_{\alpha_{i+1,m},\Psi}\big)(\lambda + \alpha_{i,m})(\mathcal{Z}_{\alpha_{i+1,k},\Psi})(\lambda + \alpha_{i+1,m})\big(\mathcal{Z}^{-1}_{\alpha_{i,k},\Psi}\big) = (y_\lambda + 1) y_\lambda^{-1},$$

$$\lambda\big(\mathcal{Z}_{\alpha_{i,k+1},\Psi} \mathcal{Z}^{-1}_{\alpha_{j,k+1},\Psi}\big)(\lambda + \alpha_{i,k+1})(\mathcal{Z}_{\alpha_{j,k},\Psi})(\lambda + \alpha_{j,k+1})\big(\mathcal{Z}^{-1}_{\alpha_{i,k},\Psi}\big) = (x_\lambda + 1) x_\lambda^{-1}.$$

The relations now follow easily from the above and (5.17b), (5.17d) with $x_\lambda = 0$ (respectively, (5.17b) and (5.17c) with $y_\lambda = 0$). Finally, in the last case $\Pi_\lambda(\alpha_{i,k+1}, \alpha_{i+1,k}) \notin \bigwedge^2 \mathfrak{n}^+_\Psi$, hence the unique path $\lambda \leftarrow \lambda + \alpha_{i,k+1} \leftarrow \lambda + \eta$ is not a relation. $\square$

5.8. Retain the notations and the assumptions of 5.4. Then by Proposition 5.4, a connected subalgebra of $\mathbf{T}_\Psi^{\mathfrak g}$ is isomorphic to the path algebra of the quiver $\Gamma_a(\boldsymbol{m},\boldsymbol{n})$ for some $\boldsymbol{m}\in(\mathbf{Z}_+\cup\{+\infty\})^r$, $\boldsymbol{n}\in(\mathbf{Z}_+\cup\{+\infty\})^s$ and $-|\boldsymbol{n}|\leqslant a\leqslant|\boldsymbol{m}|$. However, this isomorphism looses some information which is necessary for describing relations in $\mathbf{S}_\Psi^{\mathfrak g}$, since the latter depend on $\mu(\mathcal{H}_{i_p,i_{p'}-1})$, $\mu(\mathcal{H}_{j_q+1,j_{q'}})$. Given $\lambda\in P^+$, set

$$z(\lambda)_p^-=\lambda(\mathcal{H}_{i_p+1,i_{p+1}-2})+2,\qquad z_q^+(\lambda)=\lambda(\mathcal{H}_{j_q+2,j_{q+1}-1})+2.$$

These parameters are obviously constant on connected components of Δ_Ψ and can take arbitrary integer values $\geqslant 2$. Let $(\boldsymbol{x},\boldsymbol{y})=((x_1,\dots,x_r),(y_1,\dots,y_s))$ be the image of $\mu\in\Delta_\Psi[\lambda]_0$ under the isomorphism of quivers constructed in Proposition 5.4. Then we have

$$\mu(\mathcal{H}_{i_p,i_{p+1}-1})=x_p+m_{p+1}-x_{p+1}+z(\lambda)_p^-,\qquad \mu(\mathcal{H}_{j_q+1,j_{q+1}})=n_q-y_q+y_{q+1}+z(\lambda)_q^+$$

and

$$M_{p,p'}(\boldsymbol{x}):=\mu(\mathcal{H}_{i_p,i_{p'}-1})=x_p-x_{p'}+\sum_{k=p}^{p'-1}\big(m_{k+1}+z(\lambda)_k^-\big)+p'-p-1,\quad 1\leqslant p<p'\leqslant r,$$

$$N_{q,q'}(\boldsymbol{y}):=\mu(\mathcal{H}_{j_q+1,j_{q'}})=y_{q'}-y_q+\sum_{k=q}^{q'-1}\big(n_k+z(\lambda)_k^+\big)+q'-q-1,\quad 1\leqslant q<q'\leqslant s.$$

Thus, the isomorphism of the connected subalgebra of $\mathbf{T}_\Psi^{\mathfrak g}$ corresponding to $\Delta_\Psi[\lambda]$, $\lambda\in P^+$ onto $\mathbf{C}\Gamma_a(\boldsymbol{m},\boldsymbol{n})$ provided by Proposition 5.4 induces the following relations on $\Gamma_a(\boldsymbol{m},\boldsymbol{n})$. First, for all $1\leqslant p\leqslant r$, $1\leqslant q<q'\leqslant s$ and for all $(\boldsymbol{x},\boldsymbol{y})$ such that $(\boldsymbol{x}+2\boldsymbol{e}_p^{(r)},\boldsymbol{y}+\boldsymbol{e}_q^{(s)}+\boldsymbol{e}_{q'}^{(s)})\in\Gamma_a(\boldsymbol{m},\boldsymbol{n})_0$, we have a commutativity relation

$$\begin{aligned}&\big((\boldsymbol{x},\boldsymbol{y})\leftarrow\big(\boldsymbol{x}+\boldsymbol{e}_p^{(r)},\boldsymbol{y}+\boldsymbol{e}_q^{(s)}\big)\leftarrow\big(\boldsymbol{x}+2\boldsymbol{e}_p^{(r)},\boldsymbol{y}+\boldsymbol{e}_q^{(s)}+\boldsymbol{e}_{q'}^{(s)}\big)\big)\\&\quad-\big((\boldsymbol{x},\boldsymbol{y})\leftarrow\big(\boldsymbol{x}+\boldsymbol{e}_p^{(r)},\boldsymbol{y}+\boldsymbol{e}_{q'}^{(s)}\big)\leftarrow\big(\boldsymbol{x}+2\boldsymbol{e}_p^{(r)},\boldsymbol{y}+\boldsymbol{e}_q^{(s)}+\boldsymbol{e}_{q'}^{(s)}\big)\big).\end{aligned}$$

Similarly, for all $1\leqslant p<p'\leqslant r$ and for all $1\leqslant q\leqslant s$ such that $(\boldsymbol{x}+\boldsymbol{e}_p^{(r)}+\boldsymbol{e}_{p'}^{(r)},\boldsymbol{y}+2\boldsymbol{e}_q^{(s)})\in\Gamma_a(\boldsymbol{m},\boldsymbol{n})_0$, we have the commutativity relation

$$\begin{aligned}&\big((\boldsymbol{x},\boldsymbol{y})\leftarrow\big(\boldsymbol{x}+\boldsymbol{e}_p^{(r)},\boldsymbol{y}+\boldsymbol{e}_q^{(s)}\big)\leftarrow\big(\boldsymbol{x}+\boldsymbol{e}_p^{(r)}+\boldsymbol{e}_{p'}^{(r)},\boldsymbol{y}+2\boldsymbol{e}_q^{(s)}\big)\big)\\&\quad-\big((\boldsymbol{x},\boldsymbol{y})\leftarrow\big(\boldsymbol{x}+\boldsymbol{e}_{p'}^{(r)},\boldsymbol{y}+\boldsymbol{e}_q^{(s)}\big)\leftarrow\big(\boldsymbol{x}+\boldsymbol{e}_p^{(r)}+\boldsymbol{e}_{p'}^{(r)},\boldsymbol{y}+2\boldsymbol{e}_q^{(s)}\big)\big).\end{aligned}$$

Finally, for all $1\leqslant p<p'\leqslant r$, $1\leqslant q<q'\leqslant s$, let $\boldsymbol{x}'=\boldsymbol{x}+\boldsymbol{e}_p^{(r)}+\boldsymbol{e}_{p'}^{(r)}$, $\boldsymbol{y}'=\boldsymbol{y}+\boldsymbol{e}_q^{(s)}+\boldsymbol{e}_{q'}^{(s)}$. Assume that $(\boldsymbol{x}',\boldsymbol{y}')\in\Gamma_a(\boldsymbol{m},\boldsymbol{n})_0$. If $M_{p,p'}(\boldsymbol{x})\neq N_{q,q'}(\boldsymbol{y})$, we have

$$\begin{aligned}&\big(M_{p,p'}(\boldsymbol{x})+1\big)\big(N_{q,q'}(\boldsymbol{y})+2\big)\big((\boldsymbol{x},\boldsymbol{y})\leftarrow\big(\boldsymbol{x}+\boldsymbol{e}_p^{(r)},\boldsymbol{y}+\boldsymbol{e}_q^{(s)}\big)\leftarrow(\boldsymbol{x}',\boldsymbol{y}')\big)\\&\quad-\big(M_{p,p'}(\boldsymbol{x})+2\big)\big(N_{q,q'}(\boldsymbol{y})+1\big)\big((\boldsymbol{x},\boldsymbol{y})\leftarrow\big(\boldsymbol{x}+\boldsymbol{e}_{p'}^{(r)},\boldsymbol{y}+\boldsymbol{e}_{q'}^{(s)}\big)\leftarrow(\boldsymbol{x}',\boldsymbol{y}')\big)\\&\quad-\big(M_{p,p'}(\boldsymbol{x})-N_{q,q'}(\boldsymbol{y})\big)\big((\boldsymbol{x},\boldsymbol{y})\leftarrow\big(\boldsymbol{x}+\boldsymbol{e}_p^{(r)},\boldsymbol{y}+\boldsymbol{e}_{q'}^{(s)}\big)\leftarrow(\boldsymbol{x}',\boldsymbol{y}')\big)\end{aligned}$$

and

$$\begin{aligned}&M_{p,p'}(\boldsymbol{x})\big(N_{q,q'}(\boldsymbol{y})+1\big)\big((\boldsymbol{x},\boldsymbol{y})\leftarrow\big(\boldsymbol{x}+\boldsymbol{e}_p^{(r)},\boldsymbol{y}+\boldsymbol{e}_q^{(s)}\big)\leftarrow(\boldsymbol{x}',\boldsymbol{y}')\big)\\&\quad-\big(M_{p,p'}(\boldsymbol{x})+1\big)N_{q,q'}(\boldsymbol{y})\big((\boldsymbol{x},\boldsymbol{y})\leftarrow\big(\boldsymbol{x}+\boldsymbol{e}_{p'}^{(r)},\boldsymbol{y}+\boldsymbol{e}_{q'}^{(s)}\big)\leftarrow(\boldsymbol{x}',\boldsymbol{y}')\big)\\&\quad-\big(M_{p,p'}(\boldsymbol{x})-N_{q,q'}(\boldsymbol{y})\big)\big((\boldsymbol{x},\boldsymbol{y})\leftarrow\big(\boldsymbol{x}+\boldsymbol{e}_{p'}^{(r)},\boldsymbol{y}+\boldsymbol{e}_q^{(s)}\big)\leftarrow(\boldsymbol{x}',\boldsymbol{y}')\big).\end{aligned}$$

Finally, if $M_{p,p'}(\boldsymbol{x})=N_{q,q'}(\boldsymbol{y})$, we have

$$\big((\boldsymbol{x},\boldsymbol{y})\leftarrow\big(\boldsymbol{x}+\boldsymbol{e}_p^{(r)},\boldsymbol{y}+\boldsymbol{e}_q^{(s)}\big)\leftarrow(\boldsymbol{x}',\boldsymbol{y}')\big)-\big((\boldsymbol{x},\boldsymbol{y})\leftarrow\big(\boldsymbol{x}+\boldsymbol{e}_{p'}^{(r)},\boldsymbol{y}+\boldsymbol{e}_{q'}^{(s)}\big)\leftarrow(\boldsymbol{x}',\boldsymbol{y}')\big)$$

and

$$\begin{aligned}&2\big((\boldsymbol{x},\boldsymbol{y})\leftarrow\big(\boldsymbol{x}+\boldsymbol{e}_p^{(r)},\boldsymbol{y}+\boldsymbol{e}_q^{(s)}\big)\leftarrow(\boldsymbol{x}',\boldsymbol{y}')\big)-M_{p,p'}(\boldsymbol{x})\big((\boldsymbol{x},\boldsymbol{y})\leftarrow\big(\boldsymbol{x}+\boldsymbol{e}_p^{(r)},\boldsymbol{y}+\boldsymbol{e}_{q'}^{(s)}\big)\leftarrow(\boldsymbol{x}',\boldsymbol{y}')\big)\\&\quad+\big(M_{p,p'}(\boldsymbol{x})+2\big)\big((\boldsymbol{x},\boldsymbol{y})\leftarrow\big(\boldsymbol{x}+\boldsymbol{e}_{p'}^{(r)},\boldsymbol{y}+\boldsymbol{e}_q^{(s)}\big)\leftarrow(\boldsymbol{x}',\boldsymbol{y}')\big).\end{aligned}$$

Note that the coefficients in these relations, and in particular their genericity, depend on a family of $(r+s)$ positive integer parameters $z_p^-(\lambda)$, $z_q^+(\lambda)$, $1\leqslant p\leqslant r$, $1\leqslant q\leqslant s$ which are independent of $\boldsymbol{m}$, $\boldsymbol{n}$. The resulting algebra is Koszul and has global dimension at most rs. It is finite dimensional if and only if $i_1>1$ and $j_s<\ell$.

6. Type C_ℓ, $\ell\geqslant 2$

6.1. Let $\beta_{i,j}=\beta_{j,i}=\alpha_{i,\ell-1}+\alpha_{j,\ell-1}+\alpha_\ell$, $1\leqslant i\leqslant j<\ell$ and $\beta_{\ell,\ell}=\alpha_\ell$. In particular, $\beta_{1,1}=\theta$. The roots $\alpha_{i,j}$ and $\beta_{i,j}$, $i<j$ are short while the roots $\beta_{i,i}$, $i\in I$ are long and

$$R^+=\{\alpha_{i,j}\colon 1\leqslant i\leqslant j<\ell\}\cup\{\beta_{i,j}\colon i\leqslant j\in I\}.$$

In terms of fundamental weights, $\beta_{i,j}=\varpi_i+\varpi_j-\varpi_{i-1}-\varpi_{j-1}$, where we set as before $\varpi_0=0$.

Let Ψ be an extremal set of positive roots. Our first observation is that $\alpha_{i,j}\notin\Psi$ for all $1\leqslant i\leqslant j<\ell$ since $2\alpha_{i,j}=\beta_{i,i}-\beta_{j+1,j+1}$ and hence if $\alpha_{i,j}\in\Psi$ we get a contradiction by Corollary 1.1. Furthermore, since $2\beta_{i,j}=\beta_{i,i}+\beta_{j,j}$ we conclude by Corollary 1.1 that $\beta_{i,j}\in\Psi$ if and only if $\beta_{i,i},\beta_{j,j}\in\Psi$. From this observation, it is immediate that all extremal sets in R^+ are of the form $\Psi(i_1,\dots,i_k):=\{\beta_{i_r,i_s}\colon 1\leqslant r\leqslant s\leqslant k\}$, $1\leqslant i_1<\cdots<i_k\leqslant\ell$, $1\leqslant k\leqslant\ell$ (see also [7]).

Since $\varepsilon(\beta_{i,j})=\varpi_{i-1}+\varpi_{j-1}$, we immediately obtain the following

Lemma. *Let $\beta_{i,j}\in\Psi$, $i<j\in I$. Then for all $\lambda\in P^+$, $\lambda\leftarrow\lambda+\beta_{i,j}\in(\Delta_\Psi)_1$ if and only if $\lambda(h_{i-1}),\lambda(h_{j-1})>0$. Furthermore, $\lambda\leftarrow\lambda+\beta_{i,i}\in(\Delta_\Psi)_1$ if and only if $\lambda(h_{i-1})>1$.*

6.2. We now proceed to describe all paths of length 2 in Δ_Ψ. Let $\eta\in\Psi+\Psi$. It follows from 6.1 that apart from the trivial case $\eta=2\beta_{i,i}$, $\beta_{i,i}\in\Psi$, we have four cases to consider.

