A Baskakov type generalization of statistical Korovkin theory

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Received 25 June 2007
Available online 29 August 2007
Submitted by Richard M. Aron

Abstract

In this paper using the notion of \( A \)-statistical convergence, where \( A \) is a nonnegative regular summability matrix, we obtain some statistical variants of Baskakov’s results on the Korovkin type approximation theorems.

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Keywords: Statistical convergence; Positive linear operators; Korovkin theorem

1. Introduction

Baskakov [1] systematically investigated and generalized the well-known Korovkin theory (see [2]) by means of a class of linear operators containing the positive operators. Especially, he obtained various approximation theorems for the class of linear operators which converge to the derivatives of functions. In this paper, in order to get more powerful approximation theorems we get the statistical variants of Baskakov’s results by using the statistical limit operator instead of the ordinary limit operator in the approximation of functions by means of linear operators. We should note that some statistical approximation results and their applications may be found in the papers [3–5] and cited therein.

We first recall the concept of statistical convergence.

Let \((x_n)_{n\in\mathbb{N}}\) be a sequence of numbers. Then, \((x_n)_{n\in\mathbb{N}}\) is called statistically convergent to a number \(L\) if, for every \(\varepsilon > 0\),

\[
\lim_{j} \frac{\# \{ n \leq j : |x_n - L| \geq \varepsilon \} }{j} = 0,
\]

where \(\# \{ B \}\) denotes the cardinality of the subset \(B\) (see Fast [6]). We denote this statistical limit by \(st\text{-lim}_n x_n = L\).

Now let \(A = (a_{jn})\) be an infinite summability matrix. Then, the \(A\)-transform of \(x\), denoted \(Ax := ((Ax)_j)\), is given by \((Ax)_j = \sum_{n=1}^{\infty} a_{jn} x_n\), provided the series converges for each \(j\). We say that \(A\) is regular if \(\lim_j (Ax)_j = L\) whenever \(\lim_j x_j = L\) [7]. For example, the Cesáro matrix \(C_1 = (c_{jn})\) defined by

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is a regular matrix. Assume now that $A$ is a nonnegative regular summability matrix. Then, Freedman and Sember [8] introduced the notion of $A$-statistical convergence as the following way, which is a more general method of statistical convergence. The sequence $(x_n)_{n \in \mathbb{N}}$ is said to be $A$-statistically convergent to $L$ if, for every $\varepsilon > 0$,
\[
\lim_{n \rightarrow \infty} \sum_{|x_n - L| \geq \varepsilon} a_{jn} = 0 \quad (1.1)
\]
holds. This limit is denoted by $st_A^{-}\lim_n x_n = L$. It is not hard to see that if we take $A = C_1$, then $C_1$-statistical convergence coincides with the statistical convergence mentioned above, i.e., $st_{C_1}^{-}\lim_n x_n = st^{-}\lim_n x_n$. If $A$ is replaced by the identity matrix, then we get the ordinary convergence of number sequences. It is not hard to see that every convergent sequence is $A$-statistically convergent to the same value for any nonnegative regular matrix $A$. This follows from the definition (1.1) and the well-known regularity conditions of $A$ introduced by Silverman and Toeplitz (see, for instance, Hardy [9, pp. 43–45]); but its converse is not always true. Actually, if $A = (a_{jn})$ is any nonnegative regular summability matrix for which $\lim_{j} \max_{n} |a_{jn}| = 0$, then $A$-statistical convergence is stronger than convergence (see [10]). Some other results regarding statistical and $A$-statistical convergences may be found in the papers [11,12].

