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Asymptotic Representations for Root Vectors of Nonselfadjoint Operators and Pencils Generated by an Aircraft Wing Model in Subsonic Air Flow

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This paper is the second in a series of several works devoted to the asymptotic and spectral analysis of an aircraft wing in a subsonic air flow. This model has been developed in the Flight Systems Research Center of UCLA and is presented in the works by A. V. Balakrishnan. The model is governed by a system of two coupled integrodifferential equations and a two parameter family of boundary conditions modeling the action of the self-straining actuators. The differential parts of the above equations form a coupled linear hyperbolic system; the integral parts are of the convolution type. The system of equations of motion is equivalent to a single operator evolution-convolution equation in the energy space. The Laplace transform of the solution of this equation can be represented in terms of the so-called generalized resolvent operator, which is an operator-valued function of the spectral parameter. This generalized resolvent operator is a finite-meromorphic function on the complex plane having the branch cut along the negative real semi-axis. Its poles are precisely the aeroelastic modes and the residues at these poles are the projectors on the generalized eigenspaces. In the first paper and in the present one, our main object of interest is the dynamics generator of the differential parts of the system. It is a nonselfadjoint operator in the energy space with a purely discrete spectrum. In the first paper, we have shown that the spectrum consists of two branches and have derived their precise spectral asymptotics. In the present paper, we derive the asymptotical approximations for the mode shapes. Based on the asymptotical results of these first two papers, in the next paper, we will discuss the geometric properties of the mode shapes such as minimality, completeness, and the Riesz basis property in the energy space.

Key Words: flutter; aeroelastic modes; nonselfadjoint differential operator; convolution integral operator; nonselfadjoint polynomial pencil; discrete spectrum.

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1. INTRODUCTION

The present paper is the second one in a series of several papers devoted to the mathematical analysis of a certain boundary-value problem arising in the modeling of the flutter phenomenon in an aircraft wing in a subsonic airflow. In the aforementioned series of works, we are planning to give a solution of the long-standing problem devoted to the control of flutter in an aircraft wing using the so-called self-straining actuators. We use the model that has been developed in the Flight Systems Research Center of the University of California at Los Angeles. The mathematical formulation of the problem can be found in the work by Balakrishnan [4]. The model, which is used in [4], is the 2-D strip model which applies to bare wings of high aspect ratio [12]. The structure is modeled by a uniform cantilever beam which bends and twists. The aerodynamics is considered to be subsonic, incompressible, and inviscid. In addition, the author of [4] has added the self-straining actuators using a currently accepted model (see, e.g., [6–8, 13, 15, 26, 27]). We would like to mention an important paper [5], where an analytical study of continuum models of wing flutter in inviscid subsonic aerodynamics has been presented. The root locus of aeroelastic modes and precise operational definition of the flutter speed have been obtained. Also, the numerical results for the Goland wing model with the torsion mode flutter are given in [5].

As was already mentioned, this paper is the second one in the series of papers. However, the paper is self-contained and can be considered as an independent piece of work. To keep the paper self-contained, we will provide all necessary definitions and those results from the first paper which are important for the present one.

Before we recall the precise formulation of the problem, which we continue to study in the present and in subsequent papers, we would like to provide some general information about the flutter phenomenon (see, e.g., [2, 3, 12] and references therein). Flutter, which is known as a very dangerous aeroelastic development, is the onset, beyond some speed-altitude combinations, of unstable and destructive vibrations of a lifting surface in an airstream. Flutter is most commonly encountered on bodies subjected to large lateral aerodynamic loads of the lift type, such as aircraft wings, tails, and control surfaces. It is known as an aeroelastic problem. The only air forces necessary to produce flutter are those due to the deflection of the elastic structures from the undeformed state.

In many cases, one gets a realistic description of the flutter properties of a system by studying the stability of infinitesimal motions. It then suffices to analyze vibrations with exponential time dependence e^{pt} (p is complex), since all other small motions can be built up therefrom by superposition. If small deformations are dynamically unstable, it is a very undesirable situation on any piloted or automatically controlled aircraft, regardless of the stability of bigger ones. In practice, the larger displacements are stable if the smaller ones are. The flutter or critical speed u_f and frequency w_f are defined as the lowest airspeed and corresponding circular frequency at which a given structure flying at given atmospheric density and temperature will exhibit sustained, simple harmonic oscillations. Flight at u_f represents a borderline condition or neutral stability boundary, because all small motions must be stable at speeds below u_f, whereas divergent oscillations can ordinarily occur in a range of speeds (or at all speeds) above u_f. Theoretical flutter analysis often consists of assuming in advance that all dependent variables are proportional to e^{iwt} (w is real), and then finding such combinations of u and w for which this actually occur. One is thus led to complex or multiple eigenvalue problems involving eigenfunctions and associate functions. This is in contrast to free vibrations of a linear structure in vacuum, which is a real eigenvalue problem involving only eigenfunctions.

Probably, the most important type of aircraft flutter results from coupling between the bending and torsional motions of a relatively large aspect-ratio wing and tail. Precise mathematical formulation of the continuous model dynamics, which has been designed to treat the flutter caused by the aforementioned coupling, can be found in work [4].

The main objective of the aforementioned series of our papers is to find the time-domain solution of the initial-boundary value problem formulated in [4]. However, this objective requires very detailed mathematical analysis of the properties of the system. In the first paper [17], our main objective was to derive the asymptotic representations for the so-called aeroelastic modes which are associated with the discrete spectrum of the problem. (All necessary definitions will be provided at the appropriate places.) In the present paper, we will discuss asymptotical representations for the high frequency mode shapes. In the next paper, we will study the so-called geometric properties of the mode shapes, i.e., such properties as minimality (linear independence for infinitely many vectors), completeness, and the Riesz basis property. The final paper is expected to be devoted to the properties of the so-called continuous spectrum of the problem.

Now, we describe the content of the present paper. In Section 2, we recall the formulation of the initial-boundary value problem. The problem contains two continuous parameters in the boundary conditions. These parameters are introduced in order to model the action of the self-straining actuators as is accustomed in current engineering and mathematical literature. We also introduce the state space of the system (the energy space) and give reformation of the problem in the operator format in the energy space. As it has been shown in [18], the dynamics is defined by two matrix operators in this space. One of the aforementioned operators is a matrix differential operator and the second one is a matrix integral convolution-type operator. That is why in the operator setting, we have the so-called evolution-convolution problem. As it will be shown in the next paper, the aeroelastic modes (or the discrete spectrum of the problem) are asymptotically close to the discrete spectrum of the matrix differential operator while the continuous spectrum is completely determined by the matrix integral operator. *It is exactly the spectral properties of the differential operator which are of interest in the present paper*. Since the continuous spectrum is completely defined by the matrix integral operator, if the speed of an airstream $u = 0$, then the integral operators vanish and the appropriate purely structural problem has only discrete spectrum.

In Section 3, we provide the formulation of the main results from paper [17] which are essential for the present paper. In this section, we also introduce the nonselfadjoint operator-valued polynomial pencil closely related to the main matrix differential operator. More precisely, having the asymptotical approximations for the pencil eigenfunctions, we can use the explicit formulas to obtain the asymptotical approximations for the mode shapes of the problem (or, equivalently, for the eigenvectors of the main matrix differential operator).

In Sections 4 and 5, we provide detailed proofs of the asymptotical results on the pencil and the operator eigenvectors, respectively.

In the conclusion of the Introduction, we describe what kind of a control problem will be considered in connection with the flutter suppression. In the specific wing model considered in the current paper, both the matrix differential operator and the matrix integral operator contain entries depending on the speed u of the surrounding air flow. Therefore, the aeroelastic modes are functions of $u : \lambda_k = \lambda_k(u)$ $(k \in \mathbb{Z})$. The wing is stable if $\operatorname{Re}\lambda_k < 0$ for all k. However, if u is increasing, some of the modes move to the right half-plane. The flutter speed u_k^f for the kth mode is defined by the relation $\operatorname{Re}\lambda_k(u_k^f) = 0$. To understand the flutter phenomenon, it is not sufficient to trace the motion of aeroelastic modes as functions of a speed of airflow. It is also necessary to have efficient representations for the solutions of our boundary-value problem (see the next section), containing the contributions from both the discrete and the continuous parts of the spectrum. Such a representation will provide a precise description of the solution behavior. It is known that flutter cannot be eliminated completely. To successfully suppress flutter, one should design self-straining actuators (i.e., in mathematical language, to select parameters in the boundary conditions, which are the control gains β and δ in formulas (2.10) and (2.11) of Section 2), in such a way that flutter does not occur in the desired speed range. This is a highly nontrivial boundary control problem.

