Boundary point lemmas and overdetermined problems

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Abstract

Two elliptic boundary value problems are considered: a problem of mixed type in a cylindrical domain, and a Dirichlet problem in an annular domain. Under some overdetermined conditions on the boundary gradient, symmetry results for domain and solution are proved. The method of proof involves the classical boundary point lemma by Hopf, as well as a suitable adaptation of it that works well at certain corners.

Keywords: Boundary point lemma; Comparison principle; Overdetermined problem

1. Introduction

Since the classical work by Hopf [6], the well-known boundary point lemma (also told second maximum principle) has become one of the most important devices in the field of elliptic partial differential equations, especially in the study of qualitative properties of solutions.

In a sequence of recent papers [3,4,8,11] a new technique, based on the maximum principle, has been introduced and developed to the aim of studying symmetry properties of some overdetermined problems. In this paper, the technique is adapted to the study of the following two.

Problem 1 (A problem of mixed type in a cylindrical domain). In this case, the main difficulty is represented by the corners that naturally occur in a cylinder. Indeed, the clas-

The classical formulation of Hopf’s lemma requires that an interior sphere condition is satisfied. However, some variants of the lemma have been developed so far to front particular problems. Among them, the most popular is probably the one by Serrin [13]. Another type of boundary point lemma is proved here (see Lemma 4.1) to overcome the difficulty mentioned before. The final result is a generalization of theorems proved by Payne and Philippin [10] and by Henrot et al. [8].

To be more precise, let \( \omega \) be a bounded domain of class \( C^3 \) in \( \mathbb{R}^{N-1} \), \( N \geq 3 \). Alternatively, \( \omega \) can be a bounded open interval in \( \mathbb{R}^1 \) and \( N = 2 \). Let \( \varphi_0, \varphi_1 \) be real-valued functions belonging to the class \( C^2(\bar{\omega}) \) and satisfying \( \varphi_0 < \varphi_1 \) in \( \bar{\omega} \) (the closure of \( \omega \)). For \( x = (x_1, \ldots, x_N) \in \mathbb{R}^N \), denote by \( \xi = (x_1, \ldots, x_{N-1}) \) the projection of \( x \) to \( \mathbb{R}^{N-1} \).

Let the cylindrical domain \( \Omega \) and its boundary portions \( \Gamma_0, \Gamma_1, \Gamma_c \) be defined as follows:

\[
\Omega = \{ x \in \mathbb{R}^N | \xi \in \omega, \varphi_0(\xi) < x_N < \varphi_1(\xi) \},
\]

\[
\Gamma_i = \{ x \in \partial \Omega | \xi \in \omega, x_N = \varphi_i(\xi) \} \quad \text{for } i = 0, 1,
\]

\[
\Gamma_c = \{ x \in \partial \Omega | \xi \in \partial \omega, \varphi_0(\xi) < x_N < \varphi_1(\xi) \}.
\]

Furthermore, denote by \( n(x) \) the outer normal to \( \partial \Omega \) at \( x \) (when it exists). It will be convenient (cf. (HL) in Section 2) to prolongue the definition of \( n(x) \) to all of \( \Gamma_i \) by continuity. This can be done by the regularity of \( \varphi_i \), as follows:

\[
n(x) = (-1)^i \frac{(D\varphi_i(\xi), -1)}{\sqrt{1 + |D\varphi_i(\xi)|^2}} \quad \text{for } x \in \Gamma_i, \; i = 0, 1.
\]  

Consider the mixed-type problem

\[
\begin{cases}
\Delta u = 0 & \text{in } \Omega, \\
u(x) = 0 & \text{on } \Gamma_0, \\
u(x) = 1 & \text{on } \Gamma_1, \\
\partial u/\partial n(x) = 0 & \text{on } \Gamma_c.
\end{cases}
\]  

The question of existence and regularity of solutions to elliptic boundary value problems in non-smooth domains is quite delicate. The interested reader may consult, for instance, the classical monograph by Grisvard [5], as well as the book by Dauge [1] and the survey by Plamenevskij [9].

In the paper [10], Payne and Philippin examined the case when the equation above is replaced by

\[
\text{div } g(|Du|^2) Du = 0,
\]

where \( g \) is a positive \( C^2 \) function satisfying the ellipticity condition

\[
g(t) + 2tg'(t) > 0, \quad t \geq 0.
\]

They established that if problem (1.2) admits a solution \( u \) satisfying

\[
|Du(x)| = a_i = \text{const} \quad \text{on } \Gamma_i, \; i = 0, 1,
\]

then \( \varphi_0, \varphi_1 \) are constant and \( u \) depends only on \( x_N \). The case when \( u \) is harmonic and \( \varphi_0 \) is assumed to vanish identically was investigated by Henrot et al. in [8, Theorem 3]. The
authors considered solutions in $H^2(\Omega)$, and reached the same conclusion under a more general overdetermined condition on $\Gamma_1$, i.e.,

$$|Du(x)| = q(x_N),$$

where the function $q$ is non-decreasing.