(C1) Assume that $i<j\in I$. Let $\eta=\beta_{i,i}+\beta_{i,j}$. Then $\Delta_\Psi(\lambda,\lambda+\eta)=\emptyset$ unless $\lambda(h_{i-1})>2$ and

$$\Delta_\Psi(\lambda,\lambda+\eta)=\begin{cases}\{(\lambda\leftarrow\lambda+\beta_{i,i}\leftarrow\lambda+\eta),(\lambda\leftarrow\lambda+\beta_{i,j}\leftarrow\lambda+\eta)\}, & \lambda(h_{j-1})>0,\\ \{(\lambda\leftarrow\lambda+\beta_{i,i}\leftarrow\lambda+\eta)\}, \quad i=j-1,\ \lambda(h_{j-1})=0.\end{cases}$$

Similarly, if $\eta=\beta_{i,j}+\beta_{j,j}$, $\Delta_\Psi(\lambda,\lambda+\eta)=\emptyset$ unless $\lambda(h_{j-1})>1$ and $\lambda(h_{i-1})>0$. Then

$$\Delta_\Psi(\lambda,\lambda+\eta)=\begin{cases}\{(\lambda\leftarrow\lambda+\beta_{i,j}\leftarrow\lambda+\eta),(\lambda\leftarrow\lambda+\beta_{j,j}\leftarrow\lambda+\eta)\}, & \lambda(h_{j-1})>2,\\ \{(\lambda\leftarrow\lambda+\beta_{i,j}\leftarrow\lambda+\eta)\}, \quad i=j-1,\ \lambda(h_{j-1})=2.\end{cases}$$

(C2) Let $\eta = \beta_{i,i} + \beta_{j,j} = 2\beta_{i,j}$, $i \in j \in I$. Then $\Delta_\Psi(\lambda, \lambda+\eta) = \emptyset$ unless $\lambda(h_{i-1}) > 1$. If $\lambda(h_{j-1}) > 1$ then $\Delta_\Psi(\lambda, \lambda+\eta)$ contains all three possible paths. Otherwise, $\Delta_\Psi(\lambda, \lambda+\eta)$ is empty unless $i = j-1$. If $i = j-1$ we have

$$\Delta_\Psi(\lambda, \lambda+\eta) = \begin{cases} \{(\lambda \leftarrow \lambda + \beta_{i,i} \leftarrow \lambda + \eta), (\lambda \leftarrow \lambda + \beta_{i,j} \leftarrow \lambda + \eta)\}, & \lambda(h_{j-1}) = 1, \\ \{(\lambda \leftarrow \lambda + \beta_{i,i} \leftarrow \lambda + \eta)\}, & \lambda(h_{j-1}) = 0. \end{cases}$$

(C3) Assume that $i < j < k$. First, let $\eta = \beta_{i,i} + \beta_{j,k} = \beta_{i,j} + \beta_{i,k}$. Then $\Delta_\Psi(\lambda, \lambda+\eta) = \emptyset$ unless $\lambda(h_{i-1}) > 1$ and $\lambda(h_{j-1}) + \lambda(h_{k-1}) > 0$. If $\lambda(h_{j-1}), \lambda(h_{k-1}) > 0$ then we have all four possible paths. Otherwise,

$$\Delta_\Psi(\lambda, \lambda+\eta) = \begin{cases} \{(\lambda \leftarrow \lambda + \beta_{i,i} \leftarrow \lambda + \eta),\ (\lambda \leftarrow \lambda + \beta_{i,k} \leftarrow \lambda + \eta)\}, \\ \quad i = j-1,\ \lambda(h_{j-1}) = 0, \\ \{(\lambda \leftarrow \lambda + \beta_{i,j} \leftarrow \lambda + \eta)\}, \quad j = k-1, \lambda(h_{k-1}) = 0. \end{cases}$$

Next, let $\eta = \beta_{j,j} + \beta_{i,k} = \beta_{i,j} + \beta_{j,k}$. Then $\Delta_\Psi(\lambda, \lambda+\eta) = \emptyset$ unless $\lambda(h_{i-1}) > 0$ and $\lambda(h_{j-1}) + \lambda(h_{k-1}) > 1$. If $\lambda(h_{j-1}) > 1$ and $\lambda(h_{k-1}) > 0$ then we have all possible paths. Otherwise,

$$\Delta_\Psi(\lambda, \lambda+\eta) = \begin{cases} \{(\lambda \leftarrow \lambda + \beta_{i,k} \leftarrow \lambda + \eta)\}, \quad i = j-1, \lambda(h_{j-1}) = 1, \\ \{(\lambda \leftarrow \lambda + \beta_{j,j} \leftarrow \lambda + \eta),\ (\lambda \leftarrow \lambda + \beta_{i,j} \leftarrow \lambda + \eta)\}, \\ \quad j = k-1, \lambda(h_{k-1}) = 0. \end{cases}$$

Finally, if $\eta = \beta_{k,k} + \beta_{i,j} = \beta_{i,k} + \beta_{j,k}$, then $\Delta_\Psi(\lambda, \lambda+\eta) = \emptyset$ unless $\lambda(h_{i-1}) > 0$ and $\lambda(h_{j-1}) + \lambda(h_{k-1}) > 1$. If $\lambda(h_{k-1}) > 1$ and $\lambda(h_{j-1}) > 0$ then we have all possible paths. Otherwise,

$$\Delta_\Psi(\lambda, \lambda+\eta) = \begin{cases} \{(\lambda \leftarrow \lambda + \beta_{i,k} \leftarrow \lambda + \eta)\}, \quad i = j-1, \lambda(h_{j-1}) = 0, \\ \{(\lambda \leftarrow \lambda + \beta_{i,j} \leftarrow \lambda + \eta),\ (\lambda \leftarrow \lambda + \beta_{j,k} \leftarrow \lambda + \eta)\}, \\ \quad j = k-1, \lambda(h_{k-1}) = 1. \end{cases}$$

(C4) Finally, let $i < j < k < l \in I$, $\eta = \beta_{i,j} + \beta_{k,l} = \beta_{i,k} + \beta_{j,l} = \beta_{i,l} + \beta_{j,k}$. Then $\Delta_\Psi(\lambda, \lambda+\eta) = \emptyset$ unless

$$\lambda(h_{i-1}), \lambda(h_{j-1}) + \lambda(h_{k-1}), \lambda(h_{k-1}) + \lambda(h_{l-1}) > 0.$$

If $\lambda(h_{r-1}) > 0$, $r \in \{i, j, k, l\}$ we have all possible paths. Furthermore, if $\lambda(h_{k-1}) = 0$ we must have $j = k-1$ and

$$\Delta_\Psi(\lambda, \lambda+\eta) = \{(\lambda \leftarrow \lambda + \beta_{i,j} \leftarrow \eta), (\lambda \leftarrow \lambda + \beta_{j,l} \leftarrow \lambda + \eta)\}.$$

Finally, if $\lambda(h_{k-1}) > 0$ we have

$$\Delta_\Psi(\lambda, \lambda+\eta) = \begin{cases} \{(\lambda \leftarrow \lambda + \beta_{i,k} \leftarrow \lambda + \eta),\ (\lambda \leftarrow \lambda + \beta_{i,l} \leftarrow \lambda + \eta)\}, \\ \quad i = j-1,\ \lambda(h_{j-1}) = 0,\ \lambda(h_{l-1}) > 0, \\ \{(\lambda \leftarrow \lambda + \beta_{i,k} \leftarrow \lambda + \eta),\ (\lambda \leftarrow \lambda + \beta_{j,k} \leftarrow \lambda + \eta)\}, \\ \quad k = l-1, \lambda(h_{j-1}) > 0,\ \lambda(h_{l-1}) = 0, \\ \{(\lambda \leftarrow \lambda + \beta_{i,k} \leftarrow \lambda + \eta)\}, \quad k = l-1,\ i = j-1,\ \lambda(h_{j-1}) = \lambda(h_{l-1}) = 0. \end{cases}$$

The cases (C1)–(C2) (respectively, (C3), (C4)) occur if $\Psi \supset \Psi(i, j)$ (respectively, $\Psi \supset \Psi(i, j, k)$, $\Psi \supset \Psi(i, j, k, l)$). In particular, we obtain the following

Lemma. *The set $\Psi(i_1, \dots, i_k)$ is regular if and only if $i_{r+1} \neq i_r + 1$ for all $1 \leqslant r < k$.*

6.3. Retain the notations of 1.8. A straightforward induction on r shows that

$$\#\Xi_0(\boldsymbol{m})_0 = \left\lceil \frac{1}{2}(m_1+1)\cdots(m_r+1)\right\rceil, \qquad \#\Xi_1(\boldsymbol{m})_0 = \left\lfloor \frac{1}{2}(m_1+1)\cdots(m_r+1)\right\rfloor. \tag{6.1}$$

It is immediate that $\Xi_a(\boldsymbol{m}) \cong \Xi_a(\boldsymbol{m}')$ if $\boldsymbol{m}'$ is a permutation of $\boldsymbol{m}$ or is obtained from $\boldsymbol{m}$ by adding or removing zeroes. Clearly, $\Xi_a((m)) \cong \Xi_{a'}((m'))$, $a, a' \in \{0,1\}$ if and only if $\lfloor (m-a)/2\rfloor = \lfloor (m'-a')/2\rfloor$. Note also that $\Xi_0((1,1)) \cong \Xi_0((2)) \cong \Xi_0((3)) \cong \Xi_1((3)) \cong \Xi_1((4))$.

Proposition. *Let $\boldsymbol{m} = (m_1,\dots,m_r) \in \mathbf{Z}_+^r$, $m_1 \geqslant \cdots \geqslant m_r > 0$, $r > 1$. The quivers $\Xi_0(\boldsymbol{m})$ and $\Xi_1(\boldsymbol{m})$, $\boldsymbol{m} \neq (1,1)$ are connected and pairwise non-isomorphic. Furthermore, $\Xi_0(\boldsymbol{m}) \cong \Xi_1(\boldsymbol{m})^{op}$ if and only if $|\boldsymbol{m}|$ is odd.*

Proof. Observe that every vertex in $\Xi_a(\boldsymbol{m})$ is connected to a sink. Clearly, $\mathbf{0} = (0,\dots,0)$ is the unique sink in $\Xi_0(\boldsymbol{m})$, hence $\Xi_0(\boldsymbol{m})$ is connected. On the other hand, the $\boldsymbol{e}_j := \boldsymbol{e}_j^{(r)}$ are the only sinks in $\Xi_1(\boldsymbol{m})$. If $r = 2$ and $\boldsymbol{m} \neq (1,1)$, then $m_1 > 1$ and so we have $\boldsymbol{e}_1 \leftarrow 2\boldsymbol{e}_1 + \boldsymbol{e}_2 \rightarrow \boldsymbol{e}_2$. If $r > 2$ then for all $1 \leqslant i < j < k \leqslant r$, we have

$$\begin{array}{ccccc} \boldsymbol{e}_i & \longleftarrow & \boldsymbol{e}_i+\boldsymbol{e}_j+\boldsymbol{e}_k & \longrightarrow & \boldsymbol{e}_j \\ & & \big\downarrow & & \\ & & \boldsymbol{e}_k & & \end{array}$$

Thus, all sinks in $\Xi_1(\boldsymbol{m})$, $\boldsymbol{m} \neq (1,1)$ lie in the same connected component hence $\Xi_1(\boldsymbol{m})$ is connected and $\Xi_1(\boldsymbol{m}) \ncong \Xi_0(\boldsymbol{m}')$ for all $\boldsymbol{m}' = (m_1',\dots,m_k')$, $m_1' \geqslant \cdots \geqslant m_k' > 0$, $k > 1$.

Given $\boldsymbol{m} = (m_1,\dots,m_r)$, $m_1 \geqslant \cdots \geqslant m_r$, let $n_p(\boldsymbol{m}) = \#\{j\colon m_j = p\}$, $p > 0$ and $\ell(\boldsymbol{m}) = \sum_{p>0} n_p(\boldsymbol{m})$. Suppose that $\Xi_0(\boldsymbol{m})$ is isomorphic to $\Xi_0(\boldsymbol{m}')$, $\boldsymbol{m}' = (m_1',\dots,m_k')$, $m_1' \geqslant \cdots \geqslant m_k'$. Suppose first that $k = \ell(\boldsymbol{m}') > \ell(\boldsymbol{m}) = r$. Then $\#\mathbf{0}^- = r(r+1)/2 - n_1(\boldsymbol{m})$. Since $\mathbf{0}$ (respectively, $\mathbf{0}'$) is the unique sink in $\Xi_0(\boldsymbol{m})$ (respectively, in $\Xi_0(\boldsymbol{m}')$), we must have $n_1(\boldsymbol{m}') = k(k+1)/2 - r(r+1)/2 + n_1(\boldsymbol{m})$. Since $n_1(\boldsymbol{m}') \leqslant k$, this implies that $n_1(\boldsymbol{m}) \leqslant r(r+1)/2 - k(k-1) \leqslant 0$ with the equality if and only if $k = r+1$ which in turn implies that $n_1(\boldsymbol{m}') = k$. Then (6.1) implies that $\#\Xi(\boldsymbol{m})_0 \geqslant 3^r/2$, while $\#\Xi(\boldsymbol{m}')_0 = 2^r$. Since $r > 1$, it follows that $\#\Xi(\boldsymbol{m})_0 > \#\Xi(\boldsymbol{m}')_0$ which is a contradiction. Thus, $k = r$ and $n_1(\boldsymbol{m}) = n_1(\boldsymbol{m}')$.

Furthermore, note that $\boldsymbol{x} \in \Xi(\boldsymbol{m})_0$ is a source if and only if $|\boldsymbol{x}| \geqslant |\boldsymbol{m}| - 1$. It follows that if $|\boldsymbol{m}| \in 2\mathbf{Z}_+$ then $\boldsymbol{m}$ is the unique source in $\Xi_0(\boldsymbol{m})$. Otherwise, the $\boldsymbol{m} - \boldsymbol{e}_j$, $1 \leqslant j \leqslant r$ are al the sources. Therefore, $|\boldsymbol{m}| = |\boldsymbol{m}'| \pmod 2$. Since the length of any path in $\Xi_0(\boldsymbol{m})$ from a source to the unique sink is $\lfloor |\boldsymbol{m}|/2\rfloor$, it follows that $|\boldsymbol{m}| = |\boldsymbol{m}'|$. Clearly, if $n_1(\boldsymbol{m}) \geqslant r-1$ or $m_j \leqslant 2$ for all $1 \leqslant j \leqslant r$, this implies that $\boldsymbol{m} = \boldsymbol{m}'$, so we may assume that $n_1(\boldsymbol{m}) < r-1$ and $m_j > 2$ for some $1 \leqslant j \leqslant r$.

