On Dissipative Matrices*

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ABSTRACT

A matrix the imaginary part of whose Toeplitz (Cartesian) decomposition is positive semidefinite [positive definite] is said to be dissipative [strictly dissipative]. Characterizations for strictly dissipative matrices and related theory are given. Dissipative-preserving linear transformations are characterized. Finally, a natural generalization to linear transformations on matrices is developed in terms of the completely positive cone; ten characterizations follow. © Elsevier Science Inc., 1996

I. PRELIMINARIES

Let \( M_{n,q} \) denote the set of \( n \) by \( q \) complex matrices, \( M_n \) if \( n = q \), with \( \mathbb{H}_n \), PSD\(_n\), and PD\(_n\) its Hermitian, positive semidefinite, and positive definite subsets respectively. Let \( \mathcal{L}(M_n, M_q) \) be the space of all linear transformations from \( M_n \) to \( M_q \) with \( \mathcal{H} \), \( \pi \)(PSD), and \( \mathcal{C} \) its Hermitian-preserving, positive-semidefinite-preserving, and completely positive subsets. Subscripts on these sets will be used only when necessary for clarity; otherwise, they will

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be suppressed. See [22] and [2] for characterizations and other results on $SP$. A matrix $A \in \mathbb{M}_n$ can be expressed uniquely as $A = \text{Re} A + i \text{Im} A$ where $\text{Re} A, \text{Im} A \in \mathbb{R}^n$ (Toeplitz decomposition). A linear transformation $L \in \mathbb{L}(\mathbb{M}_n, \mathbb{M}_q)$ can be represented uniquely as $L = RL + i JL$ where $RL, JL \in \mathbb{HP}$ (BHH decomposition [2]).

A matrix $A \in \mathbb{M}_n$ with $\text{Im} A \in \mathbb{PD}$ (PSD) is said to be strictly dissipative (dissipative), with the sets of these matrices denoted $\mathbb{SD}$ and $\mathbb{D}$, respectively. These matrices have been studied by Fan [5–8], Bébiano and da Providência [1], Phillips [21], and Thompson [24], and in a rephrased form ($\text{Re} A = \text{Im}(\iota A)$, $\text{Re}(\iota A) = \text{Im} A$) by Johnson [13, 14] and Mathias [18]. The results in [6] are used by Gunzberger and Plemmons [9] in their study of energy-conserving norms for the solution of hyperbolic systems of partial differential equations, where the term dissipative is used in the sense that energy is nonincreasing in time. Siegel [23] uses strictly dissipative matrices in the study of the analytic theory of abelian varieties.

II. CHARACTERIZATION THEOREMS FOR STRICTLY DISSIPATIVE MATRICES

The major result in this section is a characterization theorem for strictly dissipative matrices. Combining Fan's characterizations [7] of normalizable matrices with this result gives us a list of characterizations (Theorem 2.7). A matrix $A$ is said to be normalizable (denoted by $A \in \mathbb{N}$) provided there exists a nonsingular matrix $T$ such that $T^*AT$ is normal.

**Theorem 2.1.** Let $A \in \mathbb{M}_n$. Then $A \in \mathbb{SD}$ if and only if $A \in \mathbb{N}$ and, for all nonsingular $T \in \mathbb{M}_n$, $\lambda > 0$ for all $\lambda \in \sigma(T^*AT)$.

This result may be restated in terms of the existence of a normalizing matrix $T$, viz., $A \in \mathbb{SD}$ iff $A \in \mathbb{N}$ and $\lambda > 0$ for all eigenvalues of $T^*AT$, where $T$ normalizes $A$. Four lemmas are used in our proof.

**Lemma 2.2.** The eigenvalues of a strictly dissipative [dissipative] matrix lie in the open [closed] upper half plane, i.e., $\lambda > 0$ [$\geq 0$] for all $\lambda \in \sigma(A)$.

This result immediately gives us that strictly dissipative matrices are nonsingular with the imaginary part of their traces being positive.
**Lemma 2.3.** Let $A$ be a normal matrix such that $\text{im} \lambda > 0 \ (\geq 0)$ for all $\lambda \in \sigma(A)$. Then $A \in \mathbb{S}\mathbb{D} \ [\mathbb{D}]$.

**Lemma 2.4.** Let $A$ be a normal matrix such that $\text{im} \lambda > 0 \ (\geq 0)$ for all $\lambda \in \sigma(A)$, and let $T$ be a nonsingular matrix. Then the eigenvalues of $T^*AT$ lie in the open [closed] upper half plane.

**Lemma 2.5.** Let $A \in \mathbb{S}\mathbb{D}_n \ [\mathbb{D}_n]$, and let $T$ be a nonsingular $n \times n$ matrix. Then $T^*AT \in \mathbb{S}\mathbb{D}_n \ [\mathbb{D}_n]$.

Let

$$A = \begin{bmatrix}
\frac{3}{2} & \frac{7}{10} + \frac{9}{10}i \\
\frac{9}{10} - \frac{7}{10}i & \frac{43}{50} + \frac{11}{50}i
\end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix}
6 & -5 - 5i \\
-5 + 5i & 10
\end{bmatrix}.$$

Then $P > 0$ and $PAP = \text{diag}(1 - \iota, 1 + 2\iota)$. Thus, $PAP$ is normal, which implies that $A$ is normalizable. Therefore, $A$ is an example of a normalizable matrix whose eigenvalues ($\approx 2.347 + 0.217i, \approx 0.013 + 0.003i$) have a positive imaginary part, but which is not strictly dissipative. This precludes a natural conjecture to simplify our characterization theorem by disregarding a normalizing matrix $T$.

Further,

$$A = \begin{bmatrix}
1 + \frac{1}{2}i & \iota\sqrt{2} \\
-\iota\sqrt{2} & 0
\end{bmatrix}$$

is a dissipative matrix which is not normalizable. (Let $A$ have Toeplitz decomposition $H + \iota K$. Then

$$K = \begin{bmatrix}
\frac{1}{2} & 0 \\
0 & 0
\end{bmatrix} \geq 0.$$
Equating $HPK$ and $KPH$, we have that $a = b = 0$. Thus,

$$
P = \begin{bmatrix}
0 & -c' \\
c' & d
\end{bmatrix},
$$

which has eigenvalues $d/2 \pm \frac{1}{2}\sqrt{d^2 + 4c'^2}$. Since one of the eigenvalues must be nonpositive, $P$ cannot be positive definite. Thus, there does not exist a $P > 0$ such that $HPK = KPH$. Hence, $A$ is not normalizable; cf. Theorem 2.6(a) \Rightarrow (e) below.] Since not all dissipative matrices are normalizable, Theorem 2.1 does not generalize to dissipative matrices. A characterization theorem for dissipative matrices remains an open question. However, sufficient conditions for a matrix to be dissipative are that the matrix be normalizable and that all of the eigenvalues of $T^*AT$, where $T$ is a normalizer of $A$, lie in the closed upper half plane.