In this paper, those results are extended under the assumption that the solution $u$ of problem (1.2) belongs to the class $C^2(\Omega) \cap C^1(\bar{\Omega})$. Note that if such a solution satisfies $Du(\bar{x}) \neq 0$ for an $\bar{x} \in \partial \Omega_1$ then the angle between the surfaces $\Gamma_1$ and $\Gamma_c$ at $\bar{x}$ equals $\pi/2$.

The following statement gives an account of the type of results obtained. See Section 2 for further details.

**Theorem 1.** Suppose that problem (1.2) possesses a solution $u$ of class $C^2(\Omega) \cap C^1(\bar{\Omega})$.

(A) If $u$ satisfies (for $i = 0, 1$) the overdetermined conditions

$$|Du(x)| = q_i(x_N) \quad \text{on } \Gamma_i,$$

where $q_0$ is non-increasing and $q_1$ is non-decreasing, then $\varphi_0, \varphi_1$ are constant and $u$ depends only on $x_N$.

(B) If $u$ satisfies (1.6) for $i = 1$, and if $\varphi_0 \equiv 0$, then the same conclusion holds provided $x_N q_1(x_N)$ is non-decreasing.

**Problem 2** (A Dirichlet problem in an annular domain). Let $\Omega_0, \Omega_1$ be bounded domains of class $C^2$ in $\mathbb{R}^N$, $N \geq 2$, such that $0 \in \Omega_0$ and $\bar{\Omega}_0 \subset \Omega_1$. Consider the problem

$$\begin{cases}
\Delta u = 0 & \text{in } \Omega = \Omega_1 \setminus \bar{\Omega}_0, \\
u = 0 & \text{on } \partial \Omega_0, \\
u = 1 & \text{on } \partial \Omega_1.
\end{cases}$$

(1.7)

In [8], Henrot et al. proved that if $\Omega_0$ is known in advance to be a ball centered at 0, and if problem (1.7) admits a solution $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfying the overdetermined condition

$$|Du(x)| = q(|x|) \quad \text{on } \partial \Omega_1,$$

where the function $q(r)$ is non-decreasing, then $\Omega_1$ must be a concentric ball. The same conclusion was then obtained by the author in [3] under the weaker assumption that the product $r^{N-1}q(r)$ is non-decreasing. In the same paper, the problem that arises when $\Omega_1$ is kept fixed and $\Omega_0$ is let vary was also investigated, but nothing was told for the case when both $\partial \Omega_0$ and $\partial \Omega_1$ are free boundaries. This case is considered here in Section 3. In particular, we have:

**Theorem 2.** Let $q_0, q_1$ be real-valued functions of the variable $r \in (0, +\infty)$ such that $r^{N-1}q_0(r)$ is non-increasing and $r^{N-1}q_1(r)$ is non-decreasing. If problem (1.7) possesses a solution $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfying

$$|Du(x)| = q_i(|x|) \quad \text{on } \partial \Omega_i, \ i = 0, 1,$$

(1.8)

then $\Omega_0$ and $\Omega_1$ are balls centered at 0.
For the sake of simplicity, the paper refers mainly to Laplace equation. However, the method of proof of both Theorems 1 and 2 can be applied to more general equations than (1.3), provided that a comparison principle holds. Furthermore, overdetermined conditions relating \(|Du(x)|\) to the direction of the normal \(n(x)\) and (in Problem 2) to the curvatures of the boundary can be taken into account. See the following sections for details.

2. A symmetry result in cylinders

The proof of Theorem 1 of Section 1 is based on the following facts.

(SS) Let \(C\) be the cylinder \(C = \omega \times (h, k), h < k\). If \(\Omega = C\) then problem (1.2) possesses a solution \(u\) depending only on \(x_N\). Such a solution will be called a symmetric solution, and the derivative \(du/dx_N\) will be denoted by \(u'\).