2. STATEMENT OF PROBLEM: OPERATOR SETTING IN ENERGY SPACE

In this section, we give a precise formulation of the initial-boundary value problem. Namely, we will study asymptotic and spectral properties of the system of two coupled damped integro-differential equations, which occur in mathematical description of the flutter suppression phenomenon in aircraft wings using self-straining actuators. Following [4], let us introduce the dynamical variables

$$X(x, t) = \begin{pmatrix} h(x, t) \\ \alpha(x, t) \end{pmatrix}, \qquad -L \leq x \leq 0, t \geq 0, \tag{2.1}$$

where $h(x, t)$ is the bending and $\alpha(x, t)$ the torsion angle. The model, which we will investigate, can be described by the linear system

$$(M_s - M_a)\ddot{X}(x,t) + (D_s - uD_a)\dot{X}(x,t) + (K_s - u^2 K_a)X = \begin{bmatrix} f_1(x,t) \\ f_2(x,t) \end{bmatrix}. \tag{2.2}$$

From now on, we will use the notation . (dot) to denote the differentiation with respect to t. We use the subscripts s and a to distinguish the structural and aerodynamical parameters, respectively. All 2 by 2 matrices in Eq. (2.2) are given by the formulas

$$M_s = \begin{bmatrix} m & S \\ S & I \end{bmatrix}, \qquad M_a = (-\pi\rho)\begin{bmatrix} 1 & -a \\ -a & (a^2 + 1/8) \end{bmatrix}, \tag{2.3}$$

where m is the density of the flexible structure (mass per unit length), S is the mass moment, I is the moment of inertia, ρ is the density of air, and a is the linear parameter of the structure ($-1 \leq a \leq 1$).

$$D_s = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad D_a = (-\pi\rho)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{2.4}$$

$$K_s = \begin{bmatrix} E\dfrac{\partial^4}{\partial x^4} & 0 \\ 0 & -G\dfrac{\partial^2}{\partial x^2} \end{bmatrix}, \qquad K_a = (-\pi\rho)\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \tag{2.5}$$

where E is the bending stiffness and G is the torsion stiffness. The parameter u in Eq. (2.2) denotes the stream velocity. The right hand side of system (2.2) can be represented as the following system of two convolution-type integral operations:

$$f_1(x, t) = -2\pi\rho \int_0^t \left[uC_2(t - \sigma) - \dot{C}_3(t - \sigma)\right]g(x, \sigma)d\sigma, \tag{2.6}$$

$$f_2(x,t) = -2\pi\rho \int_0^t \Big[1/2C_1(t-\sigma) - auC_2(t-\sigma) + a\dot{C}_3(t-\sigma) + uC_4(t-\sigma) + 1/2\dot{C}_5(t-\sigma)\Big] g(x,\sigma)d\sigma, \quad (2.7)$$

$$g(x,t) = u\dot{\alpha}(x,t) + \ddot{h}(x,t) + (1/2-a)\ddot{\alpha}(x,t). \quad (2.8)$$

The aerodynamical functions C_i, $i = 1, \ldots, 5$, are defined in the following ways,

$$\begin{aligned} \widehat{C}_1(\lambda) &\equiv \int_0^\infty e^{-\lambda t} C_1(t)dt = \frac{u}{\lambda} \frac{e^{-\lambda/u}}{K_0(\lambda/u) + K_1(\lambda/u)}, \qquad \operatorname{Re}\lambda > 0, \\ C_3(t) &= \int_0^t C_1(t-\sigma)(u\sigma - \sqrt{u^2\sigma^2 + 2u\sigma})d\sigma, \\ C_2(t) &= \int_0^t C_1(\sigma)d\sigma, \qquad C_4(t) = C_2(t) + C_3(t), \\ C_5(t) &= \int_0^t C_1(t-\sigma)\big((1+u\sigma)\sqrt{u^2\sigma^2+2u\sigma} - (1+u\sigma)^2\big)d\sigma, \end{aligned} \quad (2.9)$$

where K_0 and K_1 are the modified Bessel functions of the zero and first orders, respectively [16]. These formulas for the aerodynamical functions have been derived in [9]. It is known that the self-straining control actuator action can be modeled by the following boundary conditions,

$$Eh''(0,t) + \beta\dot{h}'(0,t) = 0, \qquad h'''(0,t) = 0, \quad (2.10)$$

$$G\alpha'(0,t) + \delta\dot{\alpha}(0,t) = 0, \qquad \beta, \delta \in \overline{\mathbb{C}^+} \cup \{\infty\}, \quad (2.11)$$

where $\overline{\mathbb{C}^+}$ is the closed right half-plane. Note that we essentially have tip "rate" controllers of the kind studied in [7, 10]. We consider the following boundary conditions at $x = -L$:

$$h(-L,t) = h'(-L,t) = \alpha(-L,t) = 0. \quad (2.12)$$

In Eqs. (2.10)–(2.12) and below, we use the prime for the derivative with respect to x. Let the initial state of the system be given as

$$\begin{aligned} h(x,0) &= h_0(x), \qquad \dot{h}(x,0) = h_1(x), \\ \alpha(x,0) &= \alpha_0(x), \qquad \dot{\alpha}(x,0) = \alpha_1(x). \end{aligned} \quad (2.13)$$

We will consider the solution of the problem given by Eqs. (2.2) and conditions (2.10)–(2.13) in the energy space $\mathscr{H}$. To introduce the metric of $\mathscr{H}$, we assume that the parameters satisfy the following two conditions:

$$\det\begin{bmatrix} m & S \\ S & I \end{bmatrix} > 0, \qquad 0 < u \le \frac{\sqrt{2G}}{L\sqrt{\pi\rho}}. \quad (2.14)$$

We note that (2.14) has a physical interpretation: it means that the flow velocities must be below the "divergence" or static aeroelastic instability velocity for the system.

Let $\mathscr{H}$ be the set of 4-component vector valued functions $\Psi = (h, \dot{h}, \alpha, \dot{\alpha})^T \equiv (\psi_0, \psi_1, \psi_2, \psi_3)^T$ (T means the transposition) obtained as a closure of smooth functions satisfying the conditions

$$\psi_0(-L) = \psi_0'(-L) = \psi_2(-L) = 0 \tag{2.15}$$

in the following energy norm:

$$\|\Psi\|_{\mathscr{H}}^2 = 1/2 \int_{-L}^{0} \big[E|\psi_0''(x)|^2 + G|\psi_2'(x)|^2 + \tilde{m}|\psi_1(x)|^2 + \widetilde{I}|\psi_3(x)|^2 \\ + \widetilde{S}(\psi_3(x)\bar{\psi}_1(x) + \bar{\psi}_3(x)\psi_1(x)) - \pi\rho u^2|\psi_2(x)|^2\big]dx. \tag{2.16}$$

As shown in [17], under conditions (2.14), the norm (2.16) is well defined. To rewrite the original initial-boundary value problem in the space $\mathscr{H}$, we have to complete preliminary steps.

Let $\{\widetilde{C}_i\}_{i=1}^2$ be the kernels in the convolution operations in (2.6), (2.7), i.e.,

$$\widetilde{C}_1(t) = -2\pi\rho(uC_2(t) - \dot{C}_3(t)), \tag{2.17}$$

$$\widetilde{C}_2(t) = -2\pi\rho\big(1/2C_1(t) - auC_2(t) + a\dot{C}_3(t) \\ + uC_4(t) + 1/2\dot{C}_5(t)\big), \tag{2.18}$$

and let M, D, K be the matrices

$$M = M_s - M_a, \qquad D = D_s - uD_a, \qquad K = K_s - u^2K_a. \tag{2.19}$$

Then Eq. (2.2) can be written in the form

$$M\ddot{X}(x,t) + D\dot{X}(x,t) + KX(x,t) = (\mathscr{F}\dot{X})(x,t)), \qquad t \geq 0, \tag{2.20}$$

where the matrix integral operator $\mathscr{F}$ is given by the formula

$$\mathscr{F} = \begin{bmatrix} \int_0^t \widetilde{C}_1(t-\sigma)\left(\frac{d}{d\sigma}\cdot\right)d\sigma & \int_0^t \widetilde{C}_1(t-\sigma)\left[u\cdot+(1/2-a)\left(\frac{d}{d\sigma}\cdot\right)\right]d\sigma \\ \int_0^t \widetilde{C}_2(t-\sigma)\left(\frac{d}{d\sigma}\cdot\right)d\sigma & \int_0^t \widetilde{C}_2(t-\sigma)\left[u\cdot+(1/2-a)\left(\frac{d}{d\sigma}\cdot\right)\right]d\sigma \end{bmatrix}$$

$$= \begin{bmatrix} \widetilde{C}_1*\left(\frac{d}{d\sigma}\cdot\right) & \widetilde{C}_2*\left(u\cdot+(1/2-a)\left(\frac{d}{d\sigma}\cdot\right)\right) \\ \widetilde{C}_2*\left(\frac{d}{d\sigma}\cdot\right) & \widetilde{C}_1*\left(u\cdot+(1/2-a)\left(\frac{d}{d\sigma}\cdot\right)\right) \end{bmatrix}. \tag{2.21}$$

In (2.21), we use the standard notation * for the convolution. We will call M, D, and K the spatial operators and $\mathcal{F}$ the time operator. Our goal is to rewrite Eq. (2.20) as the first order in time evolution-convolution equation in the energy space. As the first step, we will represent Eq. (2.20) in the form

$$\ddot{X} + M^{-1} D \dot{X} + M^{-1} K X = M^{-1} \mathcal{F} \dot{X}. \tag{2.22}$$

Below we provide explicit formulas for all operators entering Eq. (2.22).

$$\begin{aligned} M^{-1} &= \Delta^{-1} \begin{bmatrix} \widetilde{I} & -\widetilde{S} \\ -\widetilde{S} & \tilde{m} \end{bmatrix}, \quad \Delta = \det M, \\ M^{-1} D &= \frac{\pi \rho u}{\Delta} \begin{bmatrix} \widetilde{S} & \widetilde{I} \\ -\tilde{m} & -\widetilde{S} \end{bmatrix}, \\ M^{-1} K &= \frac{1}{\Delta} \begin{bmatrix} E\widetilde{I} \frac{\partial^4}{\partial x^4} & \widetilde{S} G \frac{\partial^2}{\partial x^2} + \widetilde{S} \pi \rho u^2 \\ -E\widetilde{S} \frac{\partial^4}{\partial x^4} & -\tilde{m} G \frac{\partial^2}{\partial x^2} - \tilde{m} \pi \rho u^2 \end{bmatrix}, \end{aligned} \tag{2.23}$$

where

$$\tilde{m} = m + \pi \rho, \qquad \widetilde{S} = S - a\pi\rho, \qquad \widetilde{I} = I + \pi\rho(a^2 + 1/8). \tag{2.24}$$

Note, due to the first condition in (2.14), M^{-1} exists. The initial-boundary value problem defined by Eq. (2.22) and conditions (2.10)–(2.13) can be represented in the form

$$\dot{\Psi} = i\mathcal{L}_{\beta\delta} \Psi + \widetilde{\mathcal{F}} \dot{\Psi}, \qquad \Psi = (\psi_0, \psi_1, \psi_2, \psi_3)^T, \qquad \Psi|_{t=0} = \Psi_0. \tag{2.25}$$