2. Statistical Korovkin theory

In this section, with the help of $A$-statistical convergence, we get various approximation results by means of a family of positive linear operators. Now consider the following sequence of linear operators:
\[
L_n(f; x) = \int_{a}^{b} f(y) d\varphi_n(x, y), \quad n \in \mathbb{N}, \quad (2.1)
\]
defined for $f \in C[a, b]$, where $\varphi_n(x, y)$ is, for every $n$ and for every fixed $x \in [a, b]$, a function of bounded variation with respect to the variable $y$ on the interval $[a, b]$. Notice that if $\varphi_n(x, y)$ is nondecreasing function with respect to the variable $y$, then the operators (2.1) will be positive. We denote by $E_{2k}$, $k \geq 1$, the class of operators (2.1) such that, for each fixed $x \in [a, b]$ and for each $n \in \mathbb{N}$, the integrals
\[
I_{2k,n}^{(1)}(y) := \int_{a}^{y} \int_{a}^{y_1} \cdots \int_{a}^{y_{2k-1}} d\varphi_n(x, y_{2k}) \cdots dy_2 dy_1 \quad \text{for } a \leq y \leq x,
\]
\[
I_{2k,n}^{(2)}(y) := \int_{y}^{b} \int_{y}^{y_1} \cdots \int_{y}^{y_{2k-1}} d\varphi_n(x, y_{2k}) \cdots dy_2 dy_1 \quad \text{for } x \leq y \leq b
\]
have a constant sign for all $y \in [a, b]$, which may depend on $n \in \mathbb{N}$. We note that these conditions were first considered by Baskakov [1].

Now we begin with the following theorem.

**Theorem 2.1.** Let $A = (a_{jn})$ be a nonnegative regular summability matrix. If the operators (2.1) belong to the class $E_{2k}$, $k \geq 1$, and if
\[
st_A^{-}\lim_n \|L_n(e_i) - e_i\| = 0, \quad i = 0, 1, \ldots, 2k, \quad (2.2)
\]
where $e_i(x) = x^i$, $i = 0, 1, \ldots, 2k$, then, for every function $f$ having a continuous derivative of order $2k$ on the interval $[a, b]$, we have
\[
st_A^{-}\lim_n \|L_n(f) - f\| = 0. \quad (2.3)
\]
Proof. By similarity it is enough to prove for the case of $k = 1$. Setting $\Psi(y) = y - x$ for each $x \in [a, b]$, we have

$$L_n(\Psi^2; x) = \int_a^b \Psi^2(y) d\varphi_n(x, y)$$

$$= \int_a^b (y - x) d\varphi_n(x, y) - 2x \int_a^b e_1(y) d\varphi_n(x, y) + \int_a^b e_0(y) d\varphi_n(x, y),$$

which implies

$$\|L_n(\Psi^2)\| \leq \|L_n(e_2) - e_2\| + 2c\|L_n(e_1) - e_1\| + c^2\|L_n(e_0) - e_0\|,$$

where $c = \max\{|a|, |b|\}$. Thus, for every $\varepsilon > 0$, consider the following subsets of the natural numbers:

$$D := \{n: \|L_n(\Psi^2)\| \geq \varepsilon\},$$

$$D_1 := \{n: \|L_n(e_2) - e_2\| \geq \frac{\varepsilon}{3}\},$$

$$D_2 := \{n: \|L_n(e_1) - e_1\| \geq \frac{\varepsilon}{6c}\},$$

$$D_1 := \{n: \|L_n(e_0) - e_0\| \geq \frac{\varepsilon}{3c^2}\}.$$  \hspace{1cm} (2.4)

Then, by (2.4), we get

$$D \subseteq D_1 \cup D_2 \cup D_3.$$  \hspace{1cm} (2.5)

This inclusion yields, for every $j \in \mathbb{N}$, that

$$\sum_{n \in D} a_{jn} \leq \sum_{n \in D_1} a_{jn} + \sum_{n \in D_2} a_{jn} + \sum_{n \in D_3} a_{jn}.$$  \hspace{1cm} (2.6)

Now letting $j \to \infty$ in (2.5) and using (2.2) we have

$$\lim_{j \to \infty} \sum_{n \in D} a_{jn} = 0,$$

which means

$$\text{st}_A\lim_{n \to \infty} \|L_n(\Psi^2)\| = 0.$$  \hspace{1cm} (2.7)

By hypothesis, it is not hard to see that

$$\text{st}_A\lim_{n \to \infty} \|L_n(\Psi)\| = 0.$$  \hspace{1cm} (2.8)