The inertia of a matrix with respect to the real axis, denoted $In'(A)$, is given by $In'(A) = \{\pi'(A), \nu'(A), \delta'(A)\}$, where $\pi'(A)$, $\nu'(A)$, and $\delta'(A)$ denote the numbers of eigenvalues (counted with their multiplicities) with positive, negative, and zero imaginary parts, respectively [15, pp. 449-450].

In [7] Fan characterizes normalizable operators. The next theorem expands Fan's characterizations. The set of normal matrices will be denoted by $\mathcal{N}$.

**Theorem 2.6.** Let $A \in \mathcal{M}_n$ with $H + iK$ the Toeplitz decomposition of $A$. Then the following are equivalent:

(a) $A$ is normalizable.
(b) There exists $P > 0$ such that $A^*PA = APA^*$.
(b') There exists a nonsingular $T \in \mathcal{M}_n$ such that $ATT^*A^* = A^*TT^*A$.
(c) There exists $Q > 0$ such that $\|QAx\| = \|QA^*x\|$ for all $x \in \mathbb{C}^n$.
(d) There exist two bases $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$ of $\mathbb{C}^n$ satisfying the biorthonormal condition $\langle x_i, y_j \rangle = \delta_{ij}$, $i, j = 1, 2, \ldots, n$, and complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $Ax_j = \alpha_j y_j$ ($1 < j < n$).
(e) There exists $P > 0$ such that $HPK = KPH$.
(f) There exists $P > 0$ such that $APA^* = HPH + KPK$.
(g) $A = PNP$ for some $N \in \mathcal{N}$ and some $P > 0$.
(h) There exists $Q > 0$ such that $QAQ \in \mathcal{N}$.
(i) There exists $P > 0$ such that $APH = HPA$.
(j) There exists $P > 0$ such that $APK = KPA$.
(k) There exists $P > 0$ such that $APH + HPA^* = 2HPH$.
(l) There exists $P > 0$ such that $APK - KPA^* = 2iKPK$. 

Combining Theorems 2.1 and 2.6, we obtain the following characterizations:

**Theorem 2.7.** Let $A \in \mathcal{M}_n$ with Toeplitz decomposition $H + iK$. Then the following are equivalent:

(a) $A \in \mathcal{D}$.  
(b) There exists a nonsingular $T$ such that $T^*AT \in \mathcal{N}$ and $\text{In}'(T^*AT) = (n, 0, 0)$.  
(c) There exists a positive definition matrix $P$ ($P > 0$) such that $A^*PA = APA^*$ and $\text{In}'(T^*AT) = (n, 0, 0)$ for all nonsingular $T \in \mathcal{M}_n$.  
(d) There exists a nonsingular $T \in \mathcal{M}_n$ such that $ATT^*A^* = A^*TT^*A$ and $\text{In}'(T^*AT) = (n, 0, 0)$.  
(e) There exists $Q > 0$ such that $\|QAx\| = \|QA^*x\|$ for all $x \in \mathbb{C}^n$ and $\text{In}'(T^*AT) = (n, 0, 0)$ for all nonsingular $T \in \mathcal{M}_n$.  
(f) There exist bases $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$ of $\mathbb{C}^n$ satisfying the biorthonormal condition $\langle x_i, y_j \rangle = \delta_{ij}$, $i, j = 1, 2, \ldots, n$, and complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $Ax_j = \alpha_j y_j$ ($1 \leq j \leq n$) and $\text{In}'(T^*AT) = (n, 0, 0)$ for all nonsingular $T \in \mathcal{M}_n$.  
(g) There exists $P > 0$ such that $HPK = KPH$ and $\text{In}'(T^*AT) = (n, 0, 0)$ for all nonsingular $T \in \mathcal{M}_n$.  
(h) There exists $P > 0$ such that $APA^* = HPH + KPK$ and $\text{In}'(T^*AT) = (n, 0, 0)$ for all nonsingular $T \in \mathcal{M}_n$.  
(i) $A = PNP$ for some $N \in \mathcal{N}$ and some $P > 0$, and $\text{In}'(T^*AT) = (n, 0, 0)$ for all nonsingular $T \in \mathcal{M}_n$.  
(j) There exists $Q > 0$ such that $QAQ \in \mathcal{N}$ and $\text{In}'(T^*AT) = (n, 0, 0)$ for all nonsingular $T \in \mathcal{M}_n$.  
(k) There exists $P > 0$ such that $APH = HPA$ and $\text{In}'(T^*AT) = (n, 0, 0)$ for all nonsingular $T \in \mathcal{M}_n$.  
(l) There exists $P > 0$ such that $APK = KPA$ and $\text{In}'(T^*AT) = (n, 0, 0)$ for all nonsingular $T \in \mathcal{M}_n$.  
(m) There exists $P > 0$ such that $APH + HPA^* = 2HPH$ and $\text{In}'(T^*AT) = (n, 0, 0)$ for all nonsingular $T \in \mathcal{M}_n$.  
(n) There exists $P > 0$ such that $APK - KPA^* = 2iKPK$ and $\text{In}'(T^*AT) = (n, 0, 0)$ for all nonsingular $T \in \mathcal{M}_n$.  
(o) $K^{-1}H$ is similar to a Hermitian matrix, and $\text{In}'(T^*AT) = (n, 0, 0)$ for all nonsingular $T \in \mathcal{M}_n$.

**Proof of Lemma 2.2.** Let $A \in \mathcal{D}$ with Toeplitz decomposition $H + iK$, and let $\lambda$ be an eigenvalue of $A$ corresponding to the eigenvector
\[ x \in \mathbb{C}^n, \text{i.e., } Ax = \lambda x, \text{ where } x \neq 0. \text{ Then } \lambda \langle x, x \rangle = \langle Ax, x \rangle = \langle Hx, x \rangle + \langle Kx, x \rangle. \text{ Since } \langle Hx, x \rangle \in \mathbb{R}, \langle x, x \rangle > 0, \text{ and } \langle Kx, x \rangle > 0 [\geq 0], \text{ we have } \operatorname{im}(\lambda) > 0 [\geq 0]. \]

**Proof of Lemma 2.3.** Suppose that \( A \) is normal. Then there exists a unitary \( U \in \mathbb{M}_n \) such that \( A = UDU^* \), where \( D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \). Thus, \( A^* = UDU^* \). Hence,

\[
\frac{A - A^*}{2i} = \frac{UDU^* - UDU^*}{2i} = U \left( \frac{D - D^*}{2i} \right) U^* = U\hat{D}U^*,
\]

where \( \hat{D} = \text{diag}(\text{im } \lambda_1, \text{im } \lambda_2, \ldots, \text{im } \lambda_n) \). Let \( H + iK \) to be the Toeplitz decomposition of \( A \), and let \( 0 \neq x \in \mathbb{C}^n \). Then

\[
\langle Kx, x \rangle = \left\langle \frac{A - A^*}{2i} x, x \right\rangle = \langle UDU^* x, x \rangle = \langle \hat{D}U^* x, U^* x \rangle.
\]

Let \( y = U^* x \). Then \( \langle Kx, x \rangle = \langle \hat{D}y, y \rangle = \sum_{j=1}^{n} (\text{im } \lambda_j) |y_j|^2 > 0 [\geq 0], \) since \( \text{im } \lambda_j > 0 [\geq 0] \) for \( j = 1, 2, \ldots, n \).

