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# A note on well-posedness for Camassa–Holm equation

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**Abstract**

In this note, we investigate the problem of well-posedness for a shallow water equation with data having critical regularity. Our results are based on the use of Besov spaces $B^s_{2,r}$ (which generalize the Sobolev spaces $H^s$) with critical index $s = 3/2$.


## 0. Introduction

In the past few years, a large amount of literature has been devoted to the following one-dimensional nonlinear dispersive equation:

$$\partial_t u - \partial^3_{txx} u + 3u\partial_x u = 2\partial_x u \partial^2_{xx} u + u\partial^3_{xxx} u. \tag{1}$$

In the above equation, the function $u = u(t, x)$ stands for the fluid velocity at time $t \geqslant 0$ in the $x$ direction. Eq. (1), commonly called Camassa–Holm equation, has been derived independently by Fokas and Fuchssteiner [FF], and by Camassa and Holm in [CH] (see also [F]). Like the celebrated KdV equation, Camassa–Holm equation describes the unidirectional propagation of waves at the free surface of shallow water under the influence of gravity. It turns out that it is also a model for the propagation of nonlinear waves in cylindrical hyperelastic rods (see [Dai]).

The problem of finding the largest spaces $E$ for which (1) with initial data in $E$ is well-posed in the sense of Hadamard has retained a lot of attention recently. Before

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going further into details, let us define what we mean by "local well-posedness in the sense of Hadamard":

**Definition 1.** Let $E$ be a Banach space. System (1) is said to be locally well-posed in $E$ in the sense of Hadamard if for any $u_0 \in E$, there exists a neighborhood $V$ of $u_0$ and a $T > 0$ such that for any $v_0 \in V$, system (1) has a unique solution $v \in C([0,T];E)$ with initial datum $v_0$, and if in addition the map $v_0 \mapsto v$ is continuous from $V$ into $C([0,T];E)$.

In most of papers devoted to (1), only Sobolev spaces $H^s$ are considered. It has been stated by different methods that (1) is well-posed in $H^s$ for $s > 3/2$ (see [Dan,HM,LO,RB]). On the other hand, counterexamples to well-posedness in the case $s < 3/2$ have been exhibited by Himonas and Misiołek [HM] (actually what they do prove there is that *uniform* continuity with respect to the data cannot hold in $H^s$ with $s < 3/2$). Therefore, in the Sobolev spaces framework, $s = 3/2$ seems to be the critical value for local well-posedness.

Let us mention in passing that Xin and Zhang showed recently that global weak solutions do exist for any data in $H^1$ (see [XZ]), but local well-posedness (or even uniqueness) for such rough initial data is unlikely to be true.

That $s = 3/2$ is critical for well-posedness has to do with the fact that it is also the critical value for the embedding $H^s \hookrightarrow \mathrm{Lip}$ to be true (Lip here denotes the bounded lipschitz functions). Indeed, denoting $P(D) = -\partial_x(1-\partial_x^2)^{-1}$, Eq. (1) may be rewritten as a nonlinear transport equation:

$$\begin{cases} \partial_t u + u\partial_x u = P(D)(u^2 + \frac{1}{2}(\partial_x u)^2), \\ u_{|t=0} = u_0. \end{cases} \tag{CH}$$

For most "reasonable" Banach spaces $E$ embedded in Lip, one can prove a priori estimates for linear transport equations

$$\partial_t v + a\partial_x v = f,$$

with $v(0) \in E$ and, say, $a(t)$ and $f(t)$ uniformly bounded in $E$. Therefore, $H^s$ has to be embedded in Lip in order that (CH) may be solved by a standard iterative scheme.

In [Dan], we gave examples of spaces $E$ which are not related to Sobolev spaces and for which local well-posedness is true. For example, any Besov space $B^s_{p,r}$ with $s > \max(3/2, 1+1/p)$ does.

In the present paper, we aim at gleaning as much information as possible on the critical value 3/2. Due to the lack of embedding $H^{\frac{3}{2}} \hookrightarrow \mathrm{Lip}$ however, we will be induced to use *Besov spaces* $B^s_{2,r}$ which are closely related to $H^s$. To simplify the presentation, we shall assume throughout that $x$ belongs to $\mathbb{R}$. That point is not essential: as our results are based on Fourier analysis, slight modifications of our proofs would enable us to handle the periodic case $x \in \mathbb{T}$ (a reading of [Dan] should convince the doubtful reader).

In the case $x \in \mathbb{R}$, those Besov spaces may be defined as follows:

**Definition 2.** Let $s \in \mathbb{R}$ and $r \in [1, +\infty]$. Denote

$$\|u\|_{B^s_{2,r}} \stackrel{\text{def}}{=} \left[ \left( \int_{-1}^{1} (1+\xi^2)^s |\hat{u}(\xi)|^2 \, d\xi \right)^{\frac{r}{2}} + \sum_{q \in \mathbb{N}} \left( \int_{2^q \leqslant |\xi| \leqslant 2^{q+1}} (1+\xi^2)^s |\hat{u}(\xi)|^2 \, d\xi \right)^{\frac{r}{2}} \right]^{\frac{1}{r}}$$

with an obvious modification if $r = +\infty$.

We then define the Besov space $B^s_{2,r}$ as follows: $B^s_{2,r} \stackrel{\text{def}}{=} \{u \in \mathscr{S}' \mid \|u\|_{B^s_{2,r}} < +\infty\}$.

Obviously, $H^s = B^s_{2,2}$. Moreover, the spaces $H^s$ and $B^s_{2,r}$ are very close. We shall often use the following chain of continuous embedding for $s' < 3/2 < s$:

$$H^s \hookrightarrow B^{\frac{3}{2}}_{2,1} \hookrightarrow H^{\frac{3}{2}} \hookrightarrow B^{\frac{3}{2}}_{2,\infty} \hookrightarrow H^{s'}.$$

In practical computations however, the spectral cut-off $1_{2^q \leqslant |\xi| \leqslant 2^{q+1}}$ is too rough and has to be replaced by a smoother one. This may be achieved by introducing a *Littlewood–Paley decomposition*, that is a dyadic partition of unity in Fourier variables.