On the other hand, breaking up the integral

$$L_n(\Psi^2; x) = \int_a^b (y - x)^2 d\varphi_n(x, y)$$

into two integrals over the intervals $[a, x]$ and $[x, b]$ and integrating twice by parts, we conclude that

$$L_n(\Psi^2; x) = 2 \left\{ \int_a^x \int_a^y \int_a^y d\varphi_n(x, y_2) dy_1 dy + \int_a^b \int_a^b \int_a^b d\varphi_n(x, y_2) dy_1 dy \right\}.$$  \hspace{1cm} (2.9)

By the definition of the class $E_2$, under the signs of the exterior integrals, we obtain expressions which have a constant sign. Thus, by (2.6) and (2.8), we see that
\[ st_A\lim_n \sup_{x \in [a, b]} \left( \int_a^x \int_a^y d\varphi_n(x, y_2) dy_1 + \int_y^b \int_y^b d\varphi_n(x, y_2) dy_1 \right) = 0. \] (2.9)

Furthermore, since the function \( f \) has a continuous second derivative on the interval \([a, b]\), it follows from the well-known Taylor’s formula that
\[ f(y) = f(x) + f'(x)(y - x) + \int_x^y f''(t)(y - t) dt. \] (2.10)

Now using the linearity of the operators \( L_n \) we have
\[ L_n(f; x) - f(x) = f(x)\left( L_n(e_0; x) - e_0(x)\right) + f'(x)L_n(\Psi; x) + R_n(x), \] (2.11)
where \( R_n(x) \) is given by
\[ R_n(x) := \int_a^b \int_y^x f''(y) d\varphi_n(x, y_2) dy_1 dy. \]

Breaking up this integral into two integrals over the intervals \([a, x]\) and \([x, b]\) and integrating twice by parts, we obtain that
\[ R_n(x) = \int_a^x \int_a^y \int_a^y f''(y) d\varphi_n(x, y_2) dy_1 dy + \int_y^b \int_y^b \int_y^b f''(y) d\varphi_n(x, y_2) dy_1 dy, \]
which gives that
\[ \|R_n\| \leq M_1 \sup_{x \in [a, b]} \left( \int_a^x \int_a^y \int_a^y d\varphi_n(x, y_2) dy_1 dy + \int_y^b \int_y^b \int_y^b d\varphi_n(x, y_2) dy_1 dy \right), \] (2.12)
where \( M_1 = \|f''\| \). Thus, by (2.9) and (2.12), we get
\[ st_A\lim_n \|R_n\| = 0. \] (2.13)

From (2.11), we may write that
\[ \|L_n(f) - f\| \leq M_2 \|L_n(e_0) - e_0\| + M_3 \|L_n(\Psi)\| + \|R_n\|, \] (2.14)
where \( M_2 = \|f\| \) and \( M_3 = \|f'\| \). Now, for a given \( \varepsilon > 0 \), define the following sets:
\[ E := \{n: \|L_n(f) - f\| \geq \varepsilon\}, \]
\[ E_1 := \{n: \|L_n(e_0) - e_0\| \geq \frac{\varepsilon}{3M_2}\}, \]
\[ E_2 := \{n: \|L_n(\Psi)\| \geq \frac{\varepsilon}{3M_3}\}, \]
\[ E_3 := \{n: \|R_n\| \geq \frac{\varepsilon}{3}\}. \]

Then, by (2.14), observe that
\[ E \subseteq E_1 \cup E_2 \cup E_3. \]

So, we have, for each \( j \in \mathbb{N} \),
\[ \sum_{n \in E} a_{jn} \leq \sum_{n \in E_1} a_{jn} + \sum_{n \in E_2} a_{jn} + \sum_{n \in E_3} a_{jn}. \]
Taking limit as \( j \to \infty \) and using (2.2), (2.7), (2.13) we conclude that
\[
\lim_{j} \sum_{n \in E} a_{jn} = 0,
\]
which implies (2.3). Therefore, the proof is completed. \( \square \)

If one replaces the matrix \( A \) by the Cesáro matrix, then the next result follows from Theorem 2.1 at once.