(CP) This version of the strong comparison principle holds. Define \(\tilde{C} = \omega \times (\inf_{\omega} \varphi_0, \inf_{\omega} \varphi_1), \tilde{C} = \omega \times (\sup_{\omega} \varphi_0, \sup_{\omega} \varphi_1)\). If \(\tilde{u}, \tilde{u}\) are symmetric solutions, in the sense given above, to (1.2) in the cylinders \(\tilde{C}, \hat{C}\), and if \(u\) solves (1.2) in \(\Omega\), then

\[
\tilde{u} \geq u \quad \text{in } \hat{C} \cap \Omega, \quad u \geq \tilde{u} \quad \text{in } \tilde{C} \cap \Omega,
\]

where the inequalities are strict unless \(\Omega = \tilde{C} = \hat{C}\).

(HL) Hopf’s lemma holds on \(\overline{T}_1\), in the sense that if \(u, \tilde{u}, \Omega, \tilde{C}\) are as above and if \(\tilde{x} \in \partial T_1 \cap \partial \tilde{C}\) then \(\partial \tilde{u}/\partial x_N(\tilde{x}) > \tilde{u}'\), unless \(\Omega = \tilde{C}\).

Remark 2.1. In case \(\tilde{x} \in \Gamma_1\), assertion (HL) is well known. However, if \(\tilde{x} \in \partial \Gamma_1\) then (HL) does not follow from the classical Hopf’s lemma because the interior sphere condition is not satisfied. The proof of this extended version of the lemma can be found in Section 4.

We shall take into consideration overdetermined conditions of the form

\[
|Du(x)| = q_i(x_N, n(x)) \quad \text{on } \Gamma_i, \ i = 0, 1,
\]

where \(q_0, q_1\) are positive functions defined on \(\mathbb{R} \times S^{N-1}_+\) and on \(\mathbb{R} \times S^{N-1}_-\), respectively. Here \(S^{N-1}_+\) denotes the upper half-sphere \(S^{N-1}_+ = \mathbb{R}^N_+ \cap \partial B(0, 1)\). Similarly, \(S^{N-1}_-\) is the lower half-sphere. In the remainder, \(e_N\) and \(e_{N-1}\) denote the unit vectors \((0, \ldots, 0, 1), (0, \ldots, 0, 1, 0) \in \mathbb{R}^N\), as usual.

Theorem 2.1. Let \(q_0, q_1\) be as above, and suppose that \(q_0(t, -e_N)\) and \(q_1(t, e_N)\) are non-increasing and non-decreasing in \(t\), respectively. If problem (1.2)–(2.2) admits a solution \(u \in C^2(\Omega) \cap C^1(\overline{\Omega})\) then \(q_0\) and \(q_1\) are constant and \(u\) depends only on \(x_N\).

Proof. Let \(\tilde{C}\) and \(\hat{C}\) be as in (CP). Choose \(\tilde{P}_i \in \overline{T}_i \cap \partial \tilde{C}\) and \(\hat{P}_i \in \overline{T}_i \cap \partial \hat{C}\), \(i = 0, 1\). If none of the four points is on \(\partial \Gamma_0 \cup \partial \Gamma_1\) then we immediately arrive at

\[
n(\tilde{P}_0) = n(\tilde{P}_1) = e_N, \quad n(\hat{P}_0) = n(\hat{P}_1) = -e_N.
\]

The same equalities also hold in the general case because \(q_0, q_1\) are positive and \(u \in C^1(\overline{\Omega})\). Since \(u(\tilde{P}_i) = \tilde{u}(\tilde{P}_i), u(\hat{P}_i) = \hat{u}(\hat{P}_i), i = 0, 1,\) and by (2.1), we obtain
\[
q_0 \left( \inf_{\omega} \varphi_0, -e_N \right) \leq \tilde{u}' \leq q_1 \left( \inf_{\omega} \varphi_1, e_N \right),
\]
\[
q_1 \left( \sup_{\omega} \varphi_1, e_N \right) \leq \hat{u}' \leq q_0 \left( \sup_{\omega} \varphi_0, -e_N \right),
\]
where (2.2) has been taken into account. The monotonicity assumption on \(q_0, q_1\) implies that the inequalities above are indeed equalities, and the conclusion follows from (HL), where \(\bar{x} = \bar{P}_1\).

\[\blacksquare\]

**Example 2.1.** If problem (1.2) is overdetermined by the following conditions
\[
\begin{align*}
|Du(x)| &= -a_0 n(x) \cdot e_N & \text{on } \Gamma_0, \\
|Du(x)| &= a_1 n(x) \cdot e_N & \text{on } \Gamma_1,
\end{align*}
\]
where \(a_0, a_1 > 0\), then Theorem 2.1 is applicable.

When the lower boundary \(\Gamma_0\) is known in advance to be flat, the conclusion is reached under a less restrictive assumption on \(q_1\).