Let $(\chi, \varphi)$ be a couple of $C^\infty$ functions with $\operatorname{Supp} \chi \subset \{|\xi| \leqslant 4/3\}$, $\operatorname{Supp} \varphi \subset \{3/4 \leqslant |\xi| \leqslant 8/3\}$ and

$$\chi(\xi) + \sum_{q \in \mathbb{N}} \varphi(2^{-q}\xi) = 1 \quad \text{for } \xi \in \mathbb{R}.$$

Denote $\varphi_q(\xi) = \varphi(2^{-q}\xi)$, $h_q = \mathscr{F}^{-1}\varphi_q$ and $\check{h} = \mathscr{F}^{-1}\chi$. We then define the dyadic blocks as

$$\Delta_q u \stackrel{\text{def}}{=} 0 \quad \text{if } q \leqslant -1, \quad \Delta_{-1} u \stackrel{\text{def}}{=} \chi(D)u = \int_{\mathbb{R}} \check{h}(y) u(x-y)\, dy,$$

$$\Delta_q u \stackrel{\text{def}}{=} \varphi(2^{-q}D)u = \int_{\mathbb{R}} h_q(y) u(x-y)\, dy \quad \text{if } q \geqslant 0.$$

Now, for $s \in \mathbb{R}$ and $r \in [1, +\infty]$, the Besov spaces $B^s_{2,r}$ may be alternatively defined by

$$B^s_{2,r} \stackrel{\text{def}}{=} \{u \in \mathscr{S}'(\mathbb{R}) \mid \|\tilde{u}\|_{B^s_{2,r}} < +\infty\},$$

where we denoted

$$\|\tilde{u}\|_{B^s_{2,r}} \stackrel{\text{def}}{=} \left( \sum_{q\geqslant -1} (2^{sq}\|\Delta_q u\|_{L^2})^r \right)^{\frac{1}{r}} \quad \text{if } 1\leqslant r < +\infty,$$

$$\text{and} \quad \|\tilde{u}\|_{B^s_{2,\infty}} \stackrel{\text{def}}{=} \sup_{q\geqslant -1} 2^{sq}\|\Delta_q u\|_{L^2}.$$

Of course, the norm above is equivalent to the one in Definition 2 so that we shall merely denote it by $\|\cdot\|_{B^s_{2,\infty}}$ instead of $\|\tilde{\cdot}\|_{B^s_{2,\infty}}$.

Let us now state our main result:

**Theorem 1.** *The exponent* $3/2$ *is critical in the following sense*:

- *System* (CH) *is locally well-posed in* $B^{\frac{3}{2}}_{2,1}$ *in the sense of Hadamard.*
- *System* (CH) *is not locally well-posed in* $B^{\frac{3}{2}}_{2,\infty}$.

Let us emphasize that local well-posedness in $B^{\frac{3}{2}}_{2,1}$ is a new result. In [Dan], existence only was stated. Note that using Besov spaces $B^s_{p,1}$ in hydrodynamics has been done before by Vishik [V] for the incompressible Euler equations (there, the critical value of $s$ in dimension 2 is $1+2/p$). The motivation there was the same as ours: getting the critical regularity exponent for local well-posedness.

The counterexample to well-posedness in $B^{\frac{3}{2}}_{2,\infty}$ is inspired by the one in [HM].

We have no definitive answer to the intermediate cases $B^{\frac{3}{2}}_{2,r}$. In particular, local well-posedness in $H^{\frac{3}{2}}$ remains an open question: our counterexample is closely related to the space $B^{\frac{3}{2}}_{2,\infty}$ (the corresponding initial data do not belong to any $B^{\frac{3}{2}}_{2,r}$ with $r<+\infty$) so that it does not provide us with any information on well-posedness in $B^{\frac{3}{2}}_{2,r}$ with $r<+\infty$. Besides, owing to the lack of embedding in Lip when $r>1$, well-posedness in $B^{\frac{3}{2}}_{2,r}$ for $r>1$ is unlikely to be proved by standard iterative methods. On the other hand, combining Proposition 1 below with our existence results in [Dan] yields local existence and uniqueness in *any* space $B^{\frac{3}{2}}_{2,r}\cap\text{Lip}$.

Once uniqueness has been stated for a given Banach space $E$, one can actually get quite a lot of information on the solution. In the case $B^{\frac{3}{2}}_{2,1}$, a rereading of the proof of theorem 0.1 in [Dan] yields the following:

**Theorem 2.** *Let* $u_0$ *be in* $B^{\frac{3}{2}}_{2,1}$. *There exists a maximal* $T^\star_{u_0}$ *such that* (1) *has a unique solution in* $C([0,T^\star_{u_0}[;B^{\frac{3}{2}}_{2,1})\cap C^1([0,T^\star_{u_0}[;B^{\frac{1}{2}}_{2,1})$ *with constant* $H^1$ *norm. Moreover, the* *lifespan* $T^\star_{u_0}$ *satisfies*

$$T^\star_{u_0} \geqslant T_{u_0} \stackrel{\text{def}}{=} -\frac{2}{\|u\|_{H^1}} \arctan\left(\frac{\|u_0\|_{H^1}}{\inf_{x\in\mathbb{R}} \partial_x u_0(x)}\right),$$

*that estimate being optimal in general, and*

$$T^\star_{u_0} < +\infty \Rightarrow \int_0^{T^\star_{u_0}} \left(\inf_{x\in\mathbb{R}} \partial_x u(t,x)\right) dt = -\infty.$$

*If the potential* $y_0 \stackrel{\text{def}}{=} u_0 - \partial^2_{xx} u_0$ *has a sign then* $T^\star_{u_0} = +\infty$ *and* $\operatorname{sgn}(u(t) - \partial^2_{xx} u(t)) = \operatorname{sgn} y_0$.

