Discrete Applied Mathematics 39 (1992) 113-123
North-Holland

The complexity of lifted inequalities for the knapsack problem

David Hartvigsen and Eitan Zemel

Department of Managerial Economics and Decision Science, J.L. Kellogg Graduate School of Management, Northwestern University, Evanston, IL 60208, USA

Received 22 February 1989
Revised 27 August 1990

Abstract

Hartvigsen, D. and E. Zemel, The complexity of lifted inequalities for the knapsack problem, Discrete Applied Mathematics 39 (1992) 113-123.

It is well known that one can obtain facets and valid inequalities for the knapsack polytope by lifting simple inequalities associated with minimal covers. We study the complexity of lifting. We show that recognizing integral lifted facets or valid inequalities can be done in $O(n^2)$ time, even if the minimal cover from which they are lifted is not given. We show that the complexities of recognizing nonintegral lifted facets and valid inequalities are similar, respectively, to those of recognizing general (not necessarily lifted) facets and valid inequalities. Finally, we show that recognizing valid inequalities is in co-NPC while recognizing facets is in D^p. The question of whether recognizing facets is complete for D^p is open.

Keywords. Knapsack polytope, facets, valid inequalities, computational complexity, lifting, polynomial algorithms.

1. Introduction

The facial structure of combinatorial optimization problems has been studied extensively in the last few decades. Indeed, there has been a great deal of progress during this period in our level of familiarity with, and ability to use, the facets of problems such as the matching problem [6,7], the travelling salesman problem (e.g. [5,22]), as well as the set packing and covering problems, the knapsack problem, etc. For surveys of various aspects of this research see [9,11,21].

Recently, the scope of these studies has been extended to include computational complexity issues. A major contribution in this direction is the work of [10,13,17],

Correspondence to: Professor E. Zemel, Department of Managerial Economics and Decision Sciences, J.L. Kellogg Graduate School of Management, Northwestern University, Evanston, IL 60208, USA.

which indicates that the problem of separating a given point from the convex hull of solutions is in the same complexity class as the underlying optimization problem. More recently, the complexity of recognizing facets is studied in [18,19] for the travelling salesman polytope. It is shown that the task of recognizing a facet for this polytope is complete for D^p—a complexity class higher than NP and defined specifically to deal with facets of polytopes.

The facets of the knapsack polytope have been studied extensively for the last 15 years and their structural properties are well understood. Facets and valid inequalities for this polytope are useful, since for any 0-1 integer programming problem, each constraint individually, and each individual aggregation of several constraints, can be regarded as a knapsack inequality. Thus, facets and valid inequalities for the knapsack polytope can be used as strong cutting plane inequalities for the general integer programming problem. This approach was utilized effectively, for example, in [4]. Naturally, in this context, the computational difficulty of computing or recognizing a facet or a valid inequality of a given type is a critical issue.

A first step towards examining the computational complexity of facets and valid inequalities for the knapsack problem was taken in [25]. It was shown that for a *given* minimal cover S, one can easily compute or recognize a certain restricted family of facets derived from S by a procedure called sequential lifting [1,2,12,15,16,20,24]. These facets are characterized by having a certain *integrality* property, and are generally recognized as useful for cutting plane algorithms. However, the results of [25] are restricted in two major ways. First, [25] requires that the minimal cover S be specified in advance. In addition, the analysis is limited to integral inequalities. Both these assumptions are quite restrictive since, in the context of a cutting plane algorithm, one would like to have as much flexibility as possible in the choice of a cutting plane inequality.

In this paper we examine the effects of these two restrictions on the computational complexity of recognizing facets and valid inequalities. Roughly speaking, our findings are that the first restriction can be easily handled while the second has severe computational implications. In addition, our analysis yields some surprising (although computationally irrelevant) relations between the complexity of recognizing facets, valid inequalities and (not necessarily valid) inequalities of certain types.