**Corollary 2.2.** If the operators (2.1) belong to the class \( E_{2k} \), \( k \geq 1 \), and if
\[
\overline{st}-\lim_{n} \| L_n(e_i) - e_i \| = 0, \quad i = 0, 1, \ldots, 2k,
\]
then, for every function \( f \) having a continuous derivative of order \( 2k \) on the interval \([a, b]\), we have
\[
\overline{st}-\lim_{n} \| L_n(f) - f \| = 0.
\]

Furthermore, considering the identity matrix instead of any nonnegative regular summability matrix in Theorem 2.1, we get the following result which was first introduced by Baskakov [1].

**Corollary 2.3.** (See [1].) If the operators (2.1) belong to the class \( E_{2k} \), \( k \geq 1 \), and if the sequence \( \{L_n(e_i)\}_{n \in \mathbb{N}} \) is uniformly convergent to \( e_i \) (\( i = 0, 1, \ldots, 2k \)) on the interval \([a, b]\), then, for every function \( f \) with a continuous derivative of order \( 2k \) on the interval \([a, b]\), the sequence \( \{L_n(f)\}_{n \in \mathbb{N}} \) converges uniformly to \( f \) on \([a, b]\).

**Remark 2.1.** Let \( A = (a_{jn}) \) be a nonnegative regular matrix summability satisfying \( \lim_j \max_n a_{jn} = 0 \). In this case it is known that \( A \)-statistical convergence is stronger than ordinary convergence [10]. So we can choose a sequence \( (u_n)_{n \in \mathbb{N}} \) which is \( A \)-statistically convergent to zero but nonconvergent. Without loss of generality we may assume that \( (u_n)_{n \in \mathbb{N}} \) is nonnegative. Otherwise we replace \( (u_n)_{n \in \mathbb{N}} \) by \( (|u_n|)_{n \in \mathbb{N}} \). Now let \( L_n \) be the operators given by (2.1) belonging to the class \( E_{2k} \) for \( k \geq 1 \). Assume further that the operators \( L_n \) satisfy the conditions of Corollary 2.3. Consider the following operators:
\[
T_n(f; x) = (1 + u_n)L_n(f; x) = (1 + u_n) \int_{a}^{b} f(y) \, d\varphi_n(x, y).
\]
Then observe that all conditions of Theorem 2.1 hold for the operators \( T_n \). So we have
\[
\overline{st}_A-\lim_{n} \| T_n(f) - f \| = 0.
\]
However, since \( (u_n)_{n \in \mathbb{N}} \) is nonconvergent, the sequence \( \{T_n(f)\}_{n \in \mathbb{N}} \) is not uniformly convergent to \( f \) (in the usual sense). So, this demonstrates that our Theorem 2.1 is a nontrivial generalization of its classical case Corollary 2.3.

Now we recall that the \( A \)-density of a subset \( K \) of \( \mathbb{N} \) is defined by
\[
\delta_A(K) = \lim_{j} \sum_{n \in K} a_{jn}
\]
provided the limit exists, where \( A = (a_{jn}) \) is a nonnegative regular summability matrix. Actually, if we take \( A = C_1 \), the Cesáro matrix, then \( \delta_{C_1}(K) \) is denoted by \( \delta(K) \), the so-called (asymptotic) density, and given by
\[
\delta(K) = \lim_{j} \frac{\#(n \leq j: n \in K)}{j}
\]
provided the limit exists.

**Theorem 2.4.** Let \( A = (a_{jn}) \) be a nonnegative regular summability matrix. If, for the operators (2.1) belonging to the class \( E_{2k} \), \( k \geq 1 \), the conditions of Theorem 2.1 hold, and if
\[ \delta_A \left\{ n: \int_a^b |d\varphi_n(x, y)| \geq M \right\} = 0 \quad (2.15) \]

for some absolute constant \( M > 0 \), then, for every function \( f \in C[a, b] \), we have

\[ \text{st}_A \lim_n \| L_n(f) - f \| = 0. \]