$\mathcal{L}_{\beta\delta}$ is the following matrix differential operator in $\mathcal{H}$,

$$\mathcal{L}_{\beta\delta} = -i \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{E\widetilde{I}}{\Delta} \frac{d^4}{dx^4} & -\frac{\pi\rho u \widetilde{S}}{\Delta} & -\frac{\widetilde{S}}{\Delta}\left(G \frac{d^2}{dx^2} + \pi\rho u^2 \right) & -\frac{\pi\rho u \widetilde{I}}{\Delta} \\ 0 & 0 & 0 & 1 \\ \frac{E\widetilde{S}}{\Delta} \frac{d^4}{dx^4} & \frac{\pi\rho u \tilde{m}}{\Delta} & \frac{\tilde{m}}{\Delta}\left(G \frac{d^2}{dx^2} + \pi\rho u^2 \right) & \frac{\pi\rho u \widetilde{S}}{\Delta} \end{bmatrix} \tag{2.26}$$

defined on the domain

$$\begin{aligned} \mathcal{D}(\mathcal{L}_{\beta\delta}) = \Big\{ &\Psi \in \mathcal{H}\colon\ \psi_0 \in H^4(-L, 0), \psi_1 \in H^2(-L, 0), \psi_2 \in H^2(-L, 0), \\ &\psi_3 \in H^1(-L, 0); \psi_1(-L) = \psi_1'(-L) = \psi_3(-L) = 0; \psi_0'''(0) = 0; \\ &E\psi_0''(0) + \beta\psi_1'(0) = 0, G\psi_2'(0) + \delta\psi_3(0) = 0 \Big\}, \end{aligned} \tag{2.27}$$

where H^i, $i=1,2,4$, are the standard Sobolev spaces [1]. $\widetilde{\mathscr{F}}$ is a linear integral operator in $\mathscr{H}$ given by the formula

$$\widetilde{\mathscr{F}}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & [\widetilde{I}(\widetilde{C}_1*)-\widetilde{S}(\widetilde{C}_2*)] & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & [-\widetilde{S}(\widetilde{C}_1*)+\tilde{m}(\widetilde{C}_2*)]\end{bmatrix} \times\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 1 & u & (1/2-a)\\ 0 & 0 & 0 & 0\\ 0 & 1 & u & (1/2-a)\end{bmatrix}. \tag{2.28}$$

The main goal of the first paper from the aforementioned series was to derive the asymptotics of the spectrum of the operator $\mathscr{L}_{\beta\delta}$. It turns out that the spectral properties of both the differential operator $\mathscr{L}_{\beta\delta}$ and the integral operator $\widetilde{\mathscr{F}}$ are of crucial importance for the representation of the solution. Namely, as it will be shown in our next paper, the discrete spectrum of the entire problem is asymptotically close to the discrete spectrum of the operator $\mathscr{L}_{\beta\delta}$ and the continuous spectrum of it is completely determined by the operator $\widetilde{\mathscr{F}}$.

Remark 2.1. We would like to mention that Eq. (2.20) occurs actually in aeroelastic problems (see the classic textbook [12]) if one ignores the aeroelastic forces. However, the boundary conditions in [12] which complemented the system of equations (2.20) with $u=0$ are totally different. To the best of our knowledge, the whole structural problem consisting of Eqs. (2.20) ($u=0$) and boundary conditions (2.10)–(2.12) has been considered only in one paper by Balakrishnan [11].

Remark 2.2. The aircraft wing model, considered in the present paper, can be described by the evolution-convolution type equation of the form

$$\dot{\Psi}(t)=iA\Psi(t)+\int_0^t F(t-\tau)\dot{\Psi}(\tau)d\tau. \tag{2.29}$$

Here $\Psi(\cdot)\in\mathscr{H}$ is the energy space of the system, Ψ is a 4-component vector-valued function, $A(A=\mathscr{L}_{\beta\delta})$ is a matrix differential operator, and $F(\cdot)$ is a matrix-valued function.

Equation (2.29) does not define an evolution semigroup and does not have a dynamics generator. In terms of spectral analysis of Eq. (2.29), we understand the following. Let us take the Laplace transformation of both parts of Eq. (2.29). Formal solution in the Laplace representation can be given by the formula

$$\widehat{\Psi}(\lambda)=\big(\lambda I-iA-\lambda\widehat{F}(\lambda)\big)^{-1}\big(I-\widehat{F}(\lambda)\big)\Psi_0, \tag{2.30}$$

where Ψ_0 is the initial state, i.e., $\Psi(0)=\Psi_0$, and the symbol ˆ is used to denote the Laplace transform. It is an extremely nontrivial problem to understand the precise meaning of Eq. (2.30) and, most importantly, to calculate the inverse Laplace transform of Eq. (2.30) in order to have the representation of the solution in the space-time domain. To do this, it is necessary to investigate the generalized resolvent operator.

$$R(\lambda)=\left(\lambda I-iA-\lambda\widehat{F}(\lambda)\right)^{-1}. \tag{2.31}$$

In the case of the 1-dim wing model, $R(\lambda)$ is an operator-valued meromorphic function on the complex plane with a branch cut along the negative real semi-axis. The poles of $R(\lambda)$ are called the eigenvalues, or *the aeroelastic modes*. The residues of $R(\lambda)$ at the poles are precisely the projectors on the corresponding generalized eigenspaces. The branch cut corresponds to the continuous spectrum.

As has already been mentioned in [17], we have obtained the asymptotic formulas for the aeroelastic modes. In the present paper, we will derive asymptotics of the eigenvectors and in the next paper, we will prove that the generalized eigenvectors of the discrete spectrum form a Riesz basis in their closed linear span in the energy space.

However, to find the space-time domain solution, we have to solve the following remaining problems. (a) To obtain asymptotic formulas for the eigenfunctions of the continuous spectrum. (These eigenfunctions can be expressed in terms of the jump of the kernel of the generalized resolvent $R(\lambda)$ across the branch cut.) (b) To obtain an expansion theorem with respect to the eigenfunctions of the continuous spectrum.

Combining (a) and (b) with already known results, we will be able to "calculate" the inverse Laplace transform in Eq. (2.30) and, thus, to obtain the desired solution of the initial-boundary value problem.

3. STATEMENT OF MAIN THEOREMS: AUXILIARY RESULTS FROM [17]

In the present section, we give precise formulation of the main results of the paper. But before this we have to reproduce the main spectral and asymptotical results from [17]. We start with the general properties of the operator $\mathscr{L}_{\beta\delta}$.

THEOREM 3.1. (a) *$\mathscr{L}_{\beta\delta}$ is a closed linear operator in $\mathscr{H}$ whose resolvent is compact, and therefore, the spectrum is discrete* [14, 27]. (b) *Operator $\mathscr{L}_{\beta\delta}$ is nonselfadjoint unless β and δ are purely imaginary. If* $\operatorname{Re}\beta\geq 0$ *and* $\operatorname{Re}\delta\geq 0$, *then this operator is dissipative, i.e.*, $\operatorname{Im}(\mathscr{L}_{\beta\delta}\Psi,\Psi)\geq 0$ *for all* $\Psi\in\mathscr{D}(\mathscr{L}_{\beta\delta})$. *The adjoint operator $\mathscr{L}^*_{\beta\delta}$ is given by the matrix differential expression* (2.26) *on the* *domain obtained from* (2.27) *by replacing parameters* β *and* δ *with* $(-\bar{\beta})$ *and* $(-\bar{\delta})$, *respectively*. (c) *When* $\mathscr{L}_{\beta\delta}$ *is dissipative, then it is maximal*; *i.e., it does not admit any more dissipative extensions*.

In the next theorem, we provide the spectral asymptotics for the operator $\mathscr{L}_{\beta\delta}$.

THEOREM 3.2. (a) *The operator* $\mathscr{L}_{\beta\delta}$ *has a countable set of complex eigenvalues. If*

$$|\delta| \neq \sqrt{G\widetilde{I}}, \tag{3.1}$$

then the set of eigenvalues is located in a strip parallel to the real axis.

(b) *The entire set of eigenvalues asymptotically splits into two disjoint subsets. We call them the* β*-branch and the* δ*-branch and denote them by* $\{\lambda_n^{\beta}\}_{n\in\mathbb{Z}}$ *and* $\{\lambda_n^{\delta}\}_{n\in\mathbb{Z}}$, *respectively. If* $\operatorname{Re}\beta \geq 0$ *and* $\operatorname{Re}\delta > 0$ *then the* δ*-branch is asymptotically close to some horizontal line in the closed upper half-plane. If* $\operatorname{Re}\beta > 0$ *and* $\operatorname{Re}\delta = 0$, *then both horizontal asymptotes coincide with the real axis. If* $\operatorname{Re}\beta = \operatorname{Re}\delta = 0$, *then the operator* $\mathscr{L}_{\beta\delta}$ *is selfadjoint and, thus, its spectrum is real. The entire set of eigenvalues may have only two points of accumulation*: $+\infty$ *and* $-\infty$ *in the sense that* $\operatorname{Re}\lambda_n^{\beta(\delta)} \longrightarrow \pm\infty$ *and* $\operatorname{Im}\lambda_n^{\beta(\delta)} \longrightarrow$ *const as* $n \longrightarrow \pm\infty$ (*see formulas* (3.2) *and* (3.3) *below*).