Note that if $\boldsymbol{x} \in \Xi(\boldsymbol{m})_0$ satisfies $\#\boldsymbol{x}^+ = 1$ then either $\boldsymbol{x} = 2x\boldsymbol{e}_j$ for some $1 \leqslant j \leqslant r$, $0 \leqslant x \leqslant m_j/2$, or $\boldsymbol{x} = \boldsymbol{e}_j + \boldsymbol{e}_k$, $1 \leqslant j < k \leqslant r$. Given $1 \leqslant j \leqslant r$ with $m_j > 2$ consider a path

$$\mathbf{0} \leftarrow 2\boldsymbol{e}_j \leftarrow \cdots \leftarrow 2k\boldsymbol{e}_j, \quad k = \lfloor m_j/2\rfloor$$

in $\Xi_0(\boldsymbol{m})$. Then its image in $\Xi_0(\boldsymbol{m}')$ under our isomorphism of quivers must be

$$\mathbf{0} \leftarrow 2\boldsymbol{e}_{j'} \leftarrow \cdots \leftarrow 2k\boldsymbol{e}_{j'},$$

for some $1 \leqslant j' \leqslant r$ with $\lfloor m_{j'}/2\rfloor = \lfloor m_j/2\rfloor$. Furthermore, it is easy to check that

$$\boldsymbol{x} \in \Xi_0(\boldsymbol{m})_0, \qquad \#\boldsymbol{x}^+ \leqslant 3 \quad\Longrightarrow\quad \boldsymbol{x} = x_i\boldsymbol{e}_i + x_j\boldsymbol{e}_j, \quad 1 \leqslant i < j \leqslant r. \tag{6.2}$$

Suppose first that $r = 2$ and $n_1(\boldsymbol{m}) = 0$. Since $m_1 + m_2 = m_1' + m_2'$, we may assume, without loss of generality, that $m_1 > m_1'$. By the above, we must have $\lfloor m_1/2\rfloor = \lfloor m_1'/2\rfloor$ hence $m_1 = 2a+1$, $m_1' = 2a$,

$a \geqslant 1$ and so $m_2 = 2b$, $m_2' = 2b+1$, $b \geqslant 1$. Since $\#\Xi_0((m_1,m_2))_0 = \#\Xi_0((m_1',m_2'))_0$, we conclude that $a = b$, which is a contradiction since $m_1' \geqslant m_2'$.

Suppose now that $r > 2$. For $1 \leqslant i < j \leqslant s$ fixed let $\Xi_0^{i,j}(\boldsymbol{m})$ be the full subquiver of $\Xi_0(\boldsymbol{m})$ defined by $\{x_i\boldsymbol{e}_i + x_j\boldsymbol{e}_j\colon\ x_i \leqslant m_i,\ x_j \leqslant m_j,\ x_i + x_j \in 2\mathbf{Z}_+\}$. Clearly for all $\boldsymbol{x} \in \Xi_0^{i,j}(\boldsymbol{m})_0$ the set of direct successors of $\boldsymbol{x}$ in $\Xi_0(\boldsymbol{m})$ is contained in $\Xi_0^{i,j}(\boldsymbol{m})_0$, hence $\Xi_0^{i,j}(\boldsymbol{m})$ is a convex connected subquiver of $\Xi_0(\boldsymbol{m})$. It is clearly isomorphic to $\Xi_0((m_i,m_j))$. It follows from (6.2) that the isomorphism of quivers $\Xi_0(\boldsymbol{m}) \to \Xi_0(\boldsymbol{m}')$ induces an isomorphism of quivers $\Xi_0^{i,j}(\boldsymbol{m}) \to \Xi_0^{i',j'}(\boldsymbol{m}')$ for some $1 \leqslant i' < j' \leqslant r$ which by the $r = 2$ case implies that $m_i = m'_{i'}$, $m_j = m'_{j'}$. Therefore, $\boldsymbol{m} = \boldsymbol{m}'$.

Suppose that $\Xi_1(\boldsymbol{m}) \cong \Xi_1(\boldsymbol{m}')$. Since $\Xi_1(\boldsymbol{m})$ contains $\ell(\boldsymbol{m})$ sinks, it follows that $\ell(\boldsymbol{m}) = \ell(\boldsymbol{m}') = r$. Furthermore, we have

$$\left|\boldsymbol{e}_i^-\right| = \begin{cases} \binom{r}{2} - n_1(\boldsymbol{m}), & m_i = 1, \\ \binom{r}{2} + r - 1 - n_1(\boldsymbol{m}), & m_i = 2, \\ \binom{r}{2} + r - n_1(\boldsymbol{m}), & m_i > 2. \end{cases}$$

It follows that $n_p(\boldsymbol{m}) = n_p(\boldsymbol{m}')$, $p = 1, 2$. Since $\Xi_1(\boldsymbol{m})$ contains a unique source if $|\boldsymbol{m}|$ is odd and r sources otherwise, it follows that $|\boldsymbol{m}| = |\boldsymbol{m}'|$ (mod 2). Since the length of a path from a source to a sink is $(|\boldsymbol{m}| - 1)/2$ if $|\boldsymbol{m}|$ is odd and $|\boldsymbol{m}|/2 - 1$ if $|\boldsymbol{m}|$ is even, it follows that $|\boldsymbol{m}| = |\boldsymbol{m}'|$. Furthermore, note that $\boldsymbol{x} \in \Xi_1(\boldsymbol{m})$, $\#\boldsymbol{x}^+ \leqslant 3$ implies that $\boldsymbol{x} \in \Xi_1^{i,j}(\boldsymbol{m})$ for some $1 \leqslant i < j \leqslant r$ or $\boldsymbol{x} = \boldsymbol{e}_i + \boldsymbol{e}_j + \boldsymbol{e}_k$, $1 \leqslant i < j < k \leqslant r$. On the other hand, a vertex of the second type is connected to three sinks in $\Xi_1(\boldsymbol{m})$ by arrows, while a vertex of the first type can be connected to at most two sinks. Thus, we conclude as before that the image of $\Xi_1^{i,j}(\boldsymbol{m})$ under the isomorphism $\Xi_1(\boldsymbol{m}) \to \Xi_1(\boldsymbol{m}')$ is contained in $\Xi_1^{i',j'}(\boldsymbol{m}')$ for some $1 \leqslant i' < j' \leqslant r$. The rest of the argument is similar to that in the "even" case and is omitted.

To prove the last assertion, note that if $|\boldsymbol{m}|$ is odd, then at least one of the m_r is odd, hence $\#\Xi_0(\boldsymbol{m})_0 = \#\Xi_1(\boldsymbol{m})_0$ and the map $\Xi_0(\boldsymbol{m})_0 \to \Xi_1(\boldsymbol{m})_0$, $\boldsymbol{x} \mapsto \boldsymbol{m} - \boldsymbol{x}$, is a bijection. This map induces the desired isomorphism of quivers. Conversely, if $|\boldsymbol{m}|$ is even, then $\Xi_1(\boldsymbol{m})$ contains $\ell(\boldsymbol{m}) > 1$ sources. Since $\Xi_0(\boldsymbol{m})$ has a unique sink, $\Xi_0(\boldsymbol{m})$ and $\Xi_1(\boldsymbol{m})^{op}$ cannot be isomorphic. □

6.4. We can now describe all connected components of Δ_Ψ for Ψ regular.

Proposition. *Let $\Psi = \Psi(i_1, \dots, i_k)$, $1 \leqslant i_1 < \cdots < i_k \leqslant \ell$ and suppose that Ψ is regular. Let $\lambda \in P^+$ and assume that $|\lambda^+ \cup \lambda^-| > 0$. Then $\Delta_\Psi[\lambda]$ is isomorphic to the quiver $\Xi_a(\boldsymbol{m})$ where $\boldsymbol{m} = (m_1, \dots, m_k) \in (\mathbf{Z}_+ \cup \{+\infty\})^k$, $m_r = \lambda(h_{i_r-1}) + \lambda(h_{i_r})$, $1 \leqslant r \leqslant k$ and $a = \lambda(h_{i_1}) + \cdots + \lambda(h_{i_k})$ (mod 2).*

Proof. Let $J = \{i_r\colon\ 1 \leqslant r \leqslant n\} \cup \{i_r - 1\colon\ 1 \leqslant r \leqslant n\}$. Suppose that $\mu \in \Delta_\Psi[\lambda]_0$. Since $\Delta_\Psi[\lambda]_0 \subset (\lambda + \mathbf{Z}\Psi) \cap P^+$, we have

$$\mu(h_{i_r}) = \lambda(h_{i_r}) + \sum_{s=1}^{r-1} x_{s,r} + 2x_{r,r} + \sum_{s=r+1}^{k} x_{r,s},$$
$$\mu(h_{i_r-1}) = \lambda(h_{i_r-1}) - \sum_{s=1}^{r-1} x_{s,r} - 2x_{r,r} - \sum_{s=r+1}^{k} x_{r,s},$$

where $x_{p,q} \in \mathbf{Z}$, $1 \leqslant p \leqslant q \leqslant k$. It follows that

$$\mu(h_{i_r-1}) + \mu(h_{i_r}) = \lambda(h_{i_r-1}) + \lambda(h_{i_r}), \quad 1 \leqslant r \leqslant k, \qquad \mu(h_j) = \lambda(h_j), \quad j \notin J, \tag{6.3a}$$
$$\sum_{r=1}^{k} \mu(h_{i_r}) = \sum_{r=1}^{k} \lambda(h_{i_r}) \pmod 2. \tag{6.3b}$$

Let $S(\lambda)$ be the set of $\mu \in P^+$ satisfying these conditions. Then $\Delta_\Psi[\lambda]_0 \subset S(\lambda)$ and for all $\mu \in S(\lambda)$, $\mu^- \subset S(\lambda)$. Thus, $S(\lambda)$ defines a convex subquiver Γ of Δ_Ψ with $\Gamma_0 = S(\lambda)$ and $\Delta_\Psi[\lambda]$ is a full connected subquiver of Γ.

Let $m_r = \lambda(h_{i_r-1}) + \lambda(h_{i_r})$, $a = \lambda(h_{i_1}) + \cdots + \lambda(h_{i_k}) \pmod 2$. Then we have a bijective map

$$\Gamma_0 \to \Xi_a(\boldsymbol{m})_0,$$
$$\mu \mapsto \big(\mu(h_{i_1}), \ldots, \mu(h_{i_k})\big).$$

It is easy to see that this induces an isomorphism of quivers $\Gamma \to \Xi_a(\boldsymbol{m})$. To complete the argument, observe that the assumption that $|\lambda^+ \cup \lambda^-| > 0$ implies that we cannot have $a = 1$ and $m_r = \delta_{r,p} + \delta_{r,q}$ for some $1 \leqslant p < q \leqslant k$. Then $\Xi_a(\boldsymbol{m})$ is connected by Proposition 6.4. Therefore, Γ is connected hence $\Delta_\Psi[\lambda] = \Gamma$. □

6.5. Fix root vectors $e_{\beta_{i,j}} \in \mathfrak{g}_{\beta_{i,j}} \setminus \{0\}$, $1 \leqslant i \leqslant j \leqslant \ell$ so that

$$\begin{aligned}
&[e_i, e_{\beta_{j,k}}] = \delta_{i,j-1} e_{\beta_{i,k}} + \delta_{i,k-1}(1 + \delta_{i,j}) e_{\beta_{j,i}}, \quad j < k, \\
&[e_i, e_{\beta_{j,j}}] = \delta_{i,j-1} e_{\beta_{i,j}},
\end{aligned} \tag{6.4}$$

and

$$\begin{aligned}
&[f_i, e_{\beta_{j,k}}] = \delta_{i,j}(1 + \delta_{j+1,k}) e_{\beta_{j+1,k}} + \delta_{i,k} e_{\beta_{j,k+1}}, \quad j < k, \\
&[f_i, e_{\beta_{j,j}}] = \delta_{i,j} e_{\beta_{j,j+1}}.
\end{aligned} \tag{6.5}$$

For example, we can use the standard presentation of $\mathfrak{g}$ as the matrix Lie algebra $\mathfrak{sp}_{2\ell}$. The subalgebra $\mathfrak{g}_J$ of $\mathfrak{g}$ with $J = I \setminus \{\ell\}$ is of course a simple Lie algebra of type $A_{\ell-1}$. Note that $[e_\ell, e_{\beta_{i,j}}] = 0 = [f_\ell, e_{\beta_{i,j}}]$, $1 \leqslant i \leqslant j \leqslant \ell$. Due to this observation, we can perform our computations in $U(\mathfrak{g}_J)$.

6.6. Retain the notations of 4.3. Fix $1 \leqslant i \leqslant j < \ell$. Given any pair $1 \leqslant r \leqslant s$ such that $r \leqslant i + 1$, $s \leqslant j + 1$ set

$$\mathcal{U}_{r,s,i,j} = e_{s-1} \cdots e_r \mathcal{X}^-_{r,i} \mathcal{X}^-_{r,j}.$$

In particular, $\mathcal{U}_{r,r,i,j} = \mathcal{X}^-_{r,i}\mathcal{X}^-_{r,j} \in U(\mathfrak{b})$. Clearly, $\mathcal{U}_{r,s,i,j} \in U(\mathfrak{g})_{-\alpha_{r,i}-\alpha_{s,j}}$. We set $\mathcal{U}_{r,s,i,j} = 0$ if $r > i + 1$.

Lemma. *The elements $\mathcal{U}_{r,s,i,j}$ satisfy*

$$e_s \mathcal{U}_{r,s,i,j} = \mathcal{U}_{r,s+1,i,j}, \tag{6.6a}$$

$$e_r \mathcal{U}_{r,s,i,j} = (1 + \delta_{r+1,s}) \mathcal{U}_{r+1,s,i,j}(\mathcal{H}_{r,i} - \delta_{i,j})\mathcal{H}_{r,j} + U(\mathfrak{g})\mathfrak{n}^+, \quad r < s, \tag{6.6b}$$

$$e_k \mathcal{U}_{r,s,i,j} \in U(\mathfrak{g})\mathfrak{n}^+, \quad k \neq r, s. \tag{6.6c}$$

Proof. The first identity is obvious. To prove (6.6b) and (6.6c), we need to show first that

$$e_k \mathcal{U}_{r,r,i,j} \in U(\mathfrak{g}) e_k, \quad k \neq r, \tag{6.7a}$$

$$e_r^2 \mathcal{U}_{r,r,i,j} = 2\mathcal{U}_{r+1,r+1,i,j}(\mathcal{H}_{r,i} - \delta_{i,j})\mathcal{H}_{r,j} + U(\mathfrak{g}) e_r, \tag{6.7b}$$

$$e_{r-1} e_r \mathcal{U}_{r,r,i,j} \in U(\mathfrak{g})\mathfrak{n}^+. \tag{6.7c}$$

Using Lemma 4.3 we immediately deduce (6.7a) and the following identity

$$\mathcal{U}_{r,r+1,i,j} = e_r\mathcal{U}_{r,r,i,j} = \mathcal{X}^-_{r+1,i}\mathcal{X}^-_{r,j}(\mathcal{H}_{r,i} - 1 - \delta_{i,j}) + \psi_{\alpha_r}\big(\mathcal{X}^-_{r,i}\big)\mathcal{X}^-_{r+1,j}\mathcal{H}_{r,j} + U(\mathfrak{g})e_r. \tag{6.8}$$

Then (6.7b) and (6.7c) are easy to obtain using (4.2a). To prove (6.6b) for $s > r$, note that the case $s = r+1$ is immediate from (6.7b). Assume that $s > r+1$. Clearly e_r commutes with the e_t, $r+1 < t \leqslant s-1$. Since $(\operatorname{ad} e_a)^2 e_b = 0$ for all $1 \leqslant a \neq b < \ell$ with $|a-b| = 1$, we have in $U(\mathfrak{g})$,

$$e_a^2 e_b - 2e_a e_b e_a + e_b e_a^2 = 0. \tag{6.9}$$

Therefore,

$$e_r\mathcal{U}_{r,s,i,j} = e_{s-1}\cdots e_{r+2}e_r e_{r+1}e_r\mathcal{U}_{r,r,i,j} = e_{s-1}\cdots e_{r+1}\mathcal{U}_{r+1,r+1,i,j}(\mathcal{H}_{r,i} - \delta_{i,j})\mathcal{H}_{r,j} + U(\mathfrak{g})\mathfrak{n}^+,$$

where we used (6.7a) and (6.7b). To prove (6.6c) for $s > r$, note that for $k < r-1$ or $k > s$ this is an immediate consequence of (6.7a). Thus, if $s = r+1$ there is nothing to do. Assume that $s > r+1$. If $k = r-1$, the assertion follows from (6.7c). If $k = s-1$, it follows from (6.9) that

$$e_{s-1}\mathcal{U}_{r,s,i,j} = e_{s-1}^2 e_{s-2}\cdots e_r\mathcal{U}_{r,r,i,j} = -e_{s-2}\cdots e_r e_{s-1}^2\mathcal{U}_{r,r,i,j} + 2e_{s-1}\cdots e_r e_{s-1}\mathcal{U}_{r,r,i,j}$$

which is contained in $U(\mathfrak{g})\mathfrak{n}^+$ by (6.7a). If $r < k < s-1$ we can write, using (6.9)

$$\begin{aligned} e_k\mathcal{U}_{r,s,i,j} &= e_{s-1}\cdots e_k e_{k+1}e_k e_{k-1}\cdots e_r\mathcal{U}_{r,r,i,j} \\ &= \frac{1}{2}e_{s-1}\cdots e_{k+1}e_k^2 e_{k-1}\cdots e_r\mathcal{U}_{r,r,i,j} + \frac{1}{2}e_{s-1}\cdots e_k^2 e_{k+1}e_{k-1}\cdots e_r\mathcal{U}_{r,r,i,j}. \end{aligned}$$

The second term is in $U(\mathfrak{g})\mathfrak{n}^+$ by (6.7a) since e_{k+1} commutes with the e_t, $t \leqslant k-1$. Applying (6.9) again, we obtain

$$e_k\mathcal{U}_{r,s,i,j} = \left(e_{s-1}\cdots e_r e_k - \frac{1}{2}e_{s-1}\cdots e_{k+1}e_{k-1}\cdots e_r e_k^2\right)\mathcal{U}_{r,r,i,j} + U(\mathfrak{g})\mathfrak{n}^+ \in U(\mathfrak{g})\mathfrak{n}^+,$$

where we used (6.7a). □

6.7. For our purposes, we need to find the projection $\bar{\mathcal{U}}_{r,s,i,j}$ of $\mathcal{U}_{r,s,i,j}$ onto $U(\mathfrak{h})$.