**Proof.** Since \( \{e_0, e_1, e_2, \ldots\} \) is a fundamental system of \( C[a, b] \) (see, for instance, [2]), for a given \( f \in C[a, b] \), we can find a polynomial \( P \) given by

\[ P(x) = a_0 e_0(x) + a_1 e_1(x) + \cdots + a_{2k} e_{2k}(x) \]

such that for any \( \varepsilon > 0 \) the inequality

\[ \| f - P \| < \varepsilon \quad (2.16) \]

is satisfied. Setting

\[ K := \left\{ n: \int_a^b |d\varphi_n(x, y)| \geq M \right\}, \]

we see from (2.15) that \( \delta_A [\mathbb{N} \setminus K] = 1 \). By linearity and monotonicity of the operators \( L_n \), we have

\[ \| L_n(f) - L_n(P) \| = \| L_n(f - P) \| \leq \| L_n \| \| f - P \|. \quad (2.17) \]

Since

\[ \| L_n \| = \int_a^b |d\varphi_n(x, y)|, \]

it follows from (2.16) and (2.17) that, for all \( n \in \mathbb{N} \setminus K \),

\[ \| L_n(f) - L_n(P) \| \leq M \varepsilon. \quad (2.18) \]

On the other hand, since

\[ L_n(P; x) = a_0 L_n(e_0; x) + a_1 L_n(e_1; x) + \cdots + a_{2k} L_n(e_{2k}; x), \]

we obtain, for every \( n \in \mathbb{N} \), that

\[ \| L_n(P) - P \| \leq C \sum_{i=0}^{2k} \| L_n(e_i) - e_i \|, \quad (2.19) \]

where \( C = \max \{|a_1|, |a_2|, \ldots, |a_{2k}|\} \). Thus, for every \( n \in \mathbb{N} \setminus K \), we get from (2.16), (2.18) and (2.19) that

\[ \| L_n(f) - f \| \leq \| L_n(f) - L_n(P) \| + \| L_n(P) - P \| + \| f - P \| \]

\[ \leq (M + 1) \varepsilon + C \sum_{i=0}^{2k} \| L_n(e_i) - e_i \|. \quad (2.20) \]

Now, for a given \( r > 0 \), choose \( \varepsilon > 0 \) such that \( 0 < (M + 1) \varepsilon < r \). Then define the following sets:

\[ H := \{ n \in \mathbb{N} \setminus K: \| L_n(f) - f \| \geq r - (M + 1) \varepsilon \}, \]

\[ H_i := \{ n \in \mathbb{N} \setminus K: \| L_n(e_i) - e_i \| \geq \frac{r - (M + 1) \varepsilon}{(2k + 1)C} \}, \quad i = 0, 1, \ldots, 2k. \]

From (2.20), we easily check that
\[ H \subseteq \bigcup_{i=0}^{2k} H_i, \]
which yields, for every \( j \in \mathbb{N} \),

\[
\sum_{n \in H} a_{jn} \leq \sum_{i=0}^{2k} \sum_{n \in H_i} a_{jn}. \quad (2.21)
\]

If we take limit as \( j \to \infty \) using the hypothesis (2.2) we see that

\[
\lim_{j} \sum_{n \in H} a_{jn} = 0.
\]

So we have

\[
st_A^{-}\lim_n \| L_n(f) - f \| = 0
\]
which completes the proof. \( \square \)

The following two results are obtained from Theorem 2.4 by taking the Cesáro matrix and the identity matrix, respectively.

**Corollary 2.5.** If, for the operators (2.1) belonging to the class \( E_{2k} \), \( k \geq 1 \), the conditions of Corollary 2.2 hold, and if

\[
\delta \left\{ n: \int_a^b |d\varphi_n(x, y)| \geq M \right\} = 0
\]
for some absolute constant \( M > 0 \), then, for every function \( f \in C[a, b] \), we have

\[
st^{-}\lim_n \| L_n(f) - f \| = 0.
\]

**Corollary 2.6.** (See [1].) If, for the operators (2.1) belonging to the class \( E_{2k} \), \( k \geq 1 \), the conditions of Corollary 2.3 hold, and if the condition

\[
\int_a^b |d\varphi_n(x, y)| \leq M
\]
holds, where \( M \) is a positive absolute constant, then, for every function \( f \in C[a, b] \), the sequence \( \{L_n(f)\}_{n \in \mathbb{N}} \) converges uniformly to \( f \) on \( [a, b] \).