Our paper is organized as follows. In Section 1, we prove estimates in $L^\infty(0,T;B^{\frac{1}{2}}_{2,\infty})$ for the difference of two solutions of (CH) belonging to $L^\infty(0,T;B^{\frac{3}{2}}_{2,\infty}\cap \mathrm{Lip})\cap C([0,T];B^{\frac{1}{2}}_{2,\infty})$. This in particular yields uniqueness of solutions with data in $B^{\frac{3}{2}}_{2,1}$. In Section 2, we address the question of continuity in $C([0,T];B^{\frac{3}{2}}_{2,1})$ for data in $B^{\frac{3}{2}}_{2,1}$. In Section 3, we show that continuity in $C([0,T];B^{\frac{3}{2}}_{2,\infty})$ with respect to data in $B^{\frac{3}{2}}_{2,\infty}$ cannot be expected in general.

## 1. Uniqueness for critical regularity index

Uniqueness is a corollary of the following.

**Proposition 1.** *Let* $u$ (*resp.* $v$) *be a solution to* (CH) *with initial datum* $u_0$ (*resp.* $v_0$). *Assume that* $u_0$ *and* $v_0$ *belong to* $B^{\frac{3}{2}}_{2,\infty}\cap\mathrm{Lip}$, *and that* $u$ *and* $v$ *belong to* $L^\infty(0,T;B^{\frac{3}{2}}_{2,\infty}\cap\mathrm{Lip})\cap C([0,T];B^{\frac{1}{2}}_{2,\infty})$. *Let* $w \stackrel{\text{def}}{=} v-u$ *and* $w_0 \stackrel{\text{def}}{=} v_0-u_0$. *There exists a constant* $C$ *such that if, for some* $T^\star \leqslant T$,

$$\sup_{t\in[0,T^\star]} \left(e^{-C\int_0^t \|\partial_x u\|_{B^{\frac{1}{2}}_{2,\infty}\cap L^\infty} d\tau} \|w(t)\|_{B^{\frac{1}{2}}_{2,\infty}}\right) \leqslant 1 \tag{2}$$

*then, denoting* $L(z) \stackrel{\text{def}}{=} z\log(e+z)$, *the following inequality holds true for* $t\in[0,T^\star]$:

$$\frac{\|w(t)\|_{B^{\frac{1}{2}}_{2,\infty}}}{e} \leqslant e^{C\int_0^t \|\partial_x u\|_{B^{\frac{1}{2}}_{2,\infty}\cap L^\infty} d\tau} \left(\frac{\|w_0\|_{B^{\frac{1}{2}}_{2,\infty}}}{e}\right)^{\exp\left[-C\int_0^t L\left(\|u\|_{B^{\frac{3}{2}}_{2,\infty}\cap\mathrm{Lip}} + \|v\|_{B^{\frac{3}{2}}_{2,\infty}\cap\mathrm{Lip}}\right)d\tau\right]}. \tag{3}$$

*In particular*, (3) *holds true on* $[0,T]$ *provided that*

$$\|w_0\|_{B^{\frac{1}{2}}_{2,\infty}} \leqslant e^{1-\exp\left[C\int_0^T L\left(\|u\|_{B^{\frac{3}{2}}_{2,\infty}\cap \mathrm{Lip}}+\|v\|_{B^{\frac{3}{2}}_{2,\infty}\cap \mathrm{Lip}}\right)dt\right]}. \tag{4}$$

**Proof.** Clearly, $w$ solves the following linear transport equation:

$$\partial_t w + u\partial_x w = -w\partial_x v + P(D)(w(u+v) + \frac{1}{2}\partial_x w\,\partial_x(u+v)).$$

By virtue of estimate (A.1) in [Dan], the following inequality holds true:

$$\begin{aligned}\|w(t)\|_{B^{\frac{1}{2}}_{2,\infty}} \leqslant{}& \|w_0\|_{B^{\frac{1}{2}}_{2,\infty}} e^{C\int_0^t \|\partial_x u\|_{B^{\frac{1}{2}}_{2,\infty}\cap L^\infty} d\tau} \\ &+ \int_0^t e^{C\int_\tau^t \|\partial_x u\|_{B^{\frac{1}{2}}_{2,\infty}\cap L^\infty} d\tau'} \\ &\times \left(\|w\partial_x v\|_{B^{\frac{1}{2}}_{2,\infty}} + \|P(D)(w(u+v) + \frac{1}{2}\partial_x w\,\partial_x(u+v))\|_{B^{\frac{1}{2}}_{2,\infty}}\right) d\tau. \end{aligned} \tag{5}$$

Bounding the right-hand side will be made possible thanks to the following five properties:

(i) The space $B^{\frac{1}{2}}_{2,1}$ is continuously embedded in $B^{\frac{1}{2}}_{2,\infty}\cap L^\infty$.

(ii) The space $B^{\frac{1}{2}}_{2,\infty}\cap L^\infty$ is an algebra.

(iii) The usual product is continuous from $B^{-\frac{1}{2}}_{2,1}\times(B^{\frac{1}{2}}_{2,\infty}\cap L^\infty)$ to $B^{-\frac{1}{2}}_{2,\infty}$.

(iv) For any $s\in\mathbb{R}$ and $r\in[1,+\infty]$, the operator $P(D)$ maps continuously $B^s_{2,r}$ into $B^{s+1}_{2,r}$.

