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New Characteristics of Some Polynomial Sequences in Combinatorial Theory

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Some new necessary and sufficient conditions on basic sets, Sheffer sets, cross-sequences, and Steffensen sequences are established. © 1993 Academic Press, Inc.

INTRODUCTION

Rota, Kahaner, and Odlyzko [1–2], in their development of the umbral calculus, have introduced the notions of polynomial sequences of binomial type, of Sheffer sequences, of cross-sequences, and Steffensen sequences. Each of these sequences of polynomials is defined by suitable functional equations, harking back to the original statement of the binomial theorem. We show in this paper that each class of polynomial sequences can be characterized by conditions that are considerably weaker than those used by the above authors in their definition.

1. ON BASIC SETS AND SHEFFER SETS

In this section, our main result is the following.

THEOREM 1.1. *Assume $p_n(x)$ and $s_n(x)$ with $s_0(x) \neq 0$ are two polynomial sequences. Then $p_n(x)$ and $s_n(x)$ are a basic set and a Sheffer set relative to some delta operator Q, respectively, if and only if the identity*

$$s_n(x+y) = \sum_{k \geq 0} \binom{n}{k} s_k(x)\, p_{n-k}(y) \tag{1}$$

holds for all n, x, and y.

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First, we prove the following.

PROPOSITION 1.2. *Let $s_n(x)$ be a Sheffer set relative to a delta operator Q and $p_n(x)$ a polynomial sequence. If Eq. (1) holds, then $p_n(x)$ is a basic set for the same operator Q.*

Proof. Setting $y=0$ in Eq. (1), we have

$$s_n(x)=s_n(x)\,p_0(0)+ns_{n-1}(x)\,p_1(0)+\cdots+s_0(x)\,p_n(0). \tag{2}$$

Comparing the coefficients of x^n yields $p_0(0)=p_0(x)=1$.

Then setting $n=1$ in (2),

$$s_1(x)=s_1(x)+s_0(x)\,p_1(0).$$

Since $s_0(x)=c\neq 0$, we have $p_1(0)=0$. By induction, now we assume $p_k(0)=0$, $k=1, 2, ..., n-1$. Then, from (2)

$$s_n(x)=s_n(x)+s_0(x)\,p_n(0),$$

it follows that $p_n(0)=0$.

Now turn to prove $p_n(x)$ satisfies the equation

$$Qp_n(x)=np_{n-1}(x).$$

First, applying Q to the two sides of the equation

$$s_1(x)=s_0(0)\,p_1(x)+s_1(0),$$

we obtain

$$s_0(x)=Qs_1(x)=s_0(0)\,Qp_1(x)+Qs_1(0)=s_0(0)\,Qp_1(x).$$

Since

$$s_0(x)=c\neq 0,$$

then

$$Qp_1(x)=1=1\cdot p_0(x).$$

Again by induction, assume that $Qp_k(x)=kp_{k-1}(x)$, $k=1, 2, ..., n-1$, hold. Then, applying Q to the two sides of (1) yields that

$$\begin{aligned}
Qs_n(x+y) &= QE^x s_n(y) = E^x Qs_n(y) \\
&= E^x ns_{n-1}(y) = ns_{n-1}(x+y) \\
&= \sum_{k \geqslant 0} \binom{n}{k} s_{n-k}(x)\, Qp_k(y) \\
&= s_0(x)\, Qp_n(y) + \sum_{k=0}^{n-1} \binom{n}{k} s_{n-k}(x)\, k \cdot p_{k-1}(y) \\
&= s_0(x)\, Qp_n(y) + n \sum_{k=1}^{n-1} \binom{n-1}{k-1} s_{n-1-(k-1)}(x)\, p_{k-1}(y) \\
&= s_0(x)\, Qp_n(y) + n \sum_{k=0}^{n-2} \binom{n-1}{k} s_{n-1-k}(x)\, p_k(y). \qquad (3)
\end{aligned}$$