**Proof of Lemma 2.4.** Let \( A \in \mathbb{N} \) such that \( \text{im } \lambda > 0 [\geq 0] \) for all \( \lambda \in \sigma(A) \), let \( H + iK \) be the Toeplitz decomposition of \( A \), and let \( T \) be a nonsingular matrix. Then \( T^*HT + iT^*KT \) is the Toeplitz decomposition of \( T^*AT \). Let \( 0 \neq y \in \mathbb{C}^n \) and \( x = Ty \). Then \( \langle T^*KTy, y \rangle = \langle KTy, Ty \rangle = \langle Kx, x \rangle \). Since \( y \neq 0 \), \( x \) must be a nonzero vector. By Lemma 2.3, \( A \in \mathbb{SD}[\mathbb{D}] \). Thus, \( K > 0 [\geq 0] \), which implies that \( \langle Kx, x \rangle > 0 [\geq 0] \). Thus, \( \langle T^*KTy, y \rangle > 0 [\geq 0] \), whence \( T^*AT \) is strictly dissipative [dissipative]. Thus, by Lemma 2.2, \( \hat{\lambda} > 0 [\geq 0] \) for all \( \hat{\lambda} \in \sigma(T^*AT) \).

**Proof of Lemma 2.5.** Let \( A \in \mathbb{SD}_n [\mathbb{D}_n] \) with Toeplitz decomposition \( H + iK \), and let \( T \) be a nonsingular \( n \times n \) matrix. Then the Toeplitz decomposition of \( T^*AT \) is \( T^*HT + iT^*KT \). Let \( 0 \neq x \in \mathbb{C}^n \), and let \( y = Tx \). Then \( \langle T^*KTx, x \rangle = \langle KTx, Tx \rangle = \langle Ky, y \rangle \). Since \( T \) is nonsingular, \( y \neq 0 \). Since \( K > 0, \langle Ky, y \rangle > 0 [\geq 0] \). Hence, \( \langle T^*KTx, x \rangle > 0 [\geq 0] \), whence \( A \in \mathbb{SD}_n [\mathbb{D}_n] \).

**Proof of Theorem 2.1.** Let \( A \in \mathbb{M}_n \).

\[ \Rightarrow: \text{Suppose } A \in \mathbb{SD}. \text{ Then } A \in \mathbb{N} [7]. \text{ Let } T \in \mathbb{M}_n \text{ be nonsingular. By Lemma 2.5, } T^*AT \in \mathbb{SD}. \text{ By Lemma 2.2, } \text{im } \lambda > 0 \text{ for all } \lambda \in \sigma(T^*AT). \]
Suppose $A \in \mathbb{N}$ and, for all nonsingular $T \in \mathbb{M}_n$, $\text{im} \lambda > 0$ for all $\lambda \in \sigma(T^*AT)$. Since $A \in \mathbb{N}$, there exists a nonsingular matrix $S$ such that $S^*AS \in \mathcal{N}$. From the hypothesis, $\text{im} \mu > 0$ for all $\mu \in \sigma(S^*AS)$. Hence, $S^*AS \in \mathbb{SD}$ by Lemma 2.3. Since $S^*AS \in \mathbb{SD}$ and $A = (S^{-1})^*(S^*AS)S^{-1}$, we have $A \in \mathbb{SD}$ by Lemma 2.5.

Proof of Theorem 2.6. Fan proved the equivalence of parts (a) through (l) in [7]. It suffices to show (l) $\iff$ (i) $\iff$ (k) and (li) $\iff$ (j) $\iff$ (l):

$\exists Q > 0$ such that $QAQ \in \mathcal{N}$

$\iff$ $\exists Q > 0$ such that $(QAQ)[\text{Re}(QAQ)] = [\text{Re}(QAQ)](QAQ)$

$\iff$ $\exists Q > 0$ such that $(QAQ)(QHQ) = (QHQ)(QAQ)$

$\iff$ $\exists P > 0$ such that $APH = HPA$  \hspace{1cm} (let $P = Q^2$)

$\iff$ $\exists P > 0$ such that $HPA^* = A^*PH$

$\iff$ $\exists P > 0$ such that $APH + HPA^* = 2HPH$.

since $APH + A^*PH = 2HPH$. Also,

$\exists Q > 0$ such that $QAQ \in \mathcal{N}$

$\iff$ $\exists Q > 0$ such that $(QAQ)[\text{Im}(QAQ)] = [\text{Im}(QAQ)](QAQ)$

$\iff$ $\exists Q > 0$ such that $(QAQ)(QKQ) = (QKQ)(QAQ)$

$\iff$ $\exists P > 0$ such that $APK = KPA$  \hspace{1cm} (let $P = Q^2$)

$\iff$ $\exists P > 0$ such that $KPA^* = A^*PK$

$\iff$ $\exists P > 0$ such that $APK - KPA^* = 2iKPK$,

since $APK - A^*PK = 2iKPK$.

III. FURTHER PROPERTIES OF (STRICTLY) DISSIPATIVE MATRICES

If we let $A = (1 + \varepsilon)I_2$, then $A$ is strictly dissipative and hence dissipative. But $-A$ is not dissipative and hence not strictly dissipative. Thus, $\mathbb{SD}_n$
and $\mathbb{D}_n$ are not closed under scalar multiplication. Hence, they are not subspaces of $\mathbb{M}_n$. However, the set of dissipative [strictly dissipative] matrices is closed under matrix addition and scalar multiplication by a nonnegative [positive] scalar. Thus, $\mathbb{D}_n$ is a cone in $\mathbb{M}_n$.

Since $0 \not\in S\mathbb{D}_n$, $S\mathbb{D}_n$ is not a cone in $\mathbb{M}_n$. If we define the set $S_0\mathbb{D}_n = S\mathbb{D}_n \cup \{0\}$, then $S_0\mathbb{D}_n$ is a pointed cone in $\mathbb{M}_n$. The following theorem yields (among other results) the fact that $\mathbb{D}_n$ is not pointed.

**Theorem 3.1.** $\mathbb{D}_n \cap (-\mathbb{D}_n) = \mathbb{M}_n$.

**Proof.** $\iff$ Suppose $A \in \mathbb{M}_n$. Then $\text{Im} A = 0$. Hence, $A \in \mathbb{D}_n \cap (-\mathbb{D}_n)$

$\Rightarrow$ Suppose $A \in \mathbb{D}_n \cap (-\mathbb{D}_n)$. Let $H + iK$ be the Toeplitz decomposition of $A$. Since $A \in \mathbb{D}_n$, $\langle Kx, x \rangle \geq 0$ for all $x \in \mathbb{C}^n$. Since $A \in -\mathbb{D}_n$, $\langle Kx, x \rangle \leq 0$ for all $x \in \mathbb{C}^n$. Thus, $\langle Kx, x \rangle = 0$ for all $x \in \mathbb{C}^n$, which implies that $K = 0$. Thus, $A = H$ is Hermitian.

