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Polytopes related to interval vectors and incidence matrices

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ABSTRACT

In this short note we investigate polytopes associated with families of interval vectors, i.e., (0, 1)-vectors with consecutive ones. Using a linear transformation we show a connection to "extended" incidence matrices of acyclic directed graphs and the convex hull of their columns. This leads to complete linear descriptions of the corresponding polytopes.

1. Introduction

An *interval vector* is a (0, 1)-vector $x \in \mathbb{R}^n$ such that its ones (if any) occur consecutively, for example, if $x_i = x_k = 1$ for some $i < k$, then $x_i = x_{i+1} = \cdots = x_k = 1$. These vectors are of interest in, e.g. scheduling applications where the ones indicate the duration of an uninterrupted activity (see [1] for models of certain job-shop scheduling problems). There is also an interesting class of matrices related to interval vectors. An *interval matrix* (see [4]) is a (0, 1)-matrix whose columns are interval vectors. Each such matrix is totally unimodular and the corresponding linear optimization problems may be solved as network flow problems (again, see [4]).

Let $\mathcal{I}$ be an arbitrary set of interval vectors (in $\mathbb{R}^n$). A goal of this paper is to find a complete linear description of the polytope $P_{\mathcal{I}} = \text{conv}(\mathcal{I})$, i.e., the convex hull of the interval vectors in $\mathcal{I}$. To this end we transform the problem and consider incidence matrices A of acyclic directed graphs extended in the sense that some columns correspond to unit vectors. For such a matrix A we investigate the convex hull of its columns and find a complete linear description of such polytopes.

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For the theory of incidence matrices of graphs we refer to [3] and for concepts and results in polyhedral theory see [4,6]. Vectors are treated as column vectors, and they are identified with the corresponding n-tuples. The zero vector is denoted by O.

2. The results

First we transform our problem. The idea is simply that each (nonzero) interval vector has a first and a last position of its ones. Let $L = [l_{ij}] \in \mathbb{R}^{n\times n}$ be the lower triangular (0, 1)-matrix where $l_{ij} = 1$ if $i \geqslant j$ and $l_{ij} = 0$ otherwise (so L is a special interval matrix). The corresponding linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ given by $T(x) = Lx$ is an isomorphism as L is invertible. We write $T(S)$ for the image of a set S under T, i.e. $T(S) = \{T(x) : x \in S\}$. Let e_i be the ith unit vector (with a 1 in position i, otherwise zeros) and define $e_{ij} = e_i - e_j$ for each $i < j$. Vectors e_i and e_{ij}, as well as the zero vector O, will be called *elementary* vectors.

Recall from Section 1 that $\mathcal{I}$ is a given (but arbitrary) set of interval vectors in $\mathbb{R}^n$. Note that

$$T(e_i) = e_i + e_{i+1} + \cdots + e_n \quad (i \leqslant n)$$

and

$$T(e_{ij}) = e_i + e_{i+1} + \cdots + e_{j-1} \quad (i < j)$$

so these are interval vectors. Also $T(O) = O$. From this we see that there is a *unique* set $\mathcal{E}$ of elementary vectors such that $T(\mathcal{E}) = \mathcal{I}$; namely $\mathcal{E} = T^{-1}(\mathcal{I})$.

We show below that it suffices to find a complete linear description of $P_{\mathcal{E}} = \text{conv}(\mathcal{E})$, the convex hull of the elementary vectors $\mathcal{E}$ associated with $\mathcal{I}$. Let C be the $(0, -1, 1)$-matrix with the vectors in $\mathcal{E}$ as its columns (in some order); then clearly $P_{\mathcal{E}}$ is the convex hull of the columns of C. This gives a connection to incidence matrices of directed graphs which will be exploited in the following.

Consider the matrix L above. Note that the inverse of L is given by $L^{-1} = [m_{ij}]$ where $m_{ij} = 1$ for $i = j$, $m_{i,i-1} = -1$ for $i \geqslant 2$, and $m_{ij} = 0$ otherwise. We remark that the essence of the following lemma is true for invertible linear transformations of polyhedra in general (as the proof shows).