On the other hand,

$$ns_{n-1}(x+y) = ns_0(x)\, p_{n-1}(y) + n \sum_{k=0}^{n-2} \binom{n-1}{k} s_{n-1-k}(x)\, p_k(y). \qquad (4)$$

Combining (3) and (4) follows that

$$Qp_n(y) = np_{n-1}(y). \quad \blacksquare$$

Proof of Theorem 1.1. In view of the Binomial Theorem of [2, p. 22] and Proposition 1.2, it is sufficient to prove that $s_n(x)$ is a Sheffer set relative to some delta operator under the condition (1).

Define an operator Q by setting

$$Q1 = 0, \qquad Qs_n(x) = ns_{n-1}(x) \qquad (5)$$

and extending Q by linearity. Write

$$s_1(x) = a_1 x + a_0, \qquad a_1 \neq 0.$$

From (5),

$$0 \neq s_0(x) = Qs_1(x) = a_1 Qx + a_0 Q1 = a_1 Qx.$$

Hence

$$Qx \neq 0.$$

Now, applying Q to the two sides of (1) and regarding x as a variable, we obtain

$$QE^y s_n(x) = Qs_n(x+y) = \sum_{k\geq 0} \binom{n}{k} Qs_k(x)\, p_{n-k}(y)$$
$$= \sum_{k\geq 0} \binom{n}{k} ks_{k-1}(x)\, p_{n-k}(y) = n \sum_{k\geq 1} \binom{n-1}{k-1} s_{k-1}(x)\, p_{n-k}(y)$$
$$= ns_{n-1}(x+y) = nE^y s_{n-1}(x) = E^y Qs_n(x).$$

From the above it is easy to obtain

$$QE^y p(x) = E^y Qp(x)$$

for all polynomials $p(x)$. That means Q is shift-invariant. Therefore, Q is a delta operator and $s_n(x)$ is a Sheffer set relative to the operator Q. Using Proposition 1.2, it follows that $p_n(x)$ is a basic set relative to the same delta operator Q. ∎

COROLLARY 1.3. *A polynomial sequence $p_n(x)$ is a basic set if and only if there exists a polynomial sequence $s_n(x)$ with $s_0(x) \neq 0$ such that* (1) *holds.*

2. ON CROSS-SEQUENCES AND STEFFENSEN SEQUENCES

The main result of this section is the following.

THEOREM 2.1. *For every real number λ and μ, let $p_n^{[\lambda]}(x)$ and $s_n^{[\mu]}(x)$ be two polynomial sequences. Then, $p_n^{[\lambda]}(x)$ is a cross-sequence and $s_n^{[\mu]}(x)$ is a Steffensen sequence if and only if the identities*

$$s_n^{[\lambda+\mu]}(x+y) = \sum_{k\geq 0} \binom{n}{k} s_k^{[\lambda]}(x)\, p_{n-k}^{[\mu]}(y) \tag{6}$$

holds for all n, λ, μ, x, and y.

Proof. In view of the definition of the Steffensen sequence we only need to prove the "if" part. Assume (6) holds. Setting $\lambda = \mu = 0$ in (6) yields

$$s_n^{[0]}(x+y) = \sum_{k\geq 0} \binom{n}{k} s_k^{[0]}(x)\, p_{n-k}^{[0]}(y). \tag{7}$$

Applying Theorem 1.1, it follows immediately that there exists a delta operator Q and polynomial sequences $p_n^{[0]}(x)$ and $s_n^{[0]}(x)$ are basic set and Sheffer set relative to the same operator Q, respectively.

In the identities (6) setting $\mu = 0$, we have

$$s_n^{[\lambda]}(x+y) = \sum_{k \geq 0} \binom{n}{k} s_k^{[\lambda]}(x) \, p_{n-k}^{[0]}(y).$$

Since $p_n^{[0]}(x)$ is a basic set relative to the delta operator Q, then for every λ, $s_n^{[\lambda]}(x)$ is a Sheffer set for the same Q according to Theorem 1.1.