The matrix $(-1 + i)I_2$ is an example which shows us that the square (and, hence, the product) of two strictly dissipative matrices is not necessarily strictly dissipative. The strictly dissipative matrices for which the square of matrix is strictly dissipative may be characterized as in the next theorem. First we establish two lemmas about normalizable matrices.

**Lemma 3.2.** Let $A \in \mathbb{N}$, where $N$ is the matrix of Theorem 2.6(g). Then $A$ is normal iff there exists a $Q > 0$ such that $NQN^* = N^*QN$.

**Proof.** Let $A \in \mathbb{N}$ with $N \in \mathbb{N}$ as above. Then, by Theorem 2.6(g) $\iff$ (a), there exists $P > 0$ such that $A = PNP$. Now

$A$ is normal $\iff$ $AA^* = A^*A$

$\iff$ $\exists P > 0$ such that $PNP^2N^*P = PNP^2NP$

$\iff$ $\exists P > 0$ such that $NQN^* = N^*QN$, where $Q = P^2$. 

\[\Box\]
**Lemma 3.3.** Let $A \in \mathbb{N}$, $A$ nonsingular. Then $A^2 \in \mathbb{N}$ iff $A \in \mathcal{N}$.

**Proof.** Let $A$ be a nonsingular normalizable matrix. Then there exists a nonsingular $T$ such that $A = T^*NT$. Then $A^2 = T^*N TT^*NT$. Now

$$A^2 \in \mathbb{N} \iff \exists \text{ nonsingular } S \text{ such that } S^*A^2S \text{ is normal}$$

$$\iff \exists \text{ nonsingular } S \text{ such that } S^*(T^*N TT^*NT)S \text{ is normal}$$

$$\iff \exists \text{ nonsingular } S \text{ such that } NTT^*NTSS^*T^*N = N^*TT^*N^*TSS^*T^*NT^*N$$

$$\iff \exists P > 0 \text{ such that } NPN^* = N^*PN, \text{ where } P = (TT^*NTS)(TT^*NTS)^*$$

$$\iff A \in \mathcal{N}, \quad \text{by Lemma 3.2.}$$

**Theorem 3.4.** Let $A \in \mathbb{S}\mathbb{D}$. Then $A^2 \in \mathbb{S}\mathbb{D}$ iff $A \in \mathcal{N}$ with $\mathbb{R}e \lambda > 0$ for all $\lambda \in \sigma(A)$.

**Proof.** Let $A \in \mathbb{S}\mathbb{D}$. Then $A \in \mathbb{N}$ and $\mathbb{I}m \lambda > 0$ for all $\lambda \in \sigma(A)$.

$\Leftarrow$: Suppose $A \in \mathcal{N}$ with $\mathbb{R}e \lambda > 0$ for all $\lambda \in \sigma(A)$. Then $A$ is unitarily similar to a diagonal matrix of its eigenvalues. Thus, $A^2$ is unitarily similar to a diagonal matrix of the squares of the eigenvalues of $A$. Let $\mu$ be an eigenvalue of $A^2$. Then $\mu = \lambda^2$ for some $\lambda \in \sigma(A)$. Thus, $\mathbb{I}m \mu = \mathbb{I}m \lambda^2 = 2(\mathbb{R}e \lambda(\mathbb{I}m \lambda)) > 0$, since $\mathbb{I}m \lambda > 0$. Since $A^2 \in \mathcal{N}$, we have $A^2 \in \mathbb{S}\mathbb{D}$.

$\Rightarrow$: Suppose $A^2 \in \mathbb{S}\mathbb{D}$. Then $A^2 \in \mathbb{N}$. Thus, $A \in \mathcal{N}$, by Lemma 3.3. Let $\lambda \in \sigma(A)$. Then $\lambda^2 \in \sigma(A^2)$. Then $\mathbb{I}m \lambda^2 > 0$, whence $2(\mathbb{I}m \lambda)(\mathbb{R}e \lambda) > 0$. Since $\mathbb{I}m \lambda > 0$, we have $\mathbb{R}e \lambda > 0$.

Masser and Neumann [17] use the angular field of values, $\Gamma(A) = \{x^*Ax : 0 \neq x \in \mathbb{C}^n\}$, to obtain the existence of the square root of a strictly dissipative matrix as follows:

**Theorem 3.5 [17].** Let $A \in \mathcal{M}_n$, and suppose that $\Gamma(A) \cap (-\infty, 0) = \emptyset$. Then there exists a unique $B \in \mathcal{M}_n(\mathbb{C})$ with $\mathbb{R}e B > 0$ such that $B^2 = A$. In particular, if $\mathbb{I}m A > 0$, then $\mathbb{I}m B > 0$, while if $\mathbb{I}m A < 0$, then $\mathbb{I}m B < 0$.

Since all eigenvalues of a strictly dissipative matrix lie in the open upper half plane, Theorem 3.5 yields the following result. Note that it generalizes...
the fact that the square root of a complex number with positive imaginary part has a unique square root in the first quadrant.

**Corollary 3.6.** Let $A \in \mathbb{S}\mathbb{D}$. Then there exists a unique $B \in \mathbb{S}\mathbb{D}$ with $\text{Re } B > 0$ such that $A = B^2$.

Horn and Johnson [12, p. 399] give a result on the invariance of the cones of positive semidefinite and positive definite matrices under conjunctivity ($\sim$ congruence), which leads to the following, where $\sim$ denotes set difference:

**Theorem 3.7.** Let $A \in \mathbb{D} \sim \mathbb{S}\mathbb{D}$ and $C \in \mathbb{A}_n$. Then $C^*AC \in \mathbb{D} \sim \mathbb{S}\mathbb{D}$.

**Proof.** Let $C \in \mathbb{A}_n$ and $A \in \mathbb{D} \sim \mathbb{S}\mathbb{D}$ with $H + iK$ the Toeplitz decomposition of $A$. Then $K \in \text{PSD} \sim \text{PD}$; hence, $C^*AC = C^*HC + iC^*KC$ is the Toeplitz decomposition of $C^*AC$. Then $C^*KC \in \text{PSD} \sim \text{PD}$. Thus, $C^*AC \in \mathbb{D} \sim \mathbb{S}\mathbb{D}$.

As one might expect, $\mathbb{D}$ and $\mathbb{S}\mathbb{D}$ are related as follows:

**Proposition 3.8.** Let $A \in \mathbb{A}_n$. Then $A \in \mathbb{D}$ iff $A + i\varepsilon I \in \mathbb{S}\mathbb{D}$ for all $\varepsilon > 0$.