Lemma. *Let $1 \leqslant i \leqslant j < \ell$ and suppose that $r < s$, $s \leqslant j$, $r \leqslant i$. Then*

$$\begin{aligned} \bar{\mathcal{U}}_{r,s,i,j} &= \mathcal{X}^-_{s,i}\mathcal{X}^-_{r,j}\prod_{t=r}^{s-1}(\mathcal{H}_{t,i} - \delta_{r,t} - \delta_{i,j}) + \psi_{\alpha_{r,s-1}}\big(\mathcal{X}^-_{r,i}\big)\mathcal{X}^-_{s,j}\prod_{t=r}^{s-1}\mathcal{H}_{t,j} \\ &\quad - \mathcal{X}^-_{s,i}\sum_{t=r+1}^{s-1}\psi_{\alpha_{r,t-1}}\big(\mathcal{X}^-_{r,t-1,i}\big)\mathcal{X}^-_{t,j}\prod_{p=r}^{t-1}\mathcal{H}_{p,j}\prod_{p=t+1}^{s-1}(\mathcal{H}_{p,i} - \delta_{i,j}). \end{aligned} \tag{6.10}$$

In particular,

$$\bar{\mathcal{U}}_{i+1,s,i,j} = \mathcal{X}^-_{s,j}\prod_{t=i+1}^{s-1}\mathcal{H}_{t,j}, \quad i+1 \leqslant s \leqslant j+1 \tag{6.11}$$

and $\mathcal{U}_{r,s,i,j}(\mu) := \pi_\mu(\bar{\mathcal{U}}_{r,s,i,j})$ *is given by the following formulae*

$$\mathcal{U}_{r,r,i,j}(\mu) = \mathcal{X}^-_{r,i}(\mu - \varpi_j)\mathcal{X}^-_{r,j}(\mu),$$

$$\mathcal{U}_{r,s,i,j}(\mu) = \mathcal{X}^-_{s,i}(\mu - \varpi_j)\mathcal{X}^-_{r,j}(\mu)\prod_{t=r}^{s-1}\big(\mu(\mathcal{H}_{t,i}) - \delta_{r,t} - \delta_{i,j}\big) + \mathcal{X}^-_{r,i}(\mu - \varpi_j)\mathcal{X}^-_{s,j}(\mu)\prod_{t=r}^{s-1}\mu(\mathcal{H}_{t,j})$$
$$- \mathcal{X}^-_{s,i}(\mu - \varpi_j)\sum_{t=r+1}^{s-1}\mathcal{X}^-_{r,t-1,i}(\mu - \varpi_j)\mathcal{X}^-_{t,j}(\mu)\prod_{p=r}^{t-1}\mu(\mathcal{H}_{p,j})\prod_{p=t+1}^{s-1}\big(\mu(\mathcal{H}_{p,i}) - \delta_{i,j}\big),$$
$$r < s.$$

Proof. The elements $\bar{\mathcal{U}}_{r,s,i,j}$ are uniquely determined by the conditions that $\bar{\mathcal{U}}_{r,s,i,j} = \mathcal{U}_{r,s,i,j} + U(\mathfrak{g})\mathfrak{n}^+$ and $\bar{\mathcal{U}}_{r,s,i,j} \in U(\mathfrak{b})$. The argument is by induction on $s - r$, the induction base being (6.8). To prove the inductive step, note that by Lemma 4.3 and the induction hypothesis we have

$$\mathcal{U}_{r,s+1,i,j} = e_s\mathcal{U}_{r,s,i,j}$$
$$= \mathcal{X}^-_{s+1,i}\mathcal{X}^-_{r,j}\prod_{k=r}^{s}(\mathcal{H}_{k,i} - \delta_{r,k} - \delta_{i,j}) + e_s\psi_{\alpha_{r,s-1}}\big(\mathcal{X}^-_{r,i}\big)\mathcal{X}^-_{s,j}\prod_{k=r}^{s-1}\mathcal{H}_{k,j}$$
$$- e_s\mathcal{X}^-_{s,i}\sum_{t=r+1}^{s-1}\psi_{\alpha_{r,t-1}}\big(\mathcal{X}^-_{r,t-1,i}\big)\mathcal{X}^-_{t,j}\prod_{p=r}^{t-1}\mathcal{H}_{p,j}\prod_{p=t+1}^{s-1}(\mathcal{H}_{p,i} - \delta_{i,j}) + U(\mathfrak{g})\mathfrak{n}^+. \tag{6.12}$$

Applying Lemma 4.3 to the second term, we obtain

$$e_s\psi_{\alpha_{r,s-1}}\big(\mathcal{X}^-_{r,i}\big)\mathcal{X}^-_{s,j}\prod_{t=r}^{s-1}\mathcal{H}_{t,j} = \psi_{\alpha_{r,s-1}}\big(\mathcal{X}^-_{s+1,i}\mathcal{X}^-_{r,s-1,i}\big)(\alpha_{r,s-1})(h_s) + \psi_{\alpha_{r,s}}\big(\mathcal{X}^-_{r,i}\big)e_s\mathcal{X}^-_{s,j}\prod_{t=r}^{s-1}\mathcal{H}_{t,j}$$
$$= -\mathcal{X}^-_{s+1,i}\psi_{\alpha_{r,s-1}}\big(\mathcal{X}^-_{r,s-1,i}\big)\mathcal{X}^-_{s,j}\prod_{t=r}^{s-1}\mathcal{H}_{t,j} + \psi_{\alpha_{r,s}}\big(\mathcal{X}^-_{r,i}\big)\mathcal{X}^-_{s+1,j}\prod_{t=r}^{s}\mathcal{H}_{t,j}$$
$$+ U(\mathfrak{g})\mathfrak{n}^+,$$

where we noted that $\psi_{\alpha_{r,s-1}}(\mathcal{X}^-_{s+1,i}) = \mathcal{X}^-_{s+1,i}$. Finally, the last term in (6.12) can be written as follows

$$-\mathcal{X}^-_{s+1,i}\mathcal{H}_{s,i}\sum_{t=r+1}^{s-1}\psi_{\alpha_{r,t-1}}\big(\mathcal{X}^-_{r,t-1,i}\big)\mathcal{X}^-_{t,j}\prod_{p=r}^{t-1}\mathcal{H}_{p,j}\prod_{p=t+1}^{s-1}(\mathcal{H}_{p,i} - \delta_{i,j})$$
$$- \psi_{\alpha_s}\big(\mathcal{X}^-_{s,i}\big)\sum_{t=r+1}^{s-1}e_s\psi_{\alpha_{r,t-1}}\big(\mathcal{X}^-_{r,t-1,i}\big)\mathcal{X}^-_{t,j}\prod_{p=r}^{t-1}\mathcal{H}_{p,j}\prod_{p=t+1}^{s-1}(\mathcal{H}_{p,i} - \delta_{i,j}).$$

Since $\mathcal{X}^-_{r,t-1,i} = \sum_{\tau\in\Sigma(r,t-1)}\mathbf{f}_\tau c^-_\tau(i)$, it follows that $e_s\psi_\eta(\mathcal{X}^-_{r,t-1,i}) = \psi_{\eta+\alpha_s}(\mathcal{X}^-_{r,t-1,i})e_s$ and so we get

$$\psi_{\alpha_s}\big(\mathcal{X}^-_{s,i}\big)e_s\psi_{\alpha_{r,t-1}}\big(\mathcal{X}^-_{r,t-1,i}\big)\mathcal{X}^-_{t,j} = \psi_{\alpha_{r,t-1}+\alpha_s}\big(\mathcal{X}^-_{r,t-1,i}\big)\psi_{\alpha_s}\big(\mathcal{X}^-_{t,j}\big)e_s \in U(\mathfrak{g})\mathfrak{n}^+.$$

Thus the last term in (6.12) equals

$$-\mathcal{X}^-_{s+1,i}\sum_{t=r+1}^{s-1}\psi_{\alpha_{r,t-1}}\big(\mathcal{X}^-_{r,t-1,i}\big)\mathcal{X}^-_{t,j}\prod_{p=r}^{t-1}\mathcal{H}_{p,j}\prod_{p=t+1}^{s}(\mathcal{H}_{p,i}-\delta_{i,j})+U(\mathfrak{g})\mathfrak{n}^+.$$

The inductive step is now straightforward. □

Corollary. *For all* $1\leqslant i\leqslant j<\ell$, $1\leqslant r\leqslant i+1$, $r\leqslant s\leqslant j+1$, $e_\ell\bar{\mathcal{U}}_{r,s,i,j}\in U(\mathfrak{g})\mathfrak{n}^+$.

6.8. We can now construct adapted families for all $\beta\in\Psi$. Suppose that $\beta_{i,j}\in\Psi$, $i\leqslant j\in I$. If $\lambda\leftarrow\lambda+\beta_{i,j}\in(\Delta_\Psi)_1$, we have $\lambda(\mathcal{H}_{t,i-1})\geqslant\lambda(h_{i-1})>0$, $1\leqslant t\leqslant i-1$ and $\lambda(\mathcal{H}_{r,j-1})\geqslant\lambda(h_{j-1})>0$, $1\leqslant r\leqslant j-1$. Furthermore, if $i=j$, $\lambda(h_{i-1})>1$, hence $\lambda(\mathcal{H}_{t,i-1}-1)>0$. Therefore,

$$\mathcal{H}_{t,i-1}-\delta_{i,j},\mathcal{H}_{r,j-1}\in F_{\beta_{i,j}}(\mathfrak{h})^\times,\quad 1\leqslant t\leqslant i-1,\ 1\leqslant r\leqslant j-1.$$

Clearly, $\{\gamma\in R^+:\ \beta_{i,j}\leqslant\gamma\}=\{\beta_{r,s}:\ 1\leqslant r\leqslant i,\ \ r\leqslant s\leqslant j\}$. Define the elements $\mathbf{u}_{\beta_{i,j},\beta_{r,s}}\in U(\mathfrak{b})_{\beta_{i,j}-\beta_{r,s}}\otimes_{S(\mathfrak{h})}F_{\beta_{i,j}}(\mathfrak{h})$ by

$$\mathbf{u}_{\beta_{i,j},\beta_{r,s}}=(-1)^{i+j+r+s}\frac{1+\delta_{r,s}}{1+\delta_{i,j}}\bar{\mathcal{U}}_{r,s,i-1,j-1}\otimes\prod_{t=r}^{i-1}(\mathcal{H}_{t,i-1}-\delta_{i,j})^{-1}\prod_{t=r}^{j-1}\mathcal{H}^{-1}_{t,j-1}. \tag{6.13}$$

Lemma. *Let* $\beta_{i,j}\in\Psi$. *Then* $\{\mathbf{u}_{\beta_{i,j},\beta_{r,s}}:\ 1\leqslant r\leqslant i,\ r\leqslant s\leqslant j\}$ *is an adapted family for* $\beta_{i,j}$.

Proof. We have

$$\pi_{\lambda,\beta_{i,j}}(\mathbf{u}_{\beta_{i,j},\beta_{r,s}})=C_{r,s}(i,j,\lambda)\mathcal{U}_{r,s,i-1,j-1}(\lambda),$$

where

$$C_{r,s}(i,j,\lambda)=(-1)^{i+j+r+s}\frac{1+\delta_{r,s}}{1+\delta_{i,j}}\prod_{t=r}^{i-1}\big(\lambda(\mathcal{H}_{t,i-1})-\delta_{i,j}\big)^{-1}\prod_{t=r}^{j-1}\big(\lambda(\mathcal{H}_{t,j-1})\big)^{-1}. \tag{6.14}$$

To shorten the notation, we denote $C_{r,s}(i,j,\lambda)$ (respectively, $\mathcal{U}_{r,s,i-1,j-1}(\lambda)$) by $C_{r,s}$ (respectively, $\mathcal{U}_{r,s}$). Observe that for all $r\leqslant s<j$ and for all $k<i\leqslant s$ we have

$$C_{r,s}=-\frac{1+\delta_{r,s}}{1+\delta_{r,s+1}}C_{r,s+1},\qquad C_{k+1,s}=-\frac{1+\delta_{k+1,s}}{1+\delta_{k,s}}\big(\lambda(\mathcal{H}_{k,i-1})-\delta_{i,j}\big)\lambda(\mathcal{H}_{k,j-1}). \tag{6.15}$$

Set

$$u=\sum_{\gamma\in R^+:\ \beta_{i,j}\leqslant\gamma}e_\gamma\otimes\pi_{\lambda,\beta_{i,j}}(\mathbf{u}_{\beta_{i,j},\gamma})\in\big(\mathfrak{g}\otimes V(\lambda+\beta_{i,j})\big)_{\lambda+\beta_{i,j}}.$$

Since by (6.11), $\mathbf{u}_{\beta_{i,j},\beta_{i,j}}=1$, it remains to show that $e_ku=0$ for all $1\leqslant k\leqslant\ell$. For $k\geqslant j$ this is immediate from (6.6c) and (6.4). Suppose that $i\leqslant k<j$. Using (6.4) and (6.6a) we obtain

$$e_ku=\sum_{r=1}^{i}\big(C_{r,k+1}(1+\delta_{r,k})+C_{r,k}\big)e_{\beta_{r,k}}\otimes\mathcal{U}_{r,k+1}v_\lambda,$$

which equals zero by (6.15). Furthermore, if $k < i$, it follows from (6.4), (6.6a) and (6.6b) that

$$e_k u = \sum_{s=k+1}^{j} \big(C_{k+1,s} + C_{k,s}(1+\delta_{k+1,s})\big(\lambda(\mathcal{H}_{k,i-1}) - \delta_{i,j}\big)\lambda(\mathcal{H}_{k,j-1})\big) e_{\beta_{k,s}} \otimes \mathcal{U}_{k+1,s} v_\lambda + \sum_{r=1}^{k} \big(C_{r,k+1}(1+\delta_{r,k}) + C_{r,k}\big) e_{\beta_{r,k}} \otimes \mathcal{U}_{r,k+1} v_\lambda.$$

Using (6.15) it is easy to see that $e_k u = 0$. □

6.9. Now we have all necessary ingredients to describe the relations. We set

$$\mathcal{Z}_{\beta_{i,j},\Psi} = \prod_{t<i:\ \beta_{t,i}\in\Psi} (\mathcal{H}_{t,i-1} - \delta_{i,j}) \prod_{t<j:\ \beta_{t,j}\in\Psi} \mathcal{H}_{t,j-1} \in F_{\beta_{i,j}}(\mathfrak{h})^\times$$

and fix the isomorphism $\mathbf{T}_\Psi^{\mathfrak{g}} \to \mathbf{C}\Delta_\Psi$ corresponding to the image of $(\mathcal{Z}_{\beta,\Psi})_{\beta\in\Psi} \in \prod_{\beta\in\Psi} F_\beta(\mathfrak{h})^\times$ in G_Ψ (cf. 5.6).

