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Dynamic homogenization of acoustic metamaterials with coupled field response

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Abstract

Acoustic metamaterials (AMM) are heterogeneous materials with dynamic subwavelength structures that can generate useful effective responses of interest to ultrasonic imaging applications such as negative refraction and zero index. Traditional effective medium models fail to capture details of frequency dependent AMM response and can give non-causal properties. This work derives non-local expressions for effective properties for an infinite periodic lattice of heterogeneities in an isotropic fluid using conservation of mass and momentum and the equation of state. The resulting model correctly predicts a causal effective material response by considering coupling between the ensemble-averaged volume strain and momentum fields.

Keywords: acoustic metamaterial; homogenization; Willis-solid; bianisotropy

1. Introduction

Acoustic metamaterials (AMM) are heterogeneous materials that often consist of dynamic subwavelength structures in a background medium. AMM are of interest in ultrasonic imaging applications because of their potential to generate negative refraction and zero-index. However, traditional effective medium models fail to properly capture the frequency dependent response of these materials. Generally, simple unit cell volume averaging

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techniques are employed to estimate the effective density and compressibility (or bulk modulus). This procedure leads to mass density and compressibility estimates that neglect non-local contributions to the overall response. Because AMM often employ sub-wavelength resonances to generate extreme parameters, non-local effects can be non-negligible and should be considered. Further, when either of these parameters become large, the wavelength becomes small and the complex field amplitudes in the medium can no longer approximated by a simple volume average of the microscopic field. Recent dynamic homogenization schemes in heterogeneous electromagnetic (EM) and elastic materials indicate that EM bianisotropy and elastic momentum-strain and stress-velocity field coupling is required to correctly describe the effective behavior of metamaterials [1-3]. Notably, the determination of material coupling terms in EM resolves apparent violations of causality and passivity that is present in earlier models [4]. This work employs a source-driven homogenization scheme to derive expressions for effective properties of an infinite periodic lattice of fluid heterogeneities in fluid medium from first principles. The result is a physically meaningful effective material response that explicitly shows the existence of coupling between the ensemble-averaged volumetric strain and momentum fields.

2. Microscopic and averaged fields

Consider a lossless homogeneous fluid characterized by mass density, ρ_0, and adiabatic compressibility, β_0, containing externally controlled, distributed sources with complex time-harmonic body forces, $\vec{f}_{\mathrm{ext}}$, and volume sources, q_{ext}. In all cases, $e^{i\vec{k}\cdot\vec{r}}e^{-i\omega t}$ plane wave dependence is assumed. The resulting complex amplitudes for the acoustic pressure and particle velocity fields are determined from the momentum and mass conservation equations:

$$i\vec{k}p_{\mathrm{ext}} = i\omega\rho_0\vec{u}_{\mathrm{ext}} + \vec{f}_{\mathrm{ext}}, \qquad i\vec{k}\cdot\vec{u}_{\mathrm{ext}} = i\omega\beta_0 p_{\mathrm{ext}} + q_{\mathrm{ext}}. \tag{1}$$

Space and time dependencies have been suppressed. Now introduce a periodic array of scatters into the fluid containing the sources. The conservation equations allow for the exact evaluation of the microscopic fields at any point in the medium which can be expressed in terms of monopole and dipole polarizations, B and $\vec{\mathrm{P}}$, respectively,

$$\nabla p(\vec{r}) = i\omega\rho_0\vec{u}(\vec{r}) + i\omega\vec{\mathrm{P}}(\vec{r}) + \vec{f}_{\mathrm{ext}}e^{i\vec{k}\cdot\vec{r}}, \qquad \nabla\cdot\vec{u}(\vec{r}) = i\omega\beta_0 p(\vec{r}) - i\omega\mathrm{B}(\vec{r}) + q_{\mathrm{ext}}e^{i\vec{k}\cdot\vec{r}}. \tag{2}$$