**IV. DISSIPATIVE-PRESERVING LINEAR TRANSFORMATIONS**

A linear transformation $L \in \mathcal{L}(\mathbb{A}_n)$ is said to be dissipative-preserving [strictly dissipative-preserving] if $L(A)$ is dissipative [strictly dissipative] whenever $A$ is dissipative [strictly dissipative]. The set of dissipative-preserving linear transformations and the set of strictly dissipative-preserving linear transformations will be denoted $\mathbb{D}\mathbb{P}$ and $\mathbb{S}\mathbb{D}\mathbb{P}$, respectively, and $\mathbb{S}_0\mathbb{D}\mathbb{P}$ will denote the set of linear transformations which preserve the set $\mathbb{S}_0\mathbb{D}$. This section characterizes the cones $\mathbb{D}\mathbb{P}$ and $\mathbb{S}\mathbb{D}\mathbb{P}$.

We observe that the set of strictly dissipative-preserving linear transformations is a subset of the set of dissipative-preserving linear transformations.

**Theorem 4.1.** $\mathbb{S}\mathbb{D}\mathbb{P} \subset \mathbb{D}\mathbb{P}$.

**Proof.** Let $L \in \mathbb{S}\mathbb{D}\mathbb{P}$ with BHH decomposition $\mathbb{A}L + i\mathbb{A}L$, and let $A \in \mathbb{D}$ with Toeplitz decomposition $H + iK$. By Theorem 3.5, $A + i\varepsilon I \in \mathbb{D}$
\[ S \subseteq D \text{ for all } \epsilon > 0. \] Since \( \text{Im} \, (A + i \epsilon I) = \mathcal{J}(H) + \mathcal{R}(K + i \epsilon I) \), we have \( \mathcal{J}(H) + \mathcal{R}(K + i \epsilon I) > 0 \). Let \( \lambda_1, \ldots, \lambda_n \) be the eigenvalues of \( \mathcal{J}(H) + \mathcal{R}(K + i \epsilon I) \) ordered so that \( \lambda_n \geq \lambda_{n-1} \geq \cdots \geq \lambda_1 > 0 \). Let \( \lambda_1, \ldots, \lambda_n \) be the eigenvalues of \( \mathcal{J}(H) + \mathcal{R}(K) \) ordered so that \( \lambda_n \geq \lambda_{n-1} \geq \cdots \geq \lambda_1 \). As \( \epsilon \to 0, \mathcal{J}(H) + \mathcal{R}(K + i \epsilon I) \to \mathcal{J}(H) + \mathcal{R}(K) \) and \( \lambda_i \to \lambda_i \) for \( i = 1, 2, \ldots, n \), since the eigenvalues of a complex matrix depend continuously upon its entries. Thus, \( \lambda_1 \geq 0 \). Hence, \( \mathcal{J}(H) + \mathcal{R}(K) > 0 \) and \( \mathcal{J}(H) + \mathcal{R}(K) = \text{Im} \, L(A) \), which implies that \( L(A) \in D \), yielding \( L \in \mathbb{DP}. \)

By defining the set \( \mathbb{P}_0 \mathbb{D} \) to be the set \( \mathbb{PD} \cup \{0\} \), and the set \( \pi(\mathbb{P}_0 \mathbb{D}) \) to be the set of \( \mathbb{P}_0 \mathbb{D} \)-preserving linear transformations, an analogous proof gives us the following:

**Proposition 4.4.** Let \( L \in \mathbb{L}(\mathbb{M}_n) \). Then \( L \in \mathbb{S}_0 \mathbb{D} \) if and only if \( \mathcal{R}L \in \pi(\mathbb{P}_0 \mathbb{D}) \) and \( \mathcal{J}(C) \in \text{PSD} \) for all \( C \in \mathbb{H}_n \).
Theorem 4.5. \( DP = \pi(\text{PSD}) \).

Proof. Let \( L \in \pi(\text{PSD}) \), and let \( A \in D \) with Toeplitz decomposition \( H + iK \). Then \( L \in \mathcal{H}P \) and \( \text{Im} \, L(A) = L(K) \). Since \( K \geq 0 \), we have \( L(K) \geq 0 \), and hence \( L \in DP \).

Now suppose that \( L \in DP \). Let \( C \in \mathcal{H}_n \). By Theorem 4.3, \( \mathcal{J}L(C) \geq 0 \). Since \(-C\) is also Hermitian, \( \mathcal{J}L(-C) \geq 0 \), whence \( \mathcal{J}L(C) \leq 0 \). Hence, \( \mathcal{J}L(C) = 0 \) for all \( C \in \mathcal{H}_n \). Thus, \( \mathcal{J}L = 0 \). Hence, \( L - \mathcal{J}L \in \pi(\text{PSD}) \), which gives us that \( DP = \pi(\text{PSD}) \).

This theorem tells us that the characterizing of dissipative-preserving linear transformations is the same problem as characterizing \( \pi(\text{PSD}) \), which is an open problem. An analogous proof gives us the equality between the \( P_0 \)D-preserving linear transformations and the \( S_0DP \)-preserving linear transformations.

Proposition 4.6. \( S_0DP = \pi(P_0D) \).

We note that since the writing of this paper, we have found a version of our Theorem 4.5 in [16].

V. \( CP \)-Dissipative Linear Transformations

For the notation of this section (bijections, matrix rearrangements, etc.), we will utilize the notation of Poluikis and Hill [22], of Oxenrider and Hill [20], and of Hill and Waters [11]. Two bijections are defined from \( \{(i, j) : i = 1, \ldots, n; j = 1, \ldots, q\} \) to \( \{1, \ldots, nq\} \) by \( [i,j] = (i-1)n + j \) and \( (i,j) = (j-1)q + i \). These correspond to the lexicographical ordering \( (i,j) < (r,s) \) iff \( i < r \) or \( i = r \) and \( j < s \)] and the antilexicographical ordering \( (i,j) < (r,s) \) iff \( j < s \) or \( j = s \) and \( i < r \), respectively, on \( \{(i, j) : i = 1, \ldots, n; j = 1, \ldots, q\} \). We will use \( \langle T \rangle \) to denote the matrix representation of the linear transformation \( T \) with respect to the standard antilexicographically ordered basis of \( \mathcal{M}_n \). Four rearrangements of \( \langle T \rangle \in \mathcal{M}_{q^2,n^2} \) are given by the following: \( \Gamma(\langle T \rangle)_{ij} = t_{[i,j],[r,s]} \), \( \Psi(\langle T \rangle)_{ij} = t_{(i,j),(r,s)} \), \( \Theta(\langle T \rangle)_{rs} = t_{(r,s),(i,j)} \), and \( \Omega(\langle T \rangle)_{rs} = t_{[r,s],[i,j]} \).

For results on \( HP \) and \( CP \) see [2], [3], [4], [10], [20], and [22].

Choi [3] contends that the pointed cone \( CP \) gives us the natural generalization for positivity for Hermitian-preserving linear transformations [rather than \( \pi(\text{PSD}) \)]. The results of Poluikis and Hill [22] and Barker, Hill,
and Haertel [2] support this contention. This suggests that linear transformations $L$ for which $\mathcal{J}L$ (in the BHH decomposition) is completely positive is “the” natural generalization for dissipativity for linear transformations on matrices. We thus define a linear transformation $L : \mathcal{M}_n \to \mathcal{M}_q$ to be $\mathcal{CP}$-dissipative if $\mathcal{J}L \in \mathcal{CP}$, where $\mathcal{J}L + i\mathcal{J}L$ is the BHH decomposition of $L$. Note how the following theorems exploit the Poluikis-Hill characterizations of $\mathcal{HP}$ and $\mathcal{CP}$.