(v) There exists a constant $C>0$ such that holds the following interpolation inequality:

$$\|f\|_{B^{\frac{1}{2}}_{2,1}} \leqslant C\|f\|_{B^{\frac{1}{2}}_{2,\infty}} \log\left(e + \frac{\|f\|_{B^{\frac{3}{2}}_{2,\infty}}}{\|f\|_{B^{\frac{1}{2}}_{2,\infty}}}\right).$$

From (i)–(iv) we readily get

$$\|w\partial_x v\|_{B^{\frac{1}{2}}_{2,\infty}} \leqslant C\|w\|_{B^{\frac{1}{2}}_{2,\infty}\cap L^\infty}\|\partial_x v\|_{B^{\frac{1}{2}}_{2,\infty}\cap L^\infty} \leqslant C\|w\|_{B^{\frac{1}{2}}_{2,1}}\|v\|_{B^{\frac{3}{2}}_{2,\infty}\cap \mathrm{Lip}},$$

$$\|P(D)(w(u+v) + \frac{1}{2}\partial_x w\,\partial_x(u+v))\|_{B^{\frac{1}{2}}_{2,\infty}} \leqslant C\|w\|_{B^{\frac{1}{2}}_{2,1}}(\|u\|_{B^{\frac{3}{2}}_{2,\infty}\cap \mathrm{Lip}} + \|v\|_{B^{\frac{3}{2}}_{2,\infty}\cap \mathrm{Lip}}).$$

Plugging the above inequalities in (5) and using the logarithmic interpolation (v), we infer that

$$
\begin{aligned}
e^{-C\int_0^t \|\partial_x u\|_{B^{\frac12}_{2,\infty}\cap L^\infty}\,d\tau}\|w(t)\|_{B^{\frac12}_{2,\infty}} \leqslant \|w_0\|_{B^{\frac12}_{2,\infty}}
&+ C\int_0^t e^{-C\int_0^\tau \|\partial_x u\|_{B^{\frac12}_{2,\infty}\cap L^\infty}\,d\tau'}\|w\|_{B^{\frac12}_{2,\infty}}\\
&\times (\|u\|_{B^{\frac32}_{2,\infty}\cap \mathrm{Lip}} + \|v\|_{B^{\frac32}_{2,\infty}\cap \mathrm{Lip}})\\
&\times \log\left(e + \frac{\|w\|_{B^{\frac32}_{2,\infty}}}{\|w\|_{B^{\frac12}_{2,\infty}}}\right)d\tau. \qquad (6)
\end{aligned}
$$

Denote $W(t) \stackrel{\text{def}}{=} e^{-C\int_0^t \|\partial_x u(\tau)\|_{B^{\frac12}_{2,\infty}\cap L^\infty}\,d\tau}\|w(t)\|_{B^{\frac12}_{2,\infty}}$ and $Z(t) \stackrel{\text{def}}{=} \|u(\tau)\|_{B^{\frac32}_{2,\infty}} + \|v(\tau)\|_{B^{\frac32}_{2,\infty}}$. Since for $x \in (0,1]$ and $\alpha > 0$,

$$\log(e + \alpha/x) \leqslant \log(e+\alpha)(1 - \log x),$$

inequality (6) rewrites

$$W(t) \leqslant W(0) + C\int_0^t Z(\tau)\log(e + Z(\tau))\, W(\tau)(1 - \log W(\tau))\,d\tau$$

provided that $W \leqslant 1$ on $[0,t]$.

Combining hypothesis (2) with a Gronwall type argument (see e.g. Lemma 5.2.1 in [Che]) yields

$$\frac{W(t)}{e} \leqslant \left(\frac{W(0)}{e}\right)^{\exp[-C\int_0^t Z(\tau)\,\log(e+Z(\tau))\,d\tau]}$$

which is the desired result. Of course (4) implies that (2) holds with $T^\star = T$. □

We still have to justify facts (i)–(v).

Property (i) is standard (see e.g. (81), p. 30 in [RS]). That $B^{\frac12}_{2,\infty} \cap L^\infty$ is an algebra may be found in [RS]. Since

$$\mathscr{F}(P(D)u)(\xi) = -\frac{i\xi}{1+\xi^2}\mathscr{F}u(\xi),$$

property (iv) may be easily deduced from Definition 2.

The proof of (iii) lies on (elementary) paradifferential calculus, a tool introduced by Bony [Bo]. What we really need here is the paraproduct.

Introducing the following low-frequency cut-off $S_q u \stackrel{\text{def}}{=} \sum_{p \leqslant q-1} \Delta_p u$, the paraproduct between $f$ and $g$ is defined by

$$T_f g \stackrel{\text{def}}{=} \sum_{q \in \mathbb{N}} S_{q-1} f \Delta_q g.$$

We have the following so-called Bony's decomposition:

$$fg = T_f g + T_g f + R(f,g) \text{ with } R(f,g) \stackrel{\text{def}}{=} \sum_{q \geqslant -1} \Delta_q f (\Delta_{q-1} + \Delta_q + \Delta_{q+1}) g.$$

Therefore, we only have to prove that the paraproducts $T_f g$ and $T_g f$, and the remainder $R(f,g)$ are continuous from $B^{-\frac{1}{2}}_{2,1} \times (B^{\frac{1}{2}}_{2,\infty} \cap L^\infty)$ to $B^{-\frac{1}{2}}_{2,\infty}$. According to estimates (7) and (9) p. 166, and (17) p. 168 in [RS], we have for a constant $C$ independent of $f$ and $g$:

$$\begin{aligned} \|T_f g\|_{B^{-\frac{1}{2}}_{2,\infty}} &\leqslant C\|f\|_{B^{-\frac{1}{2}}_{2,\infty}} \|g\|_{L^\infty} \leqslant C\|f\|_{B^{-\frac{1}{2}}_{2,1}} \|g\|_{L^\infty}, \\ \|T_g f\|_{B^{-\frac{1}{2}}_{2,\infty}} &\leqslant C\|f\|_{B^{-\frac{1}{2}}_{2,\infty}} \|g\|_{L^\infty} \leqslant C\|f\|_{B^{-\frac{1}{2}}_{2,1}} \|g\|_{L^\infty}, \\ \|R(f,g)\|_{B^0_{1,\infty}} &\leqslant C\|f\|_{B^{-\frac{1}{2}}_{2,1}} \|g\|_{B^{\frac{1}{2}}_{2,\infty}}, \end{aligned}$$

which by virtue of the embedding $B^0_{1,\infty} \hookrightarrow B^{-\frac{1}{2}}_{2,\infty}$ (see e.g. [RS(2), p. 31]) clearly entails (iii).