(c) *The following asymptotics is valid for the* β*-branch of the spectrum*,

$$\lambda_n^{\beta} = (\operatorname{sgn} n)(\pi^2/L^2)\sqrt{E\widetilde{I}/(\tilde{m}\widetilde{I} - \widetilde{S}^2)}\,(n-1/4)^2 + \kappa_n(w),$$
$$|n| \longrightarrow \infty, \tag{3.2}$$

where $w = |\beta|^{-1} + |\delta|^{-1}$. *The complex-valued sequence* $\{\kappa_n(w)\}_{n\in\mathbb{Z}}$ *is bounded in the following sense*:

$$\sup_{n\in\mathbb{Z}}\{|\kappa_n(w)|\} = C(w), \qquad C(w) \longrightarrow 0 \text{ as } w \longrightarrow 0.$$

This branch may have a finite number of multiple eigenvalues of a finite algebraic multiplicity each. For such an eigenvalue, the geometric multiplicity may be less than the corresponding algebraic multiplicity; *i.e., in addition to the eigenvector or eigenvectors, there may be the associate vectors.* (*Recall that* Φ *is an associate vector of an operator* A *of the order* m *corresponding to the eigenvalue* λ *if* $\Phi \neq 0$, $(A-\lambda I)^m\Phi \neq 0$, *and* $(A-\lambda I)^{m+1}\Phi = 0$. *If* $m = 0$, *then* Φ *is an eigenvector.*)

(d) *The following asymptotics is valid for the* δ*-branch of the spectrum*:

$$\lambda_n^{\delta} = \frac{\pi n}{L\sqrt{\widetilde{I}/G}} + \frac{i}{2L\sqrt{\widetilde{I}/G}}\ln\frac{\delta + \sqrt{G\widetilde{I}}}{\delta - \sqrt{G\widetilde{I}}} + O(|n|^{-1/2}), \quad |n| \longrightarrow \infty. \tag{3.3}$$

In (3.3), ln *means the principle value of the logarithm. In this branch, there may be only a finite number of multiple eigenvalues of a finite multiplicity each. Therefore, only a finite number of the associate vectors may exist.*

To formulate the main results of the current paper, we have to introduce a two-parameter family of nonselfadjoint operator pencils. Let $\mathscr{P}_{\beta\delta}(\lambda)$ be the 4th order polynomial pencil defined by the formula

$$\begin{aligned}(\mathscr{P}_{\beta\delta}(\lambda)\Psi)(x) = {} & EG\Psi^{VI}(x) + E(\lambda^2\widetilde{I} + \pi\rho u^2)\Psi^{V}(x) \\ & - \lambda^2\tilde{m}G\Psi''(x) - (\lambda^4\Delta - \lambda^2(\pi\rho u)^2 \\ & + \lambda^2\pi\rho u^2\tilde{m})\Psi(x), \qquad x \in [-L, 0],\end{aligned} \tag{3.4}$$

on the domain

$$\begin{aligned}\mathscr{D}(\mathscr{P}_{\beta\delta}(\lambda)) = \{ & \Psi \in H^6 \colon \Psi(-L) = \Psi'(-L) = 0, \\ & G\Psi'''(-L) + (\lambda^2\widetilde{I} + \pi\rho u^2)\Psi'(-L) = 0, \\ & GE\Psi^{IV}(0) + E(\lambda^2\widetilde{I} + \pi\rho u^2)\Psi''(0) \\ & \quad + i\lambda\beta G\Psi'''(0) + i\lambda\beta(\lambda^2\widetilde{I} + \pi\rho u^2)\Psi'(0) = 0, \\ & G\Psi'(0) + i\lambda\delta\Psi(0) = 0, \\ & G\Psi^{V}(0) + (\lambda^2\widetilde{I} + \pi\rho u^2)\Psi'''(0) = 0\}.\end{aligned} \tag{3.5}$$

In formulas (3.4) and (3.5), by Ψ^{VI}, Ψ^{V}, and Ψ^{IV} we denote the sixth, fifth, and fourth order derivatives of the function Ψ. The pencil $\mathscr{P}_{\beta\delta}(\lambda)$ is closely related to the main operator $\mathscr{L}_{\beta\delta}$. Namely, let us consider the spectral equation for the operator $\mathscr{L}_{\beta\delta}$

$$\mathscr{L}_{\beta\delta}\Phi = \lambda\Phi, \;\; \Phi = (\psi_0, \psi_1, \psi_2, \psi_3)^T. \tag{3.6}$$

Equation (3.6) can be reduced to a linear system of four equations with respect to four unknown functions $\{\psi_i\}, i = 0, \ldots, 3$. If we exclude from the system three unknown functions ψ_0, ψ_1, and ψ_3, then for the function ψ_2, we obtain the equation

$$\mathscr{P}_{\beta\delta}(\lambda)\psi_2 = 0. \tag{3.7}$$

By a direct calculation, it can be verified that the component ψ_2 satisfies the boundary conditions given by (3.5). Let us use the following definition

DEFINITION 3.1. If there exists a $\lambda = \lambda_0$ such that the equation

$$\mathscr{P}_{\beta\delta}(\lambda_0)\Psi = 0 \tag{3.8}$$

has a nontrivial solution, then this value of $\lambda = \lambda_0$ and the solution Ψ will be called an eigenvalue and an eigenfunction of the pencil $\mathscr{P}_{\beta\delta}(\lambda)$.

Therefore, the function Ψ is an eigenfunction of the pencil $\mathscr{P}_{\beta\delta}(\lambda)$ corresponding to the eigenvalue λ if and only if this function coincides with the component ψ_2 of an eigenvector $\Phi=(\psi_0,\psi_1,\psi_2,\psi_3)^T$ of the operator $\mathscr{L}_{\beta\delta}$ corresponding to the same eigenvalue λ. (Note that the relationship between associate vectors of the pencil $\mathscr{P}_{\beta\delta}(\lambda)$ and the operator $\mathscr{L}_{\beta\delta}$ is much more complicated. We do not discuss it here since there could be at most a finite number of associate vectors.) It is important that the spectrum of the operator $\mathscr{L}_{\beta\delta}$ coincides with the spectrum of the pencil and the spectral multiplicities also coincide. Moreover, if we know the pencil eigenfunction Ψ and identify it with the component ψ_2 of the eigenvector Φ of the operator $\mathscr{L}_{\beta\delta}$, then the following formulas are valid:

$$\Phi=\left(\frac{1}{i\lambda}\psi_0,\psi_0,\frac{1}{i\lambda}\psi_2,\psi_2\right)^T,$$

$$\text{where } \psi_0=-\frac{G}{\lambda^2\widetilde{S}+i\lambda\pi\rho u}\psi_2''-\frac{\lambda^2\widetilde{I}+\pi\rho u^2}{\lambda^2\widetilde{S}+i\lambda\pi\rho u}\psi_2. \tag{3.9}$$

Thus, in order to find the asymptotical representations for the eigenfunctions of the operator $\mathscr{L}_{\beta\delta}$, it suffices to find the asymptotical representations for the eigenfunctions of $\mathscr{P}_{\beta\delta}(\lambda)$.

Our next result is concerned with asymptotical representations for the eigenfunctions of the pencil. In Theorem 3.3, we formulate the results for the practically important case when β and δ are real and positive. In this case, the spectral problem has a specific symmetry which allows us to make the proofs significantly shorter than we would have had for the general case of arbitrary complex β and δ.

Now we explain the aforementioned symmetry. First of all, we note that for real β and δ, the spectrum of the operator $\mathscr{L}_{\beta\delta}$ (as well as for the pencil $\mathscr{P}_{\beta\delta}(\lambda)$) is symmetric with respect to the imaginary axis, i.e., $\lambda_{-|n|}^{\beta(\delta)}=-\overline{\lambda}_{|n|}^{\beta(\delta)}, n\in\mathbb{Z}$. The latter fact implies that if $\Psi_n^{\beta(\delta)}$ is an eigenfunction of $\mathscr{P}_{\beta\delta}(\lambda)$, i.e., $\mathscr{P}_{\beta\delta}(\lambda_n^{\beta(\delta)})\Psi_n^{\beta(\delta)}=0, n>0$, then there exist a function denoted by $\Psi_{-n}^{\beta(\delta)}$ such that $\mathscr{P}_{\beta\delta}(-\overline{\lambda}_n^{\beta(\delta)})\Psi_{-n}^{\beta(\delta)}=0$. Moreover, it can be shown by a direct calculation that

$$\Psi_{-n}^{\beta(\delta)}(x)=\overline{\Psi}_n^{\beta(\delta)}(x), \qquad x\in[-L,0],\ n>0. \tag{3.10}$$

Therefore it suffices to derive the asymptotics for the pencil eigenfunctions only if $n>0$.

THEOREM 3.3. *For sufficiently large n, the pencil $\mathscr{P}_{\beta\delta}(\lambda)$ has only eigenfunctions (no associate functions) which can be split into two branches—the β-branch and the δ-branch.*

(a) *For* $n>0$ *and* $x\in[-L,0]$, *the asymptotical representations for the* δ*-branch eigenfunctions* $\Psi_n^{\delta}(x)$ *can be given by the formulas as* $n\to\infty$

$$\Psi_n^{\beta}(x)=-2i\sin\{(\pi n+i/2\ln(-K_+/K_-))(x/L+1)\}+O(n^{-1/2}), \quad (3.11)$$

where

$$K_{\pm}=\delta^{-1}\sqrt{G\widetilde{I}}\pm 1. \quad (3.12)$$

For $n<0$ *and* $x\in[-L,0]$, *we have*

$$\Psi^{\delta}_{-|n|}(x)=\overline{\Psi}^{\delta}_{|n|}(x). \quad (3.13)$$

(b) *For* $n>0$ *and* $x\in[-L,0]$, *the asymptotical representations for the* β*-branch eigenfunctions* $\{\Psi_n^{\beta}\}_{n\in\mathbb{Z}}$ *can be given by the formulas as* $n\to\infty$

$$\Psi_n^{\delta}(x)=-2i\sin\{f(n-1/4)^2(x/L+1)\}+\Psi_n^{\beta,0}(x)+O(n^{-1}), \quad (3.14)$$

where

$$\begin{aligned}\Psi_n^{\beta,0}(x)=&(\det \mathbb{M}^{\beta})^{-1}\left[1+K_-K_+^{-1}\exp\{-2if(n-1/4)^2\}\right]\\ &\times\left[-\exp\{i\pi(n-1/4)(x/L+1)\}+(1-i)\exp\{-\pi(n-1/4)\right.\\ &\left.\times(x/L+1)\}+i\exp\{-i\pi(n-1/4)(x/L+1)\}\right]. \quad (3.15)\end{aligned}$$

In formula (3.15), *we have introduced the following notations*:

$$\det \mathbb{M}^{\beta}=-2iK_+^{-1}\exp\{-i(n-1/4)(\pi+f(n-1/4))\}(1+O(n^{-1})),$$

and

$$f=\pi^2L^{-1}(\tilde{m}\widetilde{I}-\widetilde{S}^2)^{-1/2}\widetilde{I}E^{1/2}G^{-1/2}. \quad (3.16)$$

For $n<0$ *and* $x\in[-L,0]$, *we have*

$$\Psi^{\beta}_{-|n|}(x)=\overline{\Psi}^{\beta}_{|n|}(x). \quad (3.17)$$

All estimate $O(\cdot)$ *in formulas* (3.11)–(3.16) *are uniform with respect to* $x, x\in[-L,0]$. *Numeration in formulas* (3.11)–(3.17) *is asymptotic but not absolute*.