Proposition. *Let $\eta = \beta_{i,j} + \beta_{i,i}$, $i \neq j$. If $t_{\lambda,\eta} = 2$ then $\mathfrak{R}_\Psi(\lambda, \lambda+\eta)$ is spanned by the commutativity relation. If $t_{\lambda,\eta} = 1$ then* $\dim \mathfrak{R}_\Psi(\lambda,\lambda+\eta) = 1$. *In particular, $\mathcal{N}_\eta = \emptyset$.*

Proof. Let $i < j$. We have two different cases to consider.

1°. Let $\eta = \beta_{i,i} + \beta_{i,j} \in \Psi + \Psi$, $i < j$. Suppose that $t_{\lambda,\eta} = 2$, hence $\lambda(h_{i-1}) \geqslant 3$ and $\lambda(h_{j-1}) \geqslant 1$ by 6.2(C1). Then $\beta_{i,j} < \beta_{i,i}$ and it follows from Proposition 2.7 and Lemma 6.8 that

$$\Pi_\lambda(\beta_{i,j}, \beta_{i,i}) = e_{i,i} \otimes e_{i,j}, \qquad \Pi_\lambda(\beta_{i,i}, \beta_{i,j}) = e_{\beta_{i,j}} \otimes e_{\beta_{i,i}} + e_{\beta_{i,i}} \otimes \mathbf{u}_{\beta_{i,j},\beta_{i,i}}(\mu), e_{\beta_{i,i}}, \quad \mu = \lambda + \beta_{i,i}.$$

Using (6.5) and an argument similar to that in the proof of Proposition 5.6, we conclude that

$$\mathcal{U}_{i,i,i-1,j-1} = (-1)^{j-i-1} \prod_{t=i+1}^{j-1} \mu(\mathcal{H}_{t,j-1}) f_{j-1} \cdots f_i + \operatorname{Ann}_{U(\mathfrak{g})} v_\mu \cap \operatorname{Ann}_{U(\mathfrak{g})} e_{\beta_{i,i}}, \tag{6.16}$$

hence

$$\Pi_\lambda(\beta_{i,i}, \beta_{i,j}) = e_{\beta_{i,j}} \otimes e_{\beta_{i,i}} - 2\big(\lambda(\mathcal{H}_{i,j-1}) + 2\big)^{-1} e_{\beta_{i,i}} \otimes e_{\beta_{i,j}}. \tag{6.17}$$

To complete the computation of relations in this case, it remains to observe that

$$\lambda\big(\mathcal{Z}_{\beta_{i,i},\Psi} \mathcal{Z}^{-1}_{\beta_{i,j},\Psi}\big)(\lambda + \beta_{i,i})(\mathcal{Z}_{\beta_{i,j},\Psi})(\lambda+\beta_{i,j})\big(\mathcal{Z}^{-1}_{\beta_{i,i},\Psi}\big) = \big(\lambda(\mathcal{H}_{i,j-1}) + 2\big)\lambda(\mathcal{H}_{i,j-1})^{-1},$$

and it is now easy to see that $\mathfrak{R}_\Psi(\lambda, \lambda+\eta)$ is spanned by the commutativity relation. Furthermore, if $t_{\lambda,\eta} = 1$ then it follows from 6.2(C1) that $\lambda(\mathcal{H}_{i,j-1}) = 0$ and the path in question ($\lambda \leftarrow \lambda + \beta_{i,i} \leftarrow \lambda + \eta$) is a relation by (6.17).

2°. Let $\eta = \beta_{i,j} + \beta_{j,j}$, $i < j$ and suppose that $t_{\lambda,\eta} = 2$. Then $\beta_{j,j} < \beta_{i,j}$ and by Proposition 2.7 and Lemma 6.8,

$$\Pi_\lambda(\beta_{j,j}, \beta_{i,j}) = e_{\beta_{i,j}} \otimes e_{\beta_{j,j}},$$

$$\Pi_\lambda(\beta_{i,j}, \beta_{j,j}) = e_{\beta_{j,j}} \otimes e_{\beta_{i,j}} + e_{\beta_{i,j}} \otimes \mathbf{u}_{\beta_{j,j},\beta_{i,j}}(\nu) e_{\beta_{i,j}},$$

where $\nu = \lambda + \beta_{i,j}$. Note that $\mathcal{X}^-_{k,j-1}(\nu)e_{\beta_{i,j}} = 0$, $i < k \leqslant j-1$. It follows from Lemma 6.7 that

$$\mathcal{U}_{i,j,j-1,j-1} = \mathcal{X}^-_{i,j-1}(\nu)\prod_{k=i}^{j-1}\big(\nu(\mathcal{H}_{k,j}) - \delta_{i,k} - 1\big) + \mathcal{X}^-_{i,j-1}(\nu - \varpi_{j-1})\prod_{k=i}^{j-1}\nu(\mathcal{H}_{k,j}) + \operatorname{Ann}_{U(\mathfrak{g})} v_\nu \cap \operatorname{Ann}_{U(\mathfrak{g})} e_{\beta_{i,j}}.$$

Furthermore, it is easy to see from (6.5) that for any $\xi \in P$,

$$\mathcal{X}^-_{i,j-1}(\xi) = (-1)^{j-i-1}\prod_{k=i+1}^{j-1}\xi(\mathcal{H}_{k,j-1}) f_{j-1}\cdots f_i \quad (\operatorname{mod}\ \operatorname{Ann}_{U(\mathfrak{g})} e_{\beta_{i,j}}).$$

Since $\nu(h_i) = \lambda(h_i) + 1 > 0$, $f_{j-1}\cdots f_i \notin \operatorname{Ann}_{U(\mathfrak{g})} v_\nu$ by Corollary 4.1. Therefore,

$$\Pi_\lambda(\beta_{i,j}, \beta_{j,j}) = e_{\beta_{j,j}} \otimes e_{\beta_{i,j}} - 2\lambda(\mathcal{H}_{i,j-1})^{-1} e_{\beta_{i,j}} \otimes e_{\beta_{j,j}}.$$

Since

$$\lambda\big(\mathcal{Z}_{\beta_{i,j},\Psi}\mathcal{Z}^{-1}_{\beta_{j,j},\Psi}\big)(\lambda + \beta_{i,j})(\mathcal{Z}_{\beta_{j,j},\Psi})(\lambda + \beta_{j,j})\big(\mathcal{Z}^{-1}_{\beta_{i,j},\Psi}\big) = \lambda(\mathcal{H}_{i,j-1})\big(\lambda(\mathcal{H}_{i,j-1}) - 2\big)^{-1}$$

it is now easy to see that we again obtain the commutativity relation.

Finally, if $t_{\lambda,\eta} = 1$ then by 6.2(C1), $\lambda(\mathcal{H}_{i,j-1}) = 2$ and so $\Pi_\lambda(\beta_{i,j}, \beta_{j,j}) \in \bigwedge^2 \mathfrak{n}^+_\Psi$. Therefore, the corresponding path is a relation. □

6.10. The next case $\eta = 2\beta_{i,j}$, $i < j$ is more interesting since this is the only case in this paper where $m_\eta > 1$ for $\eta \in 2\Psi$.

Proposition. *Suppose that $\Psi \supset \Psi(i,j)$, $i < j \in I$ and let $\eta = 2\beta_{i,j} = \beta_{i,i} + \beta_{j,j} \in \Psi + \Psi$. Assume that $\lambda \in P^+$ is such that $\Delta_\Psi(\lambda, \lambda + \eta) \neq \emptyset$.*

(i) *If $t_{\lambda,\eta} = 3$, the unique relation is*

$$\lambda(\mathcal{H}_{i,j-1})^2(\lambda \leftarrow \lambda + \beta_{i,i} \leftarrow \lambda + \eta) - \big(\lambda(\mathcal{H}_{i,j-1}) + 2\big)^2(\lambda \leftarrow \lambda + \beta_{j,j} \leftarrow \lambda + \eta) + \big(\lambda(\mathcal{H}_{i,j-1}) + 1\big)(\lambda \leftarrow \lambda + \beta_{i,j} \leftarrow \lambda + \eta). \tag{6.18}$$

(ii) *If $t_{\lambda,\eta} = 2$, the unique relation is*

$$(\lambda \leftarrow \lambda + \beta_{i,i} \leftarrow \lambda + \eta) + 2(\lambda \leftarrow \lambda + \beta_{i,j} \leftarrow \lambda + \eta).$$

(iii) *If $t_{\lambda,\eta} = 1$ then $\lambda(\mathcal{H}_{i,j-1}) = 0$ and $\mathfrak{R}_\Psi(\lambda, \lambda + \eta) = 0$.*

Thus, $\mathfrak{R}_\Psi(\lambda, \lambda + \eta)$ has dimension $\lfloor t_{\lambda,\eta}/2\rfloor$ and is generic if and only if $t_{\lambda,\eta} > 1$.

Proof. Suppose first that $t_{\lambda,\eta} = 3$. We have $\beta_{j,j} < \beta_{i,j} < \beta_{i,i}$, hence

$$\Pi_\lambda(\beta_{j,j}, \beta_{i,i}) = e_{\beta_{i,i}} \otimes e_{\beta_{j,j}},$$

$$\Pi_\lambda(\beta_{i,i}, \beta_{j,j}) = e_{\beta_{j,j}} \otimes e_{\beta_{i,i}} + e_{\beta_{i,i}} \otimes \mathbf{u}_{\beta_{j,j},\beta_{i,i}}(\mu)e_{\beta_{i,i}} + e_{\beta_{i,j}} \otimes \mathbf{u}_{\beta_{j,j},\beta_{i,j}}(\mu)e_{\beta_{i,i}},$$

$$\Pi_\lambda(\beta_{i,j}, \beta_{i,j}) = e_{\beta_{i,j}} \otimes e_{\beta_{i,j}} + e_{\beta_{i,i}} \otimes \mathbf{u}_{\beta_{i,j},\beta_{i,i}}(\nu)e_{\beta_{i,j}},$$

where $\mu=\lambda+\beta_{i,i}$, $\nu=\lambda+\beta_{i,j}$. Using Lemma 6.7 and (6.16) we can write

$$\mathbf{u}_{\beta_{j,j},\beta_{i,i}}(\mu)=\big(\big(\mu(\mathcal{H}_{k,j-1})-1\big)\mu(\mathcal{H}_{k,j-1})\big)^{-1}(f_{j-1}\cdots f_i)^2+\operatorname{Ann}_{U(\mathfrak{g})}e_{\beta_{i,i}}\cap\operatorname{Ann}_{U(\mathfrak{g})}v_\mu.$$

We claim that $(f_{j-1}\cdots f_i)^2\notin\operatorname{Ann}_{U(\mathfrak{n}^-)}v_\mu$. Indeed, since $\mu(h_i)=\lambda(h_i)+2\geqslant 2$, it follows from Corollary 4.1 that $F=f_{j-2}\cdots f_i f_{j-1}\cdots f_i\notin\operatorname{Ann}_{U(\mathfrak{n}^-)}v_\mu$. Now, if ζ is the weight of Fv_μ, we have $\zeta(h_{j-1})=\lambda(h_{j-1})\geqslant 2$. Furthermore, $e_{j-1}^2Fv_\mu=0$. It now follows from the elementary $\mathfrak{sl}_2$ theory that $f_{j-1}Fv_\mu\neq 0$. Thus,

$$\mathbf{u}_{\beta_{j,j},\beta_{i,i}}(\mu)e_{\beta_{i,i}}=2\big(\big(\lambda(\mathcal{H}_{i,j-1})+1\big)\big(\lambda(\mathcal{H}_{i,j-1})+2\big)\big)^{-1}e_{\beta_{j,j}}.$$

A computation similar to that of $\Pi_\lambda(\beta_{i,j},\beta_{j,j})$ in 6.9 yields

$$\mathbf{u}_{\beta_{j,j},\beta_{i,j}}(\mu)e_{\beta_{i,i}}=-\big(\lambda(\mathcal{H}_{i,j-1})+2\big)^{-1}e_{\beta_{i,j}}.$$

Thus,

$$\begin{aligned}\Pi_\lambda(\beta_{i,i},\beta_{j,j})&=e_{\beta_{j,j}}\otimes e_{\beta_{i,i}}-\big(\lambda(\mathcal{H}_{i,j-1})+2\big)^{-1}e_{\beta_{i,j}}\otimes e_{\beta_{i,j}}\\&\quad+2\big(\big(\lambda(\mathcal{H}_{i,j-1})+1\big)\big(\lambda(\mathcal{H}_{i,j-1})+2\big)\big)^{-1}e_{\beta_{i,i}}\otimes e_{\beta_{j,j}}.\end{aligned}\tag{6.19}$$

The computation of $\Pi_\lambda(\beta_{i,j},\beta_{i,j})$ is similar to that of $\Pi_\lambda(\beta_{i,i},\beta_{i,j})$ in 6.9 and yields

$$\Pi_\lambda(\beta_{i,j},\beta_{i,j})=e_{\beta_{i,j}}\otimes e_{\beta_{i,j}}-4\big(\lambda(\mathcal{H}_{i,j-1})\big)^{-1}e_{\beta_{i,i}}\otimes e_{\beta_{j,j}}.$$

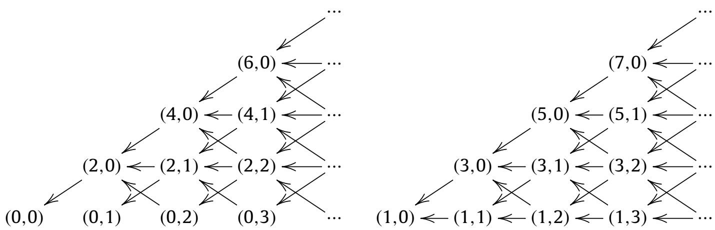
Note that none of these paths is a relation. To complete the computation of relations in this case, it remains to note that

$$\begin{aligned}\lambda(\mathcal{Z}_{\beta_{i,i},\Psi})(\lambda+\beta_{i,i})(\mathcal{Z}_{\beta_{j,j},\Psi})&=\big(\lambda(\mathcal{H}_{i,j-1})+2\big)\big(\lambda(\mathcal{H}_{i,j-1})+1\big)z,\\\lambda(\mathcal{Z}_{\beta_{i,j},\Psi})(\lambda+\beta_{i,j})(\mathcal{Z}_{\beta_{i,j},\Psi})&=\big(\lambda(\mathcal{H}_{i,j-1})\big)^2z,\\\lambda(\mathcal{Z}_{\beta_{j,j},\Psi})(\lambda+\beta_{j,j})(\mathcal{Z}_{\beta_{i,i},\Psi})&=\lambda(\mathcal{H}_{i,j-1})\big(\lambda(\mathcal{H}_{i,j-1})-1\big)z,\end{aligned}$$

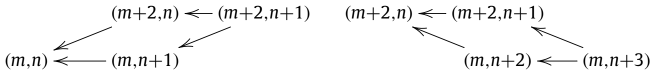
where $z\in\mathbf{C}^\times$. The relation (6.18) is now straightforward. Since all coefficients in it are positive integers, $\mathfrak{R}_\Psi(\lambda,\lambda+\eta)$ is generic.