Inequality (v) stems from the following lemma:

**Lemma 1.** *For any* $s \in \mathbb{R}$ *and* $\varepsilon \in (0,1]$, *we have*

$$\|f\|_{B^s_{2,1}} \leqslant \frac{C}{\varepsilon} \|f\|_{B^s_{2,\infty}} \log\left( e + \frac{\|f\|_{B^{s+\varepsilon}_{2,\infty}}}{\|f\|_{B^s_{2,\infty}}} \right).$$

**Proof.** Let $N \in \mathbb{N}$ be a cut-off parameter to be fixed hereafter. We have

$$\begin{aligned} \|f\|_{B^s_{2,1}} &= \sum_{q \leqslant N-1} 2^{qs} \|\Delta_1 f\|_{L^2} + \sum_{q \geqslant N} 2^{-q\varepsilon} (2^{q(s+\varepsilon)} \|\Delta_q f\|_{L^2}), \\ &\leqslant (N+1) \|f\|_{B^s_{2,\infty}} + \frac{2^{-N\varepsilon}}{1 - 2^{-\varepsilon}} \|f\|_{B^{s+\varepsilon}_{2,\infty}}. \end{aligned}$$

Choosing $N = [\frac{1}{\varepsilon} \log_2 (\frac{\|f\|_{B^{s+\varepsilon}_{2,\infty}}}{\|f\|_{B^s_{2,\infty}}})]$ yields the desired inequality. □

## 2. Continuity with respect to initial data in $B^{\frac{3}{2}}_{2,1}$

**Proposition 2.** *For any $u_0 \in B^{\frac{3}{2}}_{2,1}$, there exists a $T>0$ and a neighborhood $V$ of $u_0$ in $B^{\frac{3}{2}}_{2,1}$ such that the map*

$$\Phi : \begin{cases} V \subset B^{\frac{3}{2}}_{2,1} \to C([0,T]; B^{\frac{3}{2}}_{2,1}), \\ v_0 \mapsto v \text{ solution to } (CH) \text{ with initial datum } v_0 \end{cases}$$

*is continuous.*

The main ingredients for proving Proposition 2 are Proposition 1 and a continuity result for *linear* transport equations. More precisely, the following proposition holds true:

**Proposition 3.** *Denote $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. Let $(v^n)_{n\in\overline{\mathbb{N}}}$ be a sequence of functions belonging to $C([0,T]; B^{\frac{1}{2}}_{2,1})$. Assume that $v^n$ is the solution to*

$$\begin{cases} \partial_t v^n + a^n \partial_x v^n = f, \\ v^n_{|t=0} = v_0 \end{cases}$$

*with $v_0 \in B^{\frac{1}{2}}_{2,1}$, $f \in L^1(0,T; B^{\frac{1}{2}}_{2,1})$ and that, for some $\alpha \in L^1(0,T)$,*

$$\sup_{n\in\mathbb{N}} \|\partial_x a^n(t)\|_{B^{\frac{1}{2}}_{2,1}} \leqslant \alpha(t).$$

*If in addition $a^n$ tends to $a^\infty$ in $L^1(0,T; B^{\frac{1}{2}}_{2,1})$ then $v^n$ tends to $v^\infty$ in $C([0,T]; B^{\frac{1}{2}}_{2,1})$.*

**Proof.** Let $w^n \stackrel{\text{def}}{=} v^n - v^\infty$. We have

$$\partial_t w^n + a^n \partial_x w^n = (a^\infty - a^n)\partial_x v^\infty.$$

Let us make the additional assumption that $v_0 \in B^{\frac{3}{2}}_{2,1}$ and $f \in L^1(0,T; B^{\frac{3}{2}}_{2,1})$. In this particular case, Proposition A.1 in [Dan] insures that $v^n \in C([0,T]; B^{\frac{3}{2}}_{2,1})$ with, besides,

$$\|v^n\|_{B^{\frac{3}{2}}_{2,1}}(t) \leqslant e^{C\int_0^t \alpha(\tau)\,d\tau} \|v_0\|_{B^{\frac{3}{2}}_{2,1}} + \int_0^t e^{C\int_\tau^t \alpha(\tau')\,d\tau'} \|f(\tau)\|_{B^{\frac{3}{2}}_{2,1}}\,d\tau. \tag{7}$$

On the other hand, Proposition A.1 in [Dan] also yields

$$\|w^n(t)\|_{B^{\frac{1}{2}}_{2,1}} \leqslant \int_0^t e^{C\int_\tau^t \|\partial_x a^n(\tau')\|_{B^{\frac{1}{2}}_{2,1}}\,d\tau'} \|(a^\infty - a^n)(\tau)\partial_x v^\infty(\tau)\|_{B^{\frac{1}{2}}_{2,1}}\,d\tau$$

therefore, using that $B^{\frac{1}{2}}_{2,1}$ is an algebra and combining with (7),

$$||w^n(t)||_{B^{\frac{1}{2}}_{2,1}} \leqslant Ce^{C\int_0^t \alpha(\tau)\,d\tau}\left(||v_0||_{B^{\frac{3}{2}}_{2,1}} + \int_0^t ||f(\tau)||_{B^{\frac{3}{2}}_{2,1}}\,d\tau\right)\int_0^t ||(a^\infty - a^n)(\tau)||_{B^{\frac{1}{2}}_{2,1}}\,d\tau, \quad (8)$$

which yields the desired result of convergence.