**Theorem 5.1.** Let $L : \mathcal{M}_n \to \mathcal{M}_q$ be a linear transformation. Then the following are equivalent:

(a) $L$ is $\mathcal{CP}$-dissipative.

(b) There exist $A_1, A_2, \ldots, A_s, A_{s+1}, \ldots, A_{2s} \in \mathcal{M}_{q,n}$, $\alpha_1, \ldots, \alpha_s \in \mathbb{R}$, and nonnegative $\beta_1, \ldots, \beta_s \in \mathbb{R}$ such that $\langle L \rangle = \sum_{j=1}^{s} (\alpha_j A_j \otimes \overline{A}_j + i\beta_j A_{s+j} \otimes \overline{A}_{s+j})$.

(b') There exist $A_1, A_2, \ldots, A_s, A_{s+1}, \ldots, A_{2s} \in \mathcal{M}_{q,n}$ and $\epsilon_1, \epsilon_2, \ldots, \epsilon_s$, which assume values $\pm 1$ such that $\langle L \rangle = \sum_{j=1}^{s} (\epsilon_j A_j \otimes \overline{A}_j + iA_{s+j} \otimes \overline{A}_{s+j})$.

(c) There exist $A_1, A_2, \ldots, A_s, A_{s+1}, \ldots, A_{2s} \in \mathcal{M}_{q,n}$, $(t_{jk}) \in \mathbb{D}_s$ such that $\langle L \rangle = \sum_{j<k=1}^{s} (r_{jk} A_j \otimes \overline{A}_k + im_{jk} A_{s+j} \otimes \overline{A}_{s+k})$, where $(r_{jk}) = \Re(t_{jk})$ and $(m_{jk}) = \Im(t_{jk})$.

(d) The block matrix $(L(E_{jk}))_{1 \leq j, k \leq n}$ is dissipative.

(e) $\Gamma(\langle L \rangle)$ is dissipative.

(e') $\Gamma(\langle L \rangle^T) = \Omega(\langle L \rangle)$ is dissipative.

(f) $\Psi(\langle L \rangle)$ is dissipative.

(f') $\Psi(\langle L \rangle)^T = \Theta(\langle L \rangle)$ is dissipative.

(g) $n_{ij}^{rs} = \overline{n_{ij}^{rs}}$ and $m_{ij}^{rs} = \overline{m_{ij}^{rs}}$ for $i, r = 1, 2, \ldots, q$, $j, s = 1, 2, \ldots, n$, where $l_{ij}^{rs} = m_{ij}^{rs} + im_{ij}^{rs}$, $\langle L \rangle = ((l_{ij}^{rs}))$, $\langle \mathcal{J}L \rangle = ((n_{ij}^{rs}))$, and $\langle \mathcal{J}\mathcal{L} \rangle = ((m_{ij}^{rs}))$.

and

$$\sum_{\sigma \in \Lambda_k} m_{i_1}^{\sigma(1)} \cdots m_{i_k}^{\sigma(k)} \geq \sum_{\sigma \in S_k \sim \Lambda_k} m_{i_1}^{\sigma(1)} \cdots m_{i_k}^{\sigma(k)}$$

for all $\omega \in Q_{k,nq}$, $k = 1, 2, \ldots, nq$, where $t_s = g^{-1}(\omega(s))$ and $\Lambda_k$ is the subgroup of even permutations in (the permutation group) $S_k$. ($Q_{k,nq}$ denotes the set of all strictly increasing sequences of length $k$ chosen from the set of the first $nq$ positive integers.)

(h) $-L^*$ is $\mathcal{CP}$-dissipative.

**Proof.** (a) $\iff$ (c): Suppose $L$ is $\mathcal{CP}$-dissipative. By definition, $\mathcal{J}L \in \mathcal{HP}$ and $\mathcal{J}\mathcal{L} \in \mathcal{CP}$. By [22, Theorems 1, 2], there exist $A_1, A_2, \ldots, A_t \in \mathcal{M}_{q,n}$
and \((r_{jk}) \in \mathcal{H}_s\) such that \(\langle \mathcal{A}L \rangle = \sum_{j,k=1}^s r_{jk} A_j \otimes \bar{A}_k\), and there exist \(B_1, B_2, \ldots, B_s \in \mathbb{M}_{q,n}\) and \((m_{jk}) \in \text{PSD}_s\) such that \(\langle \mathcal{J}L \rangle = \sum_{j,k=1}^s m_{jk} B_j \otimes \bar{B}_k\). Let \(s = \max(t, u)\). Let \(r_{jk} = 0\) or \(m_{jk} = 0\) for all values of \(j\) and \(k\) between \(t\) and \(u\), whichever applies. Since the block matrix diag\((A_0, 0)\) is Hermitian if and only if \(A\) is Hermitian and the spectrum of the block matrix diag\((A, 0)\) is the union of the spectra of \(A\) and \(0\) [0 with (additional) multiplicity \(n\)], diag\((r_{jk}, 0, \ldots)\) for \(s > t\) is Hermitian and diag\((m_{jk}, 0, \ldots)\) for \(s > u\) is positive semidefinite. Thus, there exist \(A_1, A_2, \ldots, A_s \in \mathbb{M}_{q,n}\) and \((r_{jk}) \in \mathcal{H}_s\) such that \(\langle \mathcal{A}L \rangle = \sum_{j,k=1}^s r_{jk} A_j \otimes \bar{A}_k\), and there exist \(B_1, B_2, \ldots, B_s \in \mathbb{M}_{q,n}\) and \((m_{jk}) \in \text{PSD}_s\) such that \(\langle \mathcal{J}L \rangle = \sum_{j,k=1}^s m_{jk} B_j \otimes \bar{B}_k\). Let \(B_1 = A_{s+1}, \ldots, B_t = A_{s+u}\). Then there exist \(A_1, A_2, \ldots, A_s, A_{s+1}, \ldots, A_{2s} \in \mathbb{M}_{q,n}\), \((r_{jk}) \in \mathcal{H}_s\), and \((m_{jk}) \in \text{PSD}_s\) such that \(\langle L \rangle = \langle \mathcal{A}L \rangle + \epsilon \langle \mathcal{J}L \rangle = \sum_{j,k=1}^s m_{jk} A_{s+j} \otimes \bar{A}_{s+k}\). Let \(B_1 = A_{s+1}, \ldots, B_s = A_{2s}\). Then \(L_1, L_2 \in \mathcal{L}(\mathcal{M}_n)\) be defined by \(\langle L_1 \rangle = \sum_{j,k=1}^s m_{jk} A_{s+j} \otimes \bar{A}_{s+k}\). Let \(B_1 = A_{s+1}, \ldots, B_s = A_{2s}\). Then \(\langle L \rangle = \langle L_1 \rangle + \epsilon \langle L_2 \rangle = \langle L_1 \rangle + \epsilon L_2\). Hence, \(L = L_1 + \epsilon L_2\). By [22, Theorems 1, 2], \(L_1 \in \mathcal{H}_P\) and \(L_2 \in \mathcal{C}_P\). By the uniqueness of the BHH decomposition, \(L_1 = \mathcal{A}L\) and \(L_2 = \mathcal{J}L\). Hence, \(L \in \mathcal{C}_P\)-dissipative.