2. Preliminaries

Consider the inequality

$$\sum_{j\in N} a_j x_j \le a_0 \tag{1}$$

where $N=\{1,\dots,n\}$, $0<a_j\le a_0$ for $j\in N$, $\sum_{j\in N} a_j > a_0$, and the x_j are restricted to

0 or 1. The knapsack polytope P is the convex hull of 0-1 points satisfying (1). We assume that the coefficients a_j, $j \in N$, are integers, but the complexity results derived below do not depend on the size of these coefficients except via the standard assumption that arithmetic operations such as additions and comparisons could be carried out on these coefficients in constant time. Thus, all our polynomial algorithms are in fact strongly polynomial.

An inequality

$$\sum_{j \in N} b_j x_j \le b_0 \tag{2}$$

is called *valid* for P if it is satisfied by every $x \in P$. The only interesting case is where $b_j \ge 0$, $j \in N$, $b_0 > 0$, and we restrict ourselves in this paper to such inequalities. A valid inequality is a *facet* of P if it is satisfied at equality by n affinely independent points $x \in P$.

A set $S \subseteq N$ is called a *cover* for P if $\sum_{j \in S} a_j > a_0$, and is called a *minimal cover* if, in addition, no proper subset of S is a cover. We denote by s the cardinality of a minimal cover S. Let MC be the set of minimal covers for P.

Let S be a minimal cover. An inequality (2) (not necessarily valid) is called *lifted from S* if it can be scaled to the form

$$\sum_{j \in S} x_j + \sum_{j \in N \setminus S} c_j x_j \le s - 1. \tag{3}$$

We let $L(S)$ be the set of lifted inequalities from S.

An inequality (2) lifted from S is called *integral* with respect to S if all its coefficients are integers, when scaled to the form (3). Denote the set of integral inequalities with respect to S by LI(S) and let $\mathrm{LI} = \bigcup_{S \in \mathrm{MC}} \mathrm{LI}(S)$. We let the set of valid inequalities and the set of facets for P be denoted V and F, respectively. For

Table 1

Name	Class
FLI(S)	integral facets, lifted from S
VLI(S)	integral valid inequalities, lifted from S
FLI	integral facets lifted from some minimal cover S
VLI	integral valid inequalities lifted from some minimal cover S
FL(S)	facets, lifted from S
VL(S)	valid inequalities, lifted from S
FL	facets lifted from some minimal cover S
VL	valid inequalities, lifted from some minimal cover S
F	facets
V	valid inequalities

$S \in \mathrm{MC}$, let VL(S) and FL(S), be $V \cap L(S)$ and $F \cap L(S)$, respectively, and let VLI(S) and FLI(S) be $V \cap \mathrm{LI}(S)$ and $F \cap \mathrm{LI}(S)$, respectively. Finally, let VL, FL, VLI and FLI be $\bigcup_{S \in \mathrm{MC}} \mathrm{VL}(S)$, $\bigcup_{S \in \mathrm{MC}} \mathrm{FL}(S)$, $\bigcup_{S \in \mathrm{MC}} \mathrm{VLI}(S)$ and $\bigcup_{S \in \mathrm{MC}} \mathrm{FLI}(S)$, respectively.

We summarize our definitions of the different classes of inequalities in Table 1.

Let $S \in \mathrm{MC}$. Obviously, $\mathrm{VL}(S) \supseteq \mathrm{FL}(S) \supseteq \mathrm{FLI}(S)$. It is known [14] that these sets are nonempty, i.e., that there exist *integral* facets of P which are of the form (3). A *sequential* procedure for computing such facets, in which the lifting coefficients, $c_j : j \in N \setminus S$, are computed one by one, is given by Padberg [15,16]. It is known that every integral facet lifted from S can, in fact, be obtained from S by applying Padberg's procedure to some sequence of $N \setminus S$ [2]. This means that by enumerating all sequences of $N \setminus S$ and applying Padberg's procedure to each sequence, one generates exactly the class FLI(S). It is also known [2,24], that there exist lifted facets from S which are not integral, i.e., in general $\mathrm{FL}(S) \not\supseteq \mathrm{FLI}(S)$. Such nonintegral facets cannot be obtained from S by the sequential lifting algorithm of [15,16]. A generalized lifting procedure, which can account for the entire set FL(S), is given in [2,24]. This procedure can be coupled with any algorithm for enumerating MC to yield the entire set of lifted inequalities FL. In general, $\mathrm{FL} \not\subseteq F$, i.e., there may exist facets of P which are not lifted from any $S \in \mathrm{MC}$. A *lifting and complementing* procedure for generating the entire set F is given in [3].