**Theorem 2.2.** Suppose that \(\varphi_0 \equiv 0\). Suppose, further, that \(sq_1(s, e_N)\) is non-decreasing. If problem (1.2) admits a solution \(u \in C^2(\Omega) \cap C^1(\bar{\Omega})\) satisfying (2.2) for \(i = 1\), then \(\varphi_1 \equiv \text{const}\) and \(u\) depends only on \(x_N\).

The proof follows the same scheme as that of Theorem 2.1 and is therefore omitted. As for an example, we can take \(q_1(s, n) = s^\alpha n \cdot e_N\) for \(\alpha > -1\).

More general equations can be taken into account, as soon as (SS), (CP), (HL) are satisfied. For instance, let us turn to consider the case when the equation in problem (1.2) takes the form (1.3). Suppose that \(g\) is in \(C^2([0, +\infty), (0, +\infty))\) and satisfies (1.4). A meaningful example is the minimal surface equation, which occurs when \(g(t) = (1 + t)^{-1/2}\). We have:

**Proposition 2.3.** If the equation in (1.2) is replaced by (1.3), Theorem 2.1 continues to hold for \(u \in C^2(\Omega)\).

**Proof.** To prove the proposition it suffices to check that (SS), (CP), (HL) hold. Of course, (SS) is satisfied with \(u(x) = x_N/(k - h)\). In order that (CP) holds, the less immediate point is to ensure that \(w_1 = \tilde{u} - u\) (and also \(w_2 = u - \hat{u}\)) cannot attain a minimum on \(\Gamma_0\). Using the assumption that \(u \in C^2(\Omega)\), and by the smoothness of \(g\), it is possible to construct a suitable uniformly elliptic operator \(L\) such that the function \(w_1\) satisfies the linear equation \(Lu_1 = 0\) (see [2, Theorem 10.1] for details). Hence (CP) follows by means of the classical Hopf’s lemma. A similar argument also proves (HL) as long as the boundary point \(\bar{x}\) belongs to \(\Gamma_1\). The case \(\bar{x} \in \partial \Gamma_1\), instead, follows from Lemma 4.1.

\[\blacksquare\]

3. **Radial symmetry in annular domains**

The idea in the proof of Theorem 2.1 can be adapted to other shapes. In this section, for instance, an annular domain is proved to be radially symmetric. Consider problem (1.7),
overdetermined by the conditions
\[ |\nabla u(x)| = q_i(x, n_i(x), \kappa_1(x), \ldots, \kappa_{N-1}(x)) \quad \text{on} \ \partial \Omega_i, \quad i = 0, 1, \tag{3.1} \]
where \( n_i(x) \) is the outer normal to \( \Omega_i \), \( \kappa_1(x) \leq \cdots \leq \kappa_{N-1}(x) \) are the principal curvatures of \( \partial \Omega \) at \( x \), and \( q_0, q_1 \) are real-valued functions on \((\mathbb{R}^N \setminus \{0\}) \times S^{N-1} \times S^{N-1}\). The sign of the curvatures depends on the orientation of a surface, in general. Here they are intended in such a way that if for some \( i \) we have \( \Omega_i = B(0, r) \) then \( \kappa_j(x) = r^{-1} \) for \( x \in \partial \Omega_i, \ j = 1, \ldots, N - 1 \). Note, further, that the unit vector \( n_0(x) \) equals the inner normal to \( \Omega \) along \( \partial \Omega_0 \).

**Theorem 3.1.** Suppose that \( q_0, q_1 \) satisfy
\[ s^{N-1} q_0(s\xi, \xi, \hat{k}_1, \ldots, \hat{k}_{N-1}) \geq r^{N-1} q_0(r\eta, \eta, \hat{k}_1, \ldots, \hat{k}_{N-1}), \tag{3.2} \]
\[ s^{N-1} q_1(s\xi, \xi, \hat{k}_1, \ldots, \hat{k}_{N-1}) \leq r^{N-1} q_1(r\eta, \eta, \hat{k}_1, \ldots, \hat{k}_{N-1}), \tag{3.3} \]
for all unit vectors \( \xi, \eta \in S^{N-1} \), for all real numbers \( s, r \) such that \( 0 < s < r \) and all \( \hat{k}_1, \ldots, \hat{k}_{N-1} \) such that \( \hat{k}_j \leq s^{-1} \), \( \hat{k}_j \geq r^{-1} \), \( j = 1, \ldots, N - 1 \). If problem (1.7)–(3.1) has a solution \( u \in C^2(\Omega) \cap C^1(\overline{\Omega}) \) then \( \Omega_0, \Omega_1 \) are balls centered at \( 0 \).