**Remark 2.2.** Observe that the boundedness condition in (2.15), the so-called “statistical uniform boundedness,” is weaker than the (classical) uniform boundedness in (2.22). So, our Theorem 2.4 is more powerful than Corollary 2.6.

### 3. Statistical approximation to derivatives of functions

In this section we obtain various statistical approximations to derivatives of functions by means of the operators \( L_n \) defined by (2.1). We should note here that the classical versions of our results were first proved by Baskakov [1].

We first get the next result.

**Theorem 3.1.** Let \( A = (a_{jn}) \) be a nonnegative regular summability matrix. If, for the operators \( L_n \) given by (2.1) of the class \( E_{2k} \), \( k > 1 \), the conditions

\[
st_A^{-}\lim_n \left\| L_n(e_i) - e_i^{(2m)} \right\| = 0, \quad i = 0, 1, \ldots, 2k, \ m < k,
\]

(3.1)
hold, then, for every function \( f \) with a continuous derivative of order \( 2m \) on the interval \([a, b]\), we have

\[
st_A \lim_n \| L_n(f) - f'' \| = 0.
\]

**Proof.** By similarity, we give a proof for \( m = 1 \). By (2.11), we may write that

\[
L_n(f; x) = f(x)L_n(e_0; x) + f'(x)L_n(\Psi; x) + L_n^\ast(f''; x),
\]

where

\[
L_n^\ast(f''; x) := R_n(x) = \int_a^b f''(y) d\varphi_n^\ast(x, y)
\]

with

\[
d\varphi_n^\ast(x, y) := \begin{cases} 
(f_y^y \int_a^y d\varphi_n(x, y_2) dy_1) dy & \text{if } a \leq y \leq x, \\
(f_y^b \int_a^y d\varphi_n(x, y_2) dy_1) dy & \text{if } x \leq y \leq b.
\end{cases}
\]

Then, observe that the operators \( \{L_n^\ast(f''; x)\}_{n \in \mathbb{N}} \) belong to the class \( E_{2k-2} \). By (3.1), we obtain that

\[
st_A \lim_n \| L_n^\ast(e_i) - e_i \| = 0, \quad i = 0, 1, \ldots, 2k - 2.
\]

Since \( f'' \) is continuous on \([a, b]\), it follows from Theorem 2.1 that

\[
st_A \lim_n \| L_n^\ast(f'') - f'' \| = 0.
\]

Now by (3.2) one can get

\[
\| L_n(f) - f'' \| \leq M_1 \| L_n(e_0) \| + M_2 \| L_n(\Psi) \| + \| L_n^\ast(f'') - f'' \|,
\]

where \( M_1 = \| f \| \) and \( M_2 = \| f' \| \). The hypothesis (3.1) yields that

\[
st_A \lim_n \| L_n(e_0) \| = 0,
\]

\[
st_A \lim_n \| L_n(\Psi) \| = 0.
\]

For a given \( \varepsilon > 0 \), consider the following sets:

\[
U := \{n: \| L_n(f) - f'' \| \geq \varepsilon\},
\]

\[
U_1 := \{n: \| L_n(e_0) \| \geq \frac{\varepsilon}{3M_1}\},
\]

\[
U_2 := \{n: \| L_n(\Psi) \| \geq \frac{\varepsilon}{3M_2}\},
\]

\[
U_3 := \{n: \| L_n^\ast(f'') - f'' \| \geq \frac{\varepsilon}{3}\},
\]

where \( L_n^\ast \) is given by (3.3). Then, by (3.6), it is easy to see that

\[
U \subseteq U_1 \cup U_2 \cup U_3.
\]

Then one can see, for each \( j \in \mathbb{N} \), that

\[
\sum_{n \in U} a_{jn} \leq \sum_{n \in U_1} a_{jn} + \sum_{n \in U_2} a_{jn} + \sum_{n \in U_3} a_{jn}.
\]

Letting \( j \to \infty \) on the both sides of the above inequality, we have

\[
\lim_{j \to \infty} \sum_{n \in U} a_{jn} = 0,
\]

whence the result. \( \Box \)
One can also get the next results.