Define an operator $P^{-\lambda}$ by setting

$$p_n^{[\lambda]}(x) = P^{-\lambda} p_n^{[0]}(x) \tag{8}$$

and extending by linearity. Obviously, operator $P^{-\lambda}$ is invertible.

Now, from (6)

$$\begin{aligned} s_n^{[\lambda+\mu]}(x+y) &= \sum_{k \geq 0} \binom{n}{k} s_k^{[\lambda]}(x) \, p_{n-k}^{[\mu]}(y) \\ &= \sum_{k \geq 0} \binom{n}{k} s_k^{[\lambda]}(x) \, P^{-\mu} p_{n-k}^{[0]}(y) \\ &= P^{-\mu} \sum_{k \geq 0} \binom{n}{k} s_k^{[\lambda]}(x) \, p_{n-k}^{[0]}(y) \\ &= P^{-\mu} s_n^{[\lambda]}(x+y). \end{aligned} \tag{9}$$

Setting $y = 0$ yields

$$s_n^{[\lambda+\mu]}(x) = P^{-\mu} s_n^{[\lambda]}(x). \tag{10}$$

From (9) and (10) it follows that

$$\begin{aligned} P^{-\mu} E^y s_n^{[\lambda]}(x) &= P^{-\mu} s_n^{[\lambda]}(x+y) = s_n^{[\lambda+\mu]}(x+y) \\ &= E^y s_n^{[\lambda+\mu]}(x) = E^y P^{-\mu} s_n^{[\lambda]}(x). \end{aligned}$$

From the above it is easy to obtain that for all polynomials $p(x)$

$$P^{-\mu} E^y p(x) = E^y P^{-\mu} p(x).$$

Hence, the operator $P^{-\lambda}$ is shift-invariant.

Now we are going to prove

$$P^{-\lambda} \cdot P^{-\mu} = P^{-(\lambda+\mu)}. \tag{11}$$

First, applying $P^{-\lambda}$ to the two sides of (7) and using (10), we obtain

$$\begin{aligned} P^{-\lambda} s_n^{[0]}(x+y) &= \sum_{k \geq 0} \binom{n}{k} P^{-\lambda} s_k^{[0]}(x) \, p_{n-k}^{[0]}(y) \\ &= \sum_{k \geq 0} \binom{n}{k} s_k^{[\lambda]}(x) \, p_{n-k}^{[0]}(y). \end{aligned}$$

Now permuting x and y, then applying $P^{-\mu}$ to the both sides, we have

$$\begin{aligned} P^{-\mu}P^{-\lambda}s_n^{[0]}(x+y) &= \sum_{k\geq 0}\binom{n}{k} s_k^{[\lambda]}(y)\, P^{-\mu}p_{n-k}^{[0]}(x) \\ &= \sum_{k\geq 0}\binom{n}{k} s_k^{[\lambda]}(y)\, p_{n-k}^{[\mu]}(x) = s_n^{[\lambda+\mu]}(x+y) \\ &= E^y s_n^{[\lambda+\mu]}(x) = E^y P^{-(\lambda+\mu)} s_n^{[0]}(x) \\ &= P^{-(\lambda+\mu)}E^y s_n^{[0]}(x) = P^{-(\lambda+\mu)}s_n^{[0]}(x+y). \end{aligned}$$

From the above we can easily prove that (11) holds.

Finally, using Theorem 1 of [2, p. 11] and Theorem 8 of [2, p. 34], it follows that $p_n^{[\lambda]}(x)$ is a cross-sequence. Therefore, according to the definition, $s_n^{[\lambda]}(x)$ is a Steffensen sequence. ∎

COROLLARY 2.2. *For every real number λ, let $s_n^{[\lambda]}(x)$ be a polynomial sequence. Then, $s_n^{[\lambda]}(x)$ is a Steffensen sequence if and only if there is a polynomial sequence $p_n(x)$ such that*

$$s_n^{[\lambda]}(x+y) = \sum_{k\geq 0}\binom{n}{k} s_k^{[\lambda]}(x)\, p_{n-k}(y) \tag{12}$$

for every λ, n, x, and y.