**Proof.** Let us follow the same procedure as in Theorem 2.1. Define \( \hat{r}_i = \inf_{\partial \Omega_i} |x| \), \( \hat{r}_i = \sup_{\partial \Omega_i} |x| \), \( i = 0, 1 \). Let \( \hat{\Omega} \) be the lower annulus \( \hat{\Omega} = B(0, \hat{r}_1) \setminus B(0, \hat{r}_0) \), and let \( \bar{\Omega} \) be the upper annulus \( \bar{\Omega} = B(0, \hat{r}_1) \setminus B(0, \hat{r}_0) \). Denote by \( \hat{u} \) (respectively, \( \tilde{u} \)) the solution to (1.7) in \( \hat{\Omega} \) (\( \bar{\Omega} \)). Recall that \( r^{N-1}|Du(r)| = \hat{c} \) and \( r^{N-1}|D\tilde{u}(r)| = \tilde{c} \), where \( \hat{c}, \tilde{c} \) are independent of \( r \). By the comparison principle, we get:
\[ \hat{u} \geq u \quad \text{in} \ \hat{\Omega} \cap \Omega, \quad u \geq \tilde{u} \quad \text{in} \ \bar{\Omega} \cap \Omega. \tag{3.4} \]
Now, take \( \hat{P}_i = \partial \Omega_i \cap \partial \hat{\Omega} \) and \( \tilde{P}_i = \partial \Omega_i \cap \partial \bar{\Omega}, \ i = 0, 1 \). At these four points, the boundary gradient of \( u \) can be compared with that of \( \hat{u} \) or \( \tilde{u} \), and we find
\[ \hat{r}_i^{N-1}|Du(\hat{P}_i)| \leq \hat{c} \leq \hat{r}_1^{N-1}|Du(\tilde{P}_1)|, \tag{3.5} \]
\[ \hat{r}_i^{N-1}|Du(\hat{P}_i)| \leq \hat{c} \leq \hat{r}_0^{N-1}|Du(\hat{P}_0)|. \tag{3.6} \]
Observe that \( \hat{P}_i = r_i n_i(\hat{P}_i), \ i = 0, 1 \). The same equality holds with \( \hat{\cdot} \) in place of \( \hat{\cdot} \). Furthermore, we have \( \kappa_j(\hat{P}_i) \leq \hat{r}_i^{-1} \) and \( \kappa_j(\tilde{P}_i) \geq \tilde{r}_i^{-1} \) for \( j = 1, \ldots, N - 1, \ i = 0, 1 \). Hence, we can use the overdetermined conditions (3.1) and the assumptions (3.2), (3.3) to deduce that the inequalities above are indeed equalities. By the classical Hopf’s lemma, this implies \( u = \hat{u} = \tilde{u} \) and the claim follows. \( \square \)

The result extends to more general elliptic equations, as soon as the ingredients of the proof are available: knowledge of the radial solution, comparison principle, boundary point lemma. For an account of such equations, as well as for a number of examples of functions \( q_0, q_1 \) satisfying (3.2), (3.3), the reader may consult [3].
4. A boundary point lemma

Let us now turn to check the validity of assertion (HL) of Section 2 in the case when \( \bar{x} = (\bar{\xi}, \bar{x}_N) \in \partial \Gamma_1 \). In fact, a much more general statement will be proved. By assumption, \(|D\varphi_1(\bar{\xi})| < +\infty\). Hence, the domain \( \Omega \) cannot satisfy an interior sphere condition at \( \bar{x} \).

However, if \( \frac{\partial \varphi_1}{\partial \nu(\bar{\xi})} \leq 0 \), (4.1)

where \( \nu \) is the outer normal to \( \partial \omega \) in \( \mathbb{R}^{N-1} \), then a boundary point lemma still holds (see below). Of course, if \( \bar{x} \) is as in (HL) then (4.1) is satisfied. A key point in the proof of the lemma is the boundary condition \( \partial u/\partial n = 0 \) on \( \Gamma_c \), as well as the interior half-sphere condition, which is defined as follows.

**Definition 4.1.** An interior half-sphere condition is satisfied at \( \bar{x} \in \partial \Gamma_1 \) if there exists a ball \( B = B(y, R) \subset \mathbb{R}^N \), whose center \( y \) lies on \( \Gamma_c \), such that \( \bar{x} \in \partial B \) and \( B \) is divided by \( \Gamma_c \) into two half-balls, one of which is contained in \( \Omega \). Let us call this half-ball \( S \).