Lemma 1. *If $P_{\mathcal{E}} = \{x \in \mathbb{R}^n : Ax \leqslant b\}$ (where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$) is a complete linear description of $P_{\mathcal{E}}$, then a complete linear description of $P_{\mathcal{I}}$ is given by*

$$P_{\mathcal{I}} = T(P_{\mathcal{E}}) = \{y \in \mathbb{R}^n : AL^{-1}y \leqslant b\}.$$

Proof. The fact that $P_{\mathcal{I}} = T(P_{\mathcal{E}})$ follows by convexity as $T(\mathcal{E}) = \mathcal{I}$. If $y \in T(P_{\mathcal{E}})$, then $y = Lx$ for some x with $Ax \leqslant b$. Then $b \geqslant Ax = AL^{-1}Lx = AL^{-1}y$. Conversely, if y satisfies $AL^{-1}y \leqslant b$, let x be defined as $x = L^{-1}y$. Then $y = Lx$ and $Ax = AL^{-1}y \leqslant b$, so $y \in T(P_{\mathcal{E}})$. $\square$

Due to Lemma 1 we focus on finding a complete linear description of $P_{\mathcal{E}}$. To this end we exploit the connection to incidence matrices of directed graphs (see [3,4,6] for such matrices). The analysis uses the same linear algebraic properties on which the network simplex algorithm relies.

Let $G = (V, E)$ denote the directed graph with vertex set $V = \{1, 2, \ldots, n\}$ and edge set $E = \{(i, j) : e_{ij} \in \mathcal{E}\}$. Define $V_1 = \{j \in V : e_j \in \mathcal{E}\}$. Also, let k_0 denote the number of connected components of the graph G (ignoring direction of the edges) such that the component contains *no vertex* in V_1.

Theorem 2. *If $O \in \mathcal{E}$, then the dimension of $P_{\mathcal{E}}$ is $n - k_0$. In particular, $P_{\mathcal{E}}$ is full-dimensional if and only if each connected component of G contains at least one vertex in V_1. If $O \notin \mathcal{E}$, then the dimension of $P_{\mathcal{E}}$ is $n - k_0 - 1$.*

Proof. Assume first that $O \in \mathcal{E}$. The vectors in $\mathcal{E} \setminus \{O\}$ may be organized as columns of a matrix B given by

$$B = \begin{bmatrix} B_1 & B_2 \end{bmatrix},$$

where B_1 is the submatrix containing the vectors e_{ij} and B_2 contains the vectors e_j $(j \in V_1)$. Let $V^1, V^2, \ldots, V^k$ be the partition of V into the connected components of the graph G (ignoring directions). Then, by reordering rows and columns according to this partition, B_1 may be written as a direct sum $B_1 = B_1^1 \oplus \cdots \oplus B_1^k$, where B_1^t correspond to the subgraph of G induced by V^t. A similar direct sum exists for $B_2 : B_2 = B_2^1 \oplus \cdots \oplus B_2^k$ where the rows in B_2^t correspond to V^t and the columns of B_2^t are e_j $(j \in V^t \cap V_1)$. Let $n_t = |V^t|$ $(t \leqslant k)$, so $\sum_t n_t = n$. Then rank B equals the sum of the ranks of the matrices

$$B_*^t = \begin{bmatrix} B_1^t & B_2^t \end{bmatrix}.$$

By network flow theory (see e.g. [3]) rank $B_1^t = n_t - 1$ as this subgraph is connected (the rows are linearly dependent, because the sum of these vectors is the zero vector, and connectivity assures that the rank is no less than $n_t - 1$). Thus, rank B_*^t is either $n_t - 1$ or n_t. Moreover, the rank is n_t if and only if the matrix B_2^t is nonvacuous (contains at least one column). In fact, if a vector z satisfies $z^T B_*^t = O$, then $z_i - z_j = 0$ for each $(i, j) \in E$ where $i, j \in V^t$, so $z_j = \alpha$ for all $j \in V^t$ and some constant α. But if B_*^t contains a column e_j we obtain $\alpha = 0$ (from $z^T B_*^t = O$) and the rows of this matrix are linearly independent. Conversely, if there is no such column e_j, then, as noted, the rank is $n_t - 1$. It follows that rank $B = n - k_0$. Finally, as $\mathcal{E}$ contains the zero vector O, the dimension of $P_\mathcal{E}$ equals the rank of the matrix B, which proves the first two statements of the theorem. Finally, if $O \notin \mathcal{R}$, then the dimension of $P_\mathcal{E}$ is equal to the rank $B - 1$, and the proof is complete. $\square$

Consider a spanning forest in G consisting of some number r of (disjoint) trees $T^t = (V^t, E^t)$ for $t \leqslant r$, and for each t select a vertex $v^{(t)} \in V^t$. For each $j \in V^t$ let P_j^+ (resp. P_j^-) be the forward (resp. backward) edges in the unique path in T^t going from $v^{(t)}$ to vertex j, and define

$$c_j = |P_j^-| - |P_j^+| + 1 \quad (j \in V^t, t \leqslant r).$$

We may think of c_j as a "signed distance" from $v^{(t)}$ to vertex j in T^t. In particular, $c_{v^{(t)}} = 1$ $(t \leqslant r)$. Consider the corresponding inequality

$$\sum_{j=1}^{n} c_j x_j \leqslant 1. \tag{1}$$

If (1) is valid for $P_\mathcal{E}$ (meaning that each point in $\mathcal{E}$ satisfies the inequality) we call the forest (with its special vertices $v^{(t)}$) *feasible* and the corresponding inequality (1) is called a *forest inequality*. A *subset inequality* is an inequality of one of the two forms