Conversely, suppose there exist \(A_1, A_2, \ldots, A_{s+u} \in \mathbb{M}_{q,n}\), \((r_{jk}) \in \mathcal{H}_s\) such that \(\langle \mathcal{A}L \rangle = \sum_{j,k=1}^s r_{jk} A_j \otimes \bar{A}_k\), and there exist \(B_1, B_2, \ldots, B_s \in \mathbb{M}_{q,n}\) and \((m_{jk}) \in \text{PSD}_s\) such that \(\langle \mathcal{J}L \rangle = \sum_{j,k=1}^s m_{jk} B_j \otimes \bar{B}_k\). Let \(s = \max(t, u)\). Let \(r_{jk} = 0\) or \(m_{jk} = 0\) for all values of \(j\) between \(t\) and \(u\), whichever applies. Then there exist \(A_1, A_2, \ldots, A_s \in \mathbb{M}_{q,n}\), \(A_{s+1}, \ldots, A_{2s} \in \mathbb{M}_{q,n}\), \((r_{jk}) \in \mathcal{H}_s\), and \((m_{jk}) \in \text{PSD}_s\) such that \(\langle L \rangle = \sum_{j,k=1}^s m_{jk} A_{s+j} \otimes \bar{A}_{s+k}\). Let \(B_1 = A_{s+1}, \ldots, B_s = A_{2s}\). Then \(\langle L \rangle = \langle L_1 \rangle + \epsilon \langle L_2 \rangle = \langle L_1 \rangle + \epsilon L_2\). Hence, \(L = L_1 + \epsilon L_2\). By [22, Theorems 1, 2], \(L_1 \in \mathcal{H}_P\) and \(L_2 \in \mathcal{C}_P\). By the uniqueness of the BHH decomposition, \(L_1 = \mathcal{A}L\) and \(L_2 = \mathcal{J}L\). Hence, \(L \in \mathcal{C}_P\)-dissipative.

(a) \(\iff\) (b): Suppose that \(L\) is \(\mathcal{C}_P\)-dissipative. Then \(\mathcal{A}L \in \mathcal{H}_P\) and \(\mathcal{J}L \in \mathcal{C}_P\). By [22, Theorems 1, 2], there exist \(A_1, A_2, \ldots, A_s \in \mathbb{M}_{q,n}\), \(A_{s+1}, A_{s+2}, \ldots, A_{2s+u} \in \mathbb{M}_{q,n}\), \(A_{s+1}, A_{s+2}, \ldots, A_{2s} \in \mathbb{M}_{q,n}\), \((r_{jk}) \in \mathcal{H}_s\), and \((m_{jk}) \in \text{PSD}_s\) such that \(\langle \mathcal{A}L \rangle = \sum_{j=1}^s r_{jk} A_j \otimes \bar{A}_j\) and there exist \(B_1, B_2, \ldots, B_s \in \mathbb{M}_{q,n}\) and \(B_{s+1}, B_{s+2}, \ldots, B_{2s+u} \in \mathbb{M}_{q,n}\) such that \(\langle \mathcal{J}L \rangle = \sum_{j=1}^s \beta_j B_j \otimes \bar{B}_j\). Let \(s = \max(t, u)\). Let \(r_{jk} = 0\) or \(m_{jk} = 0\) for all values of \(j\) between \(t\) and \(u\), whichever applies. Then there exist \(A_1, A_2, \ldots, A_s \in \mathbb{M}_{q,n}\), \((r_{jk}) \in \mathcal{H}_s\), and \((m_{jk}) \in \text{PSD}_s\) such that \(\langle L \rangle = \sum_{j=1}^s \beta_j B_j \otimes \bar{B}_j\). Let \(B_1 = A_{s+1}, \ldots, B_s = A_{2s}\). Then \(\langle L \rangle = \langle \mathcal{A}L \rangle + \epsilon \langle \mathcal{J}L \rangle = \sum_{j=1}^s (r_{jk} A_j \otimes \bar{A}_j + \epsilon m_{jk} A_{s+j} \otimes \bar{A}_{s+j})\). Let \(B_1 = A_{s+1}, \ldots, B_s = A_{2s}\). Then \(\langle L \rangle = \sum_{j=1}^s r_{jk} A_j \otimes \bar{A}_j + \epsilon \sum_{j=1}^s \beta_j B_j \otimes \bar{B}_j\). Let \(L_1, L_2 \in \mathcal{L}(\mathcal{M}_n)\) be defined by \(\langle L_1 \rangle = \sum_{j=1}^s r_{jk} A_j \otimes \bar{A}_j\) and \(\langle L_2 \rangle = \sum_{j=1}^s \beta_j B_j \otimes \bar{B}_j\). Thus, there exist
\( \beta_1, \ldots, \beta_s \in \mathbb{R}^+ \) such that \( \langle L_2 \rangle = \sum_{j=1}^s \beta_j B_j \otimes \overline{B_j} \) for some \( t < s \) or \( \langle L_2 \rangle = 0 \). Since \( \langle L \rangle = \langle L_1 \rangle + \epsilon \langle L_2 \rangle = \langle L_1 \rangle + \epsilon L_2 \), we have \( L = L_1 + \epsilon L_2 \). By [22, Theorems 1, 2], \( L_1 \in \mathcal{AP} \) and \( L_2 \in \mathcal{CP} \). Hence, \( L \) is \( \mathcal{CP} \)-dissipative.

(b) \( \Leftrightarrow \) (b'): Statement (b') is a restatement of (b) upon absorbing \( \sqrt{\alpha_j} \) into \( A_j \) and \( \sqrt{B_j} \) into \( A_{s+j} = \beta_j \).

(a) \( \Leftrightarrow \) (e): Suppose \( L \in \mathcal{CP} \)-dissipative. Then \( \mathcal{R}L \in \mathcal{AP} \) and \( \mathcal{J}L \in \mathcal{CP} \). By [22, Theorems 1, 2], \( \Gamma(\langle \mathcal{R}L \rangle) \) is Hermitian and \( \Gamma(\langle \mathcal{J}L \rangle) \) is positive semidefinite. Thus, \( \Gamma(\langle L \rangle) = \Gamma(\langle \mathcal{R}L \rangle) + \epsilon \Gamma(\langle \mathcal{J}L \rangle) = \Gamma(\langle \mathcal{R}L \rangle) + \epsilon \Gamma(\langle \mathcal{J}L \rangle) \) is dissipative.