The last theorem in this section is devoted to the asymptotical approximations for 4-component eigenvectors of the operator $\mathscr{L}_{\beta\delta}$.

THEOREM 3.4. (a) *The asymptotical approximations for the* δ*-branch eigenvectors* $\{F_n^{\delta}\}_{n\in\mathbb{Z}}$ *of the operator* $\mathscr{L}_{\beta\delta}$ *can be represented by the formulas*

$$F_n^{\delta}(x)=(O(n^{-3/2}),O(n^{-1/2}),(i\lambda_n^{\delta})^{-1}\Psi_n^{\delta}(x),\Psi_n^{\delta}(x))^T,$$
$$x\in[-L,0], n\in\mathbb{Z}. \quad (3.18)$$

The function Ψ_n^{δ} *is given in* (3.11) *for* $n>0$ *and in* (3.13) *for* $n<0$. *Both estimates* $O(n^{-3/2})$ *and* $O(n^{-1/2})$ *are uniform with respect to* x.

(b) *The asymptotical representations for β-branch eigenvectors* $\{F_n^\beta\}_{n\in\mathbb{Z}}$ *can be represented by the formulas*

$$F_n^\beta(x) = F_{1,n}^\beta(x) + F_{2,n}^\beta(x) + F_{3,n}^\beta(x), \qquad x\in[-L,0], n\in\mathbb{Z}. \tag{3.19}$$

where

$$F_{1,n}^\beta(x) = \Psi_n^{\beta,0}\big(-\widetilde{I}(i\lambda_n^\beta\widetilde{S})^{-1}, -\widetilde{I}\widetilde{S}^{-1}, (i\lambda_n^\beta)^{-1}, 1\big)^T,$$

$\Psi_n^{\beta,0}$ *is defined in* (3.15),

$$F_{2,n}^\beta(x) = -2i\sin\{f(n-1/4)^2(x/L+1)\}((i\lambda_n^\beta)^{-1}, 1, 0, 0)^T,$$

$$F_{3,n}^\beta(x) = (O(n^{-3}), O(n^{-1}), O(n^{-3}), O(n^{-1}))^T. \tag{3.20}$$

All estimates in (3.20) *are uniform with respect to* $x\in[-L,0]$.

Remark 3.1. We would like to emphasize that all estimates in the asymptotical formulas occurring in Theorems 3.1–3.4 are uniform with respect to both parameters β and δ if these parameters change in compact subsets of the complex plane. This result is not important for the present paper. However, it will be extremely valuable in the next one devoted to the geometric properties of the generalized eigenvectors (or the root vectors) of the operator $\mathscr{L}_{\beta\delta}$. Recall that in terms of "the geometric properties," we understand such properties as minimality (linear independence in the case of infinitely many vectors), uniform minimality, completeness, and the Riesz basis property of the root vectors in the energy space.

4. PROOFS OF RESULTS ON PENCIL EIGENFUNCTIONS

In this section, we will prove Theorem 3.3. However, we have to start with some results obtained in our paper [17]. We briefly recall what major steps have been done to derive the spectral asymptotics. First, we have reduced the spectral equation

$$\mathscr{L}_{\beta\delta}\Psi = \lambda\Psi, \quad \Psi = (\psi_0, \psi_1, \psi_2, \psi_3)^T, \tag{4.1}$$

to a linear system of four coupled equations for the components. Then, instead of a system of four equations with respect to four unknown functions, we have investigated one 6th order ordinary differential equation with respect to one unknown function ψ_2. It was convenient to use the notation $\Psi(\lambda,\cdot)$ for the aforementioned unknown function ψ_2. The general solution of this 6th order ordinary differential equation has been represented in the form

$$\begin{aligned}\Psi(\lambda,x) = {}& \mathscr{A}(\lambda)e^{\gamma(\lambda)(x+L)} + \mathscr{B}(\lambda)e^{i\hat{\gamma}(\lambda)(x+L)} + \mathscr{C}(\lambda)e^{i\Gamma(\lambda)(x+L)}\\ &+ \mathscr{D}(\lambda)e^{-\gamma(\lambda)(x+L)} + \mathscr{E}(\lambda)e^{-i\hat{\gamma}(\lambda)(x+L)}\\ &+ \mathscr{F}(\lambda)e^{-i\Gamma(\lambda)(x+\lambda)},\end{aligned} \tag{4.2}$$

where

$$\begin{aligned}\gamma(\lambda)&=c\sqrt{\lambda}+c_1/\sqrt{\lambda}+O(\lambda^{-1.5}),\\ \hat{\gamma}(\lambda)&=c\sqrt{\lambda}-c_1/\sqrt{\lambda}+O(\lambda^{-1.5}),\end{aligned} \tag{4.3}$$

$$\begin{aligned}&\Gamma(\lambda)=\lambda\sqrt{R_0}+d/\lambda+O(\lambda^{-3}), \qquad R_0=\widetilde{I}/G,\\ &c=(\tilde{m}\widetilde{I}-\widetilde{S}^2)^{1/4}(E\widetilde{I})^{-1/4}, \qquad c_1=8^{-1}c^{-2}\widetilde{S}^2G\widetilde{I}^{-2}E^{-1},\\ &d=4^{-1}\widetilde{S}^2G^{1/2}\widetilde{I}^{-3/2}E^{-1}+2^{-1}\pi\rho u^2R_0^{1/2}G^{-1}.\end{aligned} \tag{4.4}$$

The function $\Psi(\lambda,\cdot)$ must satisfy the boundary conditions at both left and right ends of the interval $[-L,0]$. We have obtained that the function $\Psi(\lambda,\cdot)$ satisfies the following set of the conditions at $x=-L$,

$$\begin{aligned}\Psi(\lambda,-L)=\Psi''(\lambda,-L)=(G/\widetilde{I})\Psi'''(\lambda,-L)+\lambda^2(1+O(\lambda^{-1}))\\ \times\Psi'(\lambda,-L)=0,\end{aligned} \tag{4.5}$$

and the following set of the conditions at $x=0$,

$$\begin{aligned}&G\Psi^{V}(\lambda,0)+(\lambda^2\widetilde{I}+\pi\rho u^2)\Psi'''(\lambda,0)=0,\\ &G\Psi'(\lambda,0)+i\lambda\delta\Psi(\lambda,0)=0,\\ &GE\Psi^{IV}(\lambda,0)+\lambda^2\widetilde{I}E(1+O(\lambda^{-2}))\Psi''(\lambda,0)+i\lambda\beta G\Psi'''(\lambda,0)\\ &\quad+i\lambda^3\beta\widetilde{I}(1+O(\lambda^{-2}))\Psi'(\lambda,0)=0.\end{aligned} \tag{4.6}$$

It has been convenient for us to introduce the notations

$$\begin{pmatrix}\mathscr{A}(\lambda)\\ \mathscr{B}(\lambda)\\ \mathscr{C}(\lambda)\end{pmatrix}=X(\lambda), \qquad \begin{pmatrix}\mathscr{D}(\lambda)\\ \mathscr{E}(\lambda)\\ \mathscr{F}(\lambda)\end{pmatrix}=Y(\lambda). \tag{4.7}$$

Substituting $\Psi(\lambda,\cdot)$ in the form (4.2) into the left end boundary conditions (4.6), we have obtained the following matrix equation for two unknown 3-vectors $X(\cdot)$ and $Y(\cdot)$:

$$X(\lambda)=\begin{bmatrix} i(1+O(\lambda^{-1})) & (i-1)(1+O(\lambda^{-1})) & O(\lambda^{-1})\\ -(i+1)(1+O(\lambda^{-1})) & -i(1+O(\lambda^{-1})) & O(\lambda^{-1})\\ O(\lambda^{-1}) & O(\lambda^{-1}) & -1+O(\lambda^{-3/2})\end{bmatrix}Y(\lambda). \tag{4.8}$$

The matrix at the right hand side of Eq. (4.9) has been called the *left reflection matrix* and denoted by $\mathbb{R}_l(\lambda)$. Hence, Eq. (4.9) can be written in the form

$$X(\lambda)=\mathbb{R}_l(\lambda)Y(\lambda). \tag{4.9}$$

It has been convenient to introduce the notations

$$
\begin{aligned}
e(\lambda) &= e^{\gamma(\lambda)L}, \qquad \hat{e}(\lambda) = e^{i\hat{\gamma}(\lambda)L},\\
e_+(\lambda) &= e^{i\Gamma(\lambda)L}, \quad \mathbb{E}(\lambda) = \operatorname{diag}\{e(\lambda), \hat{e}(\lambda), e_+(\lambda)\},
\end{aligned} \tag{4.10}
$$

where $\gamma, \hat{\gamma}$, and Γ are defined in (4.3)–(4.5). Substituting the function $\Psi(\lambda,\cdot)$ into conditions (4.7), we obtain the second matrix equation for the vectors $X(\cdot)$ and $Y(\cdot)$

$$X(\lambda) = \mathbb{R}_r(\lambda)Y(\lambda), \tag{4.11}$$

with $\mathbb{R}_r$ being given by the formula

$$
\begin{aligned}
\mathbb{R}_r(\lambda) &= (\mathbb{E}(\lambda))^{-1}\\
&\times \begin{bmatrix} (1+O(\lambda^{-1/2})) & O(\lambda^{-1/2}) & O(\lambda^{-1/2}) \\ O(\lambda^{-1/2}) & (1+O(\lambda^{-1/2})) & O(\lambda^{-1/2}) \\ \dfrac{-2(1+O(\lambda^{-1/2}))}{1+\delta^{-1}G\sqrt{R_0}} & \dfrac{-2(1+O(\lambda^{-1/2}))}{1+\delta^{-1}G\sqrt{R_0}} & \dfrac{\delta^{-1}G\sqrt{R_0}-1}{\delta^{-1}G\sqrt{R_0}+1}(1+O(\lambda^{-1/2})) \end{bmatrix} (\mathbb{E}(\lambda))^{-1}.
\end{aligned} \tag{4.12}
$$

We call $\mathbb{R}_r$ *the right reflection matrix*.