If $t_{\lambda,\eta}=2$ then by 6.2(C2), $i=j-1$ and $\lambda(\mathcal{H}_{i,j-1})=1$ and we immediately obtain the relation using the above formulae. Finally, if $t_{\lambda,\eta}=1$ then $\lambda(\mathcal{H}_{i,j-1})=0$ and it is easy to see from (6.19) that the corresponding path is not a relation. □

6.11. We now present an infinite dimensional example which in particular includes the remaining rank 2 case. Let $\Psi=\Psi(1,2)$, $\ell\geqslant 2$. Since $\beta_{1,1}=2\varpi_1=\theta$, $\beta_{1,2}=\varpi_2$ and $\beta_{2,2}=-2\varpi_1+2\varpi_2$ it is clear that $\lambda,\mu\in P^+$ are in the same connected component of Δ_Ψ only if $\lambda(h_i)=\mu(h_i)$, $2<i\leqslant\ell$. Therefore, it is enough to describe the connected components of Δ_Ψ for $\ell=2$. Identify P with $\mathbf{Z}\times\mathbf{Z}$ and write $(\lambda(h_1),\lambda(h_2))$ for $\lambda\in P$. Since $\varphi(\theta)=(2,0)$, $\varphi(\beta_{1,2})=(1,1)$ and $\varphi(\beta_{2,2})=(0,2)$, we conclude that the only sinks in Δ_Ψ are $(0,0)$, $(0,1)$ and $(1,0)$. Furthermore, if (m,n) and (m',n') are in the same connected component it is immediate that $m=m'$ (mod 2). Since we have $(0,0)\leftarrow(2,0)\leftarrow(2,1)\rightarrow(0,1)$, we conclude that there are two connected components, $\Delta_\Psi[(r,0)]$, $r=0,1$ (if $\ell>2$, each of these components has infinite multiplicity). We have $\Delta_\Psi[(r,0)]_0=\{(m,n)\colon m,n\in\mathbf{Z}_+,\ m=r$ (mod 2)$\}$ and the arrows are $(m,n)\leftarrow(m+2,n)$, $m,n\in\mathbf{Z}_+$, $(m,n)\leftarrow(m,n+1)$, $m>0$, $n\in\mathbf{Z}_+$ and $(m,n)\leftarrow(m-2,n+2)$, $m\geqslant 2$, $n\in\mathbf{Z}_+$. Thus, the quivers $\Delta_\Psi[(0,r)]$, $r=0,1$ are, respectively,



Both are translation quivers with $\tau((m,n)) = (m, n-2)$, $m > 0$, $n \geqslant 2$. The relations are: the commutativity relations in



for all $m > 0$, $n \in \mathbf{Z}_+$, the zero relations $(2,n) \leftarrow (2,n+1) \leftarrow (0,n+3)$, $n \geqslant 0$ and

$$m^2\big((m,n) \leftarrow (m+2,n) \leftarrow (m,n+2)\big) - (m+2)^2\big((m,n) \leftarrow (m-2,n+2) \leftarrow (m,n+2)\big)$$
$$+ (m+1)\big((m,n) \leftarrow (m,n+1) \leftarrow (m,n+2)\big), \quad m > 1,$$

and, finally, $((1,n) \leftarrow (3,n) \leftarrow (1,n+2)) + 2((1,n) \leftarrow (1,n+1) \leftarrow (1,n+2))$. Thus, if $\ell = 2$ and $|\Psi| > 1$, the algebra $\mathbf{S}^{\mathfrak{g}}_{\Psi}$ is the direct sum of two non-isomorphic connected Koszul subalgebras of left global dimension 3.

6.12. Next, we consider $\eta \in \Psi + \Psi$ with $m_\eta = 4$. We have three possibilities here, and the computations turn out to be rather different.

Proposition. *Let $i < j < k \in I$ and let $x_\lambda = \lambda(\mathcal{H}_{i,j-1})$, $y_\lambda = \lambda(\mathcal{H}_{j,k-1})$. Assume that $\Psi \supset \Psi(i,j,k)$ and let $\eta = \beta_{i_1,i_1} + \beta_{i_2,i_3} = \beta_{i_1,i_2} + \beta_{i_1,i_3}$, $\{i_1,i_2,i_3\} = \{i,j,k\}$.*

(i) *Suppose that $i_1 = i$. If $t_{\lambda,\eta} = 4$ then $\mathfrak{R}_\Psi(\lambda, \lambda+\eta)$ is spanned by*

$$2(1+y_\lambda)(\lambda \leftarrow \lambda+\beta_{i,i} \leftarrow \lambda+\eta) - (3+x_\lambda)(x_\lambda+y_\lambda+3)(\lambda \leftarrow \lambda+\beta_{i,j} \leftarrow \lambda+\eta)$$
$$+ (2+x_\lambda)(x_\lambda+y_\lambda+4)(\lambda \leftarrow \lambda+\beta_{i,k} \leftarrow \lambda+\eta) \tag{6.20a}$$

and

$$x_\lambda(x_\lambda+y_\lambda+2)(\lambda \leftarrow \lambda+\beta_{i,i} \leftarrow \lambda+\eta) + (2x_\lambda+y_\lambda+4)(\lambda \leftarrow \lambda+\beta_{i,j} \leftarrow \lambda+\eta)$$
$$- (2+x_\lambda)(x_\lambda+y_\lambda+4)(\lambda \leftarrow \lambda+\beta_{j,k} \leftarrow \lambda+\eta). \tag{6.20b}$$

If $t_{\lambda,\eta} = 2$ then $\mathfrak{R}_\Psi(\lambda, \lambda+\eta)$ is spanned by

$$(x_\lambda+y_\lambda+1)(\lambda \leftarrow \lambda+\beta_{i,i} \leftarrow \lambda+\eta) + (x_\lambda+y_\lambda+4)(\lambda \leftarrow \lambda+\beta_{i,k} \leftarrow \lambda+\eta). \tag{6.21}$$

Finally, if $t_{\lambda,\eta} = 1$ then the unique path is a relation.

(ii) *Suppose that* $i_1 = j$. *If* $t_{\lambda,\eta} = 4$, $\mathfrak{R}_\Psi(\lambda, \lambda+\eta)$ *is spanned by*

$$(x_\lambda - 1)(2 + y_\lambda)(\lambda \leftarrow \lambda + \beta_{i,k} \leftarrow \lambda + \eta) - (1 + x_\lambda)y_\lambda(\lambda \leftarrow \lambda + \beta_{j,j} \leftarrow \lambda + \eta) \\ - (x_\lambda - y_\lambda - 1)(\lambda \leftarrow \lambda + \beta_{j,k} \leftarrow \lambda + \eta) \tag{6.22a}$$

and

$$(1 + x_\lambda)y_\lambda(\lambda \leftarrow \lambda + \beta_{i,j} \leftarrow \lambda + \eta) + 2(2 + x_\lambda + y_\lambda)(\lambda \leftarrow \lambda + \beta_{i,k} \leftarrow \lambda + \eta) \\ - (2 + x_\lambda)(1 + y_\lambda)(\lambda \leftarrow \lambda + \beta_{j,k} \leftarrow \lambda + \eta) \tag{6.22b}$$

and $\mathfrak{R}_\Psi(\lambda, \lambda+\eta)$ *is generic unless* $x_\lambda = y_\lambda + 1$. *If* $t_{\lambda,\eta} = 2$ *then* $\mathfrak{R}_\Psi(\lambda, \lambda+\eta)$ *is spanned by*

$$(x_\lambda - 1)(\lambda \leftarrow \lambda + \beta_{i,j} \leftarrow \lambda + \eta) + (x_\lambda + 2)(\lambda \leftarrow \lambda + \beta_{j,j} \leftarrow \lambda + \eta). \tag{6.23}$$

If $t_{\lambda,\eta} = 1$ *then* $\lambda(\mathcal{H}_{i,j-1}) = 1$ *and* $\mathfrak{R}_\Psi(\lambda, \lambda+\eta) = 0$.

(iii) *Suppose that* $i_1 = k$. *If* $t_{\lambda,\eta} = 1$, $\mathfrak{R}_\Psi(\lambda, \lambda+\eta)$ *is spanned by*

$$(1 + y_\lambda)(3 + x_\lambda + y_\lambda)(\lambda \leftarrow \lambda + \beta_{j,k} \leftarrow \lambda + \eta) - (2 + y_\lambda)(2 + x_\lambda + y_\lambda)(\lambda \leftarrow \lambda + \beta_{i,k} \leftarrow \lambda + \eta) \\ + 2(1 + x_\lambda)(\lambda \leftarrow \lambda + \beta_{i,j} \leftarrow \lambda + \eta) \tag{6.24a}$$

and

$$2(1 + x_\lambda)(\lambda \leftarrow \lambda + \beta_{k,k} \leftarrow \lambda + \eta) + (y_\lambda - 1)(x_\lambda + y_\lambda + 1)(\lambda \leftarrow \lambda + \beta_{j,k} \leftarrow \lambda + \eta) \\ - y_\lambda(x_\lambda + y_\lambda)(\lambda \leftarrow \lambda + \beta_{i,k} \leftarrow \lambda + \eta). \tag{6.24b}$$

If $t_{\lambda,\eta} = 2$, *then the unique relation is*

$$(x_\lambda + 4)(\lambda \leftarrow \lambda + \beta_{i,j} \leftarrow \lambda + \eta) + (x_\lambda + 1)(\lambda \leftarrow \lambda + \beta_{j,k} \leftarrow \lambda + \eta). \tag{6.25}$$

If $t_{\lambda,\eta} = 1$ *then* $\dim \mathfrak{R}_\Psi(\lambda, \lambda+\eta) = 1$.

In particular, in cases (i) *and* (iii), $\mathcal{N}_\eta = \emptyset$ *while in the case* (ii), $\{\lambda \in \mathcal{N}_\eta\colon\ t_{\lambda,\eta} > 1\} = P^+ \cap \{\xi \in \mathfrak{h}^*\colon \xi(\mathcal{H}_{i,j-1} - \mathcal{H}_{j,k-1} - 1) = 0\}$.

Proof. Suppose first that $\eta = \beta_{i,j} + \beta_{i,k}$. We have $\beta_{j,k} < \beta_{i,k} < \beta_{i,j} < \beta_{i,i}$. Suppose first that $t_{\lambda,\eta} = 4$. The first two paths are straightforward

$$\Pi_\lambda(\beta_{j,k}, \beta_{i,i}) = e_{\beta_{i,i}} \otimes e_{\beta_{j,k}}, \tag{6.26}$$

$$\Pi_\lambda(\beta_{i,k}, \beta_{i,j}) = e_{\beta_{i,j}} \otimes e_{\beta_{i,k}} - 2(x_\lambda + 1)^{-1} e_{\beta_{i,i}} \otimes e_{\beta_{j,k}}. \tag{6.27}$$

The next path is more involved. We have

$$\Pi_\lambda(\beta_{i,j}, \beta_{i,k}) = e_{\beta_{i,k}} \otimes e_{\beta_{i,j}} + e_{\beta_{i,i}} \otimes \mathbf{u}_{\beta_{i,k},\beta_{i,i}}(\mu) e_{\beta_{i,j}} + e_{\beta_{i,j}} \otimes \mathbf{u}_{\beta_{i,k},\beta_{i,j}}(\mu) e_{\beta_{i,j}},$$

where $\mu = \lambda + \beta_{i,j}$. We claim that

$$\begin{aligned}
&\mathcal{U}_{i,i,i-1,k-1}(\mu)\\
&\quad = \mathcal{X}^-_{i,k-1}(\mu)\\
&\quad = (-1)^{k-i-1}\left(\prod_{t=i+1}^{k-1}\mu(\mathcal{H}_{t,k-1})f_{k-1}\cdots f_i - \prod_{t=i+1}^{k-1}\big(\mu(\mathcal{H}_{t,k-1})+\delta_{t,j}\big)f_{j-1}\cdots f_i f_{k-1}\cdots f_j\right)\\
&\qquad + \mathrm{Ann}_{U(\mathfrak{g})}\,e_{\beta_{i,j}} \cap \mathrm{Ann}_{U(\mathfrak{g})}\,v_\mu. \qquad (6.28)
\end{aligned}$$

Indeed, suppose that $\mathbf{f}_\sigma \notin \mathrm{Ann}_{U(\mathfrak{g})}\,e_{\beta_{i,j}}$, $\sigma \in \Sigma(i,k-1)$. Then by (6.5) we must have $\sigma(i) = k-i$ or $\sigma(j) = k-i$. If $\sigma(i) = k-i$ then $\mathbf{f}_\sigma = f_{k-1}\cdots f_i$ by the definition of $\Sigma(i,k-1)$. Otherwise, we must have $\sigma(r) = k-r$, $r \leqslant j \leqslant k-1$ and so $\mathbf{f}_\sigma = \mathbf{f}_{\sigma'} f_{k-1}\cdots f_j$, where $\sigma' \in \Sigma(i,j-1)$. Then $\mathbf{f}_\sigma e_{\beta_{i,j}} = \mathbf{f}_{\sigma'} e_{\beta_{i,k}}$ hence $\mathbf{f}_{\sigma'} = f_{j-1}\cdots f_i$. Since $\mu(h_r) = \lambda(h_r) + 1 > 0$, $r = i, j$, it follows from Corollary 4.1 that the vectors $f_{k-1}\cdots f_i v_\mu$, $f_{j-1}\cdots f_i f_{k-1}\cdots f_j v_\mu$ are non-zero and linearly independent. Therefore,

$$\begin{aligned}
\mathbf{u}_{\beta_{i,k},i,i}(\mu)e_{\beta_{i,j}} &= -2\prod_{t=i}^{k-1}\mu(\mathcal{H}_{t,k-1})^{-1}\left(2\prod_{t=i+1}^{k-1}\mu(\mathcal{H}_{t,k-1}) - \prod_{t=i+1}^{k-1}\big(\mu(\mathcal{H}_{t,k-1})+\delta_{t,j}\big)\right)e_{\beta_{j,k}}\\
&= -2\big(\mu(\mathcal{H}_{j,k-1})-1\big)\big(\mu(\mathcal{H}_{i,k-1})\mu(\mathcal{H}_{j,k-1})\big)^{-1}e_{\beta_{j,k}}.
\end{aligned}$$

An already familiar computation yields $\mathbf{u}_{\beta_{i,k},\beta_{i,j}}(\mu)e_{\beta_{i,j}} = -\mu(\mathcal{H}_{j,k-1})^{-1}e_{\beta_{i,k}}$ and we obtain

$$\Pi_\lambda(\beta_{i,j},\beta_{i,k}) = e_{\beta_{i,k}}\otimes e_{\beta_{i,j}} - (y_\lambda+1)^{-1}e_{\beta_{i,j}}\otimes e_{\beta_{i,k}} - \frac{2y_\lambda}{(y_\lambda+1)(x_\lambda+y_\lambda+2)}e_{\beta_{i,i}}\otimes e_{\beta_{j,k}}. \quad (6.29)$$

Finally,

$$\begin{aligned}
\Pi_\lambda(\beta_{i,i},\beta_{j,k}) &= e_{\beta_{j,k}}\otimes e_{\beta_{i,i}} - (x_\lambda+2)^{-1}e_{\beta_{i,k}}\otimes e_{\beta_{i,j}} + 2\big((x_\lambda+2)(x_\lambda+y_\lambda+3)\big)^{-1}e_{\beta_{i,i}}\otimes e_{\beta_{j,k}}\\
&\quad - \frac{x_\lambda+1}{(x_\lambda+2)(x_\lambda+y_\lambda+3)}e_{\beta_{i,j}}\otimes e_{\beta_{i,k}}. \qquad (6.30)
\end{aligned}$$

It is now straightforward to show that $\mathfrak{R}_\Psi(\lambda,\lambda+\eta)$ is spanned by the elements (6.20a) and (6.20b) and that $\mathfrak{R}_\Psi(\lambda,\lambda+\eta)$ is generic.