**Corollary 3.2.** If the operators \( (2.1) \) belong to the class \( E_{2k}, k > 1, \) and if
\[
\text{st}_n \lim_n \| L_n(e_i) - e_i^{(2m)} \| = 0, \quad i = 0, 1, \ldots, 2k, \quad m < k,
\]
then, for every function \( f \) having a continuous derivative of order \( 2m \) on \( [a, b] \), we have
\[
\text{st}_n \lim_n \| L_n(f) - f^{(2m)} \| = 0.
\]

**Corollary 3.3.** (See [1].) If the operators \( (2.1) \) belong to the class \( E_{2k}, k > 1, \) and if the sequence \( \{ L_n(e_i) \}_{n \in \mathbb{N}} \) is uniformly convergent to \( e_i^{(2m)} \) \( (i = 0, 1, \ldots, 2k \) and \( m < k) \) on \( [a, b] \), then, for every function \( f \) with a continuous derivative of order \( 2m \) on \( [a, b] \), the sequence \( \{ L_n(f) \}_{n \in \mathbb{N}} \) converges uniformly to \( f^{(2m)} \) on \( [a, b] \).

The following theorem can easily be proved as in Theorem 2.4.

**Theorem 3.4.** Let \( A = (a_{jn}) \) be a nonnegative regular summability matrix. If, for the operators \( (2.1) \) belonging to the class \( E_{2k}, k > 1, \) the conditions of Theorem 3.1 hold, and if
\[
\delta_A \left\{ n: \int_a^b |d\varphi_n^*(x, y)| \geq M \right\} = 0
\]
for some absolute constant \( M > 0, \) where \( d\varphi_n^*(x, y) \) is given by (3.4), then, for every function \( f \) with a continuous derivative of order \( 2m, m < k, \) on the interval \( [a, b] \), we have
\[
\text{st}_A \lim_n \| L_n(f) - f^{(2m)} \| = 0.
\]

Now we denote by \( G_{2k+1}, k \geq 1, \) the class of operators \( (2.1) \) such that for each fixed \( x \in [a, b] \) and for each \( n \in \mathbb{N} \), the following integrals:
\[
J_{2k+1, n}^{(1)}(y) := \int_a^y \int_a^{y_1} \cdots \int_a^{y_{2k}} d\varphi_n(x, y, y_{2k+1}) \cdots dy_2 dy_1 \quad \text{for } a \leq y \leq x,
\]
\[
J_{2k+1, n}^{(2)}(y) := \int_y^b \int_y^{y_1} \cdots \int_y^{y_{2k}} d\varphi_n(x, y, y_{2k+1}) \cdots dy_2 dy_1 \quad \text{for } x \leq y \leq b
\]
have well-defined but opposite signs for all \( y \in [a, b] \).

Then we get the following approximation theorem.

**Theorem 3.5.** Let \( A = (a_{jn}) \) be a nonnegative regular summability matrix. If the operators \( (2.1) \) belong to the class \( G_{2k+1}, k \geq 1, \) and if
\[
\text{st}_A \lim_n \| L_n(e_i) - e_i^{(2m+1)} \| = 0, \quad i = 0, 1, \ldots, 2k + 1, \quad m < k,
\]
then, for every function \( f \) with a continuous derivative of order \( 2k + 1 \) on the interval \( [a, b] \), we have
\[
\text{st}_A \lim_n \| L_n(f) - f^{(2m+1)} \| = 0.
\]

**Proof.** It is enough to prove for \( k = 1 \) and \( m = 0 \). Assume that \( f \) has a continuous third derivative on \( [a, b] \). Then we may write, for each \( x, y \in [a, b] \), that
\[
f(y) = f(x) + \int_x^y f'(t) \, dt.
\]
So using the definition of the operators $L_n$, we get

$$L_n(f; x) = f(x)L_n(e_0; x) + \int_a^b \int_x^y f'(t) \, dt \, d\varphi_n(x, y). \quad (3.11)$$