Now we are in a position to prove our main pencil results.

Proof of Theorem 3.3. Let us rewrite Eqs. (4.9) and (4.11) as one matrix equation. We obtain the following linear system for six unknown coefficients:

$$
\begin{pmatrix} \mathscr{A}(\lambda) \\ \mathscr{B}(\lambda) \\ \mathscr{C}(\lambda) \\ \mathscr{D}(\lambda) \\ \mathscr{E}(\lambda) \\ \mathscr{F}(\lambda) \end{pmatrix} = \begin{bmatrix} 0 & \mathbb{R}_r(\lambda) \\ \mathbb{R}_l^{-1}(\lambda) & 0 \end{bmatrix} \begin{pmatrix} \mathscr{A}(\lambda) \\ \mathscr{B}(\lambda) \\ \mathscr{C}(\lambda) \\ \mathscr{D}(\lambda) \\ \mathscr{E}(\lambda) \\ \mathscr{F}(\lambda) \end{pmatrix}. \tag{4.13}
$$

By a direct computation, we obtain that the matrix $\mathbb{R}_l(\lambda)$ is asymptotically unitary; i.e., $(\mathbb{R}_l(\lambda))^{-1}$ asymptotically coincides with $\mathbb{R}_l(\lambda)$. The determinant of the homogeneous system (4.13) is

$$\mathbb{D}(\lambda) = \mathbb{I} - \mathbb{R}_l^{-1}(\lambda)\mathbb{R}_r(\lambda). \tag{4.14}$$

We already know that

$$\mathbb{D}(\lambda_n^{\beta}) = \mathbb{D}(\lambda_n^{\delta}) = 0, \qquad n \in \mathbb{Z}. \tag{4.15}$$

We will consider 2 cases corresponding to the β-branch and the δ-branch, respectively. In what follows, it is convenient to use new notations. Let ω_{ij}

be the factor of the form $(1+O((\lambda_n^{\beta(\delta)})^{-1}))$ standing on the intersection of the ith row and jth column of the matrix in (4.9) and let $\hat{\omega}_{ij}$ be the factor of the form $(1+O((\lambda_n^{\beta(\delta)})^{-1/2}))$ standing on the intersection of the ith row and jth column of the matrix in (4.12).

Case 1. *Approximations for the nontrivial solutions of Eq.* (4.13) *when* $\lambda=\lambda_n^\delta, n>0$. In this case, we assume that $\mathcal{F}=1$ and select five equations out of six given by system (4.13). If we take, e.g., three equations given by system (4.9) and two equations (the first and the third ones) from (4.11), we will have a linear system of five equations with five unknowns. Let

$$\begin{aligned}&\mathcal{A}(\lambda_n^{\beta(\delta)})=\mathcal{A}_n^{\beta(\delta)}, \quad \mathcal{B}(\lambda_n^{\beta(\delta)})=\mathcal{B}_n^{\beta(\delta)}, \quad \mathcal{C}(\lambda_n^{\beta(\delta)})=\mathcal{C}_n^{\beta(\delta)},\\ &\mathcal{D}(\lambda_n^{\beta(\delta)})=\mathcal{D}_n^{\beta(\delta)}, \qquad \mathcal{E}(\lambda_n^{\beta(\delta)})=\mathcal{E}_n^{\beta(\delta)};\end{aligned} \tag{4.16}$$

$$e(\lambda_n^{\beta(\delta)})=e_n^{\beta(\delta)}, \qquad \hat{e}(\lambda_n^{\beta(\delta)})=\hat{e}_n^{\beta(\delta)}, \qquad e_+(\lambda_n^{\beta(\delta)})=E_n^{\beta(\delta)}. \tag{4.17}$$

In terms of (4.16) and (4.17), the aforementioned system of five equations has the form

$$\begin{aligned}&\mathcal{A}_n^\delta=i\omega_{11}\mathcal{D}_n^\delta+(i-1)\omega_{12}\mathcal{E}_n^\delta+O(n^{-1}),\\ &\mathcal{B}_n^\delta=-(i+1)\omega_{21}\mathcal{D}_n^\delta-i\omega_{22}\mathcal{E}_n^\delta+O(n^{-1}),\\ &\mathcal{C}_n^\delta=O(n^{-1})\mathcal{D}_n^\delta+O(n^{-1})\mathcal{E}_n^\delta-1+O(n^{-3/2}),\\ &\mathcal{A}_n^\delta=(e_n^\delta)^{-2}\hat{\omega}_{11}\mathcal{D}_n^\delta+(e_n^\delta\hat{e}_n^\delta)^{-1}O(n^{-1/2})\mathcal{E}_n^\delta+(e_n^\delta E_n^\delta)^{-1}O(n^{-1/2}),\\ &\mathcal{C}_n^\delta=\frac{-2(e_n^\delta E_n^\delta)^{-1}}{K_+}\hat{\omega}_{31}\mathcal{D}_n^\delta-\frac{-2(\hat{e}_n^\delta E_n^\delta)^{-1}}{K_+}\hat{\omega}_{32}\mathcal{E}_n^\delta+\frac{K_-}{K_+}(E_n^\delta)^{-2}\hat{\omega}_{33}.\end{aligned} \tag{4.18}$$

Now we calculate the approximations for all exponential quantities from system (4.18). To this end, using (3.3), (4.3), and (4.4), we first obtain

$$\begin{aligned}&\gamma_n^\delta\equiv\gamma(\lambda_n^\delta)=g\sqrt{n}+O(n^{-1/2}), \qquad g=c\pi^{1/2}G^{1/4}L^{-1/2}\widetilde{I}^{-1/4}.\\ &\hat{\gamma}_n^\delta\equiv\hat{\gamma}(\lambda_n^\delta)=g\sqrt{n}+O(n^{-1/2}),\\ &\Gamma(\lambda_n^\delta)\equiv\Gamma_n^\delta=\frac{\pi n}{L}+\frac{i}{2L}\ln\frac{\delta+\sqrt{G\widetilde{I}}}{\sqrt{G\widetilde{I}}}+O\left(\frac{1}{\sqrt{n}}\right).\end{aligned} \tag{4.19}$$

Substituting (4.19) into (4.17), we have

$$\begin{aligned}&\hat{e}_n^\delta=\exp\{ig\sqrt{n}L\}(1+O(n^{-1/2})), \quad e_n^\delta=\exp\{g\sqrt{n}L\}(1+O(n^{-1/2})),\\ &E_n^\delta=\exp\{i\pi n-1/2\ln(1-K_+/K_-)\}(1+O(n^{-1/2}))\\ &\quad=(-1)^n(-K_-/K_+)^{1/2}+O(n^{-1/2}).\end{aligned} \tag{4.20}$$

Substituting (4.20) into system (4.18), we obtain the following resulting system,

$$
\begin{aligned}
&\mathscr{A}_n^\delta = i\omega_{11}\mathscr{D}_n^\delta + (i-1)\omega_{12}\mathscr{E}_n^\delta + O(n^{-1}),\\
&\mathscr{B}_n^\delta = -(1+i)\omega_{21}\mathscr{D}_n^\delta - i\omega_{22}\mathscr{E}_n^\delta + O(n^{-1}),\\
&\mathscr{C}_n^\delta = O(n^{-1})\mathscr{D}_n^\delta + O(n^{-1})\mathscr{E}_n^\delta - 1 + O(n^{-3/2}),\\
&\mathscr{A}_n^\delta = e^{-2g\sqrt{n}L}\hat{\omega}_{11}\mathscr{D}_n^\delta + e^{-(1+i)g\sqrt{n}L}O(n^{-1/2})\mathscr{E}_n^\delta + e^{-g\sqrt{n}L}O(n^{-1/2}),\\
&\mathscr{C}_n^\delta = -\frac{2\hat{\omega}_{31}}{K_+}e^{-g\sqrt{n}L}T_n^\delta\mathscr{D}_n^\delta - \frac{2\hat{\omega}_{32}}{K_+}e^{-ig\sqrt{n}L}T_n^\delta\mathscr{E}_n^\delta = \hat{\omega}_{33},
\end{aligned}
\tag{4.21}
$$

where

$$
T_n^\delta = (-1)^n(-K_+/K_-)^{1/2}. \tag{4.22}
$$

The matrix of coefficients corresponding to the system (4.21) can be given in the form

$$
\mathbb{M}^\delta = \begin{bmatrix}
-1 & 0 & 0 & i\omega_{11} & (i-1)\omega_{12}\\
-1 & 0 & 0 & e^{-2g\sqrt{n}L}\hat{\omega}_{11} & e^{-g\sqrt{n}L}O(n^{-1/2})\\
0 & -1 & 0 & -(1+i)\omega_{21} & -i\omega_{22}\\
0 & 0 & -1 & O(n^{-1}) & O(n^{-1})\\
0 & 0 & -1 & -2K_+^{-1}T_n^\delta e^{-g\sqrt{n}L} & -2K_+^{-1}T_n^\delta e^{-ig\sqrt{n}L}\hat{\omega}_{32}
\end{bmatrix}. \tag{4.23}
$$

The determinant of the matrix $\mathbb{M}^\delta$ can be approximated as

$$
\det\mathbb{M}^\delta = -2iK_+^{-1}T_n^\delta e^{-ig\sqrt{n}L}(1+O(n^{-1/2})). \tag{4.24}
$$