Proof. Assume (12) holds. For every fixed λ, using Theorem 1.1, we know that there exists a delta operator Q and that the polynomial sequences $p_n(x)$ and $s_n^{[\lambda]}(x)$ are a basic set and a Steffensen set relative to the same operator Q, respectively. That is,

$$Qs_n^{[\lambda]}(x) = ns_{n-1}^{[\lambda]}(x).$$

Define an operator $P^{-\mu}$ by setting

$$P^{-\mu}s_n^{[\lambda]}(x) = s_n^{[\lambda+\mu]}(x),$$

and extending by linearity. Then $P^{-\mu}$ is invertible and

$$\begin{aligned} E^y P^{-\mu} s_n^{[0]}(x) &= E^y s_n^{[\mu]}(x) = s_n^{[\mu]}(x+y) \\ &= \sum_{k\geq 0}\binom{n}{k} s_k^{[\mu]}(x)\, p_{n-k}(y) = P^{-\mu}\sum_{k\geq 0}\binom{n}{k} s_k^{[0]}(x)\, p_{n-k}(y) \\ &= P^{-\mu}s_n^{[0]}(x+y) = P^{-\mu}E^y s_n^{[0]}(x). \end{aligned}$$

From the above it is easy to assert that

$$E^y P^{-\mu} = P^{-\mu}E^y.$$

And we have

$$P^{-\lambda}P^{-\mu}s_n^{[0]}(x) = P^{-\mu}s_n^{[\mu]}(x) = s_n^{[\lambda+\mu]}(x) = P^{-(\lambda+\mu)}s_n^{[0]}(x).$$

Therefore we obtain

$$P^{-\lambda}P^{-\mu} = P^{-(\lambda+\mu)}.$$

Now, using Proposition 2 of [2, p. 36], it follows that $s_n^{[\lambda]}(x)$ is a Steffensen sequence. The "only if" part is obvious. ∎

COROLLARY 2.3. *$s_n^{[\lambda]}(x)$ is a Steffensen sequence if and only if there exists a cross-sequence $p_n^{[\lambda]}(x)$ and an invertible shift-invariant operator T such that*

$$s_n^{[\lambda]}(x) = T^{-1}p_n^{[\lambda]}(x).$$

This is one of the assertions of Proposition 2 of [2, p. 36].

Proof. We only prove the "if" part. Since

$$\begin{aligned}
&\sum_{k\geq 0}\binom{n}{k} s_k^{[\lambda]}(x)\, p_{n-k}^{[\mu]}(y)\\
&\quad= \sum_{k\geq 0}\binom{n}{k} T^{-1}p_k^{[\lambda]}(x)\, p_{n-k}^{[\mu]}(y)\\
&\quad= T^{-1}\sum_{k\geq 0}\binom{n}{k} p_k^{[\lambda]}(x)\, p_{n-k}^{[\mu]}(y) = T^{-1}p_n^{[\lambda+\mu]}(x+y)\\
&\quad= T^{-1}E^{y}p_n^{[\lambda+\mu]}(x) = E^{y}T^{-1}p_n^{[\lambda+\mu]}(x) = s_n^{[\lambda+\mu]}(x+y).
\end{aligned}$$

Using Corollary 2.2, it follows that $s_n^{[\lambda]}(x)$ is a Steffensen sequence. ∎

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REFERENCES

1. G.-C. ROTA, D. KAHANER, AND A. ODLYZKO, Finite operator calculus, *J. Math. Anal. Appl.* **42** (1973), 684–760.
2. G.-C. ROTA, "Finite Operator Calculus," Academic Press, San Diego, CA, 1975.