The polarizations are functions of the contrast in material density and compressibility and provide physical insight into how the inclusions alter the overall response of the heterogeneous medium, facilitating the derivation of effective parameters. Equation (3) gives the conservation relations for the ensemble-averaged fields as a function of the background material properties, ρ_0 and β_0, the appropriately averaged polarizations, and the same externally-controlled sources considered in (2), though the averaged field amplitudes are independent of the actual source distribution in general. Note that these averaged field amplitudes are not merely spatial averages of the microscopic fields over some unit cell, as these cannot account for non-local effects which become significant when extreme parameters are present. As shown in [1], true averaged field amplitudes can be analytically determined by using the Taylor expansion of the spatial dependence $e^{i\vec{k}\cdot\vec{r}}$, accounting for multiple scattering effects, and combining associated terms.

$$i\vec{k}p_{\mathrm{av}} = i\omega\rho_0\vec{u}_{\mathrm{av}} + i\omega\vec{\mathrm{P}}_{\mathrm{av}} + \vec{f}_{\mathrm{ext}} \qquad i\vec{k}\cdot\vec{u}_{\mathrm{av}} = i\omega\beta_0 p_{\mathrm{av}} - i\omega\mathrm{B}_{\mathrm{av}} + q_{\mathrm{ext}} \tag{3}$$

The dynamic equations in (3) describe the acoustic variables in an effective fluid that varies from the background fluid. In this approach, the averaged polarization terms $\vec{\mathrm{P}}_{\mathrm{av}}$ and B_{av} are used to account for material property variations. The interpretation of the polarization terms determines the constitutive relation. Then, the averaged momentum density, $\vec{\pi}_{\mathrm{av}} = \rho_0\vec{u}_{\mathrm{av}} + \vec{\mathrm{P}}_{\mathrm{av}}$, and volume strain, $\varepsilon_{\mathrm{av}} = -\beta_0 p_{\mathrm{av}} + \mathrm{B}_{\mathrm{av}}$, are defined as auxiliary acoustic fields in the homogenized media. Equation (4) presents two possible approximations for auxiliary fields. The first assumed that the auxiliary field is proportional to a single primary field. This is known as the *equivalent* parameter approximation. The second assumes that there is a contribution from both primary fields to both auxiliary fields.

This is known as the *effective* parameter approximation. The constitutive parameters allow for an anisotropic mass density tensor and general macroscopic coupling vectors: $\vec{\eta}_{\text{eff}}$ and $\vec{\gamma}_{\text{eff}}$. In the following section, these macroscopic coupling terms will be derived from properties of the lattice;

$$\tilde{\mathrm{n}}_{\text{av}} = \boldsymbol{\rho}_{\text{eq}} \cdot \vec{u}_{\text{av}} = \boldsymbol{\rho}_{\text{eff}} \cdot \vec{u}_{\text{av}} + \vec{\gamma}_{\text{eff}} p_{\text{av}}, \qquad \varepsilon_{\text{av}} = -\beta_{\text{eq}} p_{\text{av}} = \vec{\eta}_{\text{eff}} \cdot \vec{u}_{\text{av}} - \beta_{\text{eff}} p_{\text{av}}. \tag{4}$$

3. Unit cell and lattice interaction

The polarizations, $\vec{\mathrm{P}}_{\text{av}}$ and B_{av} , are assumed to be directly related to the scattered monopole and dipole fields generated at the center of the lmn^{th} unit cell. Those monopole and dipole contributions to the polarizations are denoted as b_{lmn} and $\vec{\mathrm{p}}_{lmn}$, respectively, where b_{lmn} has units of volume, and $\vec{\mathrm{p}}_{lmn}$ has units of momentum. In terms of the local fields, the overall polarizations in the unit cell centered at the origin can be expressed through effective polarizabilities, α_x , which have units of volume and characterize the response of the inclusions to the local field. This is shown explicitly in Eq. (5). Anisotropy in mass density is introduced through the second order tensor $\boldsymbol{\alpha}_u$, and coupling between moments due to asymmetries in the unit cell are accounted for via $\vec{\alpha}_{\text{c}}$. All polarizabilities must follow reciprocity as it is induced by the microscopic local fields [1].

$$V_{\text{cell}} \vec{\mathrm{P}}_{\text{av}} / \rho_0 = \vec{\mathrm{p}}_{000} / \rho_0 = \boldsymbol{\alpha}_u \cdot \vec{u}_{\text{loc}} + i\vec{\alpha}_{\text{c}} p_{\text{loc}} / Z_0 \qquad V_{\text{cell}} \mathrm{B}_{\text{av}} / \beta_0 = \mathrm{b}_{000} / \beta_0 = -\alpha_p p_{\text{loc}} + i\vec{\alpha}_{\text{c}} \cdot \vec{u}_{\text{loc}} Z_0 \tag{5}$$