Suppose now that $0 < t_{\lambda,\eta} < 4$. By 6.2(C3) we have two possibilities. If $\lambda(h_i) = 0$ and $i = j-1$ (hence $x_\lambda = 0$) we obtain from (6.27) and (6.30) that $\mathfrak{R}_\Psi(\lambda,\lambda+\eta)$ is spanned by the element (6.21). Finally, if $\lambda(h_j) = 0$ and $j = k-1$ (hence $y_\lambda = 0$), it follows from (6.29) that $\Pi_\lambda(\beta_{i,j},\beta_{i,k}) \in \bigwedge^2\mathfrak{n}^+_\Psi$ hence the unique path $(\lambda \leftarrow \lambda+\beta_{i,j} \leftarrow \lambda+\eta)$ is a relation.

We now prove (ii). Let $\eta = \beta_{i,j}+\beta_{j,k} = \beta_{i,k}+\beta_{j,j}$. The first three paths are rather easy and we obtain

$$\begin{aligned}
\Pi_\lambda(\beta_{j,k},\beta_{i,j}) &= e_{\beta_{i,j}}\otimes e_{\beta_{j,k}},\\
\Pi_\lambda(\beta_{i,k},\beta_{j,j}) &= e_{\beta_{j,j}}\otimes e_{\beta_{i,k}} - (x_\lambda+1)^{-1}e_{\beta_{i,j}}\otimes e_{\beta_{j,k}},\\
\Pi_\lambda(\beta_{j,j},\beta_{i,k}) &= e_{\beta_{i,k}}\otimes e_{\beta_{j,j}} - (y_\lambda+2)^{-1}e_{\beta_{i,j}}\otimes e_{\beta_{j,k}}.
\end{aligned}$$

The last path has some new features. We have

$$\begin{aligned}
\Pi_\lambda(\beta_{i,j},\beta_{j,k}) &= e_{\beta_{j,k}}\otimes e_{\beta_{i,j}} + e_{\beta_{j,j}}\otimes \mathbf{u}_{\beta_{j,k},\beta_{j,j}}(\nu)e_{\beta_{i,j}} + e_{\beta_{i,k}}\otimes \mathbf{u}_{\beta_{j,k},\beta_{i,k}}(\nu)e_{\beta_{i,j}}\\
&\quad + e_{\beta_{i,j}}\otimes \mathbf{u}_{\beta_{j,k},\beta_{i,j}}(\nu)e_{\beta_{i,j}},
\end{aligned}$$

where $\nu=\lambda+\beta_{i,j}$. Two terms are already familiar

$$\mathbf{u}_{\beta_{j,k},\beta_{j,j}}(\nu)e_{\beta_{i,j}}=-2\big(\nu(\mathcal{H}_{j,k-1})\big)^{-1}e_{\beta_{i,k}},$$
$$\mathbf{u}_{\beta_{j,k},\beta_{i,k}}(\nu)e_{\beta_{i,j}}=-2\big(\nu(\mathcal{H}_{i,j-1})\big)^{-1}e_{\beta_{j,j}}.$$

Furthermore, we have, modulo $\operatorname{Ann}_{U(\mathfrak{g})}e_{\beta_{i,j}}$,

$$\mathcal{U}_{i,j,j-1,k-1}(\nu)=\mathcal{X}^-_{i,k-1}(\nu)\prod_{t=i}^{j-1}\big(\nu(\mathcal{H}_{t,j-1})-\delta_{t,i}\big)+\mathcal{X}^-_{i,j-1}(\nu)\mathcal{X}^-_{j,k-1}(\nu)\prod_{t=i}^{j-1}\nu(\mathcal{H}_{t,k-1}).$$

Using (6.28), we obtain

$$\begin{aligned}\mathcal{U}_{i,j,j-1,k-1}(\nu)={}&(-1)^{k-i}\prod_{t=i}^{j-1}\big(\nu(\mathcal{H}_{t,j-1})-\delta_{t,i}\big)\\&\times\left(-\prod_{t=i+1}^{k-1}\nu(\mathcal{H}_{t,k-1})f_{k-1}\cdots f_i+\prod_{t=i+1}^{k-1}\big(\nu(\mathcal{H}_{t,k-1})+\delta_{t,j}\big)f_{j-1}\cdots f_if_{k-1}\cdots f_j\right)\\&+(-1)^{k-i}\prod_{t=i+1}^{j-1}\nu(\mathcal{H}_{t,j-1})\prod_{t=j+1}^{k-1}\nu(\mathcal{H}_{t,k-1})\prod_{t=i}^{j-1}\nu(\mathcal{H}_{t,k-1})f_{j-1}\cdots f_if_{k-1}\cdots f_j\\&+\operatorname{Ann}_{U(\mathfrak{g})}e_{\beta_{i,j}}.\end{aligned}$$

Thus,

$$\begin{aligned}&\mathbf{u}_{\beta_{j,k},\beta_{i,j}}(\nu)e_{\beta_{i,j}}\\&\quad=\big(\nu(\mathcal{H}_{i,j-1})\nu(\mathcal{H}_{i,k-1})\nu(\mathcal{H}_{j,k-1})\big)^{-1}\big(\nu(\mathcal{H}_{i,k-1})-\big(\nu(\mathcal{H}_{i,j-1})-1\big)\big(\nu(\mathcal{H}_{j,k-1})-1\big)\big)e_{\beta_{j,k}},\end{aligned}$$

hence

$$\begin{aligned}\Pi_\lambda(\beta_{i,j},\beta_{j,k})={}&e_{\beta_{j,k}}\otimes e_{\beta_{i,j}}-2(y_\lambda+1)^{-1}e_{\beta_{j,j}}\otimes e_{\beta_{i,k}}-2x_\lambda^{-1}e_{\beta_{i,k}}\otimes e_{\beta_{j,j}}\\&+\frac{(x_\lambda+y_\lambda+2-(x_\lambda-1)y_\lambda)}{x_\lambda(y_\lambda+1)(x_\lambda+y_\lambda+2)}e_{\beta_{i,j}}\otimes e_{\beta_{j,k}},\end{aligned}$$

and we obtain the relations (6.22a) and (6.22b). It is now straightforward to check that $\mathfrak{R}_\Psi(\lambda,\lambda+\eta)$ is generic if and only if $x_\lambda\neq y_\lambda+1$.

Suppose now that $0<t_{\lambda,\eta}<4$. Using 6.2(C3) and the above, we conclude that if $t_{\lambda,\eta}=2$, the unique relation is (6.23), while in the case $t_{\lambda,\eta}=1$, $x_\lambda=1$ and the unique path is not a relation.

In part (iii) we encounter some new features. We have $\beta_{k,k}<\beta_{j,k}<\beta_{i,k}<\beta_{i,j}$ and

$$\Pi_\lambda(\beta_{k,k},\beta_{i,j})=e_{\beta_{i,j}}\otimes e_{\beta_{k,k}},$$
$$\Pi_\lambda(\beta_{j,k},\beta_{i,k})=e_{\beta_{i,k}}\otimes e_{\beta_{j,k}}-2y_\lambda^{-1}e_{\beta_{i,j}}\otimes e_{\beta_{k,k}},$$
$$\Pi_\lambda(\beta_{i,k},\beta_{j,k})=e_{\beta_{j,k}}\otimes e_{\beta_{i,k}}-(x_\lambda+1)^{-1}e_{\beta_{i,k}}\otimes e_{\beta_{j,k}}-\frac{2x_\lambda}{(x_\lambda+1)(x_\lambda+y_\lambda+1)}e_{\beta_{i,j}}\otimes e_{\beta_{k,k}}.$$

The remaining path is rather interesting. Indeed, this turns out to be one of the only two cases when the last term in (6.10) does not lie in the annihilator of the corresponding root vector. As before,

$$\Pi_\lambda(\beta_{i,j},\beta_{k,k}) = e_{\beta_{k,k}}\otimes e_{\beta_{i,j}} + e_{\beta_{j,k}}\otimes \mathbf{u}_{\beta_{k,k},\beta_{j,k}}(\mu)e_{\beta_{i,j}} + e_{\beta_{i,k}}\otimes \mathbf{u}_{\beta_{k,k},\beta_{i,k}}(\nu)e_{\beta_{i,j}} + e_{\beta_{i,j}}\otimes \mathbf{u}_{\beta_{k,k},\beta_{i,j}}(\mu)e_{\beta_{i,j}},$$

where $\mu=\lambda+\beta_{i,j}$. Note that $\mu(\mathcal{H}_{r,k-1})=\lambda(\mathcal{H}_{r,k-1})+1>\lambda(h_{k-1})>2$, $r=i,j$. We immediately get

$$\mathbf{u}_{\beta_{k,k},\beta_{j,k}}(\mu)e_{\beta_{i,j}} = -\mu(\mathcal{H}_{j,k-1})^{-1}e_{\beta_{i,k}}.$$

Furthermore, by Lemma 6.7 we have

$$\begin{aligned}\mathcal{U}_{i,k,k-1,k-1}(\mu) &= \mathcal{X}^-_{i,k-1}(\mu)\prod_{t=i}^{k-1}\big(\mu(\mathcal{H}_{t,k-1})-\delta_{t,i}-1\big)+\mathcal{X}^-_{i,k-1}(\mu-\varpi_{k-1})\prod_{t=i}^{k-1}\mu(\mathcal{H}_{t,k-1})\\ &\quad -\mathcal{X}^-_{i,j,k-1}(\mu-\varpi_{k-1})\mathcal{X}^-_{j,k-1}(\mu)\prod_{t=i}^{j-1}\mu(\mathcal{H}_{t,k-1})\prod_{t=j+1}^{k-1}\big(\mu(\mathcal{H}_{t,k-1})-1\big)\\ &\quad +\operatorname{Ann}_{U(\mathfrak{g})}e_{\beta_{i,j}}.\end{aligned}$$

Using (6.28) we obtain

$$\begin{aligned}\mathbf{u}_{\beta_{k,k},\beta_{i,k}}(\mu)e_{\beta_{i,j}} &= -\frac{1}{2}\big(\mu(\mathcal{H}_{i,k-1})-1\big)^{-1}\bigg(\frac{(\mu(\mathcal{H}_{i,k-1})-2)(\mu(\mathcal{H}_{j,k-1})-1)}{\mu(\mathcal{H}_{j,k-1})\mu(\mathcal{H}_{i,k-1})}\\ &\quad +\frac{\mu(\mathcal{H}_{j,k-1})-2}{\mu(\mathcal{H}_{j,k-1})-1}+\frac{1}{\mu(\mathcal{H}_{j,k-1})(\mu(\mathcal{H}_{j,k-1})-1)}\bigg)e_{\beta_{j,k}}\\ &= -\frac{\mu(\mathcal{H}_{j,k-1})-1}{\mu(\mathcal{H}_{i,k-1})\mu(\mathcal{H}_{j,k-1})}\,e_{\beta_{j,k}}.\end{aligned}$$

To compute the remaining term observe that by Lemma 6.7,

$$\begin{aligned}&(-1)^{i+j}\mathcal{U}_{i,j,k-1,k-1}(\mu)\\ &\quad=\mathcal{X}^-_{j,k-1}(\mu-\varpi_{k-1})\mathcal{X}^-_{i,k-1}\prod_{t=i}^{j-1}\mu(\mathcal{H}_{t,k-1}-1-\delta_{t,i})\\ &\qquad+\mathcal{X}^-_{i,k-1}(\mu-\varpi_{k-1})\mathcal{X}^-_{j,k-1}(\mu)\prod_{t=i}^{j-1}\mu(\mathcal{H}_{t,k-1})+\operatorname{Ann}_{U(\mathfrak{g})}e_{\beta_{i,j}}\\ &\quad=\prod_{i\leqslant t\leqslant k-1,\ t\neq j}\mu(\mathcal{H}_{t,k-1}-1-\delta_{t,i})f_{k-1}\cdots f_j\\ &\qquad\times\bigg(\prod_{t=i+1}^{k-1}\mu(\mathcal{H}_{t,k-1})f_{k-1}\cdots f_i-\prod_{t=i+1}^{k-1}\big(\mu(\mathcal{H}_{t,k-1})+\delta_{t,j}\big)f_{j-1}\cdots f_if_{k-1}\cdots f_j\bigg)\\ &\qquad+\prod_{t=i+1}^{k-1}\big(\mu(\mathcal{H}_{t,k-1})-1\big)\prod_{i\leqslant t\leqslant k-1,t\neq j}\mu(\mathcal{H}_{t,k-1})f_{k-1}\cdots f_if_{k-1}\cdots f_j+\operatorname{Ann}_{U(\mathfrak{g})}e_{\beta_{i,j}}.\end{aligned}$$

Since $\mu(h_i) = \lambda(h_i) + 1 > 0$, $\mu(h_j) = \lambda(h_j) + 1 > 0$, the monomials

$$f_{k-2} \cdots f_i f_{k-1} \cdots f_j,\ f_{k-2} \cdots f_j f_{k-1} \cdots f_i$$

are μ-standard and hence the vectors

$$u_1 = f_{k-2} \cdots f_i f_{k-1} \cdots f_j v_\mu,\ u_2 = f_{k-2} \cdots f_j f_{k-1} \cdots f_i v_\mu$$

are linearly independent by Corollary 4.1. Clearly, $e_{k-1}^2 u_1 = e_{k-1}^2 u_2 = 0$, while $h_{k-1} u_r = (\mu(h_{k-1}) + 2) u_r$, $r = 1, 2$. Since $\mu(h_{k-1}) + 2 = \lambda(h_{k-1}) + 2 \geqslant 4$, it follows from the standard $\mathfrak{sl}_2$-theory that $f_{k-1} u_1$, $f_{k-1} u_2$ are non-zero and linearly independent. Therefore,

$$\begin{aligned} \mathbf{u}_{\beta_{k,k},\beta_{i,j}}(\mu) e_{\beta_{i,j}} &= \big(\mu(\mathcal{H}_{i,k-1}) - 1\big)^{-1} \big(\mu(\mathcal{H}_{j,k-1})\big)^{-1} \left(\frac{\mu(\mathcal{H}_{i,k-1}) - 2}{\mu(\mathcal{H}_{i,k-1})} + 1 \right) e_{\beta_{k,k}} \\ &= 2 \big(\mu(\mathcal{H}_{j,k-1}) \mu(\mathcal{H}_{i,k-1})\big)^{-1} e_{\beta_{k,k}}. \end{aligned}$$

Thus,

$$\begin{aligned} \Pi_\lambda(\beta_{i,j}, \beta_{k,k}) &= e_{\beta_{k,k}} \otimes e_{\beta_{i,j}} - (y_\lambda + 1)^{-1} e_{\beta_{j,k}} \otimes e_{\beta_{i,k}} \\ &\quad - \big((x_\lambda + y_\lambda + 2)(y_\lambda + 1)\big)^{-1} (y_\lambda e_{\beta_{i,k}} \otimes e_{\beta_{j,k}} - 2 e_{\beta_{i,j}} \otimes e_{\beta_{k,k}}). \end{aligned}$$

We immediately obtain the relations (6.24a) and (6.24b) (note that in this case $y_\lambda > 1$) and $\mathfrak{R}_\Psi(\lambda, \lambda + \eta)$ is easily seen to be generic. Finally, suppose that $0 < t_{\lambda,\eta} < 4$. Using 6.2(C3) we conclude that if $t_{\lambda,\eta} = 2$, the unique relation is (6.25), while if $t_{\lambda,\eta} = 1$, the unique path is a relation. □

6.13. Finally, we consider the case when $m_\eta = 6$, that is $\eta = \beta_{i,j} + \beta_{k,l} = \beta_{i,k} + \beta_{j,l} = \beta_{i,l} + \beta_{j,k} \in \Psi + \Psi$, $i < j < k < l \in I$. Let $x_\lambda = \lambda(\mathcal{H}_{i,j-1})$, $y_\lambda = \lambda(\mathcal{H}_{j,k-1})$, $z_\lambda = \lambda(\mathcal{H}_{k,l-1})$. Then $\lambda(\mathcal{H}_{i,k-1}) = x_\lambda + y_\lambda + 1$, $\lambda(\mathcal{H}_{j,l-1}) = y_\lambda + z_\lambda + 1$ and $\lambda(\mathcal{H}_{i,l-1}) = x_\lambda + y_\lambda + z_\lambda + 2$. Note that if $t_{\lambda,\eta} = 6$, we have $x_\lambda, y_\lambda, z_\lambda > 0$.