The computational complexity of Padberg's procedure and of recognizing the sets FLI(S) and VLI(S) was studied in [25]. Specifically, given a minimal cover S and a given sequence of $N \setminus S$, a simple dynamic programming algorithm is given for computing the corresponding integral lifted facet. The running time of this algorithm is $\mathrm{O}(ns) \le \mathrm{O}(n^2)$. More surprisingly, the two recognition problems:

- $F1$: Is (2) in FLI(S)?
- $V1$: Is (2) in VLI(S)?

can also be resolved in $\mathrm{O}(ns)$ time. The reader may note that these two recognition problems seem to require examining every sequence of $N \setminus S$ in order to check whether the inequality in question is the one corresponding to lifting in that particular sequence. However, a key observation in [25] is that if (2) is integral with respect to S, one can identify from its representation (3) a unique candidate sequence of $N \setminus S$ which needs to be examined. Applying Padberg's procedure to that sequence, one gets a facet of P. A simple comparison between the original inequality (2) (in its form (3)) and the facet obtained in this fashion enables one to recognize whether (2) is a facet or is valid. The entire computation can still be carried out in $\mathrm{O}(ns)$ time since the identification of the sequence requires linear time.

In this paper we study the computational complexity of recognizing other classes of inequalities for P. In particular, we study the complexity of the following recognition problems:

- $F2$: Is (2) in FLI?

- $V2$: Is (2) in VLI?
- $F3$: Is (2) in FL?
- $V3$: Is (2) in VL?
- $F4$: Is (2) in F?
- $V4$: Is (2) in V?

The reader may note that the first pair arises by not specifying in advance the minimal cover S, the second pair by removing the integrality requirement, and the third pair by removing the requirement that the inequality be a lifting. We find that $F2$ and $V2$ can be resolved in $O(n^2)$ time (see Section 3) and that $F3$, $V3$, $F4$ and $V4$ are closely related (see Section 4). In particular, we show that $V3$ and $V4$ are co-NP-complete and $F3$ and $F4$ are in D^P. In the concluding section we study the effect of removing the requirement of validity.

3. Recognizing integral lifted facets and valid inequalities

In this section we study the complexity of recognizing lifted integral valid inequalities and facets when the minimal cover S is not specified in advance, i.e., recognizing the classes FLI and VLI. It turns out that these tasks are closely related to the tasks of recognizing FLI(S) or VLI(S) for a given minimal cover S. The following lemma handles the cases of facets and valid inequalities in a unified way. It basically asserts that, computationally, it does not matter whether or not S is specified in advance.

Lemma 3.1. *Let Q be F or V. Let $f(n)$ be the complexity of recognizing $Q\cap L(S)$ and let $f'(n)$ be the complexity of recognizing $Q\cap L$. Similarly, let $g(n)$ be the complexity of recognizing $Q\cap \mathrm{LI}(S)$ and let $g'(n)$ be the complexity of recognizing $Q\cap \mathrm{LI}$. Then,*

(3.1.1) $f(n)-O(n)\le f'(n)\le f(n)+O(n\log n)$,

(3.1.2) $g(n)-O(n)\le g'(n)\le g(n)+O(n^2)$.