To treat the non-smooth case, one can proceed as follows. For $n \in \overline{\mathbb{N}}$ and $p \in \mathbb{N}$, we write

$$||w^n(t)||_{B^{\frac{1}{2}}_{2,1}} \leqslant ||v^n(t) - v_p^n(t)||_{B^{\frac{1}{2}}_{2,1}} + ||v_p^n(t) - v_p(t)||_{B^{\frac{1}{2}}_{2,1}} + ||v_p(t) - v(t)||_{B^{\frac{1}{2}}_{2,1}}, \quad (9)$$

where $v_p^n$ is the solution to

$$\begin{cases} \partial_t v_p^n + a^n \partial_x v_p^n = S_p f, \\ v^n_{|t=0} = S_p v_0. \end{cases}$$

Above, $S_p$ stands for the low-frequency cut-off defined in Section 1. Of course, $S_p$ is a mollifier, and in particular $S_p v_0 \in B^{\frac{3}{2}}_{2,1}$ and $S_p f \in L^1(0, T; B^{\frac{3}{2}}_{2,1})$. Hence, according to (8), we have

$$\begin{aligned} ||v_p^n(t) - v_p^\infty(t)||_{B^{\frac{1}{2}}_{2,1}} \\ \leqslant Ce^{C\int_0^t \alpha(\tau)d\tau}\left(||S_p v_0||_{B^{\frac{3}{2}}_{2,1}} + \int_0^t ||S_p f(\tau)||_{B^{\frac{3}{2}}_{2,1}}\,d\tau\right) \\ \times \int_0^t ||(a^\infty - a^m)(\tau)||_{B^{\frac{1}{2}}_{2,1}}\,d\tau. \end{aligned} \quad (10)$$

On the other hand, for any $p \in \mathbb{N}$ and $m \in \overline{\mathbb{N}}$, $v^m - v_p^m$ solves

$$\begin{cases} \partial_t u + a^m \partial_x u = f - S_p f, \\ v^n_{|t=0} = v_0 S_p v_0 \end{cases}$$

so that, applying once again Proposition A.1 in [Dan],

$$||v_p^m(t) - v^m(t)||_{B^{\frac{1}{2}}_{2,1}} \leqslant e^{C\int_0^t \alpha(\tau)\,d\tau}\left(||v_0 - S_p v_0||_{B^{\frac{1}{2}}_{2,1}} + \int_0^t ||f(\tau) - S_p f(\tau)||_{B^{\frac{1}{2}}_{2,1}}\,d\tau\right). \quad (11)$$

Plugging (10) and (11) in (9), we end up with

$$\|w^n\|_{L^\infty(0,T;B^{\frac{1}{2}}_{2,1})} \leqslant Ce^{C\int_0^T \alpha(\tau)\,d\tau}\Bigg(\|v_0 - S_p v_0\|_{B^{\frac{1}{2}}_{2,1}} + \int_0^T \|f(\tau) - S_p f(\tau)\|_{B^{\frac{3}{2}}_{2,1}}\,d\tau$$
$$+ \left(\|S_p v_0\|_{B^{\frac{3}{2}}_{2,1}} + \int_0^t \|S_p f(\tau)\|_{B^{\frac{3}{2}}_{2,1}}\,d\tau\right)\int_0^t \|(a^\infty - a^n)(\tau)\|_{B^{\frac{1}{2}}_{2,1}}\,d\tau\Bigg).$$

By virtue of Definition 2 and Lebesgue dominated convergence theorem, the first two terms of the right member may be made arbitrarily small for $p$ large enough. For fixed $p$, we then let $n$ tend to infinity so that the last term tends to zero, and we conclude that $w^n$ tends to 0 in $C([0,T];B^{\frac{1}{2}}_{2,1})$. $\square$

Let us now tackle the **proof of Proposition 2**.

*First step: Continuity in* $C([0,T];B^{\frac{1}{2}}_{2,1})$. Let us fix a $u_0 \in B^{\frac{3}{2}}_{2,1}$ and a $r>0$. We claim that there exists a $T>0$ and a $M>0$ such that for any $u_0' \in B^{\frac{3}{2}}_{2,1}$ with $\|u_0' - u_0\|_{B^{\frac{3}{2}}_{2,1}} \leqslant r$, the solution $u' = \Phi(u_0')$ of (1) associated to $u_0'$ belongs to $C([0,T];B^{\frac{3}{2}}_{2,1})$ and satisfies

$$\|u'\|_{L^\infty(0,T;B^{\frac{3}{2}}_{2,1})} \leqslant M.$$

Indeed, this is just a matter of following the proof of (2.13) in [Dan]. One can for instance take (for some suitable universal constant $C$)

$$T = C/(r + \|u_0\|_{B^{\frac{3}{2}}_{2,1}}) \quad \text{and} \quad M = 2r + 2\|u_0\|_{B^{\frac{3}{2}}_{2,1}}.$$

Now, combining the above uniform bounds with Proposition 1, we infer that

$$\frac{\|\Phi(u_0') - \Phi(u_0)\|_{L^\infty(0,T;B^{\frac{1}{2}}_{2,\infty})}}{e} \leqslant e^{CMT}\left(\frac{\|u_0' - u_0\|_{B^{\frac{1}{2}}_{2,\infty}}}{e}\right)^{\exp[-CMT\log(e+M)]}$$

provided that

$$\|u_0' - u_0\|_{B^{\frac{1}{2}}_{2,\infty}} \leqslant e^{1-\exp[CMT\log(e+M)]}.$$

Interpolating with the uniform bounds in $C([0,T];B^{\frac{3}{2}}_{2,1})$, we gather that for any $s<3/2$ and $\alpha\in[1,+\infty]$, the map $\Phi$ is (Hölder) continuous from $B^{\frac{3}{2}}_{2,1}$ into $C([0,T];B^{s}_{2,\alpha})$. In particular, $\Phi$ is (Hölder) continuous from $B^{\frac{3}{2}}_{2,1}$ into $C([0,T];B^{\frac{1}{2}}_{2,1})$.