By solving system (4.21), we obtain that the coefficients can be approximated as

$$
\begin{aligned}
&\mathscr{A}_n^\delta = O(e^{-g\sqrt{n}L}n^{-1/2}), \qquad \mathscr{B}_n^\delta = O(n^{-1/2}),\\
&\mathscr{C}_n^\delta = -1 + O(n^{-1}), \qquad \mathscr{D}_n^\delta = O(n^{-1/2}), \qquad \mathscr{E}_n^\delta = O(n^{-1/2}).
\end{aligned}
\tag{4.25}
$$

Using (4.25), we now obtain the approximations for the eigenfunctions Ψ_n^δ of the δ-branch of the pencil $\mathscr{P}_{\beta\delta(\lambda)}$. We have from (4.2)

$$
\begin{aligned}
\Psi_n^\delta(x) = {} & \mathscr{A}_n^\delta \exp\{\gamma_n^\delta(x+L)\} + \mathscr{B}_n^\delta \exp\{i\hat{\gamma}_n^\delta(x+L)\}\\
& + \mathscr{C}_n^\delta \exp\{i\Gamma_n^\delta(x+L)\} + \mathscr{D}_n^\delta \exp\{-\gamma_n^\delta(x+L)\}\\
& + \mathscr{E}_n^\delta \exp\{-i\hat{\gamma}_n^\delta(x+L)\} + \mathscr{F}_n^\delta \exp\{-i\Gamma_n^\delta(x+L)\}
\end{aligned}
$$

$$
\begin{aligned}
&= \exp\{g\sqrt{n}x\}O(n^{-1/2}) + \exp\{ig\sqrt{n}(x+L)\}O(n^{-1/2}) \\
&\quad + \exp\{-g\sqrt{n}(x+L)\}O(n^{-1/2}) + \exp\{-ig\sqrt{n}(x+L)\}O(n^{-1/2}) \\
&\quad - 2i\sin\{(\pi n + i/2ln(-K_+/K_-))(x/L+1)\} \\
&= -2i\sin\{(\pi n + i/2\ln(-K_+/K_-))(x/L+1)\} + O(n^{-1/2}). \qquad (4.26)
\end{aligned}
$$

Formula (3.11) is thus shown. As was mentioned, the eigenfunction of the δ-branch of $\mathscr{P}_{\beta\delta}(\lambda)$ corresponding to negative n can be obtained by formula (3.13).

Finally, we prove the approximation formula for the β-branch eigenfunction of $\mathscr{P}_{\beta\delta}(\lambda)$.

Case 2. Approximations for the nontrivial solutions of Eq. (4.13) *when* $\lambda = \lambda_n^\beta, n \longrightarrow \infty, n > 0$. As it was done for the δ-branch, we assume that $\mathscr{F} = 1$ and select five equations from the system of six dependent equations given by system (4.13). For this case, it is convenient to take the first four equations and the last equation from (4.13). Using notations (4.16) and (4.17), we obtain

$$
\begin{aligned}
&\mathscr{A}_n^\delta = (e_n^\beta)^{-2}\hat{\omega}_{11}\mathscr{D}_n^\beta + (e_n^\beta \hat{e}_n^\beta)^{-1} O((\lambda_n^\beta)^{-1/2})\mathscr{E}_n^\beta + (e_n^\beta E_n^\beta)^{-1} O((\lambda_n^\beta)^{-1/2}), \\
&\mathscr{B}_n^\beta = (e_n^\beta \hat{e}_n^\beta)^{-1} O((\lambda_n^\beta)^{-1/2})\mathscr{D}_n^\beta + (\hat{e}_n^\beta)^{-2}\hat{\omega}_{22}\mathscr{E}_n^\beta + (\hat{e}_n^\beta E_n^\beta)^{-1} O((\lambda_n^\beta)^{-1/2}), \\
&\mathscr{C}_n^\beta = -\frac{2\hat{\omega}_{31}}{K_+}(e_n^\beta E_n^\beta)^{-1}\mathscr{D}_n^\beta - \frac{2\hat{\omega}_{32}}{K_+}(\hat{e}_n^\beta E_n^\beta)^{-1}\mathscr{E}_n^\beta + \frac{K_-}{K_+}(E_n^\beta)^{-2}\hat{\omega}_{33}, \qquad (4.27) \\
&\mathscr{A}_n^\beta = i\omega_{11}\mathscr{D}_n^\beta + (i-1)\omega_{12}\mathscr{E}_n^\beta + O((\lambda_n^\beta)^{-1}), \\
&\mathscr{C}_n^\beta = O((\lambda_n^\beta)^{-1})\mathscr{D}_n^\beta + O((\lambda_n^\beta)^{-1})\mathscr{E}_n^\beta + (-1 + O((\lambda_n^\beta)^{-3/2}).
\end{aligned}
$$

Taking into account the asymptotics for the β-branch of the spectrum (3.2), and formulas (4.3), (4.4), we calculate

$$
\begin{aligned}
&\gamma_n^\beta \equiv \gamma(\lambda_n^\beta) = \pi/L(n-1/4) + O(n^{-1}), \\
&\hat{\gamma}_n^\beta = \pi/L(n-1/4) + O(n^{-1}), \qquad (4.28) \\
&\Gamma_n^\beta \equiv \Gamma(\lambda_n^\beta) = fL^{-1}(n-1/4)^2 + O(n^{-1}), \quad \text{where } f \text{ is given in (3.16).}
\end{aligned}
$$

With formulas (4.28), (4.17), and (4.10) we obtain that

$$
\begin{aligned}
&e_n^\beta = e^{\pi(n-1/4)}(1+O(n^{-1})), \quad \hat{e}_n^\beta = e^{i\pi(n-1/4)}(1+O(n^{-1})), \\
&E_n^\beta = e^{if(n-1/4)^2}(1+O(n^{-1})).
\end{aligned} \qquad (4.29)
$$

Substituting (4.29) into system (4.27), we obtain the following result:

$$
\begin{aligned}
\mathscr{A}_n^\beta &= e^{-2\pi(n-1/4)}(1+O(n^{-1}))\mathscr{D}_n^\beta + e^{-(1+i)(n-1/4)}O(n^{-1})\mathscr{E}_n^\beta \\
&\quad + e^{-\pi n}O(n^{-1}), \\
\mathscr{B}_n^\beta &= e^{-\pi n}O(n^{-1})\mathscr{D}_n^\beta + e^{-2\pi i(n-1/4)}(1+O(n^{-1}))\mathscr{E}_n^\beta + O(n^{-1}), \\
\mathscr{C}_n^\beta &= -\frac{2}{K_+}e^{-\pi(n-1/4)}e^{-if(n-1/4)^2}\mathscr{D}_n^\beta(1+O(n^{-1})) \\
&\quad - \frac{2}{K_+}e^{-i\pi(n-1/4)-if(n-1/4)^2}\mathscr{E}_n^\beta(1+O(n^{-1})) \\
&\quad + \frac{K_-}{K_+}e^{-2if(n-1/4)^2}(1+O(n^{-1})). \\
\mathscr{A}_n^\beta &= i(1+O(n^{-2}))\mathscr{D}_n^\beta + (i-1)(1+O(n^{-2}))\mathscr{E}_n^\beta + O(n^{-2}), \\
\mathscr{C}_n^\beta &= O(n^{-2})\mathscr{D}_n^\beta + O(n^{-2})\mathscr{E}_n^\beta - 1 + O(n^{-3}).
\end{aligned}
\tag{4.30}
$$

The matrix of the coefficients for system (4.30) can be given in the form

$$
\mathbb{M}^\beta = \begin{bmatrix}
-1 & 0 & 0 & e^{-2\pi(n-1/4)}(1+O(n^{-1})) & e^{-\pi n}O(n^{-1}) \\
-1 & 0 & 0 & i(1+O(n^{-1})) & (i-1)(1+O(n^{-1})) \\
0 & -1 & 0 & e^{-\pi n}O(n^{-1}) & i(1+O(n^{-1})) \\
0 & 0 & -1 & O(n^{-2}) & O(n^{-2}) \\
0 & 0 & -1 & \begin{matrix} -2K_+^{-1}e^{-(n-1/4)(\pi+if(n-1/4))} \\ \times(1+O(n^{-1})) \end{matrix} & \begin{matrix} -2K_+^{-1}e^{-i(n-1/4)(\pi+f(n-1/4))} \\ \times(1+O(n^{-1})) \end{matrix}
\end{bmatrix}. \tag{4.31}
$$

The determinant of the matrix $\mathbb{M}^\beta$ can be approximated as

$$
\mathbb{M}^\beta = -2iK_+^{-1}e^{-i(n-1/4)(\pi+f(n-1/4))}(1+O(n^{-1})). \tag{4.32}
$$

For the coefficients, we obtain the following asymptotic approximations:

$$
\begin{aligned}
&\mathscr{A}_n^\beta = O(e^{-\pi n}n^{-1}), \qquad \mathscr{B}_n^\beta = -V_\beta(1+O(n^{-1})), \\
&\mathscr{C}_n^\beta = -1+O(n^{-2}), \\
&\mathscr{D}_n^\beta = (1-i)V_\beta(1+O(n^{-1})), \qquad \mathscr{E}_n^\beta = iV_\beta(1+O(n^{-1})), \\
&V_\beta = (\det\mathbb{M}^\beta)^{-1}(1+K_-K_+^{-1}e^{-2if(n-1/4)^2}).
\end{aligned}
\tag{4.33}
$$

Note that $V_\beta \neq 0$ unless $\delta = \infty$. Having formulas (4.33) for the coefficients, we obtain the following approximations for the β-branch eigenfunctions of the pencil $\mathcal{P}_{\beta\delta}(\lambda)$:

$$\begin{aligned}
\Psi_n^\beta(x) = {} & \mathcal{A}_n^\beta \exp\{\pi(n-1/4)(x/L+1)\}(1+O(n^{-1})) \\
& + \mathcal{B}_n^\beta \exp\{i\pi(n-1/4)(x/L+1)\}(1+O(n^{-1})) \\
& + \mathcal{C}_n^\beta \exp\{if(n-1/4)^2(x/L+1)\}(1+O(n^{-1})) \\
& + \mathcal{D}_n^\beta \exp\{-\pi(n-1/4)(x/L+1)\}(1+O(n^{-1})) \\
& + \mathcal{E}_n^\beta \exp\{-i\pi(n-1/4)(x/L+1)\}(1+O(n^{-1})) \\
& + \exp\{if(n-1/4)^2(x/L+1)\}(1+O(n^{-1})) \\
= {} & -2i\sin\{f(n-1/4)^2(x/L+1)\} \\
& + V_\beta[-\exp\{i\pi(n-1/4)(x/L+1)\} \\
& + (1-i)\exp\{-\pi(n-1/4)(x/L+1)\} \\
& + i\exp\{-i\pi(n-1/4)(x/L+1)\}] + O(n^{-1}). \qquad (4.34)
\end{aligned}$$

Formula (4.34) implies (3.14) and (3.15).