(i) $\sum_{j \in S} x_j \geqslant 0$ for some $S \subseteq V$ such that the induced subgraph $G[S]$ is connected and no edge in G enters S, or
(ii) $\sum_{j \in S} x_j \leqslant 0$ for some $S \subseteq V$ such that $G[S]$ is connected, no edge in G leaves S and $V_1 \cap S = \emptyset$.

We are only interested in subset inequalities that are valid for $P_\mathcal{E}$ (which is easy to check).

Example. Let $n = 5$ and consider the set $\mathcal{I}$ of interval vectors shown as columns of the matrix I below. The corresponding set $\mathcal{E}$ of elementary vectors is the set of columns of the matrix C

$$I = \begin{bmatrix} 0&1&0&0&0&1&0&0&0 \\ 0&0&1&0&0&1&1&0&0 \\ 0&0&0&1&0&0&1&1&1 \\ 0&0&0&0&1&0&0&1&1 \\ 0&0&0&0&0&0&0&0&1 \end{bmatrix}, \quad C = \begin{bmatrix} 0&1&0&0&0&1&0&0&0 \\ 0&-1&1&0&0&0&1&0&0 \\ 0&0&-1&1&0&-1&0&1&1 \\ 0&0&0&-1&1&0&-1&0&0 \\ 0&0&0&0&-1&0&0&-1&0 \end{bmatrix}.$$

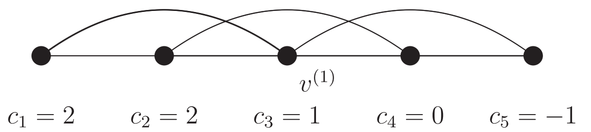

Fig. 1. The graph G and a forest inequality.

Fig. 1 shows the graph G with edge set $E = \{(i,j) : e_{ij} \in \mathcal{E}\}$; it has five vertices and seven edges (each directed from left to right). We have $V_1 = \{3\}$ (see the final column). The spanning forest consists of a single tree T^1 with root $v^{(1)}$ (the third vertex), and its edges are displayed with thick lines ((1,3), (2,3), (3,4) and (4,5)). The coefficients of the corresponding forest inequality (1) are shown. For instance, the elementary vector e_{13} satisfies this forest inequality with equality.

Theorem 3. *Assume that $k_0 = 0$ and $O \in \mathcal{E}$. Then every facet of $P_{\mathcal{E}}$ is induced by a subset inequality or a forest inequality.*

Proof. When $k_0 = 0$, by Theorem 2, $P_{\mathcal{E}}$ is full-dimensional, so each facet F of $P_{\mathcal{E}}$ has a dimension of $n-1$. Then F is induced by a valid inequality

$$\sum_{j=1}^{n} a_j x_j \leqslant \alpha, \tag{2}$$

where the coefficient vector is unique up to multiplication by a positive scalar. Let $\mathcal{E}_F = \mathcal{E} \cap F$. Consider the graph G_F with vertex vet V and edge set $E_F = \{(i,j) \in E : e_{ij} \in F\}$. Call $j \in V_1$ an *F-root* if $e_j \in F$. Let $V_F^1, V_F^2, \ldots, V_F^r$ be the partition of V corresponding to the connected components of G_F (ignoring directions), and let $k_0(F)$ be the number of these components that do not contain any F-root. We distinguish between two cases.

Case 1. F contains the zero vector O: This means that $\alpha = 0$ in (2). Note that $F = P_{\mathcal{E}_F} := \mathrm{conv}(\mathcal{E}_F)$. Therefore, we may apply Theorem 2 to the situation where $\mathcal{E}$ is replaced by $\mathcal{E}_F$ and conclude that $\dim F = n - k_0(F)$, so $k_0(F) = 1$ (as $\dim F = n-1$). We may therefore assume that component V_F^t contains an F-root, say $v^{(t)}$, $(t < r)$ while the final component V_F^r contains no F-root. Now, each point in $\mathcal{E}_F$ satisfies (2) with equality (and $\alpha = 0$). Using connectivity and the fact that $e_{v^{(t)}} \in \mathcal{E}_F$, as in the last part of the proof of Theorem 2, we conclude that

$$a_j = 0 \quad \text{for all } j \in V_F^t \quad (t < r).$$

For the final component V_F^r we "only" obtain that a_j is equal to some constant for all $j \in V_F^r$. This means that the inequality (2) by suitable scaling becomes $\sum_{j \in V_F^r} x_j \geqslant 0$ or $\sum_{j \in V_F^r} x_j \leqslant 0$ (the direction is determined by its validity). This shows that every facet F containing O is induced by a subset inequality.