*Second step: Continuity in* $C([0,T];B^{\frac{3}{2}}_{2,1})$. Let $u_0^\infty \in B^{\frac{3}{2}}_{2,1}$ and $(u_0^n)_{n\in\mathbb{N}}$ tend to $u_0^\infty$ in $B^{\frac{3}{2}}_{2,1}$. Denote by $u^n$ the solution corresponding to datum $u_0^n$. According to step one,

one can find $T, M > 0$ such that for all $n \in \mathbb{N}$, $u^n$ is defined on $[0,T]$ and

$$\sup_{n\in\overline{\mathbb{N}}} \|u^n\|_{L^\infty_T(B^{\frac{3}{2}}_{2,1})} \leqslant M. \tag{12}$$

Thanks to step one, proving that $u^n$ tends to $u^\infty$ in $C([0,T]; B^{\frac{3}{2}}_{2,1})$ amounts to proving that $v^n \overset{\text{def}}{=} \partial_x u^n$ tends to $v^\infty \overset{\text{def}}{=} \partial_x u^\infty$ in $C([0,T]; B^{\frac{1}{2}}_{2,1})$.

Note that $v^n$ solves the following linear transport equation:

$$\begin{cases} \partial_t v^n + u^n \partial_x v^n = f^n, \\ v^n_{|t=0} = \partial_x u^n_0 \end{cases}$$

with

$$f^n \overset{\text{def}}{=} -(\partial_x u^n)^2 + 2P(D)(u^n \partial_x u^n) + \partial_x P(D)[(\partial_x u)^2]/2.$$

Following Kato [K, Section 10], we decompose $v^n$ into $v^n = z^n + w^n$ with

$$\begin{cases} \partial_t z^n + u^n \partial_x z^n = f^n - f^\infty, \\ v^n_{|t=0} = \partial_x u^n_0 - \partial_x u^\infty_0 \end{cases} \quad \text{and} \quad \begin{cases} \partial_t w^n + u^n \partial_x w^n = f^\infty, \\ w^n_{|t=0} = \partial_x u^\infty_0. \end{cases}$$

Using the properties of Besov spaces exhibited in Section 1, one easily checks that $(f^n)_{n\in\overline{\mathbb{N}}}$ is uniformly bounded in $C([0,T]; B^{\frac{1}{2}}_{2,1})$. Moreover,

$$\begin{aligned} f^n - f^\infty = &\left(\frac{\partial_x P(D)}{2} - 1\right)[\partial_x u^n - \partial_x u^\infty)(\partial_x u^\infty + \partial_x u^n)] \\ &+ 2P(D)[u^n(\partial_x u^n - \partial_x u^\infty) + (u^n - u^\infty)\partial_x u^\infty], \end{aligned}$$

therefore, product laws in Besov spaces combined with Proposition A.1 in [Dan] yield

$$\begin{aligned} \|z^n(t)\|_{B^{\frac{1}{2}}_{2,1}} \leqslant e^{C\int_0^t \|u^n(\tau)\|_{B^{\frac{3}{2}}_{2,1}} d\tau} \Bigg( &\|\partial_x u^n_0 - \partial_x u^\infty_0\|_{B^{\frac{1}{2}}_{2,1}} \\ &+ C\int_0^t \|\partial_x u^\infty(\tau)\|_{B^{\frac{1}{2}}_{2,1}} \|u^n(\tau) - u^\infty(\tau)\|_{B^{\frac{1}{2}}_{2,1}} d\tau \\ &+ C\int_0^t (\|\partial_x u^n(\tau)\|_{B^{\frac{1}{2}}_{2,1}} + \|\partial_x u^\infty(\tau)\|_{B^{\frac{1}{2}}_{2,1}}) \|\partial_x u^n(\tau) - \partial_x u^\infty(\tau)\|_{B^{\frac{1}{2}}_{2,1}} d\tau \Bigg). \end{aligned} \tag{13}$$

On the other hand, since the sequence $(u^n)_{n\in\overline{\mathbb{N}}}$ is uniformly bounded in $C([0,T];B^{\frac{3}{2}}_{2,1})$ and tends to $u^\infty$ in $C([0,T];B^{\frac{1}{2}}_{2,1})$, Proposition 3 tells us that $w^n$ tends to $v^\infty=\partial_x u^\infty$ in $C([0,T];B^{\frac{1}{2}}_{2,1})$.

Let $\varepsilon>0$. Combining the above result of convergence with estimates (12) and (13), one concludes that for large enough $n\in\mathbb{N}$,

$$\begin{aligned}\|\partial_x u^n(t)-\partial_x u^\infty(t)\|_{B^{\frac{1}{2}}_{2,1}}\leqslant \varepsilon+CMe^{CMt}\Bigg(&\|\partial_x u^n_0-\partial_x u^\infty_0\|_{B^{\frac{1}{2}}_{2,1}}\\&+\int_0^t\|\partial_x u^n(\tau)-\partial_x u^\infty(\tau)\|_{B^{\frac{1}{2}}_{2,1}}\,d\tau\\&+\int_0^t\|u^n(\tau)-u^\infty(\tau)\|_{B^{\frac{1}{2}}_{2,1}}\,d\tau\Bigg).\end{aligned}$$

As $u^n$ tends to $u^\infty$ in $C([0,T];B^{\frac{1}{2}}_{2,1})$, the last term is less than $\varepsilon$ for large $n$. Hence, thanks to Gronwall lemma, we get

$$\|\partial_x u^n-\partial_x u^\infty\|_{L^\infty(0,T;B^{\frac{1}{2}}_{2,1})}\leqslant C_{M,T}(\varepsilon+\|\partial_x u^n_0-\partial_x u^\infty_0\|_{B^{\frac{1}{2}}_{2,1}})$$

for some constant $C_{M,T}$ depending only on $M$ and $T$. The proof of Proposition 2 is complete. □