The Theorem is completely shown.

Remark 4.1. As was already mentioned, in the proof of Theorem 3.3, we have used the advantage of the fact that both parameters β and δ are real (and positive). In fact, precisely this situation takes place for the flutter control problem which we are interested in. However, formal analysis can be carried out for any complex values of β and δ. In the case when at least one of the parameters is complex, the symmetry of the spectrum with respect to the imaginary axis does not hold and the relations similar to (3.10) and (3.13) do not exist. In this more general case ($\mathrm{Im}\,\beta \neq 0$ and/or $\mathrm{Im}\,\delta \neq 0$), to find the approximations for the pencil eigenfunctions $\Psi_n^{\beta(\delta)}$ with $n<0$, one should complete the steps similar to the ones that have been carried out for $n>0$.

5. ESTIMATES FOR MODE SHAPES: PROOF OF THEOREM 3.4

In this section, we complete the proof of the main asymptotical results of the paper.

Proof of Theorem 3.4. As follows from formula (3.9), to find the approximations for the eigenvectors of the operator $\mathcal{L}_{\beta\delta}$, it suffices to derive the approximations for the component ψ_0. To this end, let us derive the approximation for ψ_2''. Due to the fact that ψ_2 has been identified with Ψ, we have

$$\Psi''(\lambda,x)=\gamma^2(\lambda)e^{\gamma(\lambda)(x+L)}\mathscr{A}(\lambda)-\hat{\gamma}^2(\lambda)e^{i\hat{\gamma}(\lambda)(x+L)}\mathscr{B}(\lambda)$$
$$-\Gamma^2(\lambda)e^{i\Gamma(\lambda)(x+L)}\mathscr{C}(\lambda)+\gamma^2(\lambda)e^{-\gamma(\lambda)(x+L)}\mathscr{D}(\lambda)$$
$$-\gamma^2(\lambda)e^{i\hat{\gamma}(\lambda)(x+L)}\mathscr{E}(\lambda)-\Gamma^2(\lambda)e^{i\Gamma(\lambda)(x+L)}\mathscr{F}(\lambda). \quad (5.1)$$

We will consider separately the δ-branch and the β-branch of the eigenvectors.

Case 1. Consider the δ-branch with $n>0$. Using (4.19) and (4.20), we can rewrite (5.1) in the form

$$(\Psi_n^\delta(x))''=(\gamma_n^\delta)^2e^{\gamma_n^\delta(x+L)}\mathscr{A}_n^\delta-(\hat{\gamma}_n^\delta)^2e^{i\hat{\gamma}_n^\delta(x+L)}\mathscr{B}_n^\delta$$
$$-(\Gamma_n^\delta)^2e^{i\Gamma_n^\delta(x+L)}\mathscr{C}_n^\delta+(\gamma_n^\delta)^2e^{-\gamma_n^\delta(x+L)}\mathscr{D}_n^\delta$$
$$-(\hat{\gamma}_n^\delta)^2e^{-i\hat{\gamma}_n^\delta(x+L)}\mathscr{E}_n^\delta-(\Gamma_n^\delta)^2e^{-i\Gamma_n^\delta(x+L)}\mathscr{F}_n^\delta. \quad (5.2)$$

Taking into account formulas (4.25) for the coefficients, we simplify (5.2) and have

$$(\Psi_n^\delta(x))''=(\gamma_n^\delta)^2e^{\gamma_n^\delta x}O(n^{-1/2})+(\hat{\gamma}_n^\delta)^2e^{i\hat{\gamma}_n^\delta(x+L)}O(n^{-1/2})+(\Gamma_n^\delta)^2e^{i\Gamma_n^\delta(x+L)}$$
$$\times(1+O(n^{-1}))+(\gamma_n^\delta)^2e^{-\gamma_n^\delta(x+L)}O(n^{-1/2})+(\hat{\gamma}_n^\delta)^2e^{-i\hat{\gamma}_n^\delta(x+L)}$$
$$\times O(n^{-1/2})-(\Gamma_n^\delta)^2e^{-i\Gamma_n^\delta(x+L)}, \qquad x\in[-L,0]. \quad (5.3)$$

To make further simplification of formula (5.3), we use (4.3) and (4.4) and have

$$(\Psi_n^\delta(x))''=e^{\gamma_n^\delta x}O(n^{1/2})+e^{i\hat{\gamma}_n^\delta(x+L)}O(n^{1/2})-2i\widetilde{I}/G(\lambda_n^\delta)^2\sin\{\Gamma_n^\delta(x+L)\}$$
$$+e^{i\Gamma_n^\delta(x+L)}O(n^{1/2})+e^{-\gamma_n^\delta(x+L)}O(n^{1/2})-e^{-i\hat{\gamma}_n^\delta(x+L)}O(n^{1/2}). \quad (5.4)$$

Substituting formulas (5.4) and (3.11) into the formula for ψ_0 from (3.9) and denoting the component $\psi_0(\lambda_n^\delta,\cdot)$ by $\psi_{0,n}^\delta$, we obtain the following representation:

$$\psi_{0,n}^\delta(x)=-\frac{G}{(\lambda_n^\delta)^2\widetilde{S}}(\Psi_n^\delta(x))''(1+O(n^{-1}))-\frac{\widetilde{I}}{\widetilde{S}}\Psi_n^\delta(x)(1+O(n^{-1}))$$
$$=2i\widetilde{I}\widetilde{S}^{-1}\sin\{\Gamma_n^\delta(x+L)\}+O(n^{-3/2})-2i\widetilde{I}\widetilde{S}^{-1}$$
$$\times\sin\{\Gamma_n^\delta(x+L)\}+O(n^{-1/2})=O(n^{-1/2}). \quad (5.5)$$

To obtain the representation for the 4-component eigenvector F_n^δ of the δ-branch of the spectrum, it suffices to substitute formulas (3.11) and (5.5) into the formula for Φ from (3.9). We have for $x\in[-L,0]$

$$F_n^\delta(x)=((i\lambda_n^\delta)^{-1}\psi_{0,n}^\delta(x),\psi_{0,n}^\delta(x),(i\lambda_n^\delta)^{-1}\psi_{2,n}^\delta(x),\psi_{2,n}^\delta(x))^T, \quad (5.6)$$

where by $\psi^{\delta}_{2,n}$, we denote the function $\psi_2(\lambda^{\delta}_n, \cdot)$. (Recall that we have identified the function $\psi_2(\lambda^{\delta}_n, \cdot)$ with the function Ψ^{δ}_n.)

Using Theorem 3.3 and formula (5.5) for $\psi^{\delta}_{0,n}$, we immediately arrive at formula (3.18). Finally, we prove the result for the β-branch eigenfunctions of the operator $\mathscr{L}_{\beta\delta}$. In the notations, similar to (5.2), we can rewrite (5.1) and have

$$\begin{aligned}(\Psi^{\beta}_n(x))'' = {} & (\gamma^{\beta}_n)^2 e^{\gamma^{\beta}_n(x+L)}\mathscr{A}^{\beta}_n - (\hat{\gamma}^{\beta}_n)^2 e^{i\hat{\gamma}^{\beta}_n(x+L)}\mathscr{B}^{\beta}_n \\ & - (\Gamma^{\beta}_n)^2 e^{i\Gamma^{\beta}_n(x+L)}\mathscr{C}^{\beta}_n + (\gamma^{\beta}_n)^2 e^{-\gamma^{\beta}_n(x+L)}\mathscr{D}^{\beta}_n \\ & - (\hat{\gamma}^{\beta}_n)^2 e^{-i\hat{\gamma}^{\beta}_n(x+L)}\mathscr{E}^{\beta}_n - (\Gamma^{\beta}_n)^2 e^{-i\Gamma^{\beta}_n(x+L)}\mathscr{F}^{\beta}_n. \end{aligned} \tag{5.7}$$

Using formulas (4.33) for the coefficients, we simplify (5.7) and have

$$\begin{aligned}(\Psi^{\beta}_n(x))'' = {} & O(e^{-\pi n} n) e^{\gamma^{\beta}_n(x+L)} + O(n^2) e^{i\hat{\gamma}^{\beta}_n(x+L)} + 2i(\Gamma^{\beta}_n)^2 \\ & \times \sin\{\Gamma^{\beta}_n(x+L)\} + O(n^2) e^{-\gamma^{\beta}_n(x+L)} + O(n^2) e^{i\hat{\gamma}^{\beta}_n(x+L)}. \end{aligned} \tag{5.8}$$

Substituting formulas (5.8) and (3.15) into (3.10), we obtain the following representation for $\psi^{\beta}_{0,n}$,

$$\begin{aligned}\psi^{\beta}_{0,n}(x) &= -\frac{G}{(\lambda^{\beta}_n)^2 \widetilde{S}} (\Psi^{\beta}_n(x))''(1+O(n^{-2})) - \frac{\widetilde{I}}{\widetilde{S}} \Psi^{\beta}_n(x)(1+O(n^{-2})) \\ &= \frac{\widetilde{I}}{\widetilde{S}} \Psi^{\beta,0}_n(x) + O(n^{-1}), \end{aligned} \tag{5.9}$$

where $\Psi^{\beta,0}_n$ is defined in (3.15).

This completes the proof of the theorem.

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