All technical difficulties in computing the $\Pi_\lambda(\beta, \beta')$ which occur here have already been discussed and we omit the details. Suppose first that $t_{\lambda,\eta} = 6$. We have

$$\begin{aligned} \Pi_\lambda(\beta_{k,l}, \beta_{i,j}) &= e_{\beta_{i,j}} \otimes e_{\beta_{k,l}}, \\ \Pi_\lambda(\beta_{j,l}, \beta_{i,k}) &= e_{\beta_{i,k}} \otimes e_{\beta_{j,l}} - (y_\lambda + 1)^{-1} e_{\beta_{i,j}} \otimes e_{\beta_{k,l}}, \\ \Pi_\lambda(\beta_{j,k}, \beta_{i,l}) &= e_{\beta_{i,l}} \otimes e_{\beta_{j,k}} - (z_\lambda + 1)^{-1} e_{\beta_{i,k}} \otimes e_{\beta_{j,l}} - \frac{z_\lambda}{(z_\lambda + 1)(y_\lambda + z_\lambda + 2)} e_{\beta_{i,j}} \otimes e_{\beta_{k,l}}, \\ \Pi_\lambda(\beta_{i,l}, \beta_{j,k}) &= e_{\beta_{j,k}} \otimes e_{\beta_{i,l}} - (x_\lambda + 1)^{-1} e_{\beta_{i,k}} \otimes e_{\beta_{j,l}} - \frac{x_\lambda}{(x_\lambda + 1)(x_\lambda + y_\lambda + 2)} e_{\beta_{i,j}} \otimes e_{\beta_{k,l}}, \\ \Pi_\lambda(\beta_{i,k}, \beta_{j,l}) &= e_{\beta_{j,l}} \otimes e_{\beta_{i,k}} - (z_\lambda + 1)^{-1} e_{\beta_{j,k}} \otimes e_{\beta_{i,l}} - (x_\lambda + 1)^{-1} e_{\beta_{i,l}} \otimes e_{\beta_{j,k}} \\ &\quad + \big((x_\lambda + 1)(z_\lambda + 1)\big)^{-1} e_{\beta_{i,k}} \otimes e_{\beta_{j,l}} \\ &\quad - \frac{x_\lambda z_\lambda}{(x_\lambda + 1)(z_\lambda + 1)(x_\lambda + y_\lambda + z_\lambda + 3)} e_{\beta_{i,j}} \otimes e_{\beta_{k,l}}, \end{aligned}$$

$$\begin{aligned}\Pi_\lambda(\beta_{i,j},\beta_{k,l}) &= e_{\beta_{k,l}}\otimes e_{\beta_{i,j}} - (y_\lambda+1)^{-1}e_{\beta_{j,l}}\otimes e_{\beta_{i,k}} - \frac{y_\lambda}{(y_\lambda+1)(y_\lambda+z_\lambda+2)}\,e_{\beta_{j,k}}\otimes e_{\beta_{i,l}}\\ &\quad - \frac{y_\lambda}{(y_\lambda+1)(x_\lambda+y_\lambda+2)}\,e_{\beta_{i,l}}\otimes e_{\beta_{j,k}}\\ &\quad - \frac{x_\lambda+y_\lambda+z_\lambda+3+(x_\lambda+y_\lambda+1)(y_\lambda+1)(y_\lambda+z_\lambda+1)}{(y_\lambda+1)(x_\lambda+y_\lambda+2)(y_\lambda+z_\lambda+2)(x_\lambda+y_\lambda+z_\lambda+3)}\,e_{\beta_{i,k}}\otimes e_{\beta_{j,l}}\\ &\quad + \frac{(x_\lambda+y_\lambda+1)(y_\lambda+z_\lambda+1)+(y_\lambda+1)(x_\lambda+y_\lambda+z_\lambda+3)}{(y_\lambda+1)(x_\lambda+y_\lambda+2)(y_\lambda+z_\lambda+2)(x_\lambda+y_\lambda+z_\lambda+3)}\,e_{\beta_{i,j}}\otimes e_{\beta_{k,l}}.\end{aligned}$$

Denote the paths in $\Delta_\Psi(\lambda,\lambda+\eta)$ by $\mathbf{p}_r$, $1\leqslant r\leqslant 6$, where the numbering corresponds to the order in which they appear above. Clearly, none of these paths is a relation. A direct computation shows that $\mathfrak{R}_\Psi(\lambda,\lambda+\eta)$ is spanned by

$$\begin{aligned}\mathbf{r}_1 &= (x_\lambda+1)(y_\lambda+2)(x_\lambda-z_\lambda)(z_\lambda+1)\mathbf{p}_1 + y_\lambda(x_\lambda+y_\lambda+1)(x_\lambda-z_\lambda)(y_\lambda+z_\lambda+2)\mathbf{p}_2\\ &\quad + z_\lambda(x_\lambda+1)(y_\lambda+1)(x_\lambda+y_\lambda+2)(y_\lambda+z_\lambda+1)\mathbf{p}_3\\ &\quad - x_\lambda(y_\lambda+1)(z_\lambda+1)(x_\lambda+y_\lambda+1)(y_\lambda+z_\lambda+2)\mathbf{p}_4,\\ \mathbf{r}_2 &= (x_\lambda+1)(z_\lambda+1)(x_\lambda+y_\lambda+2)(x_\lambda+z_\lambda+2)(y_\lambda+z_\lambda+3)\mathbf{p}_1\\ &\quad + y_\lambda(x_\lambda+1)(z_\lambda+2)(x_\lambda+y_\lambda+1)(y_\lambda+z_\lambda+2)(x_\lambda+y_\lambda+z_\lambda+3)\mathbf{p}_2\\ &\quad - (y_\lambda+1)(x_\lambda+y_\lambda+2)(x_\lambda+z_\lambda+2)(y_\lambda+z_\lambda+1)(x_\lambda+y_\lambda+z_\lambda+3)\mathbf{p}_3\\ &\quad - x_\lambda(y_\lambda+1)(z_\lambda+1)(x_\lambda+y_\lambda+2)(y_\lambda+z_\lambda+2)(x_\lambda+y_\lambda+z_\lambda+2)\mathbf{p}_5,\end{aligned}$$

and

$$\begin{aligned}\mathbf{r}_3 &= (y_\lambda+2)(z_\lambda+1)(x_\lambda+y_\lambda+2)(y_\lambda+z_\lambda+3)(x_\lambda+y_\lambda+z_\lambda+3)\mathbf{p}_1\\ &\quad - (z_\lambda+2)(x_\lambda+y_\lambda+1)(y_\lambda+z_\lambda+2)(x_\lambda+2y_\lambda+z_\lambda+4)\mathbf{p}_2\\ &\quad - z_\lambda(y_\lambda+1)(x_\lambda+y_\lambda+z_\lambda+3)(x_\lambda+2y_\lambda+z_\lambda+4)\mathbf{p}_3\\ &\quad - (y_\lambda+1)(z_\lambda+1)(x_\lambda+y_\lambda+1)(y_\lambda+z_\lambda+2)(x_\lambda+y_\lambda+z_\lambda+2)\mathbf{p}_6.\end{aligned}$$

One can now check that $\mathfrak{R}_\Psi(\lambda,\lambda+\eta)$ is generic unless $x_\lambda=z_\lambda$. In the latter case the first relation reduces to $\mathbf{p}_3-\mathbf{p}_4$.

Finally, we list the relations in cases when $0<t_{\lambda,\eta}<6$. By 6.2(C4), we have three cases with $t_{\lambda,\eta}=2$. If $y_\lambda=0$, or equivalently $j=k-1$ and $\lambda(h_j)=0$, while $x_\lambda,z_\lambda>0$, the relation is

$$(z_\lambda+2)(x_\lambda+z_\lambda+4)\mathbf{p}_2+(z_\lambda+1)(x_\lambda+z_\lambda+2)\mathbf{p}_6.$$

If $x_\lambda=0$, $y_\lambda,z_\lambda>0$ the relation is

$$(y_\lambda+1)(z_\lambda+2)(y_\lambda+z_\lambda+3)\mathbf{p}_4+(y_\lambda+2)z_\lambda(y_\lambda+z_\lambda+2)\mathbf{p}_5.$$

If $z_\lambda=0$, $x_\lambda,y_\lambda>0$ the relation is

$$(x_\lambda+2)(y_\lambda+1)(x_\lambda+y_\lambda+3)\mathbf{p}_3+x_\lambda(y_\lambda+2)(x_\lambda+y_\lambda+2)\mathbf{p}_5.$$

Thus, in all these cases the space $\mathfrak{R}_\Psi(\lambda,\lambda+\eta)$ is generic. Finally, if $x_\lambda = z_\lambda = 0$, $y_\lambda > 0$ the unique path $\mathbf{p}_5$ in $\Delta_\Psi(\lambda,\lambda+\eta)$ is not a relation.

Thus, we obtain the following

Proposition. *Let $i < j < k < l \in I$. Suppose that $\eta = \beta_{i,j} + \beta_{k,l} = \beta_{i,k} + \beta_{j,l} = \beta_{i,l} + \beta_{j,k} \in \Psi + \Psi$. Then $\dim \mathfrak{R}_\Psi(\lambda,\lambda+\eta) = \lfloor |\Delta_\Psi(\lambda,\lambda+\eta)/2| \rfloor$ and*

$$\mathcal{N}_\eta \subset P^+ \cap \left\{\xi \in \mathfrak{h}^*\colon\ \xi(\mathcal{H}_{i,j-1} - \mathcal{H}_{k,l-1}) = 0\right\}$$

and coincides with the latter set if Ψ is regular.

6.14. Let $\Psi = \Psi(i_1,\dots,i_k)$ be regular. It follows from Propositions 6.9, 6.10, 6.12 and 6.13 that the coefficients in all relations in $\mathfrak{R}_\Psi(\mu,\mu+\eta)$ depend on $\mu(\mathcal{H}_{i_r,i_s-1})$, $1 \leqslant r < s \leqslant n$.

Let $\lambda \in P^+$. By Proposition 6.4, $\Delta_\Psi[\lambda]$ is isomorphic to the quiver $\Xi_a(\boldsymbol{m})$, where $a = \sum_{r=1}^k \lambda(h_{i_r})$ (mod 2), $\boldsymbol{m} = (m_1,\dots,m_k)$, $m_r = \lambda(h_{i_r-1}) + \lambda(h_{i_r})$. Let

$$\zeta_r(\lambda) = \lambda(\mathcal{H}_{i_r+1,i_{r+1}-2}) + 2, \quad 1 \leqslant r < k.$$

Let $(x_1,\dots,x_k)$ be the image of $\mu \in \Delta_\Psi[\lambda]_0$ in $\Xi_a(\boldsymbol{m})_0$. Then

$$\mu(\mathcal{H}_{i_r,i_s-1}) = x_r - x_s + \sum_{p=r+1}^{s} m_p + \zeta_{p-1}(\lambda) + r - s - 1.$$

Thus, the isomorphism of algebras $\mathbf{T}_\Psi^{\mathfrak{g}} \to \Delta_\Psi$ gives rise to a family of relations on quivers $\Xi_a(\boldsymbol{m})$. The relations, and in particular their genericity, depend on a family of positive integer parameters $\zeta_p(\lambda)$. The resulting algebras are Koszul and of global dimension at most $p(p+1)/2$, where $p = \#\{j\colon\ m_j > 0\}$. It is finite dimensional if and only if $i_1 > 1$. The explicit relations can be easily written down using Propositions 6.9, 6.10, 6.12 and 6.13.

List of notations

I	1.1	α_i, ϖ_i	1.1	ε_i, φ_i, ε, φ	1.1	R, P, R^+, P^+	1.1
$\mathfrak{n}_\Psi^\pm$, $\mathfrak{n}^\pm$, $\mathfrak{b}$	1.1	$V(\lambda)$	1.2	$\mathbf{V}$, $\mathbf{V}^\circledast$	1.2	$\mathbf{A}$, $\mathbf{T}$, $\mathbf{S}$, $\mathbf{E}$	1.2
1_λ: $\lambda \in P^+$	1.2	$\leqslant_\Psi$, $\leqslant$	1.3	$d_\Psi(\lambda,\mu)$	1.3	$\mathbf{A}_\Psi^{\mathfrak{g}}(F)$, $\mathbf{A}_\Psi^{\mathfrak{g}}$	1.3
$\leqslant_\Psi \lambda$, $\lambda \leqslant_\Psi$, $[\lambda,\mu]_\Psi$	1.3	Δ_0, Δ_1, $\bar{\Delta}$	1.4	$x^\pm$, $x \in \Delta_0$	1.4	$\mathbf{C}\Delta$	1.4
$\Delta_\Psi(F)$, Δ_Ψ	1.5	$\mathfrak{R}_\Psi(\lambda,\lambda+\eta)$	1.6	m_η, $t_{\lambda,\eta}$	1.7	$\mathcal{N}_\eta$	1.7
$\lvert\boldsymbol{x}\rvert$, $\boldsymbol{x} \in \mathbf{Z}_+^r$	1.8	$\boldsymbol{e}_i^{(r)}$	1.8	$\Xi(\boldsymbol{m})$, $\Xi_a(\boldsymbol{m})$	1.8	e_i, f_i, h_i	2.1
v_λ, $\xi_{-\lambda}$, M^λ	2.1	$\Pi_\lambda(\beta,\beta')$	2.4	$F_\beta(\mathfrak{h})$	2.6	π_λ, $\pi_{\lambda,\beta}$	2.6
$\mathbf{u}_{\beta,\gamma}$, $\mathbf{u}_{\beta,\gamma}(\lambda)$	2.7	$\alpha_{i,j}$	4.2	$\Sigma(i,j)$	4.2	$\mathbf{f}_\sigma$	4.2
ψ_η	4.3	$\mathcal{H}_{i,j}$	4.3	$\mathcal{X}_{i,j,k}^\pm$, $\mathcal{X}_{i,j}^\pm$	4.3	$\Gamma_a(\boldsymbol{m},\boldsymbol{n})$	5.3
$e_{i,j}$	5.5	$\mathbf{u}_{\alpha_{i,j},\alpha_{p,q}}$	5.5	$\mathcal{Z}_{\alpha_{i,j},\Psi}$	5.6	$\beta_{i,j}$	6.1
$e_{\beta_{i,j}}$	6.5	$\mathcal{U}_{r,s,i,j}$	6.6	$\bar{\mathcal{U}}_{r,s,i,j}$	6.7	$\mathbf{u}_{\beta_{i,j},\beta_{r,s}}$	6.8

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