## 3. A counterexample

In this section, we show that local well-posedness in $B^{\frac{3}{2}}_{2,\infty}$ fails. More precisely, we have

**Proposition 4.** *There exists a global solution $u\in L^\infty(\mathbb{R}^+;B^{\frac{3}{2}}_{2,\infty})$ to $(CH)$ such that for any positive $T$ and $\varepsilon$, there exists a solution $v\in L^\infty(0,T;B^{\frac{3}{2}}_{2,\infty})$ with*

$$\|v(0)-u(0)\|_{B^{\frac{3}{2}}_{2,\infty}}\leqslant\varepsilon\quad\textit{and}\quad\|v-u\|_{L^\infty(0,T;B^{\frac{3}{2}}_{2,\infty})}\geqslant 1.$$

**Proof.** Throughout $T$ is a fixed positive real. For $c\in\mathbb{R}$, define $u_c(x,t)\overset{\text{def}}{=}ce^{-|x-ct|}$. Recall that $u_c$ is the well-known solitary wave solution for (1). Its Fourier transform in $x$ is

$$\hat{u}_c(t,\xi)=2c\left(\frac{e^{-ict\xi}}{1+\xi^2}\right).$$

Let $c_2$ and $c_1$ be two reals to be fixed hereafter. First compute $\|u_{c_2}(0)-u_{c_1}(0)\|_{B^{\frac{3}{2}}_{2,\infty}}$ according to Definition 2. Denoting $\delta c \stackrel{\text{def}}{=} c_2-c_1$, we have

$$
\begin{aligned}
\|u_{c_2}(0)-u_{c_1}(0)\|^2_{B^{\frac{3}{2}}_{2,\infty}} &= 8\delta c^2 \max\left(\int_0^1 \frac{d\xi}{\sqrt{1+\xi^2}}, \sup_{q\in\mathbb{N}} \int_{2^q}^{2^{q+1}} \frac{d\xi}{\sqrt{1+\xi^2}}\right),\\
&= 8\delta c^2 \max\left(\log(1+\sqrt{2}),\ \sup_{q\in\mathbb{N}} \log\left(\frac{2^{q+1}+\sqrt{2^{2q+2}+1}}{2^q+\sqrt{2^{2q}+1}}\right)\right),\\
&= 8\delta c^2 \log(1+\sqrt{2}).
\end{aligned}
$$

Note that considering the particular case $c_2=c$ and $c_1=0$ yields $u_c(0)\in B^{\frac{3}{2}}_{2,\infty}$. A similar computation would also show that $\|u_c(t)\|_{B^{\frac{3}{2}}_{2,\infty}}$ does not depend on $t$. Since the part of the norm corresponding to each dyadic block does not tend to zero when $q$ tends to infinity, one also gathers that $u_c(t)$ does not belong to any other space $B^{\frac{3}{2}}_{2,r}$ (unless $c=0$).

Let us now tackle the computation of $\|u_{c_2}(t)-u_{c_1}(t)\|_{B^{\frac{3}{2}}_{2,\infty}}$. As

$$
|c_2 e^{-ic_2t\xi}-c_1e^{-ic_1t\xi}|^2 = \delta c^2 + 2c_1c_2(1-\cos(\delta c t\xi)),
$$

we infer that

$$
\begin{aligned}
\frac{\|u_{c_2}(t)-u_{c_1}(t)\|^2_{B^{\frac{3}{2}}_{2,\infty}}}{8} = \max\Bigg(&\int_0^1 \frac{\delta c^2+2c_1c_2(1-\cos\delta c t\xi)}{\sqrt{1+\xi^2}}\,d\xi,\\
&\sup_{q\in\mathbb{N}} \int_{2^q}^{2^{q+1}} \frac{\delta c^2+2c_1c_2(1-\cos\delta c t\xi)}{\sqrt{1+\xi^2}}\,d\xi\Bigg).
\end{aligned}
$$

For $q\in\mathbb{N}$, choose $c_1$ and $c_2$ so that $T\delta c = 2^{-q}\pi$. Clearly, the right-hand side above is greater than the term corresponding to frequencies of size $2^q$. Therefore,

$$
\begin{aligned}
\|u_{c_2}(T)-u_{c_1}(T)\|^2_{B^{\frac{3}{2}}_{2,\infty}} &\geqslant 16c_1c_2 \int_{2^q}^{2^{q+1}} \frac{1-\cos(2^{-q}\pi\xi)}{\sqrt{1+\xi^2}}\,d\xi,\\
&\geqslant \frac{4}{\sqrt{2}}c_1c_2. \qquad (14)
\end{aligned}
$$

Choose $c_1 = 1$ and $c_2 = 1 + 2^{-q}T^{-1}\pi$. From the above computations, we have

$$\|u_{c_2}(0) - u_{c_1}(0)\|_{B^{\frac{3}{2}}_{2,\infty}} = \frac{2\pi}{2^q T}\sqrt{2\log(1+\sqrt{2})}$$

which may be arbitrary small whereas, according to (14), $\|u_{c_2}(T) - u_{c_1}(T)\|_{B^{\frac{3}{2}}_{2,\infty}} \geqslant 2$. $\square$

**Remark 2.** Note that the use of solutions $u_c$ provides us with counterexamples to continuity in $B^{\frac{3}{2}}_{2,\infty}$ whereas it only gives counterexamples to uniform continuity in $H^s$ with $s < 3/2$ (see [HM]).

**Remark 3.** The result of Proposition 2 does not contradict the properties of orbital stability stated in [CS] for the $H^1$ norm. Indeed, the norm in $B^{\frac{3}{2}}_{2,\infty}$ is far stronger.

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