Breaking up the last integral into two integrals over $[a, x]$ and $[x, b]$ and integrating by parts we obtain that

$$\int_a^b \int_x^y f'(t) \, dt \, d\varphi_n(x, y) = -\int_x^a f'(y) \, d\varphi_n(x, y_1) \, dy + \int_x^b f'(y) \, d\varphi_n(x, y_1) \, dy$$

$$= \int_a^b f'(y) \, d\varphi_n^{**}(x, y)$$

$$=: L_n^{**}(f'; x),$$

where

$$d\varphi_n^{**}(x, y) := \begin{cases} -\left( f_a^y \, d\varphi_n(x, y_1) \right) \, dy & \text{if } a \leqslant y \leqslant x, \\ (f_y^b \, d\varphi_n(x, y_2)) \, dy & \text{if } x \leqslant y \leqslant b. \end{cases} \quad (3.12)$$

In this case, observe that all conditions of Theorem 2.1 are satisfied for the operators $L_n^{**}(f'; x)$. Since $f$ has a continuous third derivative on $[a, b]$, it follows from Theorem 2.1 that

$$\text{st}_A\lim_n \| L_n^{**}(f') - f' \| = 0. \quad (3.13)$$

On the other hand, by (3.9), it is clear that

$$\text{st}_A\lim_n \| L_n(e_0) \| = 0. \quad (3.14)$$

Now from (3.11) we have

$$L_n(f; x) - f'(x) = f(x) L_n(e_0; x) + L_n^{**}(f'; x) - f'(x),$$

which yields that

$$\| L_n(f) - f' \| \leqslant M_1 \| L_n(e_0) \| + \| L_n^{**}(f') - f' \|,$$

where $M_1 = \| f \|$. For every $\varepsilon > 0$, defining the following sets:

$$V := \{ n : \| L_n(f) - f' \| \geqslant \varepsilon \},$$

$$V_1 := \{ n : \| L_n(e_0) \| \geqslant \frac{\varepsilon}{2M_1} \},$$

$$V_2 := \{ n : \| L_n^{**}(f') - f' \| \geqslant \frac{\varepsilon}{2} \},$$

we easily get $V \subseteq V_1 \cup V_2$, which gives, for each $j \in \mathbb{N}$,

$$\sum_{n \in V} a_{jn} \leqslant \sum_{n \in V_1} a_{jn} + \sum_{n \in V_2} a_{jn}.$$

Now letting $j \to \infty$ and using (3.13) and (3.14) we have

$$\lim_j \sum_{n \in V} a_{jn} = 0.$$

This means that
\[
\text{st}_A \lim \frac{1}{n} \left\| L_n(f) - f' \right\| = 0,
\]
which completes the proof for \( l = 0 \) and \( k = 1 \). \( \square \)

By using a similar idea as in Theorems 2.4 and 3.4, we immediately get the following result.

**Theorem 3.6.** Let \( A = (a_{jn}) \) be a nonnegative regular summability matrix. If, for the operators (2.1) belonging to the class \( G_{2k+1} \), \( k \geq 1 \), the conditions of Theorem 3.5 hold, and if

\[
\delta_A \left\{ n : \int_a^x \left| J_{2k+1,n}^{(1)}(y) \right| dy + \int_x^b \left| J_{2k+1,n}^{(2)}(y) \right| dy \geq M \right\} = 0
\]

for some absolute constant \( M > 0 \), then, for every function \( f \) with a continuous derivative of order \( 2m + 1 \), \( m < k \), on the interval \([a, b]\), we have

\[
\text{st}_A \lim \frac{1}{n} \left\| L_n(f) - f^{(2m+1)} \right\| = 0.
\]

**Remark 3.1.** Finally, we note that, as in the previous corollaries, one can easily get the statistical and the classical cases of Theorems 3.4–3.6 by taking the Cesáro matrix and the identity matrix instead of the nonnegative regular matrix \( A = (a_{jn}) \).

**References**