Proof. We first prove (3.1.1). Let A be an algorithm which, given an inequality (2) and a minimal cover S, can decide in time not exceeding $f(n)$ whether or not (2) is in $Q\cap L(S)$. We wish to use this algorithm to recognize $Q\cap L$, i.e., to assert whether or not a minimal cover $S\in \mathrm{MC}$ exists such that (2) is in $Q\cap L(S)$. The general idea is to apply to (2) a preprocessing algorithm B, which identifies a unique minimal cover S if it exists and which has the following properties:

(i) If B fails, then (2) is not in $Q\cap L$, i.e., there exists no minimal cover S such that (2) is in $Q\cap L(S)$.

(ii) If B succeeds, then it produces a minimal cover S such that (2) is in $Q\cap L$ iff it is in $Q\cap L(S)$.

The running time of the algorithm B is $O(n\log n)$. Thus, in case (ii) algorithm A

can be applied to complete the recognition of $Q \cap L$. The total running time of the combined algorithm is then $f(n) + O(n \log n)$.

Algorithm B works as follows: Let $d_1, \dots, d_r$ be the distinct values in the multiset $b_1, \dots, b_n$. For each index $i = 1, \dots, r$, let $N_i = \{j \in N: b_j = d_i\}$ and $s_i = b_0/d_i + 1$. Call the index i a *success* if it satisfies the following conditions:

(a) s_i is an integer;

(b) the set T_i of s_i smallest elements in the set $\{a_j: j \in N_i\}$ is a minimal cover.

If no success is found, then algorithm B fails. Otherwise, any minimal cover T_i associated with a successful index i can be used as S of (ii). We claim that algorithm B indeed satisfies properties (i) and (ii). We first show (i), i.e., if B fails, then (2) is not in $Q \cap L$. In fact, we prove that (2) is not in $V \cap L$. Assume that (2) is valid and that there exists a minimal cover S such that (2) is in $L(S)$. Then, when algorithm B is applied, all the elements of S must be in the same subset N_i and $s_i = s$. Consider the restriction P_i of P to the index set N_i and the corresponding restriction of (2). Since (2) is valid for P, its restriction to N_i must be valid for P_i. Thus, any set of s elements in N_i is a cover. Since S is a minimal cover, and the elements of T_i are the smallest in N_i, it follows that T_i is a minimal cover and thus i is a success, contrary to the assumption that algorithm B fails. To see (ii), assume that at least two indices, say $i \neq k$, constitute a success so that (2) is in $L(T_i) \cap L(T_k)$. Both these sets are then candidates for the role of S. We have to show that it is enough to examine any one of these sets, say T_i. But this is obvious since if (2) is not in $L(T_i) \cap V$, then, being in $L(T_i)$, it cannot be in V and therefore it is not in $L(T_k) \cap V$. Thus, $f(n') \leq f(n) + O(n \log n)$.

To show the similar inequality for g we modify the definition of success in the specification of algorithm B to include the additional requirement:

(c) The coefficients b_j/d_i, $j \in N \setminus N_i$, are integers.

The only difference between cases (3.1.1) and (3.1.2) is that the verification of condition (c) may require $O(n^2)$ steps. Thus, $g'(n) \leq g(n) + O(n^2)$.

Let us show the reverse inequality for (3.1.1). Assume we are given an algorithm for recognizing $Q \cap L$ of complexity $f'(n)$. Given a minimal cover S, one can check in $O(n)$ time if (2) is in $L(S)$. If the answer is no, then (2) is not in $Q \cap L(S)$. If the answer is yes, then (2) is in $Q \cap L(S)$ iff (2) is in $Q \cap L$. This can be checked in time $f'(n)$. Hence, $f(n) - O(n) \leq f'(n)$. To show the reverse inequality for (3.1.2), we proceed as above except that we must check if (2), when in form (3) relative to s, is integral. □

It is shown in [25] that given S, one can easily recognize in $O(n^2)$ time whether an inequality (3) is valid or is a facet, provided one is restricted to integral inequalities. Theorem 3.2 which follows easily out of Lemma 3.1 implies that this remains the case even if S is not specified.