If (4.1) holds and \( N = 2 \) then an interior half-sphere condition is satisfied (see Fig. 1). If \( N \geq 3 \) then the same condition still follows from (4.1) provided \( \Gamma_c \) is flat near \( \bar{x} \).

The case when an interior half-sphere condition is not satisfied can be reduced to the previous one by means of a \( C^2 \)-diffeomorphism on \( \Omega \) (i.e., a \( C^2 \)-mapping having an inverse of class \( C^2 \)) that straightens \( \Gamma_c \) near \( \bar{x} \), and preserves the normal derivative. Such a mapping will be constructed using the assumption that \( \omega \) (hence, \( \Gamma_c \)) is of class \( C^3 \).

The final result is the following extension of the classical Hopf’s lemma. Let \( L \) be the linear operator

\[
Lu = a^{ij}(x)u_{ij} + b^i(x)u_i + c(x)u,
\]

where the coefficients \( a^{ij}(x), b^i(x), c(x) \) are real-valued functions defined in \( \Omega \). Suppose that the matrix \( a^{ij}(x) \) is symmetric and positive definite for all \( x \in \Omega \), and that its least eigenvalue \( \lambda(x) \) and its largest eigenvalue \( \Lambda(x) \) satisfy \( M^{-1} \leq \Lambda(x)/\lambda(x) \leq M \) for a positive constant \( M \) (uniform ellipticity). Suppose, further, that \( \sup_{\Omega} |b^i(x)|/\lambda(x) \leq M \) for...
all $i$, and $\sup_{\Omega} |c(x)/\lambda(x)| \leq M$. Before stating the lemma, we still need some definitions. Let $n_c$ be the unit vector defined as

$$n_c = (\nu(\bar{\xi}), 0) = \lim_{x \to \bar{x}} n(x).$$

Denote by

$$n_{\perp} = n(\bar{x}) - (n(\bar{x}) \cdot n_c)n_c,$$

the component of $n(\bar{x})$ orthogonal to $n_c$ (cf. Fig. 1). Observe, incidentally, that if (4.1) holds then by (1.1) it follows that $0 \leq n(\bar{x}) \cdot n_c < 1$ and therefore $n(\bar{x}) \cdot n_{\perp} > 0$. We are now in a position to prove the following result:

**Lemma 4.1** (A boundary point lemma). Let $\Omega$ be as in Problem 1. Let $\bar{x} = (\bar{\xi}, \bar{x}_N) \in \partial \Gamma_1$, and let $u$ be a positive function belonging to the class $C^2(\Omega) \cap C^1(\Omega \cup \Gamma_c) \cap C^0(\Omega \cup \{\bar{x}\})$ and satisfying $Lu \leq 0$ in $\Omega$ and $\partial u/\partial n = 0$ on $\Gamma_c$. If $u(\bar{x}) = 0$, and if (4.1) holds, then for all $\tau \in \mathbb{R}^N$ such that

$$\tau \cdot n_c \geq 0, \quad \tau \cdot n_{\perp} > 0,$$

(4.3)

we have $\partial u/\partial \tau(\bar{x}) < 0$ (if such derivative exists).

**Proof.** The proof rests on an adaptation of the classical idea by Hopf [6,7]. See also [2,12]. We may assume $c \leq 0$ since, if this is not the case, we can replace $c$ by $-c^-$ and still have $Lu \leq 0$.

**Step 1.** Let us consider for first the case when an interior half-sphere condition holds at $\bar{x}$, according to Definition 4.1. Let $S$, $y$ and $R$ be defined as there. The function

$$v(x) = e^{-\alpha r^2} - e^{-\alpha R^2},$$

(4.4)

where $r = |x - y|$ and $\alpha$ is a positive number, satisfies $\partial v/\partial n = 0$ on $\partial S_c = \partial S \cap \Gamma_c$. We also have $Lu \geq 0$ in $S \setminus B(y, R/2)$ for $\alpha$ large enough. Furthermore, since by assumption $\partial u/\partial n = 0$ on $\Gamma_c$, by the classical Hopf’s lemma we see that $u > 0$ on $\Gamma_c$. By a similar reason, $u - v$ cannot attain a negative minimum on $\partial S_c \setminus B(y, R/2)$. Finally, since $u > 0$ on the compact set $S \cap \partial B(y, R/2)$, we conclude by the maximum principle that

$$u - \varepsilon v \geq 0 \quad \text{in} \quad S \setminus B(y, R/2),$$

(4.5)

for a sufficiently small $\varepsilon > 0$. By the second of (4.3), and since $n_{\perp}$ has the direction of the outer normal to $B(y, R)$ at $\bar{x}$, we have

$$\frac{\partial v}{\partial \tau(\bar{x})} < 0.$$  

(4.6)

By (4.3), the direction of $\tau$ is outward with respect to $S$. Since $u(\bar{x}) = v(\bar{x}) = 0$, this and (4.5) imply $\partial(u - \varepsilon v)/\partial \tau(\bar{x}) \leq 0$. By (4.6), the conclusion follows. Before proceeding further, note that the condition $\partial u/\partial n = 0$ was not needed out of $\partial S_c$. This will be important for the next step.