The induced moments can be treated as momentum and volume source terms at each inclusion location and can be written using the Floquet condition as $\vec{\mathrm{p}}_{lmn} = \vec{\mathrm{p}}_{000} e^{i\vec{k}\cdot\vec{r}_{lmn}}$ and $\mathrm{b}_{lmn} = \mathrm{b}_{000} e^{i\vec{k}\cdot\vec{r}_{lmn}}$, where $\vec{r}_{lmn} = (ld, md, nd)$ are the locations of the inclusions, and d is the representative length of the unit cell, *i.e.* $V_{\text{cell}} = d^3$. The local field is thus the sum of the external field and the scattering from the rest of the array: *i.e.* $p_{\text{loc}} = p_{\text{array}} + p_{\text{ext}}$. The field contributions from the remainder of the array not at the origin can be calculated using the free space Green's functions and the Floquet condition, by treating the inclusions as point sources with monopole and dipole contributions. Noting that $\vec{\mathrm{p}}_{000} = \hat{\mathrm{p}} \mathrm{p}_{000}$, the local pressure and particle velocity at the origin can therefore be expressed using monopole-monopole, dipole-dipole, and coupled lattice interaction coefficients, $C_p = \sum k_0^2 g(0 \mid \vec{r}_{lmn}) e^{i\vec{k}\cdot\vec{r}_{lmn}}$, $C_u = -\sum \hat{\mathrm{p}} \cdot \nabla\nabla g(\vec{r} \mid \vec{r}_{lmn})|_{\vec{r}=0} \, e^{i\vec{k}\cdot\vec{r}_{lmn}} \cdot \hat{\mathrm{p}}$, and $C_{\text{c}} = -\sum ik_0 \nabla g(\vec{r} \mid \vec{r}_{lmn})|_{\vec{r}=0} \, e^{i\vec{k}\cdot\vec{r}_{lmn}} \cdot \hat{\mathrm{p}}$, respectively, where summations are performed over all $(l,m,n) \neq (0,0,0)$. As a result, multiple scattering effects yield,

$$\vec{u}_{\text{loc}} \cdot \hat{\mathrm{p}} = \frac{\mathrm{p}_{000}}{\rho_0} C_u - \frac{1}{Z_0} \frac{\mathrm{b}_{000}}{\beta_0} C_{\text{c}} + \vec{u}_{\text{ext}} \cdot \hat{\mathrm{p}}, \qquad p_{\text{loc}} = -\frac{\mathrm{b}_{000}}{\beta_0} C_p + Z_0 \frac{\mathrm{p}_{000}}{\rho_0} C_{\text{c}} + p_{\text{ext}}. \tag{6}$$

4. 1D periodic array

For a 1D lossless array with period d , inclusion thickness δ , and unit cell volume $V_{\text{cell}} = Sd$, reduced lattice interaction coefficients arise from subtracting (1) from (3) to eliminate source terms and substituting (6) for the external fields. The result can be solved exactly, as shown in Eq. (7). The normalized wavenumber for the array, kd , can be solved for using Bloch-wave theory or by combining (5) and (6) and setting the external sources to zero.

$$V_{\text{cell}} C_p' = V_{\text{cell}} C_u' = -\frac{k_0 d}{2}\left[i + \frac{\sin k_0 d}{\cos kd - \cos k_0 d} \right] - \frac{(k_0 d)^2}{(kd)^2 - (k_0 d)^2}, \qquad V_{\text{cell}} C_{\text{c}}' = -\frac{k_0 d}{2} \frac{\sin kd}{\cos kd - \cos k_0 d} - \hat{\mathrm{p}} \cdot \hat{k} \frac{(kd)(k_0 d)}{(kd)^2 - (k_0 d)^2} \tag{7}$$