Theorem 3.2. *The following recognition problems can be solved in* $O(n^2)$ *time*:

- $V2$: *Is* (2) *in* VLI?
- $F2$: *Is* (2) *in* FLI?

However, if we wish to consider *nonintegral* lifted inequalities, the situation is markedly different. This is established in the next section.

4. Recognizing general facets and valid inequalities

We now remove the integrality restriction and consider the complexity of computing a general lifted facet or valid inequality. As mentioned earlier, nonintegral lifting can be obtained from a minimal cover S via the generalized lifting procedure [2,24], but not via the sequential procedure of [15,16] which yields only integral inequalities.

We have already established in Lemma 3.1 that for lifted inequalities the difficulty does not depend on whether the minimal cover S is specified, i.e., that VL and VL(S) are essentially of the same complexity and similarly for FL and FL(S). We now show that the complexity of these classes is essentially the same as that of V and F, respectively. Thus, in general, the task of recognizing a general valid inequality or facet is not easier if the inequality in question is lifted, even if the minimal cover from which it is lifted is given in advance. Again, we handle facets and valid inequalities simultaneously in the following lemma.

Lemma 4.1. *Let $f(n)$ be the complexity of recognizing* VL(S) (FL(S)), *and let $f'(n)$ be the complexity of recognizing V (F). Then*

$$f(n) - O(n) \le f'(n) \le f(n+2).$$

Proof. Let P be a knapsack polytope associated with (1), and let (2) be a given inequality for P, scaled so that $b_0 = 1$. Consider the auxiliary knapsack inequality

$$\sum_{j \in N} a_j x_j + a_0 x_{n+1} + a_0 x_{n+2} \le a_0$$

with its associated polytope P'. Obviously, the set $S' = \{n+1, n+2\}$ is a minimal cover for P', with $s' = 2$. Consider the inequality

$$\sum_{j \in N} b_j x_j + x_{n+1} + x_{n+2} \le 1. \tag{2'}$$

Then (2′) is lifted from S'. However, it can be easily verified that (2) is valid for (is a facet of) P iff (2′) is the same for P'. Hence, $f'(n) \le f(n+2)$. The reverse inequality can be proved as in Lemma 3.1. □

It is easy to establish that recognizing valid inequalities is co-NP-complete. We do this by a transformation from the following Subset Sum problem, as in [8].

$P1$: *Given*: Positive integers $d_1, \ldots, d_m, e$.

Question: Does there exist a subset $U \subseteq \{1, \ldots, m\}$ such that $\sum_{i \in U} d_i = e$?

Theorem 4.2. *The following recognition problems are* co-NP-*complete*:

- $V3'$: *is* (2) *in* VL(S)?
- $V3$: *Is* (2) *in* VL?
- $V4$: *Is* (2) *in* V?

Proof. All three problems are obviously in co-NP. In view of Lemmas 3.1 and 4.1 it is sufficient to show that $V5$ is NP-hard. The Subset Sum problem $P1$ seeks the existence of a solution x_j, $j \in N$ satisfying $\sum_{j \in N} d_j x_j = e$ for arbitrary positive integer data d_j, $j \in N$, e. We transform $P1$ into the following problem concerning the validity of a given inequality. Specifically, is

$$\sum_{j \in N} b_j x_j \leq b_0 \tag{2}$$

valid for

$$\sum_{j \in N} a_j x_j \leq a_0 \tag{1}$$

where $b_j = a_j = d_j$, $j \in N$, $a_0 = e$ and $b_0 = e - 1$? It is easy to verify that the inequality (2) is valid for the knapsack problem defined by (1) iff $P1$ has no solution. □

We finally consider the complexity of facets. We have shown that the problems of recognizing integral lifted facets, namely, the sets FLI and FLI(S) are in P. We have also shown that the complexities of recognizing FL(S), FL and F are similar, i.e., a general facet is not much more difficult to recognize than a lifted one.