Conversely, suppose \( \Gamma(\langle L \rangle) \) is dissipative. Then \( \text{Im} \Gamma(\langle L \rangle) \) is positive semidefinite. However,

\[
\text{Im} \Gamma(\langle L \rangle) = \frac{1}{2\epsilon} \left( \Gamma(\langle \mathcal{R}L \rangle) - \Gamma(\langle \mathcal{J}L \rangle) \right) = \frac{1}{2\epsilon} \left[ \Gamma(\langle \mathcal{R}L \rangle) - \Gamma(\langle \mathcal{J}L \rangle) + \epsilon \Gamma(\langle \mathcal{J}L \rangle) \right]
\]

By definition, \( \mathcal{R}L, \mathcal{J}L \in \mathcal{AP} \). By [22, Theorem 1], \( \Gamma(\langle \mathcal{R}L \rangle) \) and \( \Gamma(\langle \mathcal{J}L \rangle) \) are Hermitian. Hence, \( \text{Im} \Gamma(\langle \mathcal{R}L \rangle) = 0 \) and \( \text{Re} \Gamma(\langle \mathcal{J}L \rangle) = \Gamma(\langle \mathcal{J}L \rangle) \). Thus, \( \text{Im} \Gamma(\langle L \rangle) = \Gamma(\langle \mathcal{J}L \rangle) \). Hence, \( \Gamma(\langle \mathcal{J}L \rangle) \) is positive semidefinite. By [22, Theorem 2], \( \mathcal{J}L \in \mathcal{CP} \), hence \( L \) is \( \mathcal{CP} \)-dissipative.

We observe that (a) \( \Leftrightarrow \) (e'), (a) \( \Leftrightarrow \) (f), and (a) \( \Leftrightarrow \) (f') follow from (a) \( \Leftrightarrow \) (e), \textit{mutatis mutandis}.

(d) \( \Leftrightarrow \) (f): By [22, Lemma 2], for every \( L : \mathcal{M}_n \rightarrow \mathcal{M}_q \), \( \left( L(E_{jk}) \right)_{1 \leq j, k \leq n} = \Psi(\langle L \rangle) \). The desired result follows immediately.

(e) \( \Leftrightarrow \) (g): Since \( \Gamma(\langle L \rangle) = \Gamma(\langle \mathcal{R}L \rangle) + \epsilon \Gamma(\langle \mathcal{J}L \rangle) \), we have \( l_{rs} = n_{rs} + \epsilon m_{rs} \), where \( \langle L \rangle = (l_{rs})_{rs} \), \( \langle \mathcal{R}L \rangle = (n_{rs})_{rs} \), and \( \langle \mathcal{J}L \rangle = (m_{rs})_{rs} \). Suppose \( \Gamma(\langle L \rangle) \) is dissipative. By [22, Theorems 1, 2], \( n_{rs} = \overline{n}_{rs}, m_{rs} = \overline{m}_{rs} \), and

\[
\sum_{\sigma \in A_k} \overline{m}_{rs}^{(1)} \cdots m_{rs}^{(k)} \geq \sum_{\sigma \in S_k \sim A_k} m_{rs}^{(1)} \cdots m_{rs}^{(k)}
\]

for all \( \omega \in Q_{k,nq}, k = 1, 2, \ldots, nq \), where \( t_s = g^{-1}(\omega(s)) \) and \( A_k \) is the subgroup of even permutations in \( S_k \). Hence, \( l_{rs} = n_{rs} + \epsilon m_{rs} = \overline{n}_{rs} + \epsilon \overline{m}_{rs} \).
Conversely, suppose $n_{rs}^{ij} = \pi_{rs}^{ij}$ and $m_{rs}^{ij} = \pi_{rs}^{ij}$ for $i, r = 1, 2, \ldots, q; j, s = 1, 2, \ldots, n$, where $\langle RL \rangle = ((n_{rs}^{ij}))$ and $\langle AL \rangle = ((m_{rs}^{ij}))$. Then $l_{rs}^{ij} = n_{rs}^{ij} + \epsilon m_{rs}^{ij} = \pi_{rs}^{ij} + \epsilon \pi_{rs}^{ij}$, where $\langle L \rangle = ((l_{rs}^{ij}))$; and

$$
\sum_{\sigma \in A_k} m_{t_1}^{s(1)} \cdots m_{t_k}^{s(k)} > \sum_{\sigma \in S_k \sim A_k} m_{t_1}^{s(1)} \cdots m_{t_k}^{s(k)}
$$

for all $\omega \in Q_k, nq, k = 1, 2, \ldots, nq$, where $t_s = g^{-1}(\omega(s))$ and $A_k$ is the subgroup of even permutations in $S_k$. By [22, Theorems 1, 2], $\Gamma(\langle RL \rangle)$ is Hermitian and $\Gamma(\langle AL \rangle)$ is dissipative. Hence, $\Gamma(\langle L \rangle)$ is dissipative.

(a) $\iff$ (h) Suppose $L$ is $\mathcal{CP}$-dissipative. Then $AL \in \mathcal{CP}$. Note that $-L^* = -(RL + \epsilon AL)^* = -RL^* + \epsilon AL^*$. By [22, Theorems 1, 2], $RL^* \in \mathcal{HP}$ and $AL^* \in \mathcal{HP}$. Thus, $-RL^* \in \mathcal{HP}$. Hence, $-L^*$ is $\mathcal{CP}$-dissipative. Conversely, suppose $L^*$ is $\mathcal{CP}$-dissipative. Then $(L^*)^* = L$ is $\mathcal{CP}$-dissipative.

Since the sum of two dissipative matrices is a dissipative matrix and the scalar product of a positive real number with a dissipative matrix is a dissipative matrix, Theorem 5.1 gives us that the set of $\mathcal{CP}$-dissipative linear transformations is a cone in $\mathcal{L}(\mathcal{M}_n)$. Since all Hermitian matrices are dissipative, the $\mathcal{HP}$ cone is contained in the $\mathcal{CP}$-dissipative cone. Hence, the $\mathcal{DP}$ cone is contained in the $\mathcal{CP}$-dissipative cone. However, not all $\mathcal{CP}$-dissipative matrices are Hermitian-preserving. For example, let $L \in \mathcal{L}(\mathcal{M}_2)$ be defined by

$$
\langle L \rangle = \begin{bmatrix}
-2 + \epsilon & 0 & 0 & 1 + 2\epsilon \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
2 + 3\epsilon & 0 & 0 & -1 + 4\epsilon
\end{bmatrix}.
$$

Then $\Gamma(\langle L \rangle) = \text{diag}\{-2 + \epsilon, 1 + 2\epsilon, 2 + 3\epsilon, -1 + 4\epsilon\}$, which implies that $L$ is $\mathcal{CP}$-dissipative. Moreover, by [22, Theorem 2], $L \notin \mathcal{HP}$. Since $\mathcal{DP} = \pi(\text{PSD})$, $\mathcal{DP}$ is a proper subset of the $\mathcal{CP}$-dissipative linear transformations.

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