Sufficiently thin inclusions can be modeled as sheet monopole and dipole scatterers. In addition, if the scatterer is symmetric about its geometric midpoint, then $\vec{\alpha}_{\text{c}} = 0$. In this case, the reflection coefficient from a single isotropic slab can be given by $R = ik_0(\alpha_p - \alpha_u)/2S$. Equating this with the reflection coefficient for a homogeneous slab of thickness $\delta \to 0$ [5], one finds Eq. (8) for the quasi-static polarizabilities. The imaginary parts of the inverse polarizabilities are significant. They represent the re-radiation of energy by the inclusion and in a lossless array, must cancel the imaginary parts of the interaction coefficients. These re-radiation terms differentiate the quasi-static polarizabilities from static estimates of unit cell polarizability.

$$V_{\text{cell}}\alpha_p^{-1} = \frac{d}{\delta}\frac{\beta_0}{\beta_1} - i\frac{k_0 d}{2} \qquad V_{\text{cell}}\alpha_u^{-1} = \frac{d}{\delta}\frac{\rho_0}{\rho_1} - i\frac{k_0 d}{2} \tag{8}$$

Combining Eqs. (1), (3), (5), and (6), one can solve expressions for $\vec{P}_{\text{av}}$ and B_{av} in terms of lattice interaction coefficients and inverse polarizabilities. Terms from those relations are matched with the constitutive relations in Eq. (4) to find the *effective* parameters. The resulting coupling is non-zero for finite C_c' even in the quasi-static limit, as presented in Eq. (9). Note that the non-local coupling parameters are reciprocal by being odd in $\vec{k}$.

$$c_0\vec{\gamma}_{\text{eff}}\cdot\hat{\text{p}} = -c_0\vec{\eta}_{\text{eff}}\cdot\hat{\text{p}} = c_0\chi_{\text{eff}}^{\text{o}} = \frac{V_{\text{cell}}C_c'}{\left[V_{\text{cell}}\alpha_p^{-1} - V_{\text{cell}}C_p'\right]\left[V_{\text{cell}}\alpha_u^{-1} - V_{\text{cell}}C_u'\right] - \left(V_{\text{cell}}C_c'\right)^2} \tag{9}$$

The *effective* and *equivalent* properties for a metamaterial with dynamic compressibility are presented in Fig. 1. From the *equivalent* complex density, the medium appears to be non-causal, negative slope in (a), and non-passive, negative imaginary part in (b). However, these non-physical artifacts do not appear in the *effective* properties, which account for lattice coupling [4].

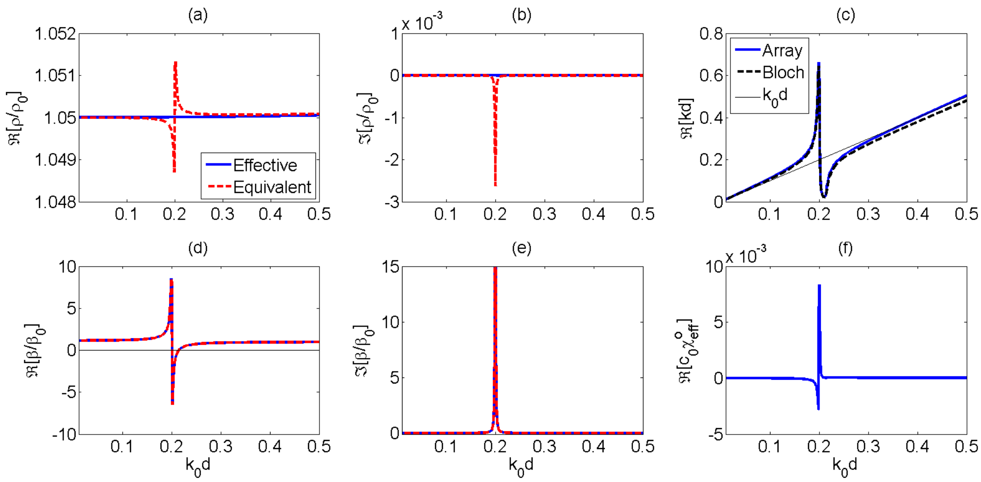

Fig. 1. 1D periodic medium containing 2% by volume of dynamic effective stiffness material slabs: (a, b) complex density, (c) comparison of eigenmodal wavenumber of array and Bloch-wavenumber, (d, e) effective and equivalent complex compressibility, and (f) coupling parameter.

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