The appropriate complexity class for handling facets, D^P, is defined in [19]. Specifically, D^P contains the intersection of pairs of languages, one from NP and the other from co-NP. (This is not the same as the intersection of NP and co-NP; in fact, D^P contains the union of NP and co-NP.) It is easy to show that FL(S), FL and F are in D^P. Papadimitriou and Wolfe [18] have shown that recognizing a facet for the travelling salesman problem is complete for D^P. We conjecture that this is the case for the knapsack polytope as well. If the conjecture is true, then, in light of Lemmas 3.1 and 4.1, recognizing nonintegral *lifted* inequalities is also D^P complete, *even if S is given*.

5. Recognizing lifted inequalities

The main thrust of this paper has been to examine the complexity of recognizing certain classes of inequalities as various requirements are relaxed. Starting with the classes FLI(S) and VLI(S), we have relaxed the assumptions that S be specified (VLI and FLI), that the inequality be integral (VL and FL) and that the inequality be lifted (V and F). In each case the inequalities studied have been valid.

In this section we examine inequalities of a different type, namely, we relax the requirement of validity. Naturally, invalid inequalities are not very useful for practical applications. We consider this class because of the interesting contrast between the results in this section and the previous sections. Specifically, relaxing the requirement of validity has the unexpected effect of converting an easy recognition problem (FLI or VLI) into an intractable one, even if we restrict our attention to integral (or even to 0-1) inequalities.

We open with a technical lemma.

Lemma 5.1. *The following recognition problem $P2$ is* NP-*complete.*

$P2$: *Given*: *A knapsack inequality* (1) *and an integer s.*

Question: *Does N contain a minimal cover S of cardinality s?*

Proof. Clearly $P2$ is in NP. To show completeness we perform a reduction from the Subset Sum problem $P1$ (see Section 4). For an arbitrary instance of $P1$ we construct the following instance of $P2$.

$$a_i = \begin{cases} d_i + 1, & i = 1, \dots, m, \\ 1, & i = m+1, \dots, 2m, \end{cases}$$

$$a_0 = e + m,$$

$$s = m + 1.$$

We show that an instance of $P1$ has an affirmative answer iff the corresponding instance of $P2$ has an affirmative answer.

For every nonempty set $U \subseteq \{1, \dots, m\}$, let $S = S(U) = U \cup \{m+1, \dots, m + (m+1-|U|)\}$. Conversely, for every set S of $m+1$ elements $S \subseteq \{1, \dots, 2m\}$, let $U = U(S) = S \cap \{1, \dots, m\}$. In both cases, we have

(i) $\sum_{i \in S} a_i = (m+1) + \sum_{i \in U} d_i$,

(ii) $\min_{i \in S} a_i = 1$,

(iii) $|S| = m+1$.

Consider the inequalities

(iv) $e + m < \sum_{i \in S} a_i \le e + m + 1$.

From (i) we get that (iv) is equivalent to

$$e - 1 < \sum_{i \in U} d_i \le e,$$

i.e., U satisfies $P1$. From (ii) we obtain that (iv) is equivalent to

$$a_0 < \sum_{i \in S} a_i$$

and

$$\sum_{i \in S} a_i - \min_{i \in S} a_i \le a_0,$$

i.e., S satisfies $P2$. □

The following theorem follows easily from this lemma. We let L denote the set of inequalities lifted from some minimal cover and we let LI denote the integral inequalities in L.

Theorem 5.2. *The following recognition problems are* NP-*complete*:

- *L*2: *Is* (2) *in* LI?
- *L*4: *Is* (2) *in* *L*?