Case 2. F does not contain O: Again due Theorem 2 (now the second part)

$$n - 1 = \dim F = n - k_0(F) - 1$$

so $k_0(F) = 0$. Therefore each component V_F^t contains an F-root $v^{(t)}$ $(t \leqslant r)$. Since $O \notin F$, the right hand side α in (2) must be nonzero, so by scaling we may assume that $\alpha = 1$. Using that each point in $\mathcal{E}_F$ satisfies (2) with equality we get

$$a_i - a_j = 1 \quad ((i,j) \in E_F)$$

and, moreover, $a_{v^{(t)}} = 1$. This implies that for each $j \in V_F^t$ and each path P from $v^{(t)}$ to j in G_F

$$a_j = |P_j^-| - |P_j^+| + 1 \quad (j \in V_F^t, t \leqslant r).$$

This shows that (2) is a forest inequality, as desired. □

We remark that several of the subset inequalities and forest inequalities may be redundant, but we do not discuss this question here.

Define an *alternating-sign vector* as a nonzero $(0, -1, 1)$-vector such that after deleting all its zeros one gets a $(-1, 1)$-vector where each pair of consecutive components have different signs. Let $\mathcal{A}_n$ be the set of alternating-sign vectors of length n.

A complete linear description of the polytope $P_{\mathcal{I}} = \text{conv}(\mathcal{I})$ may now be found.

Theorem 4. *Let $\mathcal{I}$ be a set of interval vectors (in $\mathbb{R}^n$) containing the zero vector. Let $\mathcal{E} = T^{-1}(\mathcal{I})$ and assume that $k_0 = 0$. Then every facet of $P_{\mathcal{I}}$ has one of the forms*

(i) $\sum_{j=1}^{n}(c_j - c_{j+1})x_j \leqslant 1$ *where $c_1, c_2, \ldots, c_n$ are as in (2) and $c_{n+1} = 0$,*
(ii) $w^T x \geqslant 0$ *for some $w \in \mathcal{A}_n$.*

Proof. This follows directly by combining Lemma 1 and Theorem 3: the inequalities in (i) resp. (ii) are obtained from forest inequalities resp. subset inequalities. □

We now discuss an application of Theorem 4.

Corollary 5. *A complete linear description of the convex hull of all interval vectors in $\mathbb{R}^n$ is*

$$\begin{aligned} &w^T x \leqslant 1 \quad (w \in \mathcal{A}_n), \\ &x_k \geqslant 0 \quad (k \leqslant n). \end{aligned} \tag{3}$$

Proof. We apply Theorem 4. The corresponding set $\mathcal{E}$ contains O and all elementary vectors e_j $(j \leqslant n)$ and e_{ij} $(i < j)$. Any nonredundant subset inequality must be of the form

$$x_1 + x_2 + \cdots + x_k \geqslant 0 \quad (k \leqslant n) \tag{4}$$

as each other vertex set S has an entering edge (and no subset inequality of type (ii) is valid). The transformed inequalities (Lemma 1) are trivial inequalities

$$x_k \geqslant 0 \quad (k \leqslant n).$$

Next, consider a nonredundant forest inequality. Then $c_i - c_j \leqslant 1$ $(i < j)$ and $c_i \leqslant 1$ $(i \leqslant n)$ as the inequality is valid. This implies that the forest just contains a single tree T^1 and that $v^{(1)} = 1$. But then $0 \leqslant c_j \leqslant 1$ $(j \leqslant n)$ so the nonredundant forest inequalities are

$$\sum_{j \in S} x_j \leqslant 1 \quad (S \subset V). \tag{5}$$

The transformed inequalities are

$$w^T x \leqslant 1 \quad (w \in \mathcal{A}_n).$$

Thus, a complete linear description of $P_{\mathcal{E}}$ is given by (4) and (5), and the transformed inequalities give a complete linear description of $P_{\mathcal{I}}$, as desired. □

We remark that alternating-sign vectors arise in another setting. In [5] one considers the class of $(0, -1, 1)$-matrices where each row and column is an alternating sign-vector, and one finds a complete linear description of the corresponding convex hull of these matrices (the so-called alternating sign matrix polytope). This result actually extends the result by Birkhoff and von Neumann on doubly stochastic matrices (see [2] for several results concerning doubly stochastic matrices).

Finally, it would be interesting to see if the results above may be useful in the study of scheduling formulations and algorithms. We leave this for possible future work.

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