**Step 2.** In order to reduce the remaining case (where $N \geq 3$) to the previous one, observe that since $\omega$ is of class $C^3$, there exist a ball $B = B_{N-1}(\xi, r) \subset \mathbb{R}^{N-1}$ and a
$C^3$-diffeomorphism $\psi$ on $B$ that takes $\tilde{\xi}$ to 0 and that straightens $\partial\omega$ near $\tilde{\xi}$. We immediately see that the mapping $(\xi, x_N) \mapsto (\psi(\xi), x_N)$ straightens $T_\epsilon$ near $\tilde{x}$. However, this mapping does not preserve the condition $\partial u / \partial n = 0$ on $T_\epsilon$, in general. In order to achieve this, let us construct a more suitable diffeomorphism by means of the following argument.

Let $\psi_{N-2}$ be the restriction of $\psi$ to $B \cap \partial\omega$. Without loss of generality we may assume that $\psi_{N-2}$ maps $B \cap \partial\omega$ onto the unit ball $B_{N-2}(0, 1)$ in $\mathbb{R}^N$. Furthermore, for small $\mu > 0$ the distance function $d(\xi) = \text{dist}(\xi, \partial\omega)$ is of class $C^3$ in the neighborhood $\omega_\mu = \{ \xi \in \tilde{\omega} \mid d(\xi) < \mu \}$ of $\partial\omega$. If $\xi \in \omega_\mu$ then the point $\zeta(\xi) \in \partial\omega$ such that $d(\xi) = |\zeta(\xi) - \xi|$ is uniquely determined. Recall that the function $\zeta(\xi)$ is (only) of class $C^2$, in general.

Observe that further, by reducing the radius $r$ of $B$ and the width $\mu$ of $\omega_\mu$, we can assume that $\psi_{N-2} \in C^1(B \cap \partial\omega), d \in C^2(\omega_\mu)$, and $\zeta \in C^2(\omega_\mu)$.

Let $\psi_{N-1} = \{ \xi \in \omega_\mu \setminus \partial\omega \mid \zeta(\xi) \in \partial\omega \}$. The formula

$$\psi_{N-1}(\xi) = (\psi_{N-2}(\zeta(\xi)), d(\xi))$$

defines a $C^2$-diffeomorphism $\psi_{N-1} : G_{N-1} \rightarrow B_{N-2}(0, 1) \times (0, \mu) = \tilde{\omega}$. Now let $\tilde{\omega}$ be the compound function $\tilde{\phi}_1 = \phi_1 \circ \psi_{N-1}^{-1}$, and let

$$\tilde{\Omega} = \{ x \in \mathbb{R}^N \mid \tilde{\phi}_1 < 0 < x_N < \tilde{\phi}_1(\xi) \}.$$

Finally, let $G_N = \{ x \in \Omega \mid x \in G_{N-1} \}$ and define the $C^2$-diffeomorphism $\psi_N : G_N \rightarrow \tilde{\Omega}$ as follows:

$$\psi_N(x) = (\psi_{N-1}(\xi), x_N).$$

Since the lines of steepest descent of the distance function $d$ are orthogonal to $\partial\omega$, the mapping $\psi_N$ not only straightens $T_\epsilon$ near $\tilde{x}$, but also transforms the condition $\partial u / \partial n = 0$ on $T_\epsilon$ into $\partial \tilde{u} / \partial x_{N-1} = 0$ on $\partial \tilde{\Omega} \cap \{ x_{N-1} = 0 \}$. Here and in the sequel, $\tilde{u}$ denotes the compound function $u = u \circ \psi_{N-1}^{-1}$. By the same reason as before we have $-\partial \tilde{\phi}_1 / \partial x_{N-1}(0) = \partial \phi_1 / \partial \nu(\tilde{x})$, which is $\leq 0$ by assumption. Since $-e_{N-1}$ is the outer normal to $\Omega$ along $\{ x_{N-1} = 0 \}$, we conclude that $\tilde{\Omega}$ satisfies an interior half-sphere condition at $\tilde{x} = \psi_N(\tilde{x}) = (0, \ldots, 0, \tilde{x}_N)$. Moreover, from $Lu \leq 0$ we get

$$\tilde{L} \tilde{u} = \tilde{a}^{ij}(x)\tilde{u}_{ij} + \tilde{b}^i(x)\tilde{u}_i + \tilde{c}(x)\tilde{u} \leq 0 \quad \text{in } \tilde{\Omega},$$

where $\tilde{a}^{ij}, \tilde{b}^i$ and $\tilde{c}$ are found by computation. We have $\tilde{c} \leq 0$, and, since $\psi_N \in C^2(\tilde{\Omega}_N)$, it turns out that $\tilde{L}$ is uniformly elliptic and that the ratios $|\tilde{b}^i(x)|/\lambda(x)$ and $|\tilde{c}(x)|/\lambda(x)$ are bounded.

Let $J$ be the differential of $\psi_N$ at $\tilde{x}$. Since $\tilde{\Omega}$ satisfies an interior half-sphere condition at $\tilde{x}$, to conclude the proof it suffices to derive from (4.3) the following inequalities:

$$\tilde{\tau} \cdot \tilde{n}_\epsilon > 0, \quad \tilde{\tau} \cdot \tilde{n}_\perp > 0, \quad \text{(4.7)}$$

where $\tilde{\tau} = J e, \tilde{n}_\epsilon = J n_e = -e_{N-1}, \tilde{n}_\perp = \tilde{n} - (\tilde{n} \cdot \tilde{n}_\perp)\tilde{n}_\perp$, and $\tilde{n}$ is the outer normal to the graph of $\tilde{\phi}_1$ at $\tilde{x}$, which is given by $\tilde{n} = (-D\tilde{\phi}_1(0), 1 + |D\tilde{\phi}_1(0)|^2)^{-1/2}$. To check (4.7), observe that the $(N - 1)$-dimensional hyperplane $T_\epsilon = \{ x \in \mathbb{R}^N \mid (x - \tilde{x}) \cdot n_e = 0 \}$, which is tangent to $T_\epsilon$ at $\tilde{x}$, is transformed by the affine mapping $A(x) = J(x - \tilde{x}) + \tilde{x}$ into the hyperplane $\tilde{T}_\epsilon = \{ x_{N-1} = 0 \}$. Hence, from the first of (4.3) we get the first of (4.7).
The second inequality is less immediate because $J_n \perp \tilde{n}$, in general. Indeed, $\psi_{N-2}$ (hence, $\psi_N$) may have a dilating effect along $\partial \omega$ (Fig. 2). However, the $(N-2)$-dimensional hyperplane $T^0_c = \{ x \in T_c | (x - \bar{x}) \cdot n_{\perp} = 0 \}$ is taken by $\psi$ into $\tilde{T}^0_c = \{ x \in \tilde{T}_c | (x - \tilde{x}) \cdot n_{\perp} = 0 \}$. Furthermore, the $(N-1)$-dimensional half-plane $T^{+}_c = \{ x \in T_c | (x - \bar{x}) \cdot n_{\perp} > 0 \}$ is transformed into $\tilde{T}^{+}_c = \{ x \in \tilde{T}_c | (x - \tilde{x}) \cdot \tilde{n}_{\perp} > 0 \}$. Since $\tau \cdot n_{\perp} > 0$ by assumption, the second of (4.7) follows.

Now we can apply the result established in Step 1 and deduce that $\partial \tilde{u} / \partial \tilde{\tau}(\tilde{x}) < 0$. Since $\partial u / \partial \tau(\bar{x}) = \partial \tilde{u} / \partial \tilde{\tau}(\tilde{x})$, the conclusion follows and the proof is complete. 

**Corollary 4.2.** If, in addition to the assumptions of the lemma, we also have $u \in C^1(\tilde{\Omega})$, then the same conclusion holds irrespectively of the sign of $\tau \cdot n_c$.

**Proof.** By using the lemma with $\tau = n_{\perp}$ we see that $D(\tilde{x}) \neq 0$. Since $\partial u / \partial n = 0$ on $\Gamma_c$, and by continuity, the gradient vector $Du(\tilde{x})$ must lie in the hyperplane $T_c$ defined before. Since $u$ is constant on $\Gamma_1$, we deduce that $Du(\tilde{x}) = -|Du(\tilde{x})|n_{\perp}$, and the claim follows.

**Remark 4.1.** In order to prove (HL) of Section 2, it suffices to let $\tau = e_N$ and to take into account the fact that, in that case, we have $n_{\perp} = e_N$.

**References**