Proof. Consider the inequality

$$\sum_{j\in N} x_j \le s-1. \tag{4}$$

Then (4) is in L precisely if the answer to the recognition problem of Lemma 5.1 is affirmative. Note that in this case (4) is integral. □

References

[1] E. Balas, Facets of the knapsack polytope, Math. Programming 8 (1975) 146–164.
[2] E. Balas and E. Zemel, Facets of the knapsack polytope from minimal covers, SIAM J. Appl. Math. 34 (1978) 119–148.
[3] E. Balas and E. Zemel, Lifting and complementing yields all the facets of positive zero-one programming polytopes, in: R.W. Cottle, H.L. Kelmanson and B. Korte, eds., Mathematical Programming (North-Holland, Amsterdam, 1984) 13–24.
[4] H. Crowder, E.L. Johnson and M. Padberg, Solving large scale zero-one linear programming problems, Oper. Res. 5 (1983) 803–834.
[5] H. Crowder and M. Padberg, Solving large scale symmetric traveling salesman problems to optimality, Management Sci. 26 (1980) 495–509.
[6] J. Edmonds, Covers and packings in a family of sets, Amer. Math. Soc. Bull. 68 (1962) 494–499.
[7] J. Edmonds, Maximum matching and a polyhedron with 0-1 vertices, J. Res. Nat. Bur. Standards 69B (1965) 125–130.
[8] M.R. Garey and D.S. Johnson, Computers and Intractability: A Guide to Theory of NP-Completeness (Freeman, San Francisco, CA, 1979).
[9] M. Grotschel, Polyhedral combinatorics, in: M. O'hEigeartaigh, J.K. Lenstra and A.H.G. Rinnooy Kan, eds., Combinatorial Optimization—Annotated Bibliographies (Wiley/Interscience, New York, 1985) 1–10.
[10] M. Grotschel, L. Lovasz and A. Schrijver, The ellipsoid method and its consequences in combinatorial optimization, Combinatorica 1 (1981) 169–197.
[11] M. Grotschel and M.W. Padberg, Polyhedral theory, in: E.L. Lawler, J.K. Lenstra, A.H.G. Rinnooy Kan and D.B. Shmoys, eds., The Travelling Salesman Problem (Wiley, Chichester, 1985) 251–305.
[12] P.L. Hammer, E.L. Johnson and U.N. Peled, Facets of regular 0-1 polytopes, Math. Programming 8 (1975) 179–206.
[13] R.M. Karp and C.H. Papadimitriou, On linear characterization of combinatorial optimization problems, SIAM J. Comput. 2 (1982) 620–632.
[14] G.L. Nemhauser and L.E. Trotter Jr, Properties of vertex packing and independence system polyhedra, Math. Programming 6 (1974) 48–61.

[15] M.W. Padberg, On the facial structure of set packing polyhedra, Math. Programming (1973) 198–216.
[16] M.W. Padberg, A note on zero-one programming, Oper. Res. 3 (1975) 833–837.
[17] M.W. Padberg and M.R. Rao, The Russian method for linear inequalities III: Bounded integer programming, Math. Programming Stud., to appear.
[18] C.H. Papadimitriou and D. Wolfe, The complexity of facets resolves, in: Proceedings 26th Annual Symposium on Foundations of Computer Science, Portland, OR (1985) 74–78.
[19] C.H. Papadimitriou and M. Yanakakis, The complexity of facets (and some facets of complexity), J. Comput. System Sci. 2 (1982) 244–259.
[20] U.N. Peled, Properties of facets of binary polytopes, in: Annals of Discrete Mathematics 1 (North-Holland, Amsterdam, 1977) 435–456.
[21] W.R. Pulleyblank, Polyhedral combinatorics, in: A. Bachem, M. Grotschel and B. Korte, eds., Mathematical Programming The State of the Art (Springer, Berlin, 1983) 312–345.
[22] G. Rinaldi and H. Padberg, Optimization of a 532 city symmetric travelling salesman problem by branch and cut, Oper. Res. Lett. 6 (1987) 1–7.
[23] L.A. Wolsey, Faces of linear inequalities in 0-1 variables, Math. Programming 8 (1975) 165–178.
[24] E. Zemel, Lifting the facets of 0-1 polytopes, Math. Programming 15 (1978) 268–277.
[25] E. Zemel, Easily computable facets of the knapsack problem, Math. Oper. Res. 14 